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The Exponential Function in Discrete Fractional Calculus under the Delta Operator

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THE EXPONENTIAL FUNCTION IN DISCRETE FRACTIONAL CALCULUS UNDER THE DELTA OPERATOR

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ABSTRACT

Previously the exponential problem in discrete fractional calculus under the nabla operator was solved with the discrete Mittag–Leffler function. We now show the solution to the exponential problem in discrete fractional calculus under the delta operator, providing multiple derivations of the solution with recursion and Laplace transforms. We also share some computational and numerical results of experiments with different orders of difference to display the nature of the solution.

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1. INTRODUCTION

This work was inspired and motivated by seeing if there is an application of discrete fractional calculus in quantitative finance. Some research has already been conducted to investigate the viability of using discrete fractional calculus in modeling applications. See [1, 2] for more details. As best as the author can tell, no applications have been considered in quantitative finance though. Under the traditional conventions to model security value growth $V(t)$, discrete models or continuous models are used. We refer the reader to [3] for more details.

Most are more likely to be familiar with the continuous case. Under continuous models, security values are generally considered to follow exponential growth. Thus, one could use the model and initial value

$$\frac{dV}{dt} = rV(t), \quad V(0) = V_0,$$

where r is the risk free rate and V_0 is the initial value of the security. Then as most know, using separation of variables, we find the solution

$$V(t) = V_0 e^{rt},$$

where t is the variable for continuous time.

Of course, using a Taylor series, we could also represent the security's value as

$$V(t) = V_0 \sum_{k=0}^{\infty} \frac{(rt)^k}{k!}.$$

Beyond exponential functions, Taylor series are useful in finding solutions of differential equations. The same is also true for difference equations, which are used to model securities' values in discrete time.

In discrete time, one would consider the model

$$\Delta V(\tau) = rV(\tau), \quad V(0) = V_0,$$

where, again, r is the risk free rate, V_0 is the initial value of the security, and $\tau \in \mathbb{N}$ is the number of time periods. Hence, just as in the continuous case, one is assuming the growth of the security's value is proportional to its current value.

Solving this difference equation gives the solution

$$V(\tau) = V_0(1+r)^{\tau}, \tag{1.1}$$

since

$$\begin{aligned}\Delta V_0(1+r)^{\tau} &= V_0(1+r)^{\tau+1} - V_0(1+r)^{t} \\ &= V_0(1+r)^{\tau}(1+r-1) \\ &= rV_0(1+r)^{\tau} \\ &= rV(\tau).\end{aligned}$$

As for the discrete Taylor series of this solution to (1.1), we will show this series later. Know that this Taylor series will be essential to the results of this thesis.

The focus now becomes modeling the dynamics of V using discrete fractional calculus. One could then use the nabla operator and consider solving the fractional difference equation

$$\nabla_1^{\nu} V = rV. \tag{1.2}$$

Know that (1.2) has already been solved for $|r| < 1$ and its solution is the discrete Mittag–Leffler function. For greater detail see [4, 5, 6, 7, 8]. Another choice to consider would be using the delta operator instead of the nabla operator. This would be more consistent with the traditional discrete models used in quantitative finance. Then we should solve the ν-th order Riemann–Liouville delta fractional difference equation

$$\Delta_0^{\nu} V = rV.$$

Solving this fractional difference equation is the focus of this thesis.

2. PRELIMINARIES OF ORDINARY DELTA CALCULUS

This chapter will serve as an overview and introduction to the necessary concepts of discrete fractional delta calculus from which our results are founded. Before we share any properties of discrete fractional calculus, we first share some of the elementary results of classical difference calculus. Then we can offer some of the properties and results of discrete fractional calculus.

2.1. BASICS OF DELTA CALCULUS

As most are familiar with the natural numbers $\mathbb{N} := \{1, 2, 3, \ldots\}$, we define a new set, which will serve as the domain of our functions. Let

$$\mathbb{N}_a := \{a, a+1, a+2, \ldots\},$$

where $a \in \mathbb{R}$. If we wanted to bound this set so that it is a finite set, we would have $\mathbb{N}_a^b := \{a, a+1, a+2, \ldots, b\}$, where $a, b \in \mathbb{R}$ and $b - a$ is a positive integer.

Definition 2.1.1 (See [4, Definition 1.1])**.** Assume $f : \mathbb{N}_a^b \to \mathbb{R}$. If $b > a$, then we define the **forward difference operator** Δ by

$$\Delta f(t) := f(t+1) - f(t)$$

for $t \in \mathbb{N}_a^{b-1}$.

Realize that though Δf is a function and it would be proper to write $(\Delta f)(t)$, we will use the standard convention and write $\Delta f(t)$. Understand we have recursion when taking higher order integer differences, that is the operator Δ^n, $n = 1, 2, 3, \ldots$ is defined by

$$\Delta^n f(t) = \Delta\left(\Delta^{n-1} f(t)\right),$$

for $t \in \mathbb{N}_a^{b-n}$ assuming the integer $b - a \geq n$. Lastly, by general convention, we will assume Δ^0 to be the identity operator. Thus, $\Delta^0 f(t) = f(t)$. Of course the delta difference is needed in delta calculus, but so is the delta definite integral.

Definition 2.1.2 (See [4, Definition 1.49])**.** If $f : \mathbb{N}_a \to \mathbb{R}$ and $c \leq d$ are elements of $\mathbb{N}_a$, then the **delta integral** is given as

$$\int_c^d f(t)\Delta t := \sum_{t=c}^{d-1} f(t),$$

where t takes on the values $c, c+1, c+2, \ldots, d-1$ if $d > c$.

Hence the value of the integral $\int_c^d f(t)\Delta t$ is not dependent on the value $f(d)$, as it is not part of the sum. We will use the convention on sums that $\sum_{t=c}^{c-k} f(t) := 0$ whenever k is a natural number. We assume this convention to be true even if $f(t)$ is not defined for one or more (maybe even all) values $t \in \mathbb{N}_{c-k}$.

Next we define the forward jump operator, which will be of great use in later definitions.

Definition 2.1.3 (See [4, Definition 1.2])**.** We define the **forward jump operator** σ on $\mathbb{N}_a^{b-1}$ by

$$\sigma(t) = t + 1.$$

We now look to the uses of the gamma function in delta calculus.

Definition 2.1.4 (See [4, Definition 1.4])**.** For a positive integer n, we define the **falling function**, $t^{\underline{n}}$, read t to the n falling, by

$$t^{\underline{n}} := t(t-1)(t-2)\cdots(t-n+1),$$

where we will use the standard convention that $t^{\underline{0}} := 1$.

Definition 2.1.5 (See [4, Definition 1.6])**.** The **gamma function** is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

for those complex numbers z for which the real part of z is positive.

One of the prominent characteristics of the gamma function is $\Gamma(z+1) = z\Gamma(z)$, for when the real part of z in positive. Of course this is possible to be shown when we use integration by parts so that

$$\begin{aligned} \Gamma(z+1) &= \int_0^\infty e^{-t} t^{z} dt \\ &= \left[-e^{-t} t^{z}\right]_{t\to 0+}^{t\to\infty} - \int_0^\infty \left(-e^{-t}\right) z t^{z-1} dt \\ &= z\Gamma(z) \end{aligned}$$

when the real part of z is positive. Thus, this wonderful property makes us capable of extending the domain of the $\Gamma(z)$ to any complex numbers z such that $z \in \mathbb{C} \setminus -\mathbb{N}_0$. Another prominent feature of the gamma function is that it is regarded as a generalization of the factorial function, namely it satisfies

$$\Gamma(n+1) = n! \quad \text{for } n \in \mathbb{N}_0.$$

Consequently, if we let $n \in \mathbb{N}_1$, then

$$\begin{aligned} t^{\underline{n}} &= t(t-1)\cdots(t-n+1) \\ &= \frac{t(t-1)\cdots(t-n+1)\Gamma(t-n+1)}{\Gamma(t-n+1)} \\ &= \frac{\Gamma(t+1)}{\Gamma(t-n+1)}. \end{aligned}$$

If $n, k \in \mathbb{N}$ with $0 \leq k \leq n$, then we can express the traditional binomial coefficients as

$$\binom{n}{k} := \frac{n!}{(n-k)!k!} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n^{\underline{k}}}{\Gamma(k+1)}.$$

Of course these traditional falling functions and binomial coefficients are useful in ordinary delta calculus, but we will need to generalize these for the fractional work.

Another important item that we need are (delta) Taylor monomials.

Definition 2.1.6 (See [4, Definition 1.60])**.** We define the discrete **Taylor monomials** (based at $s \in \mathbb{N}_a$), $h_n(t, s), n \in \mathbb{N}_0$ by

$$h_n(t,s) = \frac{(t-s)^{\underline{n}}}{n!}, \quad t \in \mathbb{N}_a.$$

Just as we can take differences of functions, we can also take sums of functions. Below, we define the n-th integer sum.

Definition 2.1.7 (See [4, Theorem 2.23])**.** Assume $f : \mathbb{N}_a \to \mathbb{R}$. Then the n-th **integer sum** of $f(t)$ is given as

$$\Delta_a^{-n} f(t) = \int_a^{t-n+1} h_{n-1}(t, \sigma(s)) f(s) \Delta s.$$

We can now give the discrete version of Taylor's theorem.

Theorem 2.1.8 (See [4, Theorem 1.62])**.** *Assume $f : \mathbb{N}_a \to \mathbb{R}$ and $n \in \mathbb{N}_0$. Then*

$$f(t) = p_n(t) + R_n(t), \quad t \in \mathbb{N}_a,$$

where the n-th degree Taylor polynomial, $p_n(t)$, is given by

$$p_n(t) := \sum_{k=0}^{n} \Delta^k f(a) \frac{(t-a)^{\underline{k}}}{k!} = \sum_{k=0}^{n} \Delta^k f(a) h_k(t, a)$$

and the Taylor remainder, $R_n(t)$, is given by

$$R_n(t) = \int_a^t \frac{(t-\sigma(s))^{\underline{n}}}{n!}\Delta^{n+1} f(s)\Delta s = \int_a^t h_n(t,\sigma(s))\Delta^{n+1} f(s)\Delta s,$$

for $t \in \mathbb{N}_a$.

Of course we can let $n \to \infty$ in Taylor's theorem. Then we would have a Taylor series.

Definition 2.1.9 (See [4, Definition 1.63]). If $f : \mathbb{N}_a \to \mathbb{R}$, then we call

$$\sum_{k=0}^{\infty} \Delta^k f(a)\frac{(t-a)^{\underline{k}}}{k!} = \sum_{k=0}^{\infty} \Delta^k f(a) h_k(t,a)$$

the (formal) **Taylor series** of f based $t = a$.

This series will be useful as we will be able to compare our results for fractional work with the known results of ordinary delta calculus.

2.2. THE DELTA EXPONENTIAL FUNCTION

As we are working in exponential problems, we want to consider a set of functions that is fitting. We should then consider the regressive functions.

Definition 2.2.1 (See [4, page 6]). The set of **regressive functions** is defined by

$$\mathcal{R} = \{p : \mathbb{N}_a \to \mathbb{R} \text{ such that } p(t) \neq -1 \text{ for } t \in \mathbb{N}_a\}.$$

We now suppose that p is a regressive function. Then we have the additive inverse of p defined next.

Definition 2.2.2 (See [4, Theorem 1.16]). For $p \in \mathcal{R}$, we use the following notation for the **additive inverse** of p:

$$\ominus p := \frac{-p}{1+p}.$$

We now introduce the delta exponential function $e_p(t,s)$, where $s \in \mathbb{N}_a$ and $p \in \mathcal{R}$, which is defined to be the unique solution to the initial value problem,

$$\Delta x(t) = p(t)x(t),$$
$$x(s) = 1.$$

Theorem 2.2.3 (See [4, Theorem 1.11]). *Assume $p \in \mathcal{R}$ and $s \in \mathbb{N}_a$. Then the* ***delta exponential function*** *is*

$$e_p(t,s) = \begin{cases} \prod_{\tau=s}^{t-1}[1+p(\tau)], & t \in \mathbb{N}_s \\ \prod_{\tau=t}^{s-1}[1+p(\tau)]^{-1}, & t \in \mathbb{N}_a^{s-1}. \end{cases}$$

Above, we use the standard convention in our product that for any function h

$$\prod_{\tau=s}^{s-1} h(\tau) := 1.$$

Theorem 2.2.4 (See [4, Example 1.12]). *For $p \in \mathcal{R}$ and $p(t)$ is a constant,*

$$e_p(t,s) := (1+p)^{t-s}, \quad t \in \mathbb{N}_a.$$

It is reasonable to not consider the initial condition in the prior initial value problem, but only consider the difference equation. Then we have the following theorem.

Theorem 2.2.5 (See [4, Theorem 1.14]). *Let c is an arbitrary constant. If $p \in \mathcal{R}$, then a general solution of*

$$\Delta y(t) = p(t)y(t), \quad t \in \mathbb{N}_a$$

is given by

$$y(t) = ce_p(t,a), \quad t \in \mathbb{N}_a.$$

As our work is to solve a similar equation in the fractional case, we will use the discrete exponential function's Taylor series to compare our work. This Taylor series is given now.

Theorem 2.2.6 (See [4, Theorem 1.64])**.** *Assume p is a constant. If $p \neq -1$, then*

$$e_p(t,a) = \sum_{n=0}^{\infty} p^n h_n(t,a) \tag{2.1}$$

for all $t \in \mathbb{N}_a$.

2.3. DELTA LAPLACE TRANSFORMS

As it will be shown, delta Laplace transforms will be of great use when solving our problem. Here, the basics of delta Laplace transforms are given. Know that the definition given next are in accordance with the definition of Laplace transforms in [9].

Definition 2.3.1 (See [4, Definition 2.1])**.** Assume $f : \mathbb{N}_a \to \mathbb{R}$. Then we define the (delta) **Laplace transform** of f based at a by

$$\mathcal{L}_a\{f\}(s) = \int_a^{\infty} e_{\ominus s}(\sigma(t), a) f(t) \Delta t$$

for all complex numbers $s \neq -1$ such that this improper integral converges.

To make more sense and use of this definition, we use the following theorem.

Theorem 2.3.2 (See [4, Theorem 2.2])**.** *Assume $f : \mathbb{N}_a \to \mathbb{R}$. Then*

$$\begin{aligned} \mathcal{L}_a\{f\}(s) = F_a(s) &:= \int_0^{\infty} \frac{f(a+k)}{(s+1)^{k+1}} \Delta k \\ &= \sum_{k=0}^{\infty} \frac{f(a+k)}{(s+1)^{k+1}} \end{aligned} \tag{2.2}$$

for all complex numbers $s \neq -1$ such that this improper integral (infinite series) converges.

Proof. Let $f : \mathbb{N}_a \to \mathbb{R}$. Then

$$\begin{aligned}
\mathcal{L}_a\{f\}(s) &= \int_a^\infty e_{\ominus s}(\sigma(t), a) f(t) \Delta t \\
&= \sum_{t=a}^{\infty} e_{\ominus s}(\sigma(t), a) f(t) \\
&= \sum_{t=a}^{\infty} [1 + \ominus s]^{\sigma(t)-a} f(t) \\
&= \sum_{t=a}^{\infty} \frac{f(t)}{(1+s)^{t-a+1}} \\
&= \sum_{k=0}^{\infty} \frac{f(a+k)}{(1+s)^{k+1}} \\
&= \int_0^\infty \frac{f(a+k)}{(s+1)^{k+1}} \Delta k.
\end{aligned}$$

Hence the result follows. □

One of the main questions with Laplace transforms is knowing that such transforms exist. Knowing if a function is of exponential order is helpful in determining existence.

Definition 2.3.3 (See [4, Definition 2.3])**.** We say that a function $f : \mathbb{N}_a \to \mathbb{R}$ is of **exponential order** $r > 0$ (at ∞) if there exists a constant $A > 0$ such that $|f(t)| \leq Ar^t$ for sufficiently large $t \in \mathbb{N}_a$.

We can now know when a function's Laplace transform will exist.

Theorem 2.3.4 (See [4, Theorem 2.4])**.** *Suppose $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r > 0$. Then $\mathcal{L}_a\{f\}(s)$ converges absolutely for $|s+1| > r$.*

Proof. Assume $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r > 0$. Then there is a constant $A > 0$ and an $m \in \mathbb{N}_0$ such that for each $t \in \mathbb{N}_{a+m}$, $|f(t)| \leq Ar^t$. Hence for $|s+1| > r$,

$$\begin{aligned}
\sum_{k=m}^{\infty} \left| \frac{f(k+a)}{(s+1)^{k+1}} \right| &= \sum_{k=m}^{\infty} \frac{|f(k+a)|}{|s+1|^{k+1}} \\
&\leq \sum_{k=m}^{\infty} \frac{Ar^{k+a}}{|s+1|^{k+1}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{Ar^a}{|s+1|} \sum_{k=m}^{\infty} \left(\frac{r}{|s+1|} \right)^k \\
&= \frac{Ar^a}{|s+1|} \frac{\left(\frac{r}{|s+1|} \right)^m}{1 - \left(\frac{r}{|s+1|} \right)} \\
&= \frac{A}{|s+1|^m} \frac{r^{a+m}}{|s+1| - r} \\
&< \infty.
\end{aligned}
$$

Hence, the Laplace transform of f converges absolutely for $|s+1| > r$. □

A fair question to ask is if a function's Laplace transform exists, then is it unique? The following theorem answers the question.

Theorem 2.3.5 (See [4, Theorem 2.7]). *Assume $f, g : \mathbb{N}_a \to \mathbb{R}$ and there is an $r > 0$ such that*

$$\mathcal{L}_a\{f\}(s) = \mathcal{L}_a\{g\}(s)$$

for $|s+1| > r$. Then, for all $t \in \mathbb{N}_a$, $f(t) = g(t)$.

Proof. By our hypothesis, we have that

$$\mathcal{L}_a\{f\}(s) = \mathcal{L}_a\{g\}(s)$$

for $|s+1| > r$. This implies that

$$\sum_{k=0}^{\infty} \frac{f(a+k)}{(s+1)^{k+1}} = \sum_{k=0}^{\infty} \frac{g(a+k)}{(s+1)^{k+1}}$$

for $|s+1| > r$. It follows from this that

$$f(a+k) = g(a+k), \quad k \in \mathbb{N}_0,$$

and this completes the proof. □

One of the wonderful properties of Laplace transforms is linearity. We have the following theorem.

Theorem 2.3.6 (See [4, Theorem 2.6])**.** *Suppose* $f, g : \mathbb{N}_a \to \mathbb{R}$ *and the Laplace transforms of* f *and* g *converge for* $|s+1| > r$*, where* $r > 0$*, and let* $c_1, c_2 \in \mathbb{C}$*. Then the Laplace transform of* $c_1 f + c_2 g$ *converges for* $|s+1| > r$ *and*

$$\mathcal{L}_a \{c_1 f + c_2 g\}(s) = c_1 \mathcal{L}_a\{f\}(s) + c_2 \mathcal{L}_a\{g\}(s),$$

for $|s+1| > r$*.*

Proof. Since $f, g : \mathbb{N}_a \to \mathbb{R}$ and the Laplace transforms of f and g converge for $|s+1| > r$, where $r > 0$, we have that for $|s+1| > r$,

$$\begin{aligned} c_1 \mathcal{L}_a\{f\}(s) &+ c_2 \mathcal{L}_a\{g\}(s) \\ &= c_1 \sum_{k=0}^{\infty} \frac{f(a+k)}{(s+1)^{k+1}} + c_2 \sum_{k=0}^{\infty} \frac{g(a+k)}{(s+1)^{k+1}} \\ &= \sum_{k=0}^{\infty} \frac{(c_1 f + c_2 g)(a+k)}{(s+1)^{k+1}} \\ &= \mathcal{L}_a \{c_1 f + c_2 g\}(s). \end{aligned}$$

This completes the proof. □

One of the Laplace transforms we will need is the Laplace transform of Taylor monomials.

Theorem 2.3.7 (See [4, Theorem 2.22])**.** *Let* $n \geq 0$*. Then*

$$\mathcal{L}_a \{h_n(t,a)\}(s) = \frac{1}{s^{n+1}}$$

for $|s+1| > 1$*.*

Another Laplace transform we will want to use is the Laplace transform of the N-th order difference of a function.

Theorem 2.3.8 (See [4, Theorem 2.12])**.** *Assume that f is of exponential order $r > 0$. Then for any positive integer N,*

$$\mathcal{L}_a\left\{\Delta^N f\right\}(s) = s^N F_a(s) - \sum_{j=0}^{N-1} s^j \Delta^{N-1-j} f(a)$$

for $|s+1| > r$.

We will also want to be able to take the convolution product of two functions.

Definition 2.3.9 (See [4, Definition 2.59])**.** Let $f, g : \mathbb{N}_a \to \mathbb{R}$ be given. Define the **convolution product** of f and g to be

$$(f * g)(t) := \sum_{r=a}^{t-1} f(r)g(t - \sigma(r) + a) \quad \text{for } t \in \mathbb{N}_a. \tag{2.3}$$

Realize $(f * g)(a) = 0$ by our convention on sums. Another major tool we use is the convolution theorem stated now.

Theorem 2.3.10 (See [4, Theorem 2.61])**.** *Let $f, g : \mathbb{N}_a \to \mathbb{R}$ be of exponential order $r_0 > 0$. Then*

$$\mathcal{L}_a\{f * g\}(s) = F_a(s)G_a(s) \quad \textit{for } |s+1| > r_0.$$

Proof. We have

$$\begin{aligned}
\mathcal{L}_a\{f * g\}(s) &= \sum_{k=0}^{\infty} \frac{(f * g)(a+k)}{(s+1)^{k+1}} = \sum_{k=1}^{\infty} \frac{(f * g)(a+k)}{(s+1)^{k+1}} \\
&= \sum_{k=1}^{\infty} \frac{1}{(s+1)^{k+1}} \sum_{r=a}^{a+k-1} f(r)g(a+k-\sigma(r)+a) \\
&= \sum_{k=1}^{\infty} \sum_{r=0}^{k-1} \frac{f(a+r)g(a+k-r-1)}{(s+1)^{k+1}}
\end{aligned}$$

$$= \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{f(a+r)g(a+k-r-1)}{(s+1)^{k+1}}.$$

Making the change of variables $\tau = k - r - 1$ gives us that

$$\begin{aligned}
\mathcal{L}_a\{f * g\}(s) &= \sum_{\tau=0}^{\infty} \sum_{r=0}^{\infty} \frac{f(a+r)g(a+\tau)}{(s+1)^{\tau+r+2}} \\
&= \sum_{r=0}^{\infty} \frac{f(a+r)}{(s+1)^{r+1}} \sum_{\tau=0}^{\infty} \frac{g(a+\tau)}{(s+1)^{\tau+1}} \\
&= F_a(s)G_a(s), \text{ for } |s+1| > r_0.
\end{aligned}$$

Hence the result follows. □

3. PRELIMINARIES OF FRACTIONAL DELTA CALCULUS

3.1. FRACTIONAL SUMS AND DIFFERENCES

Thus far we have covered the necessary topics of ordinary delta calculus, but now we will generalize these definitions and theorems for the fractional case. The definitions and theorems of the ordinary case should serve as motivation for the definitions and theorems in the fractional case.

We first generalize the falling function.

Definition 3.1.1 (See [4, Definition 1.7])**.** The **(generalized) falling function** is defined by

$$t^{\underline{r}} := \frac{\Gamma(t+1)}{\Gamma(t-r+1)}$$

for those values of t and r such that the right-hand side of this equation makes sense. In order to use this function when $t-r+1$ is a nonpositive integer and $t+1$ is not a nonpositive integer, we will use the convention that $t^{\underline{r}} = 0$.

This convention comes from the limit

$$\lim_{s \to t} s^{\underline{r}} = \lim_{s \to t} \frac{\Gamma(s+1)}{\Gamma(s-r+1)} = 0,$$

where $t - r + 1$ is a nonpositive integer and $t + 1$ is not a nonpositive integer.

We now generalize binomial coefficients.

Definition 3.1.2 (See [4, Definition 1.9])**.** The **(generalized) binomial coefficient** $\binom{t}{r}$ is defined by

$$\binom{t}{r} := \frac{t^{\underline{r}}}{\Gamma(r+1)}$$

for those values of t and r so that the right-hand side is well defined.

To be consistent with the prior stated conventions of the generalized falling function, we let $\binom{n}{k} = 0$ if the denominator is undefined and the numerator is defined.

We should also generalize Taylor monomials.

Definition 3.1.3 (See [4, Definition 2.24])**.** The ν-th **fractional Taylor monomial** based at s is defined by

$$h_\nu(t,s) = \frac{(t-s)^{\underline{\nu}}}{\Gamma(\nu+1)}, \tag{3.1}$$

whenever the right-hand side is well defined.

Just as we could define the ordinary sum of f with ordinary Taylor monomials, we can now define the ν-th fractional sum.

Definition 3.1.4 (See [4, Definition 2.25])**.** Assume $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$. Then the ν**-th fractional sum** of f (based at a) is defined by

$$\begin{aligned} \Delta_a^{-\nu} f(t) &:= \int_a^{t-\nu+1} h_{\nu-1}(t,\sigma(\tau)) f(\tau) \Delta\tau \\ &= \sum_{\tau=a}^{t-\nu} h_{\nu-1}(t,\sigma(\tau)) f(\tau) \end{aligned} \tag{3.2}$$

for $t \in \mathbb{N}_{a+\nu}$.

Recall the convention on sums that $\sum_{t=c}^{c-k} f(t) := 0$ whenever k is a natural number. Then let $N \in \mathbb{N}_1$ so that $N-1 < \nu \leq N$. Then the domain of $\Delta_a^{-\nu} f$ can be extended to $\mathbb{N}_{a+\nu-N}$, since

$$\Delta_a^{-\nu} f(t) = 0, \quad t \in \mathbb{N}_{a+\nu-N}^{a+\nu-1}.$$

At last, we can now give the definition of the Riemann–Liouville of the ν-th delta fractional difference.

Definition 3.1.5 (See [4, Definition 2.29]). Assume $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$. Choose a positive integer N such that $N - 1 < \nu \leq N$. Then we define the **ν-th fractional difference** by

$$\Delta_a^{\nu} f(t) := \Delta^N \Delta_a^{-(N-\nu)} f(t), \quad t \in \mathbb{N}_{a+N-\nu}.$$

Using Definition 3.1.5 of a fractional difference gives us consistency with our ordinary integer order differences. Thus, for any $\nu = N \in \mathbb{N}_0$

$$\Delta_a^{\nu} f(t) := \Delta^N \Delta_a^{-(N-\nu)} f(t) = \Delta^N \Delta_a^{-0} f(t) = \Delta^N f(t)$$

for $t \in \mathbb{N}_a$.

Though we have defined these fractional sums and differences, we need a practical way to compute them. The following theorem provides the means for such easy computation.

Theorem 3.1.6 (See [4, Theorem 2.45]). *Assume $N - 1 < \nu \leq N$ and $f : \mathbb{N}_a \to \mathbb{R}$. Then*

$$\Delta_a^{\nu} f(t) = \sum_{k=0}^{t+\nu-a} (-1)^k \binom{\nu}{k} f(t + \nu - k), \quad t \in \mathbb{N}_{a+N-\nu} \tag{3.3}$$

and

$$\begin{aligned} \Delta_a^{-\nu} f(t) &= \sum_{k=0}^{t-a-\nu} (-1)^k \binom{-\nu}{k} f(t - \nu - k) \\ &= \sum_{k=0}^{t-a-\nu} \binom{\nu + k - 1}{k} f(t - \nu - k), \quad t \in \mathbb{N}_{a+\nu}. \end{aligned} \tag{3.4}$$

Proof. Assume $f : \mathbb{N}_a \to \mathbb{R}$ and $0 \leq \nu \leq N$. Fix $t \in \mathbb{N}_{a+N-\nu}$. Then $t = a + N - \nu + m$, for some $m \in \mathbb{N}_0$. Then

$$\begin{aligned} \Delta_a^{\nu} f(t) &= \int_a^{t+\nu+1} h_{-\nu-1}(t, \sigma(\tau)) f(\tau) \Delta\tau \\ &= \sum_{\tau=a}^{t+\nu} \frac{(t - \sigma(\tau))^{\underline{-\nu-1}}}{\Gamma(-\nu)} f(\tau) \\ &= \sum_{\tau=a}^{t+\nu} \frac{\Gamma(t - \tau)}{\Gamma(t - \tau + \nu + 1)\Gamma(-\nu)} f(\tau) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\tau=a}^{a+N+m} \frac{\Gamma(a+N-\nu+m-\tau)}{\Gamma(a+N+m-\tau+1)\Gamma(-\nu)} f(\tau) \\
&= \sum_{\tau=0}^{N+m} \frac{\Gamma(N+m-\tau-\nu)}{\Gamma(N+m-\tau+1)\Gamma(-\nu)} f(a+\tau) \\
&= f(a+N+m) + \sum_{\tau=0}^{N+m-1} \frac{(N+m-1-\tau-\nu)\cdots(-\nu)}{\Gamma(N+m-\tau+1)} f(a+\tau) \\
&= f(a+N+m) + \sum_{\tau=0}^{N+m-1} (-1)^{N+m-\tau} \frac{(\nu)\cdots(\nu-(N+m-\tau)+1)}{\Gamma(N+m-\tau+1)} f(a+\tau) \\
&= \sum_{\tau=0}^{N+m} (-1)^{N+m-\tau} \binom{\nu}{N+m-\tau} f(a+\tau) \\
&= \sum_{k=0}^{N+m} (-1)^{k} \binom{\nu}{k} f(a+N+m-k) \\
&= \sum_{k=0}^{N+m} (-1)^{k} \binom{\nu}{k} f((a+N-\nu+m)+\nu-k) \\
&= \sum_{k=0}^{t-a+\nu} (-1)^{k} \binom{\nu}{k} f(t+\nu-k).
\end{aligned}
$$

Hence the result for the fractional difference. To prove the result for the fractional difference we just replace ν by $-\nu$. Lastly, we shift the domain of the fractional difference to the domain of the fractional difference. □

Our fractional sums and differences have power rules that can be stated as follows.

Theorem 3.1.7 (See [4, Theorem 2.38])**.** *If* $\mu \geq 0$ *and* $\nu \geq 0$*, then*

$$
\Delta_{a+\mu}^{-\nu}(t-a)^{\underline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t-a)^{\underline{\mu+\nu}}
$$

for $t \in \mathbb{N}_{a+\mu+\nu}$.

Theorem 3.1.8 (See [4, Theorem 2.40])**.** *Let* $t \in \mathbb{N}_{a+\mu+N-\nu}$. *If* $\mu > 0$, $\nu \geq 0$, *and* $N-1 < \nu < N$, *then*

$$
\Delta_{a+\mu}^{\nu}(t-a)^{\underline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)}(t-a)^{\underline{\mu-\nu}}.
$$

In our work, we rely on taking fractional difference and sums of Taylor monomials. The next theorem gives us these computations.

Theorem 3.1.9 (See [4, Theorem 2.42]). *Assume $\mu > 0, \nu > 0$, then the following hold:*

$$\Delta_{a+\mu}^{-\nu} h_\mu(t,a) = h_{\mu+\nu}(t,a), \quad t \in \mathbb{N}_{a+\mu+\nu}, \tag{3.5}$$

$$\Delta_{a+\mu}^{\nu} h_\mu(t,a) = h_{\mu-\nu}(t,a), \quad t \in \mathbb{N}_{a+\mu-\nu}. \tag{3.6}$$

We sometimes want to compose fractional sums together. The following theorem tells us how to perform such composition.

Theorem 3.1.10 (See [4, Theorem 2.46]). *Assume f is defined on $\mathbb{N}_a$ and μ, ν are positive numbers. Then*

$$\left[\Delta_{a+\nu}^{-\mu}\left(\Delta_a^{-\nu} f\right)\right](t) = \left(\Delta_a^{-(\mu+\nu)} f\right)(t) = \left[\Delta_{a+\mu}^{-\nu}\left(\Delta_a^{-\mu} f\right)\right] \tag{3.7}$$

for $t \in \mathbb{N}_{a+\mu+\nu}$.

Proof. For $t \in \mathbb{N}_{a+\mu+\nu}$, consider

$$\begin{aligned}
\left[\Delta_{a+\nu}^{-\mu}\left(\Delta_a^{-\nu} f\right)\right](t) &= \sum_{s=a+\nu}^{t-\mu} h_{\mu-1}(t,\sigma(s))\left(\Delta_a^{-\nu} f\right)(s) \\
&= \sum_{s=a+\nu}^{t-\mu} h_{\mu-1}(t,\sigma(s)) \sum_{r=a}^{s-\nu} h_{\nu-1}(s,\sigma(r)) f(r) \\
&= \frac{1}{\Gamma(\mu)\Gamma(\nu)} \sum_{s=a+\nu}^{t-\mu} \sum_{r=a}^{s-\nu} (t-\sigma(s))^{\underline{\mu-1}} (s-\sigma(r))^{\underline{\nu-1}} f(r) \\
&= \frac{1}{\Gamma(\mu)\Gamma(\nu)} \sum_{r=a}^{t-(\mu+\nu)} \sum_{s=r+\nu}^{t-\mu} (t-\sigma(s))^{\underline{\mu-1}} (s-\sigma(r))^{\underline{\nu-1}} f(r),
\end{aligned}$$

where in the last step we interchanged the order of summation. Letting $x = s - \sigma(r)$, we obtain

$$\begin{aligned}
\left[\Delta_{a+\nu}^{-\mu}\left(\Delta_a^{-\nu} f\right)\right](t) &= \frac{1}{\Gamma(\mu)\Gamma(\nu)} \sum_{r=a}^{t-(\mu+\nu)} \left[\sum_{x=\nu-1}^{t-\mu-r-1} (t-x-r-2)^{\underline{\mu-1}} x^{\underline{\nu-1}}\right] f(r) \\
&= \frac{1}{\Gamma(\nu)} \sum_{r=a}^{t-(\mu+\nu)} \left[\frac{1}{\Gamma(\mu)} \sum_{x=\nu-1}^{(t-r-1)-\mu} (t-r-1-\sigma(x))^{\underline{\mu-1}} x^{\underline{\nu-1}}\right] f(r) \\
&= \frac{1}{\Gamma(\nu)} \sum_{r=a}^{t-(\mu+\nu)} \left[\Delta_{\nu-1}^{-\mu} t^{\underline{\nu-1}}\right]_{t\to t-r-1} f(r).
\end{aligned}$$

But, by Theorem 3.1.7,

$$\Delta_{\nu-1}^{-\mu} t^{\underline{\nu-1}} = \frac{\Gamma(\nu)}{\Gamma(\nu+\mu)} t^{\underline{\mu+\nu-1}}$$

and therefore

$$\begin{aligned}
\left[\Delta_{a+\nu}^{-\mu}\left(\Delta_a^{-\nu} f\right)\right](t) &= \frac{1}{\Gamma(\nu)} \sum_{r=a}^{t-(\mu+\nu)} \frac{\Gamma(\nu)}{\Gamma(\mu+\nu)} (t-r-1)^{\underline{\mu+\nu-1}} f(r) \\
&= \frac{1}{\Gamma(\mu+\nu)} \sum_{r=a}^{t-(\mu+\nu)} (t-\sigma(r))^{\underline{\mu+\nu-1}} f(r) \\
&= \left(\Delta_a^{-(\mu+\nu)} f\right)(t),
\end{aligned}$$

$t \in \mathbb{N}_{a+\nu+\mu}$, which is one of the desired conclusions. Interchanging μ and ν in the above formula, we also get the result

$$\left[\Delta_{a+\mu}^{-\nu}\left(\Delta_a^{-\mu} f\right)\right](t) = \left(\Delta_a^{-(\mu+\nu)} f\right)(t)$$

for $t \in \mathbb{N}_{a+\mu+\nu}$. □

3.2. LAPLACE TRANSFORMS OF μ-TH TAYLOR MONOMIALS

We now consider a Taylor monomial where its order is not a negative integer.

Definition 3.2.1 (See [4, Definition 2.55])**.** For each $\mu \in \mathbb{R} \setminus (-\mathbb{N}_1)$, define the **$\mu$-th order Taylor monomial**, $h_\mu(t,a)$, by

$$h_\mu(t,a) := \frac{(t-a)^{\underline{\mu}}}{\Gamma(\mu+1)} \quad \text{for } t \in \mathbb{N}_a.$$

The following two theorems are essential to our work. We will want to take Laplace transforms of μ-th Taylor monomials. These theorems give us this transform.

Theorem 3.2.2 (See [4, Theorem 2.56])**.** *If $\mu \le 0$ and $\mu \notin (-\mathbb{N}_1)$, then $h_\mu(t,a)$ is bounded (and hence is of exponential order $r = 1$). If $\mu > 0$, then for every $r > 1$, $h_\mu(t,a)$ is of exponential order r.*

Proof. First consider the case that $\mu \le 0$ with $\mu \notin (-\mathbb{N}_0)$. Then for all large $t \in \mathbb{N}_a$,

$$h_\mu(t,a) = \frac{\Gamma(t-a+1)}{\Gamma(\mu+1)\Gamma(t-a+1-\mu)} \le \frac{1}{\Gamma(\mu+1)},$$

implying that h_μ is of exponential order one (i.e., bounded). Next assume that $\mu > 0$, with $N \in \mathbb{N}_0$ chosen so that $N-1 < \mu \le N$. Then for any fixed $r > 1$,

$$\begin{aligned} h_\mu(t,a) = \frac{(t-a)^{\underline{\mu}}}{\Gamma(\mu+1)} &= \frac{\Gamma(t-a+1)}{\Gamma(\mu+1)\Gamma(t-a+1-\mu)} \\ &\le \frac{\Gamma(t-a+1)}{\Gamma(\mu+1)\Gamma(t-a+1-N)} \\ &= \frac{(t-a)\cdots(t-a-N+1)}{\Gamma(\mu+1)} \\ &\le \frac{(t-a)^N}{\Gamma(\mu+1)} \\ &\le \frac{r^t}{\Gamma(\mu+1)}, \end{aligned}$$

for sufficiently large $t \in \mathbb{N}_a$. Therefore, $h_\mu(t,a)$ is of exponential order r for each $\mu \in \mathbb{R}\setminus(-\mathbb{N}_1)$ and $r > 1$. It follows from Theorem 2.3.4 that

$$\mathcal{L}_a\left\{h_\mu(t,a)\right\}(s)$$

exists for $|s+1| > 1$. □

Theorem 3.2.3 (See [4, Theorem 2.58])**.** *Let* $\mu \in \mathbb{R}\setminus(-\mathbb{N}_1)$. *Then*

$$\mathcal{L}_{a+\mu}\left\{h_\mu(t,a)\right\}(s) = \frac{(s+1)^\mu}{s^{\mu+1}} \tag{3.8}$$

for $|s+1| > 1$.

Proof. For $|s+1| > 1$, consider

$$\frac{(s+1)^\mu}{s^{\mu+1}} = \frac{1}{s+1}\left(\frac{s+1}{s}\right)^{\mu+1} = \frac{1}{s+1}\left(1-\frac{1}{s+1}\right)^{-\mu-1}.$$

Since $\left|\frac{1}{s+1}\right| < 1$, we have by the binomial theorem that

$$\begin{aligned}\frac{(s+1)^\mu}{s^{\mu+1}} &= \frac{1}{s+1}\sum_{k=0}^{\infty}(-1)^k\binom{-\mu-1}{k}\left(\frac{1}{s+1}\right)^k\\ &= \sum_{k=0}^{\infty}(-1)^k\binom{-\mu-1}{k}\frac{1}{(s+1)^{k+1}}.\end{aligned}$$

However

$$\begin{aligned}(-1)^k\binom{-\mu-1}{k} &= (-1)^k\frac{(-\mu-1)^{\underline{k}}}{k!}\\ &= (-1)^k\frac{(-\mu-1)(-\mu-2)\cdots(-\mu-k)}{k!}\\ &= \frac{(\mu+k)(\mu+k-1)\cdots(\mu+1)}{k!}\\ &= \frac{(\mu+k)^{\underline{k}}}{k!}\end{aligned}$$

$$
\begin{aligned}
&= \binom{\mu + k}{k} = \binom{\mu + k}{\mu} \\
&= \frac{(\mu + k)^{\underline{\mu}}}{\Gamma(\mu + 1)} \\
&= \frac{[(a + \mu + k) - a]^{\underline{\mu}}}{\Gamma(\mu + 1)} \\
&= h_\mu(a + \mu + k, a).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{(s+1)^\mu}{s^{\mu+1}} &= \sum_{k=0}^{\infty} \frac{h_\mu(a + \mu + k, a)}{(s+1)^{k+1}} \\
&= \mathcal{L}_{a+\mu} \left\{ h_\mu(t, a) \right\} (s)
\end{aligned}
$$

for $|s + 1| > 1$. □

3.3. LAPLACE TRANSFORMS OF DISCRETE FRACTIONAL SUM AND DIFFERENCE OPERATORS

One of the main strategies we used to solve our initial value problem was using Laplace transforms. In order to take the Laplace transform of a fractional difference, we will first want to know if this fractional difference is of an exponential order.

Theorem 3.3.1 (See [4, Theorem 2.65]). *Suppose that $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r \geq 1$, and let $\nu > 0$, $N - 1 < \nu \leq N$, be given. Then for each fixed $\varepsilon > 0$, $\Delta_a^{-\nu} f : \mathbb{N}_{a+\nu} \to \mathbb{R}$, $\Delta_a^{-\nu} f : \mathbb{N}_{a+\nu-N} \to \mathbb{R}$, and $\Delta_a^{\nu} f : \mathbb{N}_{a+N-\nu} \to \mathbb{R}$ are of exponential order $r + \varepsilon$.*

Corollary 3.3.2 (See [4, Corollary 2.66]). *Suppose that $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r \geq 1$ and let $\nu > 0$ be given with $N - 1 < \nu \leq N$. Then*

$$
\mathcal{L}_{a+\nu} \left\{ \Delta_a^{-\nu} f \right\} (s), \quad \mathcal{L}_{a+\nu-N} \left\{ \Delta_a^{-\nu} f \right\} (s), \quad \text{and} \quad \mathcal{L}_{a+N-\nu} \left\{ \Delta_a^{\nu} f \right\} (s)
$$

converge for all $|s + 1| > r$.

Now that we know the Laplace transform of fractional sums and differences exist, the question is what are these Laplace transforms. The following theorems give us the results.

Theorem 3.3.3 (See [4, Theorem 2.67])**.** *Suppose* $f : \mathbb{N}_a \to \mathbb{R}$ *is of exponential order* $r \geq 1$*, and let* $\nu > 0$ *be given with* $N - 1 < \nu \leq N$*. Then for* $|s+1| > r$*,*

$$\mathcal{L}_{a+\nu}\left\{\Delta_a^{-\nu} f\right\}(s) = \frac{(s+1)^\nu}{s^\nu} F_a(s)$$

and

$$\mathcal{L}_{a+\nu-N}\left\{\Delta_a^{-\nu} f\right\}(s) = \frac{(s+1)^{\nu-N}}{s^\nu} F_a(s).$$

Theorem 3.3.4 (See [4, Theorem 2.70])**.** *Suppose* $f : \mathbb{N}_a \to \mathbb{R}$ *is of exponential order* $r \geq 1$*, and let* $\nu > 0$ *be given with* $N - 1 < \nu \leq N$*. Then for* $|s+1| > r$*,*

$$\begin{aligned}\mathcal{L}_{a+N-\nu}\left\{\Delta_a^{\nu} f\right\}(s) =& s^\nu (s+1)^{N-\nu} F_a(s) \\ & - \sum_{j=0}^{N-1} s^j \Delta_a^{\nu-1-j} f(a+N-\nu).\end{aligned}$$

4. RESULTS

Thus far we have given no original results, but we now share the original results of this thesis. Our focus is to solve the Riemann–Liouville delta fractional difference initial value problem

$$\Delta_a^\nu f(t) = \lambda f(t+\nu-1), \quad f(a) = f_a, \tag{4.1}$$

where $0 < \nu \leq 1$, $t \in \mathbb{N}_{a+1-\nu}$, and $f : \mathbb{N}_a \to \mathbb{R}$, $f_a \in \mathbb{R}$. This chapter will show the various techniques and methods used to find the solution.

4.1. USING RECURSION

An intuitive way to solve difference equations is to use recursion and look for a pattern to form the explicit formula for the solution of the difference equation. The following displays the results of our initial effort and attempt to solve initial value problem (4.1).

Assume $N-1 \leq \nu \leq N$ for $N \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{R}$. Choose $N = 1$. Use and recall (3.3) of Theorem 3.1.6 such that

$$\Delta_a^\nu f(t) = \sum_{k=0}^{t+\nu-a} (-1)^k \binom{\nu}{k} f(t+\nu-k) \quad \text{for} \quad t \in \mathbb{N}_{a+1-\nu}.$$

Assume $\Delta_a^\nu f(t) = \lambda f(t+\nu-1)$ and suppose $f(a) = f_a \in \mathbb{R}$. Then

$$\sum_{k=0}^{t+\nu-a} (-1)^k \binom{\nu}{k} f(t+\nu-k) = \lambda f(t+\nu-1) \quad \text{for} \quad \lambda \in \mathbb{R}.$$

Therefore

$$f(t+\nu) - \nu f(t+\nu-1) + \sum_{k=2}^{t+\nu-a} (-1)^k \binom{\nu}{k} f(t+\nu-k) = \lambda f(t+\nu-1).$$

Thus

$$f(t+\nu) = (\nu+\lambda) f(t+\nu-1) - \sum_{k=2}^{t+\nu-a} (-1)^k \binom{\nu}{k} f(t+\nu-k).$$

We now proceed to use this recursive relation to look for an explicit pattern of the function. We will consider specific values of t so that we might find a solution to this initial value problem:

<u>$t = a - \nu$:</u>

$$\begin{aligned} f(a) &= f_a \frac{\Gamma(\nu)}{\Gamma(\nu)\Gamma(1)} \\ &= f_a h_{\nu-1}(a, a+1-\nu). \end{aligned}$$

<u>$t = a + 1 - \nu$:</u>

$$\begin{aligned} f(a+1) &= f_a(\nu+\lambda) \\ &= f_a \left[\nu \frac{\Gamma(\nu)}{\Gamma(\nu)\Gamma(2)} + \lambda \frac{\Gamma(2\nu)}{\Gamma(2\nu)} \right] \\ &= f_a \left[\frac{\Gamma(\nu+1)}{\Gamma(\nu)\Gamma(2)} + \lambda \frac{\Gamma(2\nu)}{\Gamma(2\nu)\Gamma(1)} \right] \\ &= f_a \left[\frac{\Gamma(1-(1-\nu)+1)}{\Gamma(\nu)\Gamma(2)} + \lambda \frac{\Gamma(1-2(1-\nu)+1)}{\Gamma(2\nu)\Gamma(1)} \right] \\ &= f_a \left[h_{\nu-1}(a+1, a+1-\nu) + \lambda h_{2\nu-1}(a+1, a+2(1-\nu) \right]. \end{aligned}$$

<u>$t = a + 2 - \nu$:</u>

$$\begin{aligned} f(a+2) &= (\nu+\lambda) f(a+1) - \binom{\nu}{2} f(a) \\ &= f_a(\nu+\lambda)^2 - \binom{\nu}{2} f_a \\ &= f_a \left[\nu^2 + 2\nu\lambda + \lambda^2 - \binom{\nu}{2} \right] \\ &= f_a \left[\nu^2 + 2\nu\lambda + \lambda^2 + \frac{\nu(1-\nu)}{2} \right] \\ &= fa \left[\lambda^2 + 2\nu\lambda + \frac{\nu^2+\nu}{2} \right] \\ &= f_a \left[\lambda^2 + 2\nu\lambda + \frac{\nu(\nu+1)}{2} \right] \end{aligned}$$

$$
\begin{aligned}
&= f_a \left[\frac{\lambda^2}{\Gamma(1)} + \frac{2\nu\lambda}{\Gamma(2)} + \frac{\nu(\nu+1)}{\Gamma(3)} \right] \\
&= f_a \left[\frac{\lambda^2}{\Gamma(1)} + \frac{2\lambda\nu\Gamma(2\nu)}{\Gamma(2\nu)\Gamma(2)} + \frac{(\nu+1)\nu\Gamma(\nu)}{\Gamma(\nu)\Gamma(3)} \right] \\
&= f_a \left[\frac{\lambda^2}{\Gamma(1)} + \frac{\lambda 2\nu\Gamma(2\nu)}{\Gamma(2\nu)\Gamma(2)} + \frac{(\nu+1)\Gamma(\nu+1)}{\Gamma(\nu)\Gamma(3)} \right] \\
&= f_a \left[\frac{\lambda^2\Gamma(3\nu)}{\Gamma(3\nu)\Gamma(1)} + \frac{\lambda\Gamma(2\nu+1)}{\Gamma(2\nu)\Gamma(2)} + \frac{\Gamma(\nu+2)}{\Gamma(\nu)\Gamma(3)} \right] \\
&= f_a \left[\frac{\lambda^2\Gamma(2-3(1-\nu)+1)}{\Gamma(3\nu)\Gamma(1)} + \frac{\lambda\Gamma(2-2(1-\nu)+1)}{\Gamma(2\nu)\Gamma(2)} + \frac{\Gamma(2-(1-\nu)+1)}{\Gamma(\nu)\Gamma(3)} \right] \\
&= f_a \left[h_{\nu-1}(a+2, a+1-\nu) + \lambda h_{2\nu-1}(a+2, a+2(1-\nu)) \right. \\
&\quad \left. + \lambda^2 h_{3\nu-1}(a+2, a+3(1-\nu)) \right].
\end{aligned}
$$

<u>$t = a + 3 - \nu$:</u>

$$
\begin{aligned}
f(a+3) &= f(a+2)(\nu+\lambda) - \binom{\nu}{2} f(a+1) + \binom{\nu}{3} f(a) \\
&= (\nu+\lambda) f_a \left[(\nu+\lambda)^2 - \binom{\nu}{2} \right] - \binom{\nu}{2} f_a(\nu+\lambda) + \binom{\nu}{3} f_a \\
&= f_a \left[(\nu+\lambda)^3 - 2\binom{\nu}{2}(\nu+\lambda) + \binom{\nu}{3} \right] \\
&= f_a \left[\nu^3 + 3\lambda^2\nu + 3\lambda\nu^2 + \nu^3 - 2\nu\binom{\nu}{2} - 2\lambda\binom{\nu}{2} + \binom{\nu}{3} \right] \\
&= f_a \left[\nu^3 + 3\nu^2\lambda + 3\nu\lambda^2 + \lambda^3 - \frac{2\lambda\nu(\nu-1)}{2} - \frac{2\nu(\nu-1)\nu}{2} + \frac{\nu(\nu-1)(\nu-2)}{6} \right] \\
&= f_a \left[\lambda^3 + 3\lambda^2\nu + 3\lambda\nu^2 + \nu^3 - \lambda\left(\nu^2 - \nu\right) - \nu^3 + \nu^2 + \frac{\nu^3 - 3\nu^2 + 2\nu}{6} \right] \\
&= f_a \left[\lambda^3 + 3\lambda^2\nu + 3\lambda\nu - \lambda\nu^2 + \lambda\nu + \nu^2 + \frac{\nu^3}{6} - \frac{\nu^2}{2} + \frac{\nu}{3} \right] \\
&= f_a \left[\lambda^3 + 3\lambda^2\nu + \lambda\nu + 2\lambda\nu^2 + \frac{\nu^3}{6} + \frac{\nu^2}{2} + \frac{\nu}{3} \right] \\
&= f_a \left[\lambda^3 + 3\lambda^2\nu + \nu\lambda + 2\nu^2\lambda + \frac{2\nu + 3\nu^2 + \nu^3}{6} \right] \\
&= f_a \left[\lambda^3 + 3\lambda^2\nu + \lambda(2\nu+1)\nu + \frac{(2+\nu)(1+\nu)\nu}{6} \right] \\
&= f_a \left[\frac{(2+\nu)(1+\nu)\nu}{\Gamma(4)} + \frac{\lambda(2\nu+1)2\nu}{\Gamma(3)} + \frac{3\lambda^2\nu}{\Gamma(2)} + \lambda^3 \right] \\
&= f_a \left[\frac{(2+\nu)(1+\nu)\nu\Gamma(\nu)}{\Gamma(\nu)\Gamma(4)} + \frac{\lambda(2\nu+1)2\nu}{\Gamma(3)} + \frac{\lambda^2 3\nu}{\Gamma(2)} + \lambda^3 \right]
\end{aligned}
$$

$$= f_a\left[\frac{(2+\nu)(1+\nu)\Gamma(1+\nu)}{\Gamma(\nu)\Gamma(4)} + \frac{\lambda(2\nu+1)(2\nu)\Gamma(2\nu)}{\Gamma(2\nu)\Gamma(3)} + \frac{\lambda^2 3\nu}{\Gamma(2)} + \lambda^3\right]$$

$$= f_a\left[\frac{(2+\nu)\Gamma(2+\nu)}{\Gamma(\nu)\Gamma(4)} + \frac{\lambda(2\nu+1)\Gamma(1+2\nu)}{\Gamma(2\nu)\Gamma(3)} + \frac{\lambda^2 3\nu\Gamma(3\nu)}{\Gamma(3\nu)\Gamma(2)} + \lambda^3\right]$$

$$= f_a\left[\frac{\Gamma(3+\nu)}{\Gamma(\nu)\Gamma(4)} + \frac{\lambda\Gamma(2+2\nu)}{\Gamma(2\nu)\Gamma(3)} + \frac{\lambda^2\Gamma(1+3\nu)}{\Gamma(3\nu)\Gamma(2)} + \frac{\lambda^3\Gamma(4\nu)}{\Gamma(4\nu)\Gamma(1)}\right]$$

$$= f_a\left[\frac{\Gamma(3-(1-\nu)+1)}{\Gamma(\nu)\Gamma(4)} + \frac{\lambda\Gamma(3-2(1-\nu)+1)}{\Gamma(2\nu)\Gamma(3)} + \frac{\lambda^2\Gamma(3-3(1-\nu)+1)}{\Gamma(3\nu)\Gamma(2)}\right.$$

$$\left. + \frac{\lambda^3\Gamma(3-4(1-\nu)+1)}{\Gamma(4\nu)\Gamma(1)}\right]$$

$$= f_a\left[h_{\nu-1}(a+3, a+1-\nu) + \lambda h_{2\nu-1}(a+3, a+2(1-\nu))\right.$$

$$\left. + \lambda^2 h_{3\nu-1}(a+3, a+3(1-\nu)) + \lambda^3 h_{4\nu-1}(a+3, a+4(1-\nu))\right].$$

<u>$t = a+4-\nu$:</u>

$$f(a+4) = f(a+3)(\lambda+\nu) - \binom{\nu}{2} f(a+2) + \binom{\nu}{3} f(a+1) - \binom{\nu}{4} f(a)$$

$$= f_a\left[(\lambda+\nu)\left[(\lambda+\nu)^3 - 2\binom{\nu}{2}(\lambda+\nu) + \binom{\nu}{3}\right]\right.$$

$$\left. -\binom{\nu}{2}\left[(\lambda+\nu)^2 - \binom{\nu}{2}\right] + \binom{\nu}{3}(\lambda+\nu) - \binom{\nu}{4}\right]$$

$$= f_a\left[(\lambda+\nu)^4 - 2\binom{\nu}{2}(\lambda+\nu)^2 + \binom{\nu}{3}(\lambda+\nu) - \binom{\nu}{2}(\lambda+\nu)^2\right.$$

$$\left. + \binom{\nu}{2}^2 + \binom{\nu}{3}(\lambda+\nu) - \binom{\nu}{4}\right]$$

$$= f_a\left[(\lambda+\nu)^4 - 3\binom{\nu}{2}(\lambda+\nu)^2 + 2\binom{\nu}{3}(\lambda+\nu) + \binom{\nu}{2}^2 - \binom{\nu}{4}\right]$$

$$= f_a\left[\lambda^4 + 4\lambda^3\nu + 6\lambda^2\nu^2 + 4\lambda\nu^3 + \nu^4 - 3\binom{\nu}{2}\left(\lambda^2 + 2\lambda\nu + \nu^2\right)\right.$$

$$\left. + 2\binom{\nu}{3}(\lambda+\nu) + \binom{\nu}{2}^2 - \binom{\nu}{4}\right]$$

$$= f_a\left[\lambda^4 + 4\lambda^3\nu + 6\lambda^2\nu^2 + 4\lambda\nu^3 + \nu^4 - \frac{3\nu(\nu-1)\left(\lambda^2 + 2\lambda\nu + \nu^2\right)}{2}\right.$$

$$\left. + \frac{2\nu(\nu-1)(\nu-2)(\lambda+\nu)}{6} + \frac{\nu^2(\nu-1)^2}{4} - \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{24}\right]$$

$$= f_a\left[\lambda^4 + 4\lambda^3\nu + 6\lambda^2\nu^2 + 4\lambda\nu^3 + \nu^4 - \frac{3\lambda^2\nu^2}{2} + \frac{3\lambda^2\nu}{2} - 3\lambda\nu^3 + 3\lambda\nu^2\right.$$

$$-\frac{3\nu^4}{2} + \frac{3\nu^3}{2} + \frac{\lambda\nu^3}{3} - \lambda\nu^2 + \frac{2\lambda\nu}{3} + \frac{\nu^4}{3} - \nu^3 + \frac{2\nu^2}{3} + \frac{\nu^4}{4} - \frac{\nu^3}{2} + \frac{\nu^2}{4}$$

$$-\frac{\nu^4}{24} + \frac{\nu^3}{4} - \frac{11\nu^2}{24} + \frac{\nu}{4}\Bigg]$$

$$= f_a\left[\lambda^4 + 4\nu\lambda^3 + \frac{9\lambda^2\nu^2}{2} + \frac{3\nu\lambda^2}{2} + \frac{4\lambda\nu^3}{3} + 2\nu^2\lambda + \frac{2\nu\lambda}{3}\right.$$

$$\left.+\frac{\nu^4}{24} + \frac{\nu^3}{4} + \frac{11\nu^2}{24} + \frac{\nu}{4}\right]$$

$$= f_a\left[\frac{\nu^4 + 6\nu^3 + 11\nu^2 + 6\nu}{24} + \lambda\frac{8\nu^3 + 12\nu^2 + 4\nu}{6} + \lambda^2\frac{9\nu^2 + 3\nu}{2}\right.$$

$$\left.+\lambda^3 4\nu + \lambda^4\right]$$

$$= f_a\left[\frac{(\nu+3)(\nu+2)(\nu+1)\nu}{24} + \frac{\lambda(2\nu+2)(2\nu+1)2\nu}{6}\right.$$

$$\left.+\frac{\lambda^2(3\nu+1)3\nu}{2} + 4\lambda^3\nu + \lambda^4\right]$$

$$= f_a\left[\frac{(\nu+3)(\nu+2)(\nu+1)\nu\Gamma(\nu)}{24\Gamma(\nu)} + \frac{\lambda(2\nu+2)(2\nu+1)2\nu\Gamma(2\nu)}{6\Gamma(2\nu)}\right.$$

$$\left.+\frac{\lambda^2(3\nu+1)3\nu\Gamma(3\nu)}{2\Gamma(3\nu)} + \frac{\lambda^3 4\nu\Gamma(4\nu)}{\Gamma(4\nu)} + \frac{\lambda^4\Gamma(5\nu)}{\Gamma(5\nu)}\right]$$

$$= f_a\left[\frac{(\nu+3)(\nu+2)(\nu+1)\Gamma(\nu+1)}{\Gamma(5)\Gamma(\nu)} + \frac{\lambda(2\nu+2)(2\nu+1)\Gamma(2\nu+1)}{\Gamma(4)\Gamma(2\nu)}\right.$$

$$\left.+\frac{\lambda^2(3\nu+1)\Gamma(3\nu+1)}{\Gamma(3)\Gamma(3\nu)} + \frac{\lambda^3\Gamma(4\nu+1)}{\Gamma(2)\Gamma(4\nu)} + \frac{\lambda^4\Gamma(5\nu)}{\Gamma(1)\Gamma(5\nu)}\right]$$

$$= f_a\left[\frac{(\nu+3)(\nu+2)\Gamma(\nu+2)}{\Gamma(5)\Gamma(\nu)} + \frac{\lambda(2\nu+2)\Gamma(2\nu+2)}{\Gamma(4)\Gamma(2\nu)}\right.$$

$$\left.+\frac{\lambda^2\Gamma(3\nu+2)}{\Gamma(3)\Gamma(3\nu)} + \frac{\lambda^3\Gamma(4\nu+1)}{\Gamma(2)\Gamma(4\nu)} + \frac{\lambda^4\Gamma(5\nu)}{\Gamma(1)\Gamma(\nu)}\right]$$

$$= f_a\left[\frac{(\nu+3)\Gamma(\nu+3)}{\Gamma(5)\Gamma(\nu)} + \frac{\lambda\Gamma(2\nu+3)}{\Gamma(4)\Gamma(2\nu)}\right.$$

$$\left.+\frac{\lambda^2\Gamma(3\nu+2)}{\Gamma(3)\Gamma(3\nu)} + \frac{\lambda^3\Gamma(4\nu+1)}{\Gamma(2)\Gamma(4\nu)} + \frac{\lambda^4\Gamma(5\nu)}{\Gamma(1)\Gamma(5\nu)}\right]$$

$$= f_a\left[\frac{\Gamma(4+\nu)}{\Gamma(5)\Gamma(\nu)} + \frac{\lambda\Gamma(2\nu+3)}{\Gamma(4)\Gamma(2\nu)}\right.$$

$$\left.+\frac{\lambda^2\Gamma(3\nu+2)}{\Gamma(3)\Gamma(3\nu)} + \frac{\lambda^3\Gamma(4\nu+1)}{\Gamma(2)\Gamma(4\nu)} + \frac{\lambda^4\Gamma(5\nu)}{\Gamma(1)\Gamma(5\nu)}\right]$$

$$= f_a\left[\frac{\Gamma(4-(1-\nu)+1)}{\Gamma(5)\Gamma(\nu)} + \frac{\lambda\Gamma(4-2(1-\nu)+1)}{\Gamma(4)\Gamma(2\nu)} + \frac{\lambda^2\Gamma(4-3(1-\nu)+1)}{\Gamma(3\nu)\Gamma(3)}\right.$$

$$+\frac{\lambda^3\Gamma(4-4(1-\nu)+1)}{\Gamma(4\nu)\Gamma(2)}+\frac{\lambda^4\Gamma(4-5(1-\nu)+1)}{\Gamma(1)\Gamma(5\nu)}\Bigg]$$

$$= f_a\left[h_{\nu-1}(a+4,a+1-\nu)+\lambda_{2\nu-1}(a+4,a+2(1-\nu))\right.$$

$$+\lambda^2 h(a+4,a+3(1-\nu))+\lambda^3 h_{4\nu-1}(a+4,a+4(1-\nu))$$

$$\left.+\lambda^4 h_{5\nu-1}(a+4,a+5(1-\nu))\right].$$

<u>$t = a+5-\nu$:</u>

$$f(a+5) = (\lambda+\nu)f(a+4)-\binom{\nu}{2}f(a+3)+\binom{\nu}{3}f(a+2)$$

$$-\binom{\nu}{4}f(a+1)+\binom{\nu}{5}f(a)$$

$$= f_a\left[(\lambda+\nu)\left[(\lambda+\nu)^4-3\binom{\nu}{2}(\lambda+\nu)^2+2\binom{\nu}{3}(\lambda+\nu)+\binom{\nu}{2}^2-\binom{\nu}{4}\right]\right.$$

$$-\binom{\nu}{2}\left[(\lambda+\nu)^3-2\binom{\nu}{2}(\lambda+\nu)+\binom{\nu}{3}\right]$$

$$\left.+\binom{\nu}{3}\left[(\lambda+\nu)^2-\binom{\nu}{2}\right]-\binom{\nu}{4}(\lambda+\nu)+\binom{\nu}{5}\right]$$

$$= f_a\left[(\lambda+\nu)^5-3\binom{\nu}{2}(\lambda+\nu)^3+2\binom{\nu}{3}(\lambda+\nu)^2+\binom{\nu}{2}^2(\lambda+\nu)\right.$$

$$-\binom{\nu}{4}(\lambda+\nu)-\binom{\nu}{2}(\lambda+\nu)^3+2\binom{\nu}{2}^2(\lambda+\nu)$$

$$\left.-\binom{\nu}{2}\binom{\nu}{3}+\binom{\nu}{3}(\lambda+\nu)^2-\binom{\nu}{2}\binom{\nu}{3}-\binom{\nu}{4}(\lambda+\nu)+\binom{\nu}{5}\right]$$

$$= f_a\left[(\lambda+\nu)^5-4\binom{\nu}{2}(\lambda+\nu)^3+3\binom{\nu}{3}(\lambda+\nu)^2\right.$$

$$\left.+\left(3\binom{\nu}{2}^2-2\binom{\nu}{4}\right)(\lambda+\nu)-2\binom{\nu}{2}\binom{\nu}{3}+\binom{\nu}{5}\right]$$

$$= f_a\left[\lambda^5+5\lambda^4\nu+10\lambda^3\nu^2+10\lambda^2\nu^3+5\lambda\nu^4+\nu^5\right.$$

$$-4\binom{\nu}{2}\left(\lambda^3+3\lambda^2\nu+3\lambda\nu^2+\nu^3\right)+3\binom{\nu}{3}\left(\lambda^2+2\lambda\nu+\nu^2\right)$$

$$\left.+\left(3\binom{\nu}{2}^2-2\binom{\nu}{4}\right)(\lambda+\nu)-2\binom{\nu}{2}\binom{\nu}{3}+\binom{\nu}{5}\right]$$

$$= f_a\left[\lambda^5+5\lambda^4\nu+\lambda^3\left[10\nu^2-4\binom{\nu}{2}\right]-\binom{\nu}{2}\binom{\nu}{3}-\binom{\nu}{4}(\lambda+\nu)+\binom{\nu}{5}\right]$$

$$+\lambda\left[5\nu^4-12\binom{\nu}{2}\nu^2+6\binom{\nu}{3}\nu+3\binom{\nu}{2}^2-2\binom{\nu}{4}\right]$$

$$+\nu^5-4\binom{\nu}{2}\nu^3+3\binom{\nu}{2}\nu^2+\nu\left(3\binom{\nu}{2}^2-2\binom{\nu}{4}\right)-2\binom{\nu}{2}\binom{\nu}{3}+\binom{\nu}{5}\Bigg]$$

$$= f_a\left[\lambda^5+5\lambda^4\nu+\lambda^3\left[10\nu^2-2\nu(\nu-1)\right]\right.$$

$$+\lambda^2\left[10\nu^3-\frac{12\nu(\nu-1)\nu}{2}+\frac{3\nu(\nu-1)(\nu-2)}{6}\right]$$

$$+\lambda\left[5\nu^4-\frac{12\nu^3(\nu-1)}{2}+\frac{6\nu^2(\nu-1)(\nu-2)}{6}\right.$$

$$\left.+3\left(\frac{\nu(\nu-1)}{2}\right)^2-\frac{2\nu(\nu-1)(\nu-2)(\nu-3)}{24}\right]$$

$$+\nu^5-2\nu^4(\nu-1)+\frac{3\nu^3(\nu-1)(\nu-2)}{6}$$

$$+\nu\left[\frac{3\nu^2(\nu-1)^2}{4}-\frac{2\nu(\nu-1)(\nu-2)(\nu-3)}{24}\right]$$

$$\left.-\frac{2\nu(\nu-1)\nu(\nu-1)(\nu-2)}{12}+\frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{120}\right]$$

$$= f_a\left[\frac{\nu^5}{120}+\frac{\nu^4}{12}+\frac{7\nu^3}{24}+\frac{5\nu^2}{12}+\frac{\nu}{5}+\lambda\left[\frac{2\nu^4}{3}+2\nu^3+\frac{11\nu^2}{6}+\frac{\nu}{2}\right]\right.$$

$$\left.+\lambda^2\left[\frac{9\nu^3}{2}+\frac{9\nu^2}{2}+\nu\right]+\lambda^3\left[8\nu^2+2\nu\right]+5\lambda^4\nu+\lambda^5\right]$$

$$= f_a\left[\frac{(4+\nu)(3+\nu)(2+\nu)(1+\nu)\nu}{120}+\frac{\lambda(3+2\nu)(2+2\nu)(1+2\nu)2\nu}{24}\right.$$

$$\left.+\frac{\lambda^2(2+3\nu)(1+3\nu)3\nu}{6}+\frac{\lambda^3(1+4\nu)4\nu}{2}+5\lambda^4\nu+\lambda^5\right]$$

$$= f_a\left[\frac{(4+\nu)(3+\nu)(2+\nu)(1+\nu)\nu\Gamma(\nu)}{\Gamma(\nu)\Gamma(6)}\right.$$

$$+\frac{\lambda(3+2\nu)(2+2\nu)(1+2\nu)2\nu\Gamma(2\nu)}{\Gamma(2\nu)\Gamma(5)}$$

$$+\frac{\lambda^2(2+3\nu)(1+3\nu)3\nu\Gamma(3\nu)}{\Gamma(3\nu)\Gamma(4)}+\frac{\lambda^3(1+4\nu)4\nu\Gamma(4\nu)}{\Gamma(4\nu)\Gamma(3)}$$

$$\left.+\frac{\lambda^4 5\nu\Gamma(5\nu)}{\Gamma(5\nu)\Gamma(2)}+\frac{\lambda^5}{\Gamma(1)}\right]$$

$$= f_a\left[\frac{(4+\nu)(3+\nu)(2+\nu)(1+\nu)\Gamma(1+\nu)}{\Gamma(\nu)\Gamma(6)}\right.$$

$$+\frac{\lambda(3+2\nu)(2+2\nu)(1+2\nu)\Gamma(1+2\nu)}{\Gamma(2\nu)\Gamma(5)}+\frac{\lambda^2(2+3\nu)(1+3\nu)\Gamma(1+3\nu)}{\Gamma(3\nu)\Gamma(4)}$$

$$
\begin{aligned}
&+\frac{\lambda^3(1+4\nu)\Gamma(1+4\nu)}{\Gamma(4\nu)\Gamma(3)}+\frac{\lambda^4\Gamma(1+5\nu)}{\Gamma(5\nu)\Gamma(2)}+\frac{\lambda^5}{\Gamma(1)}\Bigg]\\
=&\ f_a\Bigg[\frac{(4+\nu)(3+\nu)(2+\nu)\Gamma(2+\nu)}{\Gamma(\nu)\Gamma(6)}+\frac{\lambda(3+2\nu)(2+2\nu)\Gamma(2+2\nu)}{\Gamma(2\nu)\Gamma(5)}\\
&+\frac{\lambda^2(2+3\nu)\Gamma(2+3\nu)}{\Gamma(3\nu)\Gamma(4)}+\frac{\lambda^3\Gamma(2+4\nu)}{\Gamma(4\nu)\Gamma(3)}+\frac{\lambda^4\Gamma(1+5\nu)}{\Gamma(5\nu)\Gamma(2)}+\frac{\lambda^6\Gamma(6\nu)}{\Gamma(6\nu)\Gamma(1)}\Bigg]\\
=&\ f_a\Bigg[\frac{(4+\nu)(3+\nu)\Gamma(3+\nu)}{\Gamma(\nu)\Gamma(6)}+\frac{\lambda(3+2\nu)\Gamma(3+2\nu)}{\Gamma(2\nu)\Gamma(5)}\\
&+\frac{\lambda^2\Gamma(3+3\nu)}{\Gamma(3\nu)\Gamma(4)}+\frac{\lambda^3\Gamma(2+4\nu)}{\Gamma(4\nu)\Gamma(3)}+\frac{\lambda^4\Gamma(1+5\nu)}{\Gamma(5\nu)\Gamma(2)}+\frac{\lambda^6\Gamma(6\nu)}{\Gamma(6\nu)\Gamma(1)}\Bigg]\\
=&\ f_a\Bigg[\frac{(4+\nu)\Gamma(4+\nu)}{\Gamma(\nu)\Gamma(6)}+\frac{\lambda\Gamma(4+2\nu)}{\Gamma(2\nu)\Gamma(5)}+\frac{\lambda^2\Gamma(3+3\nu)}{\Gamma(3\nu)\Gamma(4)}\\
&+\frac{\lambda^3\Gamma(2+4\nu)}{\Gamma(4\nu)\Gamma(3)}+\frac{\lambda^4\Gamma(1+5\nu)}{\Gamma(5\nu)\Gamma(2)}+\frac{\lambda^6\Gamma(6\nu)}{\Gamma(6\nu)\Gamma(1)}\Bigg]\\
=&\ f_a\Bigg[\frac{\Gamma(5+\nu)}{\Gamma(\nu)\Gamma(6)}+\frac{\lambda\Gamma(4+2\nu)}{\Gamma(2\nu)\Gamma(5)}+\frac{\lambda^2\Gamma(3+3\nu)}{\Gamma(3\nu)\Gamma(4)}\\
&+\frac{\lambda^3\Gamma(2+4\nu)}{\Gamma(4\nu)\Gamma(3)}+\frac{\lambda^4\Gamma(1+5\nu)}{\Gamma(5\nu)\Gamma(2)}+\frac{\lambda^6\Gamma(6\nu)}{\Gamma(6\nu)\Gamma(1)}\Bigg]\\
=&\ f_a\Bigg[\frac{\Gamma(5-(1-\nu)+1)}{\Gamma(\nu)\Gamma(6)}+\frac{\lambda\Gamma(5-2(1-\nu)+1)}{\Gamma(2\nu)\Gamma(5)}+\frac{\lambda^2\Gamma(5-3(1-\nu)+1)}{\Gamma(3\nu)\Gamma(4)}\\
&+\frac{\lambda^3\Gamma(5-4(1-\nu)+1)}{\Gamma(4\nu)\Gamma(3)}+\frac{\lambda^4\Gamma(5-5(1-\nu)+1)}{\Gamma(5\nu)\Gamma(2)}+\frac{\lambda^5\Gamma(5-6(1-\nu)+1)}{\Gamma(6\nu)\Gamma(1)}\Bigg]\\
=&\ f_a\left[h_{\nu-1}(a+5,a+1-\nu)+\lambda h_{2\nu-1}(a+5,a+2(1-\nu))\right.\\
&+\lambda^2h_{3\nu-1}(a+5,a+3(1-\nu))+\lambda^3h_{4\nu-1}(a+5,a+4(1-\nu))\\
&\left.+\lambda^4h_{5\nu-1}(a+5,a+5(1-\nu))+\lambda^5h_{6\nu-1}(a+5,a+6(1-\nu))\right].
\end{aligned}
$$

Hence, as we have considered a few values of $t\in\mathbb{N}_{a-\nu}$, we conjecture the solution to be

$$f(t+\nu)=f_a\sum_{k=0}^{t+\nu-a}\lambda^k h_{(k+1)\nu-1}(t+\nu,a+(k+1)(1-\nu)).$$

Recall that $f : \mathbb{N}_a \to \mathbb{R}$. So, we let $\tau \in \mathbb{N}_a$. Then we can apply shifts in our guessed solution such that

$$f(\tau) = f_a \sum_{k=0}^{\tau-a} \lambda^k h_{(k+1)\nu-1}(\tau, a+(k+1)(1-\nu)).$$

We can expand this into terms of the gamma function by applying Definition 3.1.3 of fractional Taylor monomials and Definition 3.1.1 of the generalized falling function such that

$$\begin{aligned} f(\tau) &= f_a \sum_{k=0}^{\tau-a} \lambda^k h_{(k+1)\nu-1}(\tau, a+(k+1)(1-\nu)) \\ &= f_a \sum_{k=0}^{\tau-a} \lambda^k \frac{(\tau-a-(k+1)(1-\nu))^{\underline{(k+1)\nu-1}}}{\Gamma((k+1)\nu-1+1)} \\ &= f_a \sum_{k=0}^{\tau-a} \lambda^k \frac{\Gamma(\tau-a-(k+1)(1-\nu)+1)}{\Gamma((k+1)\nu)\Gamma(\tau-a-(k+1)(1-\nu)+1-(k+1)\nu+1)} \\ &= f_a \sum_{k=0}^{\tau-a} \lambda^k \frac{\Gamma(\tau-a-(k+1)(1-\nu)+1)}{\Gamma((k+1)\nu)\Gamma(\tau-a-k+1)}. \end{aligned}$$

Realize $\tau-a-k+1$ lives in the integers and $\tau-a-(k+1)(1-\nu)+1$ is not an integer unless ν is a natural number. However, ν is bounded above by 1. Then $\tau-a-(k+1)(1-\nu)+1$ is not an integer unless $\nu = 1$. Then we can assume $\tau-a+1 \geq 1$ by our convention that summations are zero if the upper bound is less than the lower bound. Furthermore, since $f : \mathbb{N}_a \to \mathbb{R}$ and $\tau \in \mathbb{N}_a$, $\tau - a \geq 0$. Now recall our convention that $\frac{\Gamma(m)}{\Gamma(M)} = 0$ if m is not a nonpositive integer and M is a nonpositive integer. Then, as $\tau-a-k+1$ is an integer,

$$\tau-a-k+1 \leq 0 \iff k \geq \tau-a+1.$$

Hence

$$\frac{\Gamma(\tau-a-(k+1)(1-\nu)+1)}{\Gamma((k+1)\nu)\Gamma(\tau-a-k+1)} = 0$$

for any $k \geq \tau - a + 1$. Therefore we can extend the upper bound in our summation and say

$$f(\tau) = f_a \sum_{k=0}^{\infty} \lambda^k \frac{\Gamma(\tau - a - (k+1)(1-\nu) + 1)}{\Gamma((k+1)\nu)\Gamma(\tau - a - k + 1)}.$$

Then we make our final proposal of the solution as a Taylor series by giving the solution in terms of fractional Taylor monomials:

$$f(\tau) = f_a \sum_{k=0}^{\infty} \lambda^k h_{(k+1)\nu-1}(\tau, a + (k+1)(1-\nu)). \tag{4.2}$$

We will show the verification of this solution after showing the use of Laplace Transforms to find the identical solution, but first realize this meets our expectations of what we would have for $\nu = 1$. Comparing (4.2) to Theorem 2.2.6 when $\nu = 1$ gives us the desired result as these are then equivalent.

4.2. USING LAPLACE TRANSFORMS

Our second technique is to use Laplace transforms to solve the initial problem

$$\Delta_a^{\nu} f(t) = \lambda f(t + \nu - 1), \quad f(a) = f_a,$$

where $0 < \nu \leq 1$, $t \in \mathbb{N}_{a+1-\nu}$, and $f : \mathbb{N}_a \to \mathbb{R}$.

We first let $t \in \mathbb{N}_{a+1-\nu}$. Then take the Laplace transform of each side such that

$$\mathcal{L}_{a+1-\nu}\{\Delta_a^{\nu} f(t)\}(s) = \lambda \mathcal{L}_{a+1-\nu}\{f(t + \nu - 1)\}(s).$$

Equivalently, by Theorem 3.3.4,

$$s^{\nu}(s+1)^{1-\nu} F_a(s) - \Delta_a^{\nu-1} f(a + 1 - \nu) = \lambda \mathcal{L}_{a+1-\nu}\{f(t + \nu - 1)\}(s).$$

Now let $g(t) = f(t+\nu-1)$. Then by Theorem 2.3.2,

$$\begin{aligned}
\mathcal{L}_{a+1-\nu}\{f(t+\nu-1)\}(s) &= \mathcal{L}_{a+1-\nu}\{g(t)\}(s) \\
&= \sum_{k=0}^{\infty} \frac{g(a+1-\nu+k)}{(s+1)^{k+1}} \\
&= \sum_{k=0}^{\infty} \frac{f(a+1-\nu+\nu-1+k)}{(s+1)^{k+1}} \\
&= \sum_{k=0}^{\infty} \frac{f(a+k)}{(s+1)^{k+1}} \\
&= \mathcal{L}_a\{f\}(s).
\end{aligned}$$

Therefore,

$$s^{\nu}(s+1)^{1-\nu}F_a(s) - \Delta_a^{\nu-1}f(a+1-\nu) = \lambda\mathcal{L}_a\{f\}(s).$$

Now, of course we have

$$\lambda\mathcal{L}_a\{f\}(s) = \lambda F_a(s).$$

Then

$$s^{\nu}(s+1)^{1-\nu}F_a(s) - \Delta_a^{\nu-1}f(a+1-\nu) = \lambda F_a(s).$$

Hence, for $\tau \in \mathbb{N}_a$,

$$\begin{aligned}
F_a(s) &= \frac{\Delta_a^{\nu-1}f(a+1-\nu)}{s^{\nu}(s+1)^{1-\nu}-\lambda} \\
&= \frac{\Delta_a^{-(1-\nu)}f(a+1-\nu)}{s^{\nu}(s+1)^{1-\nu}-\lambda} \\
&\overset{(3.4)}{=} \frac{\sum_{k=0}^{a+1-\nu-a-(1-\nu)}(-1)^k\binom{1-\nu}{k}f(a+1-\nu-(1-\nu)-k)}{s^{\nu}(s+1)^{1-\nu}-\lambda} \\
&= \frac{f(a)}{s^{\nu}(s+1)^{1-\nu}-\lambda} \\
&= \frac{f_a}{s^{\nu}(s+1)^{1-\nu}-\lambda} \\
&= f_a\frac{s^{-\nu}(s+1)^{\nu-1}}{1-\lambda s^{-\nu}(s+1)^{-(\nu-1)}}
\end{aligned}$$

$$
\begin{aligned}
&= f_a s^{-\nu}(s+1)^{\nu-1} \sum_{k=0}^{\infty} [\lambda s^{-\nu}(s+1)^{\nu-1}]^k \\
&= f_a \sum_{k=0}^{\infty} \lambda^k [s^{-\nu}(s+1)^{\nu-1}]^{k+1} \\
&= f_a \sum_{k=0}^{\infty} \lambda^k \prod_{j=0}^{k} \frac{(s+1)^{\nu-1}}{s^{\nu}} \\
&\overset{(3.8)}{=} f_a \sum_{k=0}^{\infty} \lambda^k \prod_{j=0}^{k} \mathcal{L}_{a+\nu-1}\{h_{\nu-1}(\tau,a)\}(s) \\
&\overset{(2.2)}{=} f_a \sum_{k=0}^{\infty} \lambda^k \prod_{j=0}^{k} \sum_{l=0}^{\infty} \frac{h_{\nu-1}(l+a+\nu-1,a)}{(s+1)^{l+1}} \\
&\overset{(3.1)}{=} f_a \sum_{k=0}^{\infty} \lambda^k \prod_{j=0}^{k} \sum_{l=0}^{\infty} \frac{h_{\nu-1}(l+a,a+1-\nu)}{(s+1)^{l+1}} \\
&= f_a \sum_{k=0}^{\infty} \lambda^k \prod_{j=0}^{k} \mathcal{L}_a\{h_{\nu-1}(\tau,a+1-\nu)\}(s).
\end{aligned}
$$

Therefore, as $f : \mathbb{N}_a \to \mathbb{R}$

$$
\begin{aligned}
f(\tau) =& f_a \left[h_{\nu-1}(\tau,a+1-\nu) + \lambda \left(h_{\nu-1}(\cdot,a+1-\nu) * h_{\nu-1}(\cdot,a+1-\nu) \right)(\tau) \right. \\
&\left. + \lambda^2 \left(h_{\nu-1}(\cdot,a+1-\nu) * h_{\nu-1}(\cdot,a+1-\nu) * h_{\nu-1}(\cdot,a+1-\nu) \right)(\tau) + \cdots \right]
\end{aligned}
$$

Thus we need to consider and find the k convolutions of the fractional Taylor monomial $h_{\nu-1}(\tau, a+1-\nu)$. We give this result in the following theorem.

Theorem 4.2.1. *For $f : \mathbb{N}_a \to \mathbb{R}$, any positive real number ν, and $\tau \in \mathbb{N}_a$*

$$
\underbrace{\left(h_{\nu-1}(\cdot,a+1-\nu) * h_{\nu-1}(\cdot,a+1-\nu) * h_{\nu-1}(\cdot,a+1-\nu) * \cdots \right)}_{k \text{ convolutions}}(\tau) = \Delta_a^{-k\nu} h_{\nu-1}(t_k, a+1-\nu),
$$

where $t_k \in \mathbb{N}_{a+k(\nu-1)}$.

Proof. We proceed by induction. For our initial base case let $k = 1$. So we consider

$$\begin{aligned}
&\left(h_{\nu-1}(\cdot, a+1-\nu) * h_{\nu-1}(\cdot, a+1-\nu)\right)(\tau) \\
&\overset{(2.3)}{=} \sum_{r=0}^{\tau-1} h_{\nu-1}(r, a+1-\nu) h_{\nu-1}(\tau - \sigma(r) + a, a+1-\nu) \\
&= \sum_{r=a}^{\tau-1} h_{\nu-1}(r, a+1-\nu) h_{\nu-1}(\underbrace{\tau - 1 + \nu}_{=t^* \in \mathbb{N}_{a+\nu-1}}, \sigma(r)) \\
&= \sum_{r=a}^{t^*-\nu} h_{\nu-1}(r, a+1-\nu) h_{\nu-1}(t^*, \sigma(r)) \\
&\overset{(3.2)}{=} \Delta_a^{-\nu} h_{\nu-1}(t^*, a+1-\nu).
\end{aligned}$$

Recall Definition 3.1.4 demands that the argument of the fractional sum be an element of $\mathbb{N}_{a+\nu}$. Since $t^* \in \mathbb{N}_{a+\nu-1}$, our result is fine since we have $\mathbb{N}_{a+\nu} \subset \mathbb{N}_{a+\nu-1}$ and we assume any element in $\mathbb{N}_{a+\nu-1}$ that is not in $\mathbb{N}_{a+\nu}$ has no effect on the summation or fractional sum in the last two lines. Hence the result for one convolution.

Of course only considering one convolution is sufficient for this induction proof, but it is more interesting to see at least two convolutions. Then let $k = 2$ so that there are two convolutions. Thus consider

$$\begin{aligned}
&\left(h_{\nu-1}(\cdot, a+1-\nu) * h_{\nu-1}(\cdot, a+1-\nu) * h_{\nu-1}(\cdot, a+1-\nu)\right)(\tau) \\
&= \sum_{r=a}^{\tau-1} h_{\nu-1}(\tau - \sigma(r) + a, a+1-\nu)\left(h_{\nu-1}(r, a+1-\nu) * h_{\nu-1}(r, a+1-\nu)\right) \\
&= \sum_{r=a}^{\tau-1} h_{\nu-1}(\tau - 1 + \nu, \sigma(r)) \sum_{s=a}^{r-1} h_{\nu-1}(s, a+1-\nu) h_{\nu-1}(r - \sigma(s) + a, a+1-\nu) \\
&= \sum_{r=a}^{\tau-1} h_{\nu-1}(\tau - 1 + \nu, \sigma(r)) \sum_{s=a}^{r-1} h_{\nu-1}(s, a+1-\nu) h_{\nu-1}(\underbrace{r - 1 + \nu}_{=t^* \in \mathbb{N}_{a+\nu-1}}, \sigma(s)) \\
&= \sum_{t^*=a+\nu-1}^{\tau-2+\nu} h_{\nu-1}(\tau - 1 + \nu, \sigma(t^* + 1 - \nu)) \sum_{s=a}^{t^*-\nu} h_{\nu-1}(s, a+1-\nu) h_{\nu-1}(t^*, \sigma(s))
\end{aligned}$$

$$= \sum_{t^*=a+\nu-1}^{\tau-2+\nu} h_{\nu-1}(\underbrace{\tau-2+2\nu}_{=t^{**}\in\mathbb{N}_{a+2\nu-2}}, \sigma(t^*)) \sum_{s=a}^{t^*-\nu} h_{\nu-1}(s, a+1-\nu) h_{\nu-1}(t^*, \sigma(s))$$

$$= \sum_{t^*=a+\nu-1}^{t^{**}-\nu} h_{\nu-1}(t^{**}, \sigma(t^*)) \sum_{s=a}^{t^*-\nu} h_{\nu-1}(s, a+1-\nu) h_{\nu-1}(t^*, \sigma(s))$$

$$= \sum_{t^*=a+\nu}^{t^{**}-\nu} h_{\nu-1}(t^{**}, \sigma(t^*)) \sum_{s=a}^{t^*-\nu} h_{\nu-1}(s, a+1-\nu) h_{\nu-1}(t^*, \sigma(s))$$

$$= [\Delta_{a+\nu}^{-\nu}(\Delta_a^{-\nu} h_{\nu-1}(t^{**}, a+1-\nu))](t^{**})$$

$$\overset{(3.7)}{=} \Delta_a^{-2\nu} h_{\nu-1}(t^{**}, a+1-\nu).$$

Recall Theorem 3.1.10 has the argument of the composition of fractional sums to be based at $a+2\nu$. Then our result is justified as the fractional sum's domain is $t^{**} \in \mathbb{N}_{a+2\nu-2}$ which is a subset of $\mathbb{N}_{a+2\nu}$. Again, any elements in $\mathbb{N}_{a+2\nu-2}$ that are not in $\mathbb{N}_{a+\nu}$ will have no impact on the sum and fractional sum above. Hence the result for two convolutions and we are now ready for our induction hypothesis.

Induction Hypothesis:

Assume $k = j$ so that we have j convolutions. Then

$$\underbrace{(h_{\nu-1}(\cdot, a+1-\nu) * h_{\nu-1}(\cdot, a+1-\nu) * h_{\nu-1}(\cdot, a+1-\nu) \cdots)}_{j \text{ convolutions}}(\tau) = \Delta_a^{-j\nu} h_{\nu-1}(t_j, a+1-\nu),$$

where $t_j \in \mathbb{N}_{a+j(\nu-1)}$ and $\tau \in \mathbb{N}_a$.

Let $k = j+1$ so that there are $j+1$ convolutions. Then by our induction hypothesis, we have

$$h_{\nu-1}(\tau, a+1-\nu) * \Delta_a^{-j\nu} h_{\nu-1}(t_j, a+1-\nu)$$

$$= h_{\nu-1}(\tau, a+1-\nu) * \sum_{s=a}^{t_j-j\nu} h_{j\nu-1}(t_j, \sigma(s)) h_{\nu-1}(s, a+1-\nu)$$

$$= h_{\nu-1}(\tau, a+1-\nu) * \sum_{s=a}^{\tau-j} h_{j\nu-1}(\tau - j + j\nu, \sigma(s)) h_{\nu-1}(s, a+1-\nu)$$

$$= \sum_{r=a}^{\tau-1} h_{\nu-1}(\tau - \sigma(r) + a, a+1-\nu) \sum_{s=a}^{r-j} h_{\nu-1}(s, a+1-\nu) h_{j\nu-1}(\underbrace{r - j + j\nu}_{=t_j \in \mathbb{N}_{a+j\nu-j}}, \sigma(s))$$

$$= \sum_{r=a}^{\tau-1} h_{\nu-1}(\tau - 1 + \nu, \sigma(r)) \sum_{s=a}^{r-j} h_{\nu-1}(s, a+1-\nu) h_{j\nu-1}(r - j + j\nu, \sigma(s))$$

$$= \sum_{t_j=a+j\nu-j}^{\tau-1-j+j\nu} h_{\nu-1}(\tau - 1 + \nu, \sigma(t_j + j - j\nu)) \sum_{s=a}^{t_j - j\nu} h_{\nu-1}(s, a+1-\nu) h_{j\nu-1}(t_j, \sigma(s))$$

$$= \sum_{t_j=a+j\nu-j}^{\tau-1-j+j\nu} h_{\nu-1}(\tau - 1 - j + (j+1)\nu, \sigma(t_j)) \sum_{s=a}^{t_j - j\nu} h_{\nu-1}(s, a+1-\nu) h_{j\nu-1}(t_j, \sigma(s))$$

$$= \sum_{t_j=a+j\nu-j}^{t_{j+1}-\nu} h_{\nu-1}(t_{j+1}, \sigma(t_j)) \sum_{s=a}^{t_j - j\nu} h_{\nu-1}(s, a+1-\nu) h_{j\nu-1}(t_j, \sigma(s))$$

$$= \sum_{t_j=a+j\nu}^{t_{j+1}-\nu} h_{\nu-1}(t_{j+1}, \sigma(t_j)) \sum_{s=a}^{t_j - j\nu} h_{\nu-1}(s, a+1-\nu) h_{j\nu-1}(t_j, \sigma(s))$$

$$= [\Delta_{a+j\nu}^{-\nu}(\Delta_a^{-j\nu} h_{\nu-1}(t_{j+1}, a+1-\nu))](t_{j+1})$$

$$\overset{(3.7)}{=} \Delta_a^{-(j+1)\nu} h_{\nu-1}(t_{j+1}, a+1-\nu).$$

Hence our claim is proven. □

Therefore, we can write our previous form of f in the new terms

$$\begin{aligned} f(\tau) =& f_a \left[h_{\nu-1}(\tau, a+1-\nu) + \lambda \left(h_{\nu-1}(\cdot, a+1-\nu) * h_{\nu-1}(\cdot, a+1-\nu) \right)(\tau) \right. \\ & \left. + \lambda^2 \left(h_{\nu-1}(\cdot, a+1-\nu) * h_{\nu-1}(\cdot, a+1-\nu) * h_{\nu-1}(\cdot, a+1-\nu) \right)(\tau) + \cdots \right] \\ =& f_a \sum_{k=0}^{\infty} \lambda^k \Delta_a^{-k\nu} h_{\nu-1}(t_k, a+1-\nu) \\ =& f_a \sum_{k=0}^{\infty} \lambda^k \Delta_a^{-k\nu} h_{\nu-1}(\tau + k(\nu-1), a+1-\nu) \\ =& f_a \sum_{k=0}^{\infty} \lambda^k \Delta_a^{-k\nu} h_{\nu-1}(\tau, a+(k+1)(1-\nu)). \end{aligned}$$

Now we use Theorem 3.1.9 to simplify our results. Realize the statement is still true algebraically as long as $\mu \notin (-\mathbb{N}_1)$ despite this being an unconventional use to allow $\mu < 0$. Then we have the final form of the solution, which is the same solution we found by recursion. Hence,

$$f(\tau) = f_a \sum_{k=0}^{\infty} \lambda^k h_{(k+1)\nu-1}(\tau, a + (k+1)(1-\nu)).$$

4.3. PROOF AND VALIDATION OF PROPOSED SOLUTION

We are now ready to prove our proposed solution from the methods of recursive pattern detection and using Laplace transforms.

Theorem 4.3.1. *The solution to the initial value problem*

$$\Delta_a^{\nu} f(t) = \lambda f(t + \nu - 1), \quad f(a) = f_a,$$

where $\lambda \neq -1$ is a real scalar, $0 < \nu \leq 1$, $t \in \mathbb{N}_{a+1-\nu}$, $\tau \in \mathbb{N}_a$, and $f : \mathbb{N}_a \to \mathbb{R}$ is given by

$$f(\tau) = f_a \sum_{k=0}^{\infty} \lambda^k h_{(k+1)\nu-1}(\tau, a + (k+1)(1-\nu)).$$

Proof. Assume $f :\to \mathbb{R}, \tau \in \mathbb{N}_a, t \in \mathbb{N}_{a+1-\nu}, f(a) = f_a \in \mathbb{R}$, and $0 < \nu \leq 1$. Suppose

$$f(\tau) = f_a \sum_{j=o}^{\infty} \lambda^j h_{(j+1)\nu-1}(\tau, a + (j+1)(1-\nu)).$$

Then by Theorem 3.1.6,

$$\begin{aligned}
\Delta_a^{\nu} f(t) &= \sum_{k=0}^{t+\nu-a} (-1)^k \binom{\nu}{k} f(t + \nu - k) \\
&= \sum_{k=0}^{t+\nu-a} (-1)^k \binom{\nu}{k} \left[f_a \sum_{j=o}^{\infty} \lambda^j h_{(j+1)\nu-1}(t + \nu - k, a + (j+1)(1-\nu)) \right]
\end{aligned}$$

$$= f_a \sum_{j=o}^{\infty} \lambda^j \sum_{k=0}^{t+\nu-a} (-1)^k \binom{\nu}{k} h_{(j+1)\nu-1}(t+\nu-k, a+(j+1)(1-\nu))$$

$$= f_a \sum_{j=o}^{\infty} \lambda^j \Delta_a^\nu h_{(j+1)\nu-1}(t, a+(j+1)(1-\nu))$$

$$= f_a \sum_{j=o}^{\infty} \lambda^j \Delta_a^\nu h_{(j+1)\nu-1}(t-j, a+1-(j+1)\nu)$$

$$= f_a \left[\Delta_a^\nu h_{\nu-1}(t, a+1-\nu) + \sum_{j=1}^{\infty} \lambda^j \Delta_a^\nu h_{(j+1)\nu-1}(t-j, a+1-(j+1)\nu) \right]$$

$$\overset{3.6}{=} f_a \left[\underbrace{h_{-1}(t, a+1-\nu)}_{=0} + \sum_{j=1}^{\infty} \lambda^j \Delta_a^\nu h_{(j+1)\nu-1}(t-j, a+1-(j+1)\nu) \right]$$

$$= f_a \sum_{j=1}^{\infty} \lambda^j \Delta_a^\nu h_{(j+1)\nu-1}(t-j, a+1-(j+1)\nu).$$

Then

$$\Delta_a^\nu f(t+1) = f_a \sum_{j=1}^{\infty} \lambda^j \Delta_a^\nu h_{(j+1)\nu-1}(t+1-j, a+1-(j+1)\nu)$$

$$= f_a \left[\lambda \Delta_a^\nu h_{2\nu-1}(t, a+1-2\nu) + \sum_{j=2}^{\infty} \lambda^j \Delta_a^\nu h_{(j+1)\nu-1}(t+1-j, a+1-(j+1)\nu) \right]$$

$$= f_a \left[\lambda h_{\nu-1}(t, a+1-2\nu) + \sum_{j=2}^{\infty} \lambda^j \Delta_a^\nu h_{(j+1)\nu-1}(t+1-j, a+1-(j+1)\nu) \right].$$

Hence

$$\Delta_a^\nu f(t) = f_a [\lambda h_{\nu-1}(t-1, a+1-2\nu) + \sum_{j=2}^{\infty} \lambda^j \Delta_a^\nu h_{(j+1)\nu-1}(t-j, a+1-(j+1)\nu)].$$

Now consider

$$\Delta_a^\nu f(t+2) = f_a [\lambda h_{\nu-1}(t+1, a+1-2\nu) + \lambda^2 \Delta_a^\nu h_{3\nu-1}(t, a+1-3\nu)$$

$$+\sum_{j=3}^{\infty}\lambda^j\Delta_a^{\nu}h_{(j+1)\nu-1}(t+2-j,a+1-(j+1)\nu)]$$

$$= f_a[\lambda+h_{\nu-1}(t+1,a+1-2\nu)+\lambda^2h_{2\nu-1}(t,a+1-3\nu)$$

$$+\sum_{j=3}^{\infty}\lambda^j\Delta_a^{\nu}h_{(j+1)\nu-1}(t+2-j,a+1-(j+1)\nu)].$$

Therefore

$$\Delta_a^{\nu}f(t) = f_a[\lambda h_{\nu-1}(t-1,a+1-2\nu)+\lambda^2h_{2\nu-1}(t-2,a+1-3\nu)$$
$$+\sum_{j=3}^{\infty}\lambda^j\Delta_a^{\nu}h_{(j+1)\nu-1}(t-j,a+1-(j+1)\nu)].$$

Then

$$\Delta_a^{\nu}f(t+3) = f_a[\lambda h_{\nu-1}(t+2,a+1-2\nu)+\lambda^2h_{2\nu-1}(t+1,a+1-3\nu)$$
$$+\lambda^3\Delta_a^{\nu}h_{4\nu-1}(t,a+1-4\nu)+\sum_{j=4}^{\infty}\lambda^j\Delta_a^{\nu}h_{(j+1)\nu-1}(t+3-j,a+1-(j+1)\nu)]$$
$$= f_a[\lambda h_{\nu-1}(t+2,a+1-2\nu)+\lambda^2h_{2\nu-1}(t+1,a+1-3\nu)$$
$$+\lambda^3h_{3\nu-1}(t,a+1-4\nu)+\sum_{j=4}^{\infty}\lambda^j\Delta_a^{\nu}h_{(j+1)\nu-1}(t+3-j,a+1-(j+1)\nu)].$$

Hence

$$\Delta_a^{\nu}f(t) = f_a[\lambda h_{\nu-1}(t-1,a+1-2\nu)+\lambda^2h_{2\nu-1}(t-2,a+1-3\nu)$$
$$+\lambda^3h_{3\nu-1}(t-3,a+1-4\nu)+\sum_{j=4}^{\infty}\lambda^j\Delta_a^{\nu}h_{(j+1)\nu-1}(t-j,a+1-(j+1)\nu)].$$

Therefore, based on the last three computations, we can see that

$$\Delta_a^{\nu}f(t) = f_a\left[\sum_{p=1}^{m}\lambda^p h_{p\nu-1}(t-p,a+1-(p+1)\nu)\right.$$

$$+ \sum_{j=m+1}^{\infty} \lambda^j \Delta_a^\nu h_{(j+1)\nu-1}(t-j, a+1-(j+1)\nu) \Bigg].$$

Then we can say

$$\Delta_a^\nu f(t) = f_a \sum_{p=1}^{\infty} \lambda^p h_{p\nu-1}(t-p, a+1-(p+1)\nu).$$

Re-indexing gives

$$\begin{aligned}
\Delta_a^\nu f(t) &= f_a \sum_{p=0}^{\infty} \lambda^{p+1} h_{(p+1)\nu-1}(t-(p+1), a+1-(p+2)\nu) \\
&= \lambda f_a \sum_{p=0}^{\infty} \lambda^p h_{(p+1)\nu-1}(t-(p+1), a+1-(p+2)\nu) \\
&= \lambda f_a \sum_{p=0}^{\infty} \lambda^p h_{(p+1)\nu-1}(t, a+(p+2)(1-\nu)) \\
&= \lambda f_a \sum_{p=0}^{\infty} \lambda^p h_{(p+1)\nu-1}(t+\nu-1, a+(p+1)(1-\nu)) \\
&= \lambda f(t+\nu-1).
\end{aligned}$$

Hence our solution is justified. □

4.4. NUMERICAL EXPERIMENTS

As we have now justified our result to be the solution to the initial value problem (4.1), we now show some numerical results of various initial value problems. Know that the code for the results of the following examples is in the appendix.

Example 4.4.1. Assume $f : \mathbb{N}_0 \to \mathbb{R}$ is the solution to (4.1), where $f(0) = 20$ and $\lambda = 0.35$. Table 4.1 and Figure 4.1 below show the solution to (4.1) for different values of ν.

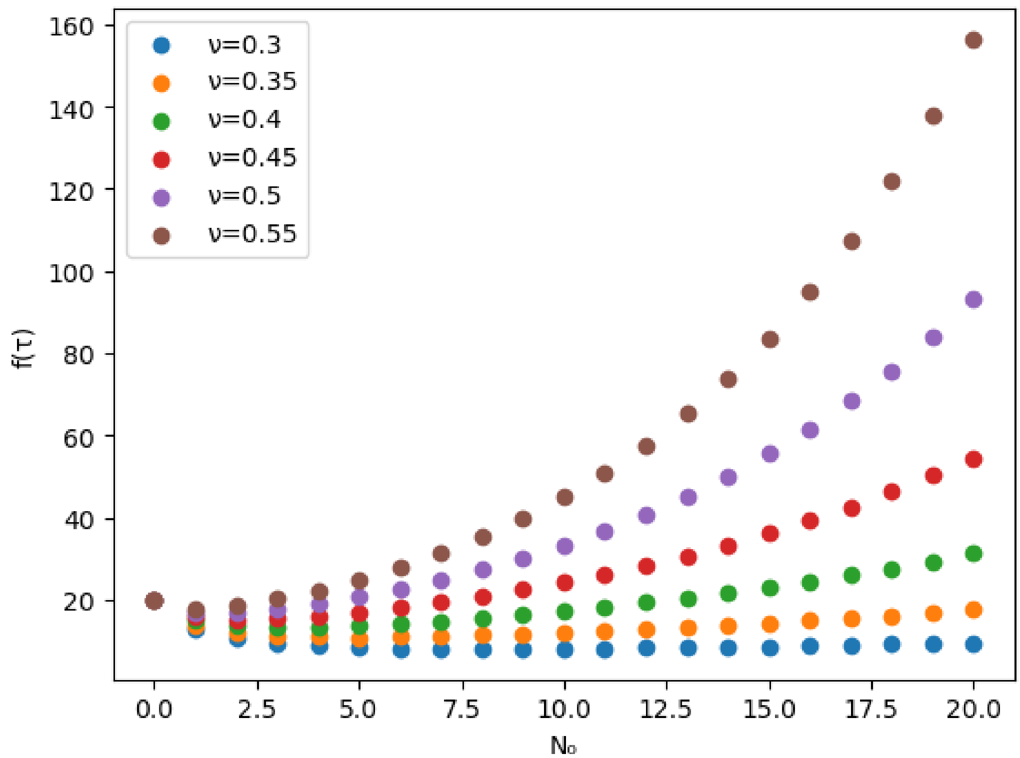

Figure 4.1. Some plotted solutions of Example 4.4.1 for different values of ν

Table 4.1. Computational results for Example 4.4.1 with various values of ν

ν	$\tau = 0.0$	$\tau = 1.0$	$\tau = 2.0$	$\tau = 3.0$	$\tau = 4.0$	$\tau = 5.0$
0.20	20.000000	11.000000	7.650000	6.047500	5.138125	4.557289
0.40	20.000000	15.000000	13.650000	13.317500	13.418125	13.758334
0.60	20.000000	19.000000	20.450000	22.827500	25.876125	29.562179
0.80	20.000000	23.000000	28.050000	34.737500	43.280125	54.078824
1.00	20.000000	27.000000	36.450000	49.207500	66.430125	89.680669

The main point we wish to emphasize is that our solution is not guaranteed to be monotonic. Perhaps this point can be better visualized in Figure 4.1. Figure 4.1 clearly indicates that when $0.35 \leq \nu \leq 0.55$, f is not a monotonic function. Another observation that can be made is that the order of the fractional difference ν seems to be serving as a growth rate or proportionality constant. As ν changes values, the difference in any two sequential values of f seems to be dependent on the value ν. This is interesting since λ is usually the only parameter that affects the growth rate of the function under traditional ordinary difference orders.

Example 4.4.2. Suppose $f : \mathbb{N}_3 \to \mathbb{R}$ is the solution to (4.1), with $f(3) = \pi$ and $\lambda = -0.85$. Table 4.2 and Figure 4.2 show the solution to (4.1) for different values of ν.

Table 4.2. Computational results for Example 4.4.2 with various values of ν

ν	$\tau = 3.0$	$\tau = 4.0$	$\tau = 5.0$	$\tau = 6.0$	$\tau = 7.0$	$\tau = 8.0$
0.60	3.141593	-0.785398	0.573341	-0.061654	0.145790	0.033651
0.70	3.141593	-0.471239	0.400553	0.033379	0.097802	0.048978
0.80	3.141593	-0.157080	0.259181	0.075006	0.067250	0.043554
0.90	3.141593	0.157080	0.149226	0.066366	0.039839	0.025674
1.00	3.141593	0.471239	0.070686	0.010603	0.001590	0.000239

Example 4.4.2 shows another interesting feature of our solution. When $\lambda < 0$, f oscillates between positive and negative values. The only exception to this observation seems to be when $\nu = 1$. This is what we would expect though given our understanding of ordinary orders of difference. Nonetheless, we see convergence of f for any $0 < \nu \leq 1$ since we can see $\lim_{\tau \to \infty} f(\tau) = 0$.

Example 4.4.3. Consider $f : \mathbb{N}_0 \to \mathbb{R}$, $f(0) = 0.1$, and $\lambda = -1$. Assume f is the solution to (4.1) and $0 < \nu < 1$. We do not let $\nu = 1$ since this would violate Theorem 2.2.6. Table 4.3 and Figure 4.3 show the solution to (4.1) for different values of ν.

The purpose of Example 4.4.3 is to show that, where we must have $\lambda \neq -1$ when $\nu = 1$, it is plausible for $\lambda = -1$ when $0 < \nu < 1$. Also realize, according to Figure 4.3 and Table 4.3, f does not become the trivial solution when $0 < \nu < 1$. This is an interesting observation since when $\nu = 1$ and $\lambda = -1$, f does become the trivial solution.

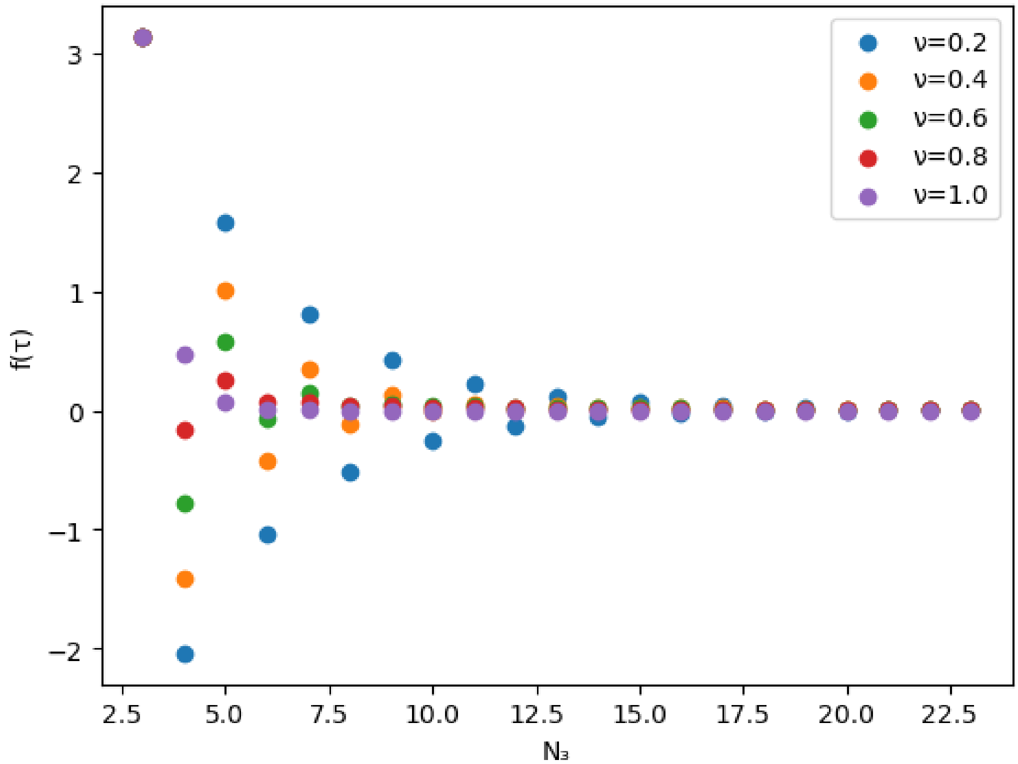

Figure 4.2. Some plotted solutions of Example 4.4.2 for different values of ν

Table 4.3. Computational results for Example 4.4.3 with various values of ν

ν	$\tau = 0.0$	$\tau = 1.0$	$\tau = 2.0$	$\tau = 3.0$	$\tau = 4.0$	$\tau = 5.0$
0.10	0.100000	-0.090000	0.085500	-0.078150	0.073684	0.067643
0.30	0.100000	-0.070000	0.059500	-0.043050	0.036234	-0.026183
0.50	0.100000	-0.050000	0.037500	-0.018750	0.014844	-0.006641
0.70	0.100000	-0.030000	0.019500	-0.004450	0.004634	-0.000028
0.90	0.100000	-0.010000	0.005500	0.000650	0.000884	0.000482

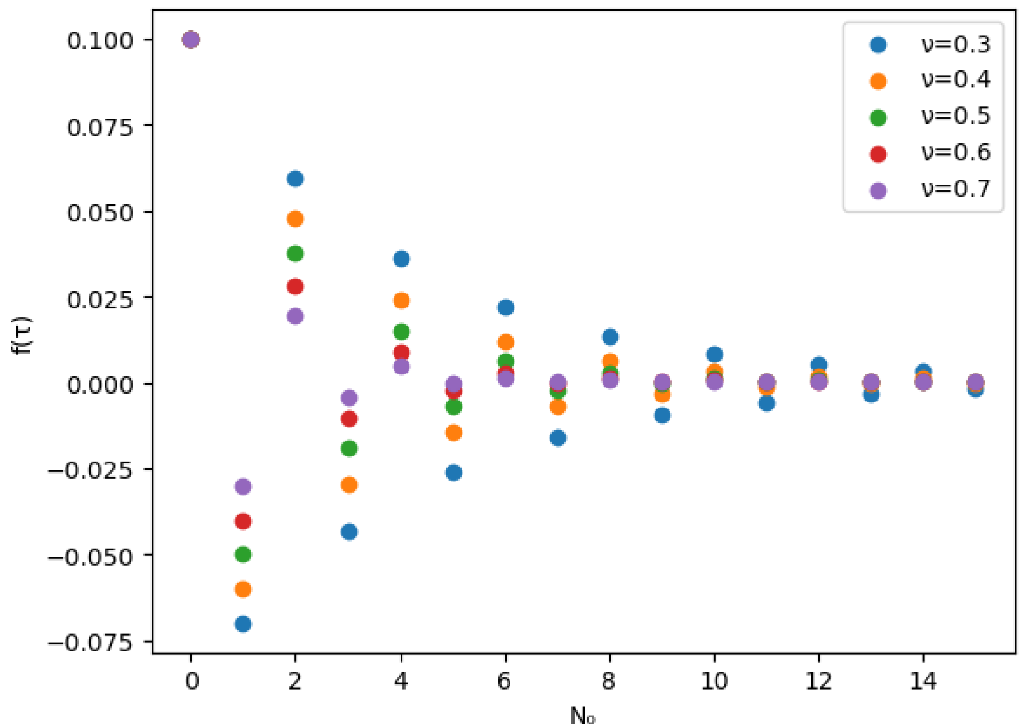

Figure 4.3. Some plotted solutions of Example 4.4.3 for different values of ν

5. CONCLUSIONS AND FUTURE WORK

In this thesis, we have introduced the preliminaries of ordinary delta calculus and the preliminaries of fractional calculus. We then shared original results, solving a constant coefficient exponential problem in discrete fractional calculus by recursive pattern finding and using Laplace transforms.

We are proud of the original results shared in the fourth chapter, but we see a great potential for future research in delta fractional calculus that can be built from this work. Rigorous analysis is needed to better understand the relationships between ν, λ, and the behavior of f. The numerical experiments show that there is a great deal of attention needed to be given to know when f will be monotonic or not. While we have solved the homogeneous initial value problem, one could easily consider a nonhomogeneous initial value problem. Another avenue of research to consider is a higher order initial value problem where $N - 1 < \nu \leq N$ for any $N \in \mathbb{N}_1$. Lastly, one could consider coefficients that are more generalized instead of being restricted to constants.

Finally, we wonder if there is an application of our work to quantitative finance. Recall that the inspiration for this work came from considering using discrete fractional calculus in quantitative finance. Now that we have solved this exponential problem we can consider using our solution to model security values.

APPENDIX

CODE FOR NUMERICAL EXPERIMENTS

We now share our code for the computation of the solution to the initial value problem (4.1). This code is in Python and we use the fact that we can compute the solution on $\mathbb{N}_0$ in stead of $\mathbb{N}_a$ since the Taylor monomials in our solution remove the shifting factor a. However, we have it so that the results are displayed on $\mathbb{N}_a$.

```
import numpy as np
import math
import pandas as pd
import matplotlib.pyplot as plt

def fractional_Taylor_monomial(MU,T,A): #parameters of Taylor monomial
    try:
        monomial=math.gamma(T-A+1)/(math.gamma(MU+1)*math.gamma(T-A+1-MU
    ))
    except:
        monomial=0
    return monomial

def theorem(t):
    tot=0
    for k in range(0,t+1):
        monomial=fractional_Taylor_monomial((k+1)*v-1, t,(k+1)*(1-v))
        tot+=Lambda**k*monomial
    return f[1,0]*tot

#Parameters
a=0.75 #initial point elemnt in domain
```

```
b=10 #end domain
v=0.5 #fractional difference order
Lambda=0.35 #lambda value
f=np.zeros([2,b+1]) #allocate space for computation
f[0,:]=np.arange(a,a+b+1,1)#domain for delta differences
f[1,0]=200 #set the inital value

#Calculuates the fractional difference with our theorem
for t in range(1,b+1):
    f[1,t]=theorem(t)
#to display results
plt.scatter(f[0,:],f[1,:], label="Fractional Delta")
plt.xlabel('Domain N_{}'.format(a))
plt.ylabel('f(t)')
plt.title('f(t) vs Domain N_{}; f_{}={}'.format(a,a,f[1,0]))
plt.legend()
plt.show
df = pd.DataFrame(np.transpose(f), columns =['Domain N_a', 'Fractional
   Delta'])
print(df)
```

Listing 6.1. Python Code to Numerical Computation of Solution

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VITA

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