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Continuous and discrete models for optimal harvesting in fisheries

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CONTINUOUS AND DISCRETE MODELS FOR OPTIMAL HARVESTING IN FISHERIES

by

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ABSTRACT

This work focuses on the logistic growth model, where the Gordon–Schaefer model is considered in continuous time. We view the Gordon–Schaefer model as a bioeconomic equation involved in the fishing business, considering biological rates, carrying capacity, and total marginal costs and revenues. In [25], the authors illustrate the analytical solution of the Schaefer model using the integration by parts method and two theorems. The theorems have many assumptions with many different strategies. Due to the nature of the problem, the optimal control system involves many equations and functions, such as the second root of the equation. We concentrate on Theorem 1, where we re-illustrate it with more details and clarifications. We present the four methods for explaining such an optimal path, where the optimal choice of the four strategies generally depends on the particular applications. Also, we provide the Schaefer model's solution by the Euler–Lagrange equation. This thesis also illustrates the Beverton–Holt model and its solution by the Euler–Lagrange equation. The Beverton–Holt model serves as a classical population model considered in the literature for the discrete-time case of the logistic model.

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TABLE OF CONTENTS

Page

LIST OF ILLUSTRATIONS

1. INTRODUCTION

Mathematical equations and functions are the scientific methods to display natural phenomena like thermal diffusion and population growth. This thesis focuses on two bioeconomic models, the Schaefer and the Beverton–Holt models. Both are considered logistic growth models in different ways. Pierre François Verhulst, Figure 1.1, adjusted the exponential growth model, see Section 2. He published some papers between 1838 and 1847, where he introduced the logistic growth model, which is defined as an accurate equation that assumes that the relative growth rate decreases when the population approaches the carrying capacity of the environment, i.e.,

$$\frac{\mathrm{d}N}{\mathrm{d}s} = rN\left(1 - \frac{N}{K}\right), \tag{1.1}$$

where $N = N(s)$ is the amount of the biomass at time s, $K > 0$ is carrying capacity of the environment, and $r > 0$ is the intrinsic growth rate. Substantially, we suppose an initial time $s = 0$, where $N_0 = N(0) > 0$, which indicates $N(0)$ is positive. Technically, it has to be a population that creates new generations. Mathematically, any actual continuous-time population model is supposed to contain a level of population, $N(s)$, which begins from a positive level, i.e., $N_0 > 0$. This level assumes to stay positive as $s \to \infty$ to be an initial condition which is used to solve the logistic growth equation, see Figure 1.2 and Figure 1.3. Conversely, if $N_0 = N(0) = 0$, or $N_0 = N(0) < 0$, then the model is used for hypothetical reason. Clarifying (1.1), we see that on the left-hand side, we have the derivative, which represents the rate of change of the population level N within time s. We see the growth calculation on the right-hand side according to the rate, the environment carrying capacity, and how much population the environment already has. By understanding both sides, we find if N is getting closer to K, then the term $\frac{N}{K}$ gets closer to 1, which means the term

Figure 1.1. Pierre François Verhulst.

$\left(1-\frac{N}{K}\right)$ approaches 0. As a result, the term $\frac{dN}{ds}$ equals to 0, i.e.,

$$0=\frac{dN}{ds}=rN\left(1-\frac{N}{K}\right),$$

which indicates the population would stop growing as soon as the population's density is getting closer to the environment's carrying capacity, as we mentioned in the definition of the logistic growth model; this case is called the equilibrium case. The main purpose for assuming the equilibrium case is to make the study simpler. There are two equilibrium points we can determine:

- $N=0$ indicates there is no population.
- $N=K$ indicates the population is equal to the maximum carrying capacity of the environment.

In conclusion, if $0<N_0<K$, then the stock level is below the level of K, which means the population keeps growing. Otherwise, the curve approaches the equilibrium level K, where $s\to\infty$, and $\frac{dN}{ds}\to 0$.

1.1. LOGISTIC FUNCTION

Let us solve (1.1). We rewrite (1.1) as

$$\frac{\mathrm{d}N}{N\left(1-\frac{N}{K}\right)} = r\mathrm{d}s$$

and use the partial fraction decomposition method on the left-hand side to get

$$\left(\frac{1}{N} - \left(\frac{-\frac{1}{K}}{1-\frac{N}{K}}\right)\right)\mathrm{d}N = r\mathrm{d}s,$$

i.e.,

$$\frac{\mathrm{d}N}{N} - \left(\frac{-\frac{1}{K}}{1-\frac{N}{K}}\right)\mathrm{d}N = r\mathrm{d}s.$$

If we integrate the left-hand side with respect to N and the right-hand side with respect to s, then we obtain

$$\ln|N| - \ln\left|1-\frac{N}{K}\right| + C_1 = rs + C_2, \tag{1.2}$$

since

$$\frac{\mathrm{d}}{\mathrm{d}N}\ln|N| = \frac{1}{N}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}N}\ln\left|1-\frac{N}{K}\right| = \frac{-\frac{1}{K}}{1-\frac{N}{K}}.$$

If $0 < N(s) < K$, then (1.2) yields

$$\ln(N) - \ln\left(1-\frac{N}{K}\right) = rs + C_3,$$

where $C_3 = C_2 - C_1$. By the properties of the natural logarithm function, we get

$$\ln\left(\frac{N}{1-\frac{N}{K}}\right) = rs + C_3.$$

Hence,

$$\frac{N}{1-\frac{N}{K}} = C_4 e^{rs},$$

where $C_4 = e^{C_3}$, since

$$e^{\ln\left(\frac{N}{1-\frac{N}{K}}\right)} = \frac{N}{1-\frac{N}{K}}.$$

Thus, we have

$$\frac{1-\frac{N}{K}}{N} = Ce^{-rs}, \tag{1.3}$$

where $C = \frac{1}{C_4}$. We use the condition $N(0) = N_0$ to find

$$C = \frac{1-\frac{N_0}{K}}{N_0} = \frac{1}{N_0} - \frac{1}{K}.$$

If we substitute C in (1.3), then we obtain

$$\frac{1-\frac{N}{K}}{N} = \left(\frac{1}{N_0} - \frac{1}{K}\right) e^{-rs},$$

i.e.,

$$1 - \frac{N}{K} = N\left(\frac{1}{N_0} - \frac{1}{K}\right) e^{-rs},$$

i.e.,

$$1 = N\left\{\frac{1}{K} + \left(\frac{1}{N_0} - \frac{1}{K}\right) e^{-rs}\right\},$$

and solving for N yields

$$N(s) = \frac{1}{\frac{1}{K} + \left(\frac{1}{N_0} - \frac{1}{K}\right) e^{-rs}} = \frac{N_0 K}{(K - N_0)e^{-rs} + N_0}. \tag{1.4}$$

If $N(s) > K$, then (1.2) yields

$$\ln(N) - \ln\left(\frac{N}{K} - 1\right) = rs + C_3,$$

where $C_3 = C_2 - C_1$. By using the properties of the natural logarithm function, we get

$$\ln \frac{N}{\frac{K}{N} - 1} = rs + C_3.$$

Hence,

$$\frac{N}{\frac{N}{K} - 1} = C_4 e^{rs},$$

where $C_4 = e^{C_3}$. Thus, we have

$$\frac{\frac{N}{K} - 1}{N} = Ce^{-rs}, \tag{1.5}$$

where $C = \frac{1}{C_4}$. We use the condition $N(0) = N_0$ to find

$$C = \frac{\frac{N_0}{K} - 1}{N_0} = \frac{1}{K} - \frac{1}{N_0}.$$

If we substitute C in (1.5), then we obtain

$$\frac{\frac{N}{K} - 1}{N} = \left(\frac{1}{K} - \frac{1}{N_0}\right) e^{-rs},$$

i.e.,

$$\frac{N}{K} - 1 = N\left(\frac{1}{K} - \frac{1}{N_0}\right) e^{-rs},$$

i.e.,

$$N\left\{\frac{1}{K} + \left(\frac{1}{N_0} - \frac{1}{K}\right) e^{-rs}\right\} = 1,$$

and solving for N yields once again (1.4), see Figure 1.2.

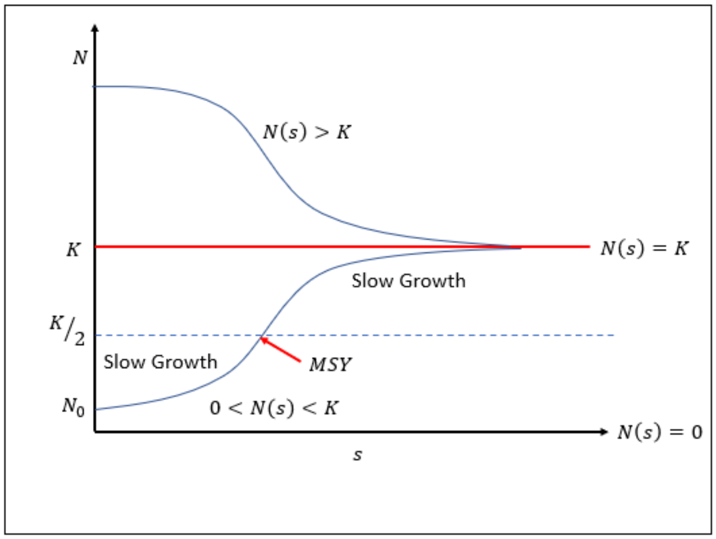

Figure 1.2. Logistic Growth Curve.

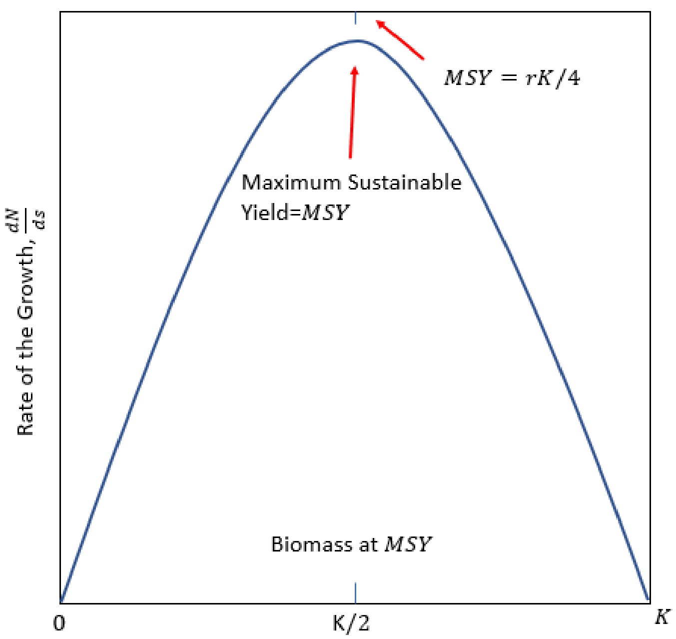

Figure 1.3. Population Growth Curve.

1.2. THE LOGISTIC MODEL WITH HARVESTING TERM (BIO-ECONOMIC MODEL)

A harvesting term, Y, is taken into account for the logistic growth model, see the Gordon–Schaefer (1954) model in [24], i.e.,

$$\frac{\mathrm{d}N}{\mathrm{d}s} = rN\left(1 - \frac{N}{K}\right) - Y \tag{1.6}$$

with

$$Y(s) = qN(s)E(s),$$

where $E(s)$ is the catching effort and q is the catchability of the stock. By assuming the equilibrium case, the population's growth amount equals to the harvesting amount, so that

$$0 = rN\left(1 - \frac{N}{K}\right) - Y.$$

This yields

$$0 = rN - \frac{rN^2}{K} - Y = -\frac{r}{K}N^2 + rN - Y,$$

i.e.,

$$N^2 - KN + \frac{KY}{r} = 0. \tag{1.7}$$

Considering the discriminant

$$\Delta = K^2 - \frac{4KY}{r},$$

we discuss the following three different scenarios.

1. If $\Delta > 0$, then (1.7) has the two roots

$$N_{\min} = \frac{K + \sqrt{K^2 - \frac{4KY}{r}}}{2}, \quad N_{\max} = \frac{K - \sqrt{K^2 - \frac{4KY}{r}}}{2}.$$

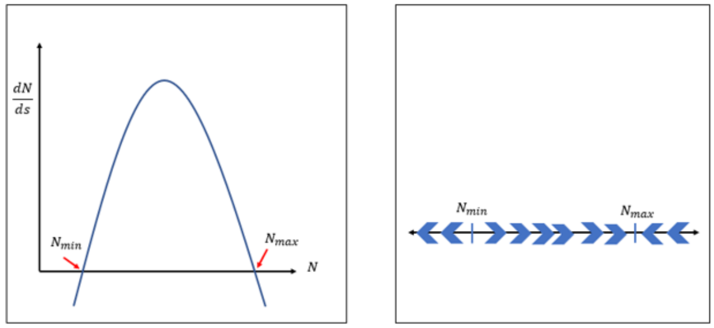

Figure 1.4. The Equilibrium Root $N_{\min}$.

As we see in Figure 1.4, if $N \in (0, N_{\min})$ and $N \in (N_{\max}, \infty)$, then $\frac{dN}{ds}$ is negative. If $N \in (N_{\min}, N_{\max})$, then $\frac{dN}{ds}$ is positive. In this case, since $\Delta > 0$, we have

$$Y < \frac{Kr}{4}.$$

This case provides an initial population condition (positive), which means the population is in equilibrium and the harvesting is moderate. We find that $N_{\min}$ is unstable, while $N_{\max}$ is asymptotically stable.

2. If $\Delta = 0$, then (1.7) has one root

$$N_* = \frac{K}{2}.$$

As Figure 1.5 illustrates, if $N_* \in (0, N_*) \cup (N_*, \infty)$, then $\frac{dN}{ds}$ is negative. In this case, since $\Delta = 0$, we have

$$Y = \frac{Kr}{4}.$$

This case means there is no conclusion observed; it is assumed the harvesting is a fixed number of an exact population amount, which is impossible in a realistic situation. In addition, there are many factors other than harvesting that impact the population, and we have no control over them.

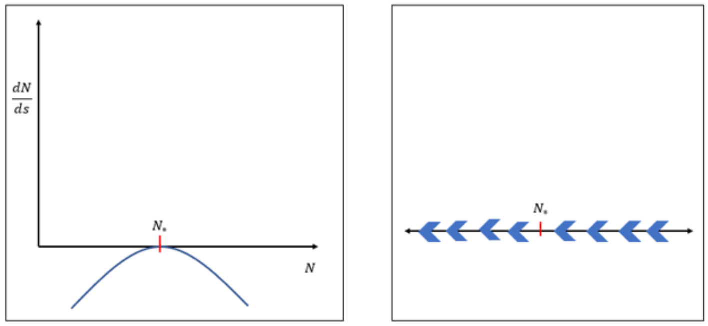

Figure 1.5. The Equilibrium Root N_*.

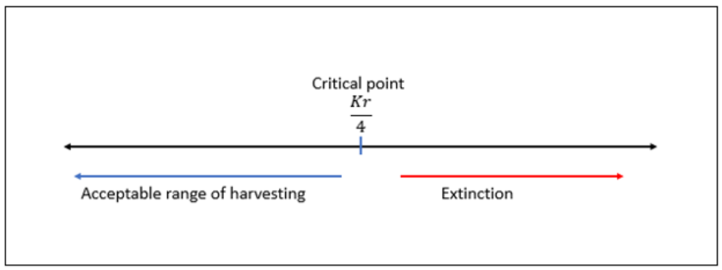

Figure 1.6. The Acceptable Range of Harvesting.

3. If $\Delta < 0$, then (1.7) has no real root. Then the population goes to the extinction situation, which indicates the harvesting is exceptionally high. In this case, since $\Delta < 0$, we have

$$Y > \frac{Kr}{4}.$$

In conclusion, there are ranges where harvesting is acceptable, but no exact harvest is observed in realistic situations. Of course, if the population gets harvested more than the acceptable range, then the population would disappear, i.e., $N_0 > N_{\max}$, see Figure 1.6.

1.3. THE PROFIT FUNCTION

In this section, we discuss the financial part of the harvesting model. We introduce the idea of the commercial variable explicitly, and we shape the harvesting by using the classical population economic Gordon–Schaefer model. Recall

$$Y(s) = qN(s)E(s),$$

where $Y(s)$ defines the harvesting term in proportion to the population. So harvesting is proportional to the size of the population, where $E(s)$ is the effort that changes over time, based on profit. The profit is total revenue T_R minus total cost of the effort T_C, i.e.,

$$\text{Profit} = T_R - T_C.$$

For clarification, we represent the profit function as

$$\text{Profit} = P(s) \cdot Y(s) - C(s) \cdot E(s),$$

where $P(s)$ is the market price of the resource, $Y(s)$ is the harvested amount, $C(s)$ is the cost per unit of effort, and $E(s)$ is the effort of the harvesting like marketing and shipping.

1.4. MAXIMUM SUSTAINABLE YIELD

MSY is also called maximum surplus production, maximum equilibrium catch, maximum constant yield, and maximum sustained yield [14, 21, 28]. MSY is the highest technical equilibrium yield obtained continuously from a stock under existing (average) environmental conditions. In other words, it is the highest catch that still leaves the population to sustain itself indefinitely through somatic growth and spawning [33], see Figure 1.3. The benefit of knowing MSY is to comprehend how much can be taken out of the population without causing it to deflate. MSY was introduced by Milner Baily Schaefer [30]. Considering the first derivative (the slope) of the curve over the corresponding biomass (the collective weight of the individuals at a particular time), we can explain the increase in biomass (termed surplus production or yield) within time, in the form of a parabolic curve. The idea of the parabola is simple. Initially, the population grows exponentially, unrestricted by environmental conditions, but as population size approaches the carrying capacity, the growth slows down and eventually seizes. At the opposite end, where the population is at the ecosystem's carrying capacity for this stock, there is no surplus production by definition, and thus again, zero yields are obtained. Removing this maximum surplus stock, in some way, limits the population from growing any further, keeping it at half of the maximum population size, producing maximum surplus forever. Hence, this is the circumstance of MSY. The resource stock level corresponding to the maximum sustainable yield, $N_{\text{MSY}}(s)$, is taken from the logistic growth equation (1.1). Differentiating (1.1) on both sides yields

$$\frac{\mathrm{d}^2N}{\mathrm{d}s^2} = rN'(s) - \frac{2rN(s)N'(s)}{K} = r\left(1 - \frac{2N(s)}{K}\right)N'(s).$$

Setting this equal to zero results in

$$1 - \frac{2N(s)}{K} = 0 \quad \text{or} \quad N'(s) = 0,$$

so that

$$N_{\text{MSY}}(s) = \frac{K}{2}.$$

In addition, Y_{MSY} is obtained by substituting N_{MSY} into (1.1), and thus, we get

$$\begin{aligned} Y_{\text{MSY}} &= rN_{\text{MSY}}\left(1 - \frac{N_{\text{MSY}}}{K}\right) \\ &= r\frac{K}{2}\left(1 - \frac{\frac{K}{2}}{K}\right) \end{aligned}$$

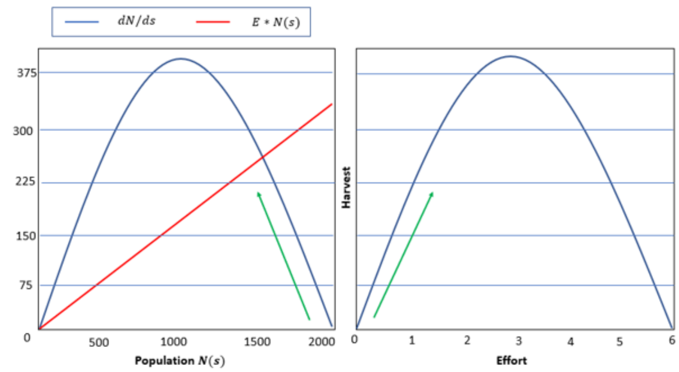

Figure 1.7. Effort Yield Curve.

$$
\begin{aligned}
&= r\frac{K}{2}\left(1-\frac{1}{2}\right)\\
&= \frac{rK}{4}.
\end{aligned}
$$

1.5. EFFORT–YIELD CURVE

If we assume the equilibrium case in (1.6), then we get

$$0 = rN(s)\left(1-\frac{N(s)}{K}\right) - Y(s),$$

i.e.,

$$Y(s) = rN(s)\left(1-\frac{N(s)}{K}\right), \tag{1.8}$$

where

$$Y(s) = qN(s)E(s). \tag{1.9}$$

If we substitute Y in (1.8), then we obtain

$$qN(s)E(s) = rN(s)\left(1-\frac{N(s)}{K}\right). \tag{1.10}$$

Solving (1.10) for $N(s) > 0$ in terms of $E(s)$ yields

$$N(s) = K\left(1 - \frac{q}{r}E(s)\right). \tag{1.11}$$

Substituting (1.11) into (1.9) gives

$$Y(s) = qK\left(1 - \frac{qE(s)}{r}\right)E(s),$$

which is known as the effort yield curve, see Figure 1.7. The effort yield curve is a reflecting object of the shape of the sustainable yield curve. In other words, each point on the effort curve is the sustainable harvest obtained by applying a specific level of effort. Technically, if the fishing effort equals to 0, then that effort reflects on the slope of the harvesting (the red line in Figure 1.7), which means there is no harvesting. Conversely, if the effort is increased, then the slope of the harvest line also is increased.

2. LITERATURE REVIEW

2.1. HISTORY OF THE LOGISTIC GROWTH MODEL

Thomas Robert Malthus suggested a simple exponential growth model, see Figure 2.1, which has been named after him. The Malthusian growth model is

$$\frac{\mathrm{d}N}{\mathrm{d}s} = rN. \tag{2.1}$$

Separating the variables in (2.1) yields

$$\frac{\mathrm{d}N}{N} = r\mathrm{d}s,$$

and integrating gives

$$\int \frac{\mathrm{d}N}{N} = \int r\mathrm{d}s,$$

i.e.,

$$\ln|N| = rs + C_1,$$

i.e., by assuming $N > 0$,

$$N = Ce^{rs},$$

where $C = e^{C_1}$. Thus, $C = N(0)$, and we get the solution of (2.1) as

$$N(s) = N_0 e^{rs}. \tag{2.2}$$

In 1798, Malthus wrote an essay [18, 22] where he mentioned, "Through the animal and vegetable kingdoms, nature has scattered the seeds of life abroad with the most profuse and liberal hand. The germs of existence contained in this spot of earth, with ample food, and ample room to expand in, would fill millions of worlds in the course of a few thousand years.

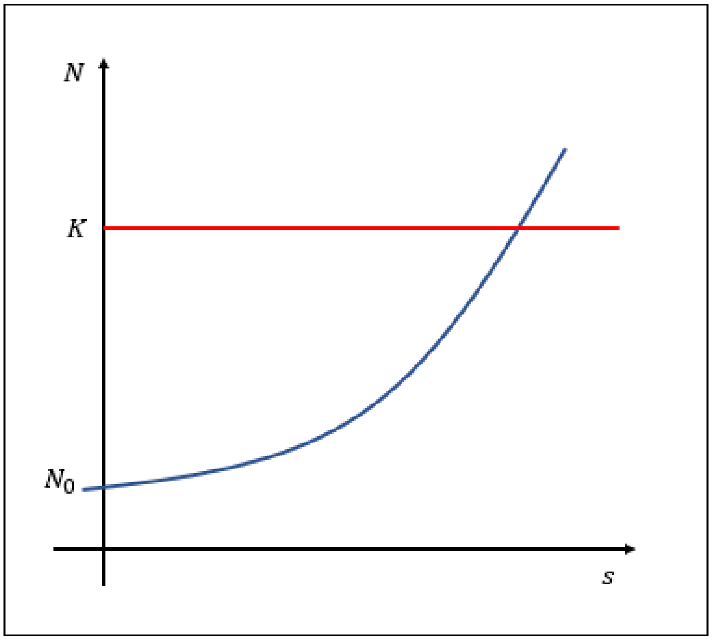

Figure 2.1. Exponential Growth Curve.

Necessity, that imperious all pervading law of nature, restrains them within the prescribed bounds. The race of plants and the race of animals shrink under this great restrictive law. And the race of man cannot, by any efforts of reason, escape from it. Among plants and animals, its effects are waste of seed, sickness, and premature death. Among mankind, misery, and vice."

Pierre François Verhulst had read Malthus' essay. In 1838, he developed a model of population growth bounded by resource limitations. Verhulst named the model as the logistic model, see (1.1).

2.2. SOME APPLICATIONS OF THE LOGISTIC MODEL

Verhulst derived his logistic equation to explain the self-limiting growth of a biological population. The model was re-created in 1911 by Anderson Gray McKendrick to progress bacteria in broth and experimentally tested applying a technique for nonlinear parameter estimation [23]. The model is also called the Verhulst–Pearl equation, following its rediscovery in 1920 by Raymond Pearl (1879–1940) and Lowell Reed (1888–1966) of Johns Hopkins University [27]. Another scientist, Alfred James Lotka, derived the equation again in 1925, calling it the law of population growth. The logistic model is used in many different fields. In this thesis, we focus on the fishery management field, and there are two more fields that we would like to mention briefly as follows.

1. In Chemotherapy (modeling the growth of tumors), a logistic differential equation with a time-varying periodic parameter is applied to model the growth of cells, especially cancer cells, with chemotherapeutic medications [26]. The chemotherapeutic effects are modeled by a periodic parameter that adjusts the growth rate of the cell tissue. A negative growth rate describes the harmful effects of the medications. A simple pattern is obtained for the performance of the chemotherapy.

2. In Marketing, the logistic growth model corresponds to the growth of the number of product users observed over time in a closed market, without the occupation of any other product [15]. The model is defined with three parameters: K is market capacity, a is the growth rate parameter, and b is the time shift parameter, i.e.,

$$L(K, a, b; t) = \frac{K}{1 + e^{-a(t-b)}}.$$

3. THE SCHAEFER MODEL (CONTINUOUS CASE)

The primary concern of fishery management is to perform superior sustainable harvesting, see [4, 8, 10, 20, 29], with less effort, could be, by guiding harvesting strategies away with no extinction possibility. As we mentioned, the existence, the extinction, and the harvesting policy are considered critical concepts. In other words, the economy is not accurate enough since there are many restrictions the harvester has no control over, such as, earthquakes, extensive tankers leaking oil, etc. The Schaefer model [9, 10, 20, 31] is of the form

$$\frac{\mathrm{d}N}{\mathrm{d}s} = rN(s)\left(1 - \frac{N(s)}{K}\right) - Y(s), \tag{3.1}$$

where $N(s)$ is the population biomass of fish at time s, r is the intrinsic growth rate of the population, K is the carrying capacity, and $Y(s)$ is the harvesting function. We suppose that $r \geq 0$, and $K > 0$ is constant, so that

$$Y(s) = qN(s)U(s), \tag{3.2}$$

where $q \geq 0$ is the catchability factor of the stock, known as the fish population, which means the fish that harvesters can hunt by effort unit. $U \geq 0$ is the effort of the human activity to the population fished. In other words, U is all kinds of costs associated with running the harvesting, such as, the cost of the gear and human labor hourly payment.

3.1. CASE STATEMENT

We consider (3.1), (3.2) subject to constraints, i.e.,

$$\begin{cases} \frac{dX}{ds} = rX(s)\left(1 - \frac{X(s)}{K}\right) - qX(s)U(s), \\ X(0) = X_0 > 0, \quad X(\bar{T}) = X_{\bar{T}} > 0, \\ 0 < X_{\min} \leq X(s) \leq K, \quad 0 \leq U(s) \leq U_{\max}, \\ \bar{J}(X,U) = \int_0^{\bar{T}} e^{-\delta s}\left[q\bar{p}(s)U(s)X(s) - c(U(s))\right] ds \longrightarrow \max, \end{cases} \tag{3.3}$$

where $\bar{J}(X,U)$ is the discounted profit, $\delta > 0$ is the discount factor, $\bar{p}$ is the price function, and c is the harvesting cost. Suppose c is a linear function, i.e., $c(U) = cU$.

3.2. EXPONENTIAL DISTRIBUTION OF THE PRICE FUNCTION

Assume the price function is given by

$$\bar{p}(s) = \begin{cases} p_0, & s < \tau, \\ p_0 + \theta, & s \geq \tau, \end{cases}$$

where $p_0 > 0$ and $\theta > 0$. Suppose τ is an exponentially distributed random variable with parameter $\gamma > 0$, i.e.,

$$P(\tau \leq s) = F(s) = 1 - e^{-\gamma s}.$$

The expected value of the price function is calculated as

$$\begin{aligned} \mathbb{E}(\bar{p}(s)) &= p_0 P(s < \tau) + (p_0 + \theta)P(s \geq \tau) \\ &= p_0(1 - F(s)) + (p_0 + \theta)F(s) \\ &= p_0 - p_0 F(s) + p_0 F(s) + \theta F(s) \\ &= p_0 + \theta F(s) \end{aligned}$$

$$= \quad p_0 + \theta(1 - e^{-\gamma s}).$$

Thus,

$$\begin{aligned}
\mathbb{E}(\bar{J}(X,U)) &= \int_0^{\bar{T}} e^{-\delta s} \left[\mathbb{E}\left(\bar{p}(s)\right) qU(s)X(s) - cU(s)\right] \mathrm{d}s \\
&= \int_0^{\bar{T}} e^{-\delta s} \left[\left(p_0 + \theta\left(1 - e^{-\gamma s}\right)\right) qU(s)X(s) - cU(s)\right] \mathrm{d}s \\
&= \int_0^{\bar{T}} e^{-\delta s} \left[p_0\left(1 + \beta_1\left(1 - e^{-\gamma s}\right)\right) qX(s) - c\right] U(s)\mathrm{d}s,
\end{aligned}$$

where

$$\beta_1 := \frac{\theta}{p_0} > 0.$$

3.3. CHANGE OF VARIABLES

Assume (X, U) solves (3.3). If we perform the change of variables

$$t = rs, \quad x(t) = \frac{X(s)}{K}, \quad u(t) = \frac{qU(s)}{r}, \quad T := r\bar{T},$$

then x satisfies

$$\begin{aligned}
x'(t) &= \frac{\frac{\mathrm{d}}{\mathrm{d}t} X\left(\frac{t}{r}\right)}{K} \\
&= \frac{\frac{1}{r} X'\left(\frac{t}{r}\right)}{K} \\
&= \frac{1}{rK}\left\{ rX\left(\frac{t}{r}\right)\left(1 - \frac{X\left(\frac{t}{r}\right)}{K}\right) - qU\left(\frac{t}{r}\right) X\left(\frac{t}{r}\right) \right\} \\
&= \frac{X(s)}{K}\left(1 - \frac{X(s)}{K}\right) - \frac{qU(s)}{r}\frac{X(s)}{K} \\
&= x(t)(1 - x(t)) - u(t)x(t),
\end{aligned}$$

so that

$$x'(t) = (1 - u(t))\, x(t) - x^2(t). \tag{3.4}$$

Notice that

$$\begin{aligned} x(0) &= \frac{X(0)}{K} = \frac{X_0}{K} =: x_0 > 0, \\ x(T) &= \frac{X\left(\frac{T}{r}\right)}{K} = \frac{X(\bar{T})}{K} = \frac{X_{\bar{T}}}{K} =: x_T \in (0, 1], \end{aligned}$$

and

$$\begin{aligned} x(t) = \frac{X(s)}{K} &\geq \frac{X_{\min}}{K} =: \beta_2 > 0, \\ 0 \leq u(t) = \frac{qU(s)}{r} &\leq \frac{qU_{\max}}{r} =: u_{\max}. \end{aligned}$$

After the new variables are set up, we substitute them into the objective function of the optimal control problem as

$$\begin{aligned} J(x,u) := \mathbb{E}(\bar{J}(X,U)) &= \int_0^{\bar{T}} e^{-\delta s} \left[p_0 \left(1 + \beta_1 \left(1 - e^{-\gamma s}\right)\right) qX(s) - c\right] U(s)\mathrm{d}s \\ &= \int_0^{T} e^{-\delta \frac{t}{r}} \left[p_0(1 + \beta_1(1 - e^{-\gamma \frac{t}{r}}))qX\left(\frac{t}{r}\right) - c\right] U\left(\frac{t}{r}\right) \frac{\mathrm{d}t}{r} \\ &= \int_0^{T} e^{-\frac{\delta t}{r}} \left[p_0(1 + \beta_1(1 - e^{-\frac{\gamma t}{r}}))qKx(t) - c\right] \frac{u(t)}{q}\mathrm{d}t, \end{aligned}$$

so that

$$J(x,u) = \int_0^T e^{-\alpha_1 t}(p(t)x(t) - B)u(t)\mathrm{d}t,$$

where

$$\begin{aligned} &B := \tfrac{c}{q} > 0, \quad p(t) = A\left[1 + \beta_1(1 - e^{-\alpha_2 t})\right], \\ &A := p_0 K > 0, \quad \alpha_1 = \tfrac{\delta}{r} > 0, \quad \alpha_2 = \tfrac{\gamma}{r} > 0. \end{aligned}$$

Summarizing, after changing the variables in (3.3), we are concerned with the phase-constrained optimal control problem

$$\begin{cases} x'(t) = (1-u(t))x(t) - (x(t))^2, \\ x(0) = x_0 > 0, \quad x(T) = x_T > 0, \\ 0 < x(t) \le \beta_2, \quad 0 \le u(t) \le u_{\max}, \\ J(x,u) = \int_0^T e^{-\alpha_1 t}(p(t)x(t) - B)u(t)\mathrm{d}t \longrightarrow \max. \end{cases} \tag{3.5}$$

Also, we suppose

$$\alpha_2 > 1, \quad 0 < \beta_2 < x_0 < 1,$$
$$\beta_2 < x_T \le x_0, \quad u_{\max} \ge 1,$$
$$p(t)x(t) - B > 0, \quad t \in [0,T].$$

3.4. EULER–LAGRANGE EQUATION

Solving (3.4) for u, we obtain

$$u = 1 - x - \frac{x'}{x}, \tag{3.6}$$

and therefore,

$$\begin{aligned} J(x,u) &= \int_0^T e^{-\alpha_1 t}(p(t)x(t) - B)u(t)\mathrm{d}t, \\ &= \int_0^T e^{-\alpha_1 t}\,(p(t)x(t) - B)\left(1 - x(t) - \frac{x'(t)}{x(t)}\right)\mathrm{d}t \\ &= \int_0^T L(t, x(t), x'(t))\mathrm{d}t, \end{aligned}$$

where

$$L(t,x,v) := e^{-\alpha_1 t}\,(p(t)x - B)\left(1 - x - \frac{v}{x}\right).$$

To solve (3.5), we must solve the Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t} L_v(t, x(t), x'(t)) = L_x(t, x(t), x'(t)). \tag{3.7}$$

To do this, we first calculate

$$L_x(t, x, v) = e^{-\alpha_1 t} \left(p(t) - 2p(t)x + B - \frac{Bv}{x^2} \right)$$

and

$$L_v(t, x, v) = -e^{-\alpha_1 t} \left(p(t) - \frac{B}{x} \right).$$

Hence

$$L_x(t, x(t), x'(t)) = e^{-\alpha_1 t} \left(p(t) - 2p(t)x(t) + B - \frac{Bx'(t)}{x^2(t)} \right) \tag{3.8}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} L_v(t, x(t), x'(t)) &= \frac{\mathrm{d}}{\mathrm{d}t} e^{-\alpha_1 t} \left(\frac{B}{x(t)} - p(t) \right) \\ &= -\alpha_1 e^{-\alpha_1 t} \left(\frac{B}{x(t)} - p(t) \right) + e^{-\alpha_1 t} \left(-\frac{Bx'(t)}{x^2(t)} - p'(t) \right) \\ &= e^{-\alpha_1 t} \left(\alpha_1 p(t) - p'(t) - \alpha_1 \frac{B}{x(t)} - \frac{Bx'(t)}{x^2(t)} \right). \end{aligned} \tag{3.9}$$

By the Euler–Lagrange equation (3.7), (3.8) is equal to (3.9), and thus,

$$p(t) - 2p(t)x(t) + B - \frac{Bx'(t)}{x^2(t)} = \alpha_1 p(t) - p'(t) - \alpha_1 \frac{B}{x(t)} - \frac{Bx'(t)}{x^2(t)},$$

i.e.,

$$(1 - \alpha_1)p(t) + p'(t) + B - 2p(t)x(t) + \alpha_1 \frac{B}{x(t)} = 0,$$

i.e.,

$$a(t)x^2(t) + b(t)x(t) + \alpha_1 B = 0, \tag{3.10}$$

where

$$a(t) := -2p(t) < 0$$

and

$$\begin{aligned} b(t) &:= (1-\alpha_1)p(t) + p'(t) + B \\ &= (1-\alpha_1)A\left[1+\beta_1(1-e^{-\alpha_2 t})\right] + \alpha_2 A\beta_1 e^{-\alpha_2 t} + B \\ &= A\beta_1(\alpha_1+\alpha_2-1)e^{-\alpha_2 t} + A(1-\alpha_1)(1+\beta_1) + B > 0. \end{aligned}$$

By solving the quadratic equation (3.10), we find that it has two roots

$$\begin{aligned} x_1(t) &:= \frac{b(t) - \sqrt{b^2(t) - 4\alpha_1 Ba(t)}}{-2a(t)}, \\ x_2(t) &:= \frac{b(t) + \sqrt{b^2(t) - 4\alpha_1 Ba(t)}}{-2a(t)}. \end{aligned}$$

We know

$$-2a(t) > 0, \quad -4\alpha_1 Ba(t) > 0,$$

so that

$$\sqrt{b^2(t) - 4\alpha_1 Ba(t)} > \sqrt{b^2(t)} = b(t).$$

Thus, $x_1(t) < 0$ for all t, and hence x_1 does not satisfy the condition $x(t) \geq \beta_2 > 0$. Thus, the considered root in this work is

$$x_2(t) = \frac{b(t) + \sqrt{b^2(t) - 4\alpha_1 Ba(t)}}{-2a(t)} = F_2(t) + \beta_2,$$

where

$$F_2(t) := \frac{b(t) + \sqrt{b^2(t) - 4\alpha_1 B a(t)}}{-2a(t)} - \beta_2.$$

3.5. AUXILIARY RESULTS

Lemma 1. *x_2 is strictly decreasing.*

Proof. Recall that x_2 solves (3.10). Differentiating both sides of (3.10), we obtain

$$a'x^2 + 2axx' + b'x + bx' = 0.$$

Hence,

$$\begin{aligned} 0 &= \left(a'x^2 + 2axx' + b'x + bx'\right) a \\ &= a'ax^2 + ab'x + a(2ax + b)x' \\ &\overset{(3.10)}{=} a'(-bx - \alpha_1 B) + ab'x + a(2ax + b)x' \\ &= (ab' - a'b)x - \alpha_1 B a' + a(2ax + b)x', \end{aligned}$$

and thus,

$$a(2ax + b)x' = (a'b - b'a)x + \alpha_1 B a'. \tag{3.11}$$

In order to prove x_2 is strictly decreasing, we proceed in five steps as follows.

1. $\alpha_1 B a' = -2\alpha_1 B p' < 0$.

2. Note that

$$\begin{aligned} a(t) &= -2p(t), \\ a'(t) &= -2p'(t), \\ b(t) &= (1 - \alpha_1)p(t) + p'(t) + B, \end{aligned}$$

$$\begin{aligned} b'(t) &= (1-\alpha_1)p'(t) + p''(t), \\ p(t) &= A\left[1+\beta_1\left(1-e^{-\alpha_2 t}\right)\right] \\ &= A + A\beta_1 - A\beta_1 e^{-\alpha_2 t} > 0, \\ p'(t) &= \alpha_2 A\beta_1 e^{-\alpha_2 t} > 0, \\ p''(t) &= -A\beta_1 {\alpha_2}^2 e^{-\alpha_2 t} < 0. \end{aligned}$$

Hence,

$$\begin{aligned} a'b - b'a &= -2p'[(1-\alpha_1)p + p' + B] + 2p[(1-\alpha_1)p' + p''] \\ &= -2pp'(1-\alpha_1) - 2(p')^2 - 2p'B + 2pp'(1-\alpha_1) + 2pp'' \\ &= -2\left[(p')^2 + p'B - pp''\right] < 0. \end{aligned}$$

3. $(a'b - b'a)x_2 + \alpha_1 Ba' < 0$, since $x_2 > 0$.

4. $2ax_2 + b = -\sqrt{b^2 - 4\alpha_1 Ba} < 0$.

5. $a(2ax_2 + b) > 0$, since $a < 0$.

By (3.11), we conclude $x_2' < 0$, i.e., x_2 is strictly decreasing. □

Lemma 2. *If $F_2(0) > 0$ and $F_2(T) < 0$, then there exists a unique $\sigma \in (0, T)$ such that $F_2(\sigma) = 0$.*

Proof. By Lemma 1, F_2 is strictly decreasing. Therefore, such $\sigma \in (0, T)$ exists due to the intermediate value theorem. □

3.6. MAIN RESULTS

Now we put

$$x_{20} := x_2(0)$$

and we give our three main results.

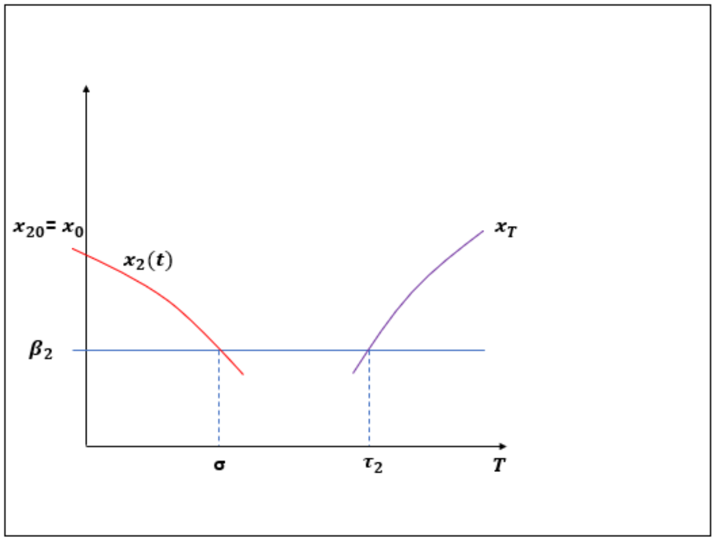

Figure 3.1. $x_{20} = x_0$.

Theorem 1. *If* $x_{20} = x_0$, $\sigma < \tau_2$, *and* $\beta_2 < x_T < 1$, *then the solution* (x_*, u_*) *of* (3.5) *is*

$$x_*(t) = \begin{cases} F_2(t) + \beta_2, & 0 \le t \le \sigma, \\ \beta_2, & \sigma \le t \le \tau_2, \\ \frac{1}{1+\gamma_T e^{T-t}}, & \tau_2 \le t \le T, \end{cases} \quad u_*(t) = \begin{cases} F_1(t), & 0 \le t \le \sigma, \\ 1 - \beta_2, & \sigma \le t \le \tau_2, \\ 0, & \tau_2 \le t \le T, \end{cases}$$

where

$$\gamma_T := \frac{1-x_T}{x_T} > 0,$$
$$\tau_2 := T - \ln\left(\frac{1-\beta_2}{\beta_2\gamma_T}\right),$$

and

$$F_1(t) := 1 - x_2(t) - \frac{x_2'(t)}{x_2(t)}.$$

Proof. Since $x_0 = x_{20}$, we start the trajectory with $x_2(t)$, which decreases (Lemma 1) and will hit β_2 at σ (Lemma 2). See Figure 3.1. So

$$x_*(t) = x_2(t) = F_2(t) + \beta_2, \quad t \in [0, \sigma].$$

Substituting this in (3.6), we obtain the control

$$u_*(t) = F_1(t), \quad t \in [0, \sigma].$$

From there, we stay on β_2, until τ^* to be determined subsequently, i.e.,

$$x_*(t) = \beta_2, \quad t \in [\sigma, \tau^*].$$

Substituting this in (3.6), we obtain the control

$$u_*(t) = 1 - \beta_2, \quad t \in [\sigma, \tau^*].$$

We have to end up in x_T at time T, so we should go up from β_2 as steep as possible (i.e., $u = 0$, see (3.6)). So we should solve

$$x' = x - x^2, \quad x(T) = x_T. \tag{3.12}$$

By Lemma 3 (following this proof), the unique solution of (3.12) is

$$x(t) = \frac{1}{1 + \gamma_T e^{T-t}},$$

which we put equal to β_2 to find τ^*:

$$\frac{1}{1 + \gamma_T e^{T-\tau^*}} = \beta_2,$$

i.e.,

$$1 + \gamma_T e^{T-\tau^*} = \frac{1}{\beta_2},$$

i.e.,

$$\left(\frac{1}{\beta_2} - 1\right)\frac{1}{\gamma_T} = e^{T-\tau^*},$$

i.e.,

$$\ln\frac{1-\beta_2}{\beta_2\gamma_T} = T - \tau^*,$$

i.e.,

$$\tau^* = T - \ln\left(\frac{1-\beta_2}{\beta_2\gamma_T}\right) = \tau_2.$$

Note that $\tau_2 \in (\sigma, T)$ as $\sigma < \tau_2$ was assumed and

$$\frac{1-\beta_2}{\beta_2\gamma_T} > \frac{1-x_T}{\beta_2\gamma_T} = \frac{x_T}{\beta_2} > 1$$

implies $\tau_2 < T$. Hence

$$x_*(t) = \frac{1}{1+\gamma_T e^{T-t}}, \quad t \in [\tau_2, T].$$

We recall that the corresponding control was

$$u_*(t) = 0, \quad t \in [\tau_2, T].$$

This completes the proof. □

Lemma 3. *The unique solution of* (3.12) *is*

$$x(t) = \frac{1}{1+\gamma_T e^{T-t}}.$$

Proof. Suppose x solves (3.12). For

$$\tilde{x} = \frac{1}{x},$$

we obtain

$$\tilde{x}' = -\frac{x'}{x^2} = -\frac{x-x^2}{x^2} = -\frac{1}{x} + 1 = 1 - \tilde{x}.$$

Hence, we get

$$e^t = e^t \tilde{x}'(t) + e^t \tilde{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^t \tilde{x}(t)\right).$$

By integrating both sides, we obtain

$$e^t \tilde{x}(t) = e^t + C,$$

i.e.,

$$\tilde{x}(t) = 1 + Ce^{-t}. \tag{3.13}$$

Using the condition $x(T) = x_T$ from (3.12), we have

$$\frac{1}{x_T} = \tilde{x}(T) = 1 + Ce^{-T},$$

so

$$C = \left(\frac{1}{x_T} - 1\right) e^T = \frac{1 - x_T}{x_T} e^T = \gamma_T e^T.$$

Substituting this C into (3.13), we find

$$x(t) = \frac{1}{\tilde{x}(t)} = \frac{1}{1 + Ce^{-t}} = \frac{1}{1 + \gamma_T e^T e^{-t}} = \frac{1}{1 + \gamma_T e^{T-t}}.$$

Conversely, it can be checked that this x solves (3.12) as

$$\begin{aligned} x(T) &= \frac{1}{1 + \gamma_T e^{T-T}} = \frac{1}{1 + \gamma_T} = x_T, \\ x'(t) &= \frac{\gamma_T e^{T-t}}{(1 + \gamma_T e^{T-t})^2}, \end{aligned}$$

and

$$x(t) - x^2(t) = \frac{1}{1 + \gamma_T e^{T-t}} - \frac{1}{(1 + \gamma_T e^{T-t})^2}$$

$$= \quad \frac{\gamma_T e^{T-t}}{(1+\gamma_T e^{T-t})^2} = x'(t).$$

This completes the proof. □

Theorem 2. *If* $x_{20} > x_0$*,* $\sigma < \tau_2$*, and* $\beta_2 < x_T \le x_0 < 1$*, then the solution* (x_*, u_*) *of* (3.5) *is*

$$x_*(t) = \begin{cases} \frac{1}{1+\gamma_0 e^{-t}}, & 0 \le t \le \tau, \\ F_2(t) + \beta_2, & \tau \le t \le \sigma, \\ \beta_2, & \sigma \le t \le \tau_2, \\ \frac{1}{1+\gamma_T e^{T-t}}, & \tau_2 \le t \le T, \end{cases} \qquad u_*(t) = \begin{cases} 0, & 0 \le t \le \tau, \\ F_1(t), & \tau \le t \le \sigma, \\ 1-\beta_2, & \sigma \le t \le \tau_2, \\ 0, & \tau_2 \le t \le T, \end{cases}$$

where

$$\gamma_0 := \frac{1-x_0}{x_0} > 0$$

and $\tau \in (0, \sigma)$ *is the unique root of*

$$L(t) := F_2(t) + \beta_2 - \frac{1}{1+\gamma_0 e^{-t}}.$$

Proof. In this case, we start the trajectory at $x_0 < x_{20}$ and go up to reach $x_2(t)$ as fast as possible (i.e., $u = 0$, see (3.6)). See Figure 3.2. So we should solve

$$x' = x - x^2, \quad x(0) = x_0. \tag{3.14}$$

By Lemma 4 (following this proof), the unique solution of (3.14) is

$$x(t) = \frac{1}{1+\gamma_0 e^{-t}}.$$

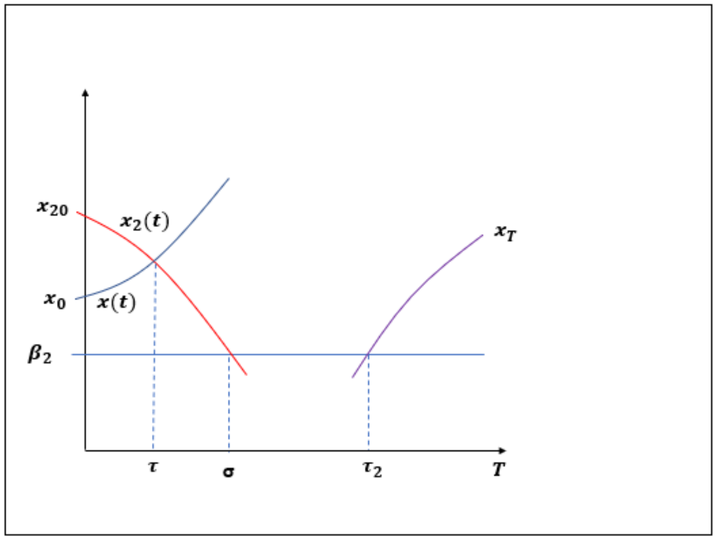

Figure 3.2. $x_{20} > x_0$.

This $x(t)$ intersects $x_2(t)$ when

$$\frac{1}{1+\gamma_0 e^{-t}} = F_2(t) + \beta_2,$$

i.e., when

$$L(t) = 0.$$

By Lemma 5 (after this proof), L has a unique root $\tau \in (0, \sigma)$, and thus

$$x_*(t) = \frac{1}{1+\gamma_0 e^{-t}}, \quad t \in [0, \tau].$$

We recall that the corresponding control was

$$u_*(t) = 0, \quad t \in [0, \tau].$$

The rest of the proof is exactly the same as in the proof of Theorem 1, namely the trajectory now continues along $x_2(t)$, decreasing until it hits β_2, staying on β_2, then increasing again along the solution of (3.12) until it reaches its required destination x_T. Hence

$$\begin{aligned} x_*(t) &= x_2(t), \quad t \in [\tau, \sigma], \\ u_*(t) &= F_1(t), \quad t \in [\tau, \sigma], \\ x_*(t) &= \beta_2, \quad t \in [\sigma, \tau_2], \\ u_*(t) &= 1 - \beta_2, \quad t \in [\sigma, \tau_2], \\ x_*(t) &= \frac{1}{1 + \gamma_T e^{T-t}}, \quad t \in [\tau_2, T], \\ u_*(t) &= 0, \quad t \in [\tau_2, T]. \end{aligned}$$

This completes the proof. □

Lemma 4. *The unique solution of* (3.14) *is*

$$x(t) = \frac{1}{1 + \gamma_0 e^{-t}}.$$

Proof. Proceeding as in the proof of Lemma 3, we find (3.13). This time using the condition $x(0) = x_0$ from (3.14), we have

$$\frac{1}{x_0} = \tilde{x}(0) = 1 + C,$$

so

$$C = \frac{1}{x_0} - 1 = \frac{1 - x_0}{x_0} = \gamma_0.$$

Substituting this C into (3.13), we find

$$x(t) = \frac{1}{\tilde{x}(t)} = \frac{1}{1 + Ce^{-t}} = \frac{1}{1 + \gamma_0 e^{-t}}.$$

Conversely, it can be checked that this x solves (3.14) as

$$\begin{aligned} x(0) &= \frac{1}{1+\gamma_0} = x_0, \\ x'(t) &= \frac{\gamma_0 e^{-t}}{(1+\gamma_0 e^{-t})^2}, \end{aligned}$$

and

$$\begin{aligned} x(t) - x^2(t) &= \frac{1}{1+\gamma_0 e^{-t}} - \frac{1}{(1+\gamma_0 e^{-t})^2} \\ &= \frac{\gamma_0 e^{-t}}{(1+\gamma_0 e^{-t})^2} = x'(t). \end{aligned}$$

This completes the proof. □

Lemma 5. *If* $x_{20} > x_0$ *and*

$$L(t) = F_2(t) + \beta_2 - \frac{1}{1+\gamma_0 e^{-t}},$$

then L *has a unique root* $\tau \in (0, \sigma)$.

Proof. Since L is strictly decreasing (use Lemma 1) and

$$L(0) = F_2(0) + \beta_2 - \frac{1}{1+\gamma_0} = x_2(0) - x_0 = x_{20} - x_0 > 0$$

and

$$\begin{aligned} L(\sigma) &= F_2(\sigma) + \beta_2 - \frac{1}{1+\gamma_0 e^{-\sigma}} \\ &= \beta_2 - \frac{1}{1+\gamma_0 e^{-\sigma}} < \beta_2 - \frac{1}{1+\gamma_0} = \beta_2 - x_0 < 0, \end{aligned}$$

the existence of a unique $\tau \in (0, \sigma)$ with $L(\tau) = 0$ is ensured. □

Theorem 3. *If* $x_{20} < x_0$, $\beta_2 < x_0 < 1$, $u_{\max} = 1$, *and*

(H_1) *there exists* $s_2 \in (0, \sigma)$ *with* $K_2(s_2) = 0$ *and* $K_2(t) < 0$ *for* $t \in [0, s_2)$*, where*

$$K_2(t) := F_2(t) + \beta_2 - \frac{1}{t + \frac{1}{x_0}},$$

then the solution (x_*, u_*) *of* (3.5) *is*

$$x_*(t) = \begin{cases} \frac{1}{t+\frac{1}{x_0}}, & 0 \le t \le s_2, \\ F_2(t) + \beta_2, & s_2 \le t \le \sigma, \\ \beta_2, & \sigma \le t \le \tau_2, \\ \frac{1}{1+\gamma_T e^{T-t}}, & \tau_2 \le t \le T, \end{cases} \qquad u_*(t) = \begin{cases} 1, & 0 \le t \le s_2, \\ F_1(t), & s_2 \le t \le \sigma, \\ 1 - \beta_2, & \sigma \le t \le \tau_2, \\ 0, & \tau_2 \le t \le T, \end{cases}$$

while if (H_1) *does not hold, then the solution* (x_*, u_*) *of* (3.5) *is*

$$x_*(t) = \begin{cases} \frac{1}{t+\frac{1}{x_0}}, & 0 \le t \le \tau_1', \\ \beta_2, & \tau_1' \le t \le \tau_2, \\ \frac{1}{1+\gamma_T e^{T-t}}, & \tau_2 \le t \le T, \end{cases} \qquad u_*(t) = \begin{cases} 1, & 0 \le t \le \tau_1', \\ 1 - \beta_2, & \tau_1' \le t \le \tau_2, \\ 0, & \tau_2 \le t \le T, \end{cases}$$

where we assume $\tau_1' < \tau_2$ *with*

$$\tau_1' := \frac{1}{\beta_2} - \frac{1}{x_0} > 0.$$

Proof. First, assume (H_1) holds. In this case, we start the trajectory at $x_0 > x_{20}$, this time going down to reach $x_2(t)$ as fast as possible (i.e., $u = u_{\max} = 1$, see (3.6)). See Figure 3.3. So we should solve

$$x' = -x^2, \quad x(0) = x_0. \tag{3.15}$$

By Lemma 6 (following this proof), the unique solution of (3.15) is

$$x(t) = \frac{1}{t + \frac{1}{x_0}}.$$

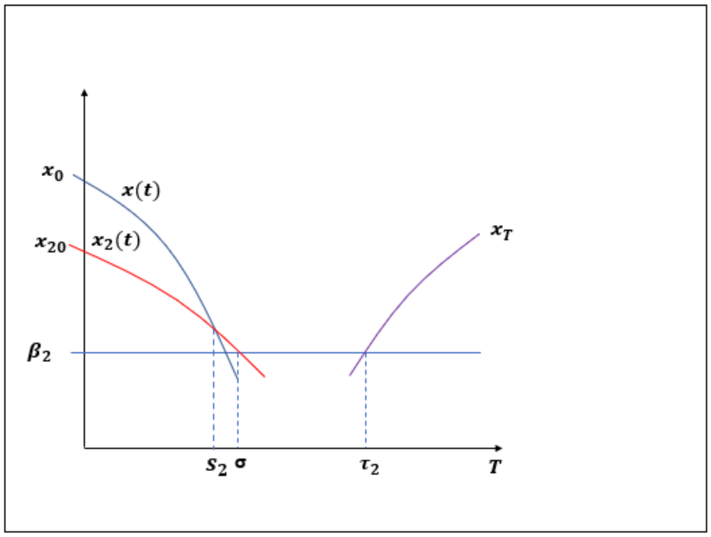

Figure 3.3. $x_{20} < x_0$, when (H_1) holds.

This $x(t)$ intersects $x_2(t)$ when

$$\frac{1}{t + \frac{1}{x_0}} = F_2(t) + \beta_2,$$

i.e., when

$$K_2(t) = 0.$$

Due to (H_1), $s_2 \in (0, \sigma)$ satisfies $K_2(s_2) = 0$ uniquely, and thus

$$x_*(t) = \frac{1}{t + \frac{1}{x_0}}, \quad t \in [0, s_2].$$

We recall that the corresponding control was

$$u_*(t) = u_{\max} = 1, \quad t \in [0, s_2].$$

The rest of the proof is exactly the same as in the proof of Theorem 2, namely the trajectory now continues along $x_2(t)$, decreasing until it hits β_2, staying on β_2, then increasing again along the solution of (3.12) until it reaches its required destination x_T. Hence

$$\begin{aligned}
x_*(t) &= x_2(t), \quad t \in [s_2, \sigma], \\
u_*(t) &= F_1(t), \quad t \in [s_2, \sigma], \\
x_*(t) &= \beta_2, \quad t \in [\sigma, \tau_2], \\
u_*(t) &= 1 - \beta_2, \quad t \in [\sigma, \tau_2], \\
x_*(t) &= \frac{1}{1 + \gamma_T e^{T-t}}, \quad t \in [\tau_2, T], \\
u_*(t) &= 0, \quad t \in [\tau_2, T].
\end{aligned}$$

Second, assume (H_1) does not hold. In this case, we start the trajectory at $x_0 > x_{20}$, this time going down to reach β_2 as fast as possible (i.e., $u = u_{\max} = 1$, see (3.6)). See Figure 3.4. So we should solve (3.15). By Lemma 6 (following this proof), the unique solution of (3.15) is

$$x(t) = \frac{1}{t + \frac{1}{x_0}}.$$

This $x(t)$ hits β_2 when

$$\frac{1}{t + \frac{1}{x_0}} = \beta_2,$$

i.e., when

$$t + \frac{1}{x_0} = \frac{1}{\beta_2},$$

i.e., when

$$t = \frac{1}{\beta_2} - \frac{1}{x_0} = \tau_1'.$$

Thus,

$$x_*(t) = \frac{1}{t + \frac{1}{x_0}}, \quad t \in [0, \tau_1'].$$

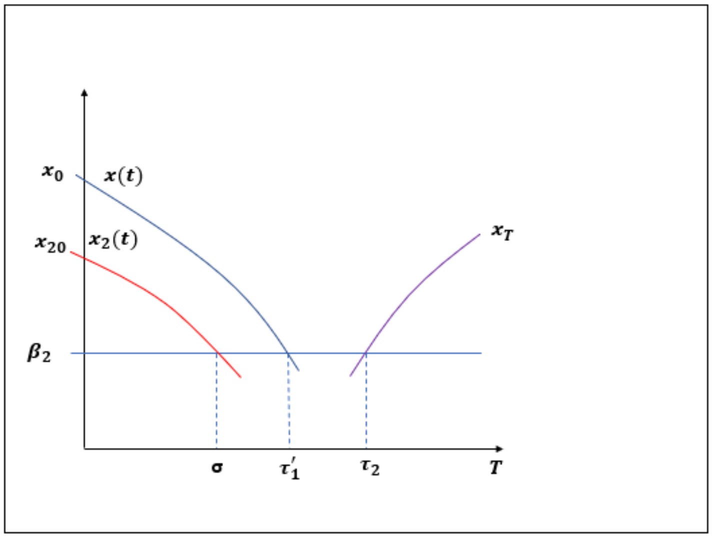

Figure 3.4. $x_{20} < x_0$, when (H_1) and (H_2) do not hold.

We recall that the corresponding control was

$$u_*(t) = u_{\max} = 1, \quad t \in [0, \tau_1'].$$

We proceed now exactly as in the first part of this proof, staying on β_2, then increasing again along the solution of (3.12) until reaching x_T. Hence (note that $\sigma < \tau_1' < \tau_2$)

$$\begin{aligned} x_*(t) &= \beta_2, \quad t \in [\tau_1', \tau_2], \\ u_*(t) &= 1 - \beta_2, \quad t \in [\tau_1', \tau_2], \\ x_*(t) &= \frac{1}{1 + \gamma_T e^{T-t}}, \quad t \in [\tau_2, T], \\ u_*(t) &= 0, \quad t \in [\tau_2, T]. \end{aligned}$$

This completes the proof. □

Lemma 6. *The unique solution of* (3.15) *is*

$$x(t) = \frac{1}{t + \frac{1}{x_0}}.$$

Proof. Suppose x solves (3.15). For

$$\tilde{x} = \frac{1}{x},$$

we get

$$\tilde{x}' = -\frac{x'}{x^2} = \frac{x^2}{x^2} = 1.$$

Thus

$$\tilde{x}(t) = t + C. \tag{3.16}$$

Using the condition $x(0) = x_0$ from (3.15), we have

$$\frac{1}{x_0} = \tilde{x}(0) = C.$$

Substituting this C into (3.16), we obtain

$$x(t) = \frac{1}{\tilde{x}(t)} = \frac{1}{t + C} = \frac{1}{t + \frac{1}{x_0}}.$$

Conversely, it can be checked that this x solves (3.15) as

$$\begin{aligned} x(0) &= \frac{1}{0 + \frac{1}{x_0}} = x_0, \\ x'(t) &= -\frac{1}{\left(t + \frac{1}{x_0}\right)^2} = -x^2(t). \end{aligned}$$

This completes the proof. □

Theorem 4. *If* $x_{20} < x_0$, $u_{\max} > 1$, *and*

(H_2) *there exists* $s_1 \in (0, \sigma)$ *with* $K_1(s_1) = 0$ *and* $K_1(t) < 0$ *for* $t \in [0, s_1)$, *where*

$$\begin{aligned} K_1(t) &:= F_2(t) + \beta_2 - \frac{u_{\max} - 1}{(\gamma_2 + 1)\, e^{(u_{\max}-1)t} - 1}, \\ \gamma_2 &:= \frac{u_{\max} - 1}{x_0} > 0, \end{aligned}$$

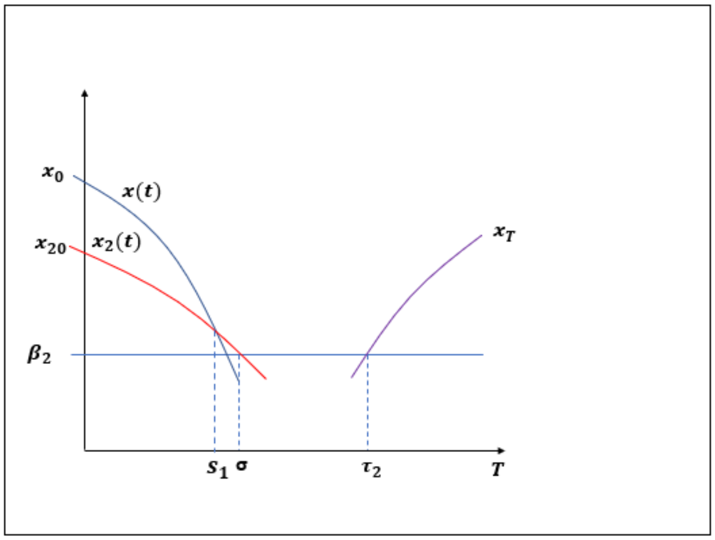

Figure 3.5. $x_{20} < x_0$ when (H_2) holds.

then the solution (x_*, u_*) of (3.5) is

$$x_*(t) = \begin{cases} \frac{u_{\max}-1}{(\gamma_2+1)e^{(u_{\max}-1)t}-1}, & 0 \le t \le s_1, \\ F_2(t) + \beta_2, & s_1 \le t \le \sigma, \\ \beta_2, & \sigma \le t \le \tau_2, \\ \frac{1}{1+\gamma_T e^{T-t}}, & \tau_2 \le t \le T, \end{cases} \qquad u_*(t) = \begin{cases} u_{\max}, & 0 \le t \le s_1, \\ F_1(t), & s_1 \le t \le \sigma, \\ 1-\beta_2, & \sigma \le t \le \tau_2, \\ 0, & \tau_2 \le t \le T, \end{cases}$$

while if (H_2) does not hold, then the solution (x_*, u_*) of (3.5) is

$$x_*(t) = \begin{cases} \frac{u_{\max}-1}{(\gamma_2+1)e^{(u_{\max}-1)t}-1}, & 0 \le t \le \tau_1', \\ \beta_2, & \tau_1' \le t \le \tau_2, \\ \frac{1}{1+\gamma_T e^{T-t}}, & \tau_2 \le t \le T, \end{cases} \qquad u_*(t) = \begin{cases} u_{\max}, & 0 \le t \le \tau_1', \\ 1-\beta_2, & \tau_1' \le t \le \tau_2, \\ 0, & \tau_2 \le t \le T, \end{cases}$$

where we assume $\tau_1' < \tau_2$ with

$$\tau_1' := \frac{1}{u_{\max}-1} \ln\left(\frac{\gamma_1+1}{\gamma_2+1}\right), \quad \gamma_1 := \frac{u_{\max}-1}{\beta_2} > 0.$$

Proof. First, assume (H_2) holds. In this case, we start the trajectory at $x_0 > x_{20}$, again going down to reach $x_2(t)$ as fast as possible (i.e., $u = u_{\max} > 1$, see (3.6)). See Figure 3.5. So we should solve

$$x' = (1 - u_{\max})\, x - x^2, \quad x(0) = x_0. \tag{3.17}$$

By Lemma 7 (following this proof), the unique solution of (3.17) is

$$x(t) = \frac{u_{\max} - 1}{(\gamma_2 + 1)\, e^{(u_{\max}-1)t} - 1}.$$

This $x(t)$ intersects $x_2(t)$ when

$$\frac{u_{\max} - 1}{(\gamma_2 + 1)\, e^{(u_{\max}-1)t} - 1} = F_2(t) + \beta_2,$$

i.e., when

$$K_1(t) = 0.$$

Due to (H_2), $s_1 \in (0, \sigma)$ satisfies $K_1(s_1) = 0$, and thus

$$x_*(t) = \frac{u_{\max} - 1}{(\gamma_2 + 1)\, e^{(u_{\max}-1)t} - 1}, \quad t \in [0, s_1].$$

We recall that the corresponding control was

$$u_*(t) = u_{\max}, \quad t \in [0, s_1].$$

The rest of the proof is exactly the same as in the proof of Theorem 3, namely the trajectory again continues along $x_2(t)$, decreasing until it hits β_2, staying on β_2, then increasing again along the solution of (3.12) until it reaches x_T. Hence

$$\begin{aligned} x_*(t) &= x_2(t), \quad t \in [s_1, \sigma], \\ u_*(t) &= F_1(t), \quad t \in [s_1, \sigma], \end{aligned}$$

$$
\begin{aligned}
x_*(t) &= \beta_2, \quad t \in [\sigma, \tau_2], \\
u_*(t) &= 1 - \beta_2, \quad t \in [\sigma, \tau_2], \\
x_*(t) &= \frac{1}{1 + \gamma_T e^{T-t}}, \quad t \in [\tau_2, T], \\
u_*(t) &= 0, \quad t \in [\tau_2, T].
\end{aligned}
$$

Second, assume (H_2) does not hold. In this case, we start the trajectory at $x_0 > x_{20}$, this time going down to reach β_2 as fast as possible (i.e., $u = u_{\max} > 1$, see (3.6)). See Figure 3.4. So we should solve (3.17). By Lemma 7 (following this proof), the unique solution of (3.17) is

$$x(t) = \frac{u_{\max} - 1}{(\gamma_2 + 1)\, e^{(u_{\max}-1)t} - 1}.$$

This $x(t)$ hits β_2 when

$$\frac{u_{\max} - 1}{(\gamma_2 + 1)\, e^{(u_{\max}-1)t} - 1} = \beta_2,$$

i.e., when

$$(\gamma_2 + 1)\, e^{(u_{\max}-1)t} = \frac{u_{\max} - 1}{\beta_2} + 1 = \gamma_1 + 1,$$

i.e., when

$$e^{(u_{\max}-1)t} = \frac{\gamma_1 + 1}{\gamma_2 + 1},$$

i.e., when

$$t = \frac{1}{u_{\max} - 1} \ln\left(\frac{\gamma_1 + 1}{\gamma_2 + 1}\right) = \tau_1'$$

Thus,

$$x_*(t) = \frac{u_{\max} - 1}{(\gamma_2 + 1)\, e^{(u_{\max}-1)t} - 1}, \quad t \in [0, \tau_1'].$$

We recall that the corresponding control was

$$u_*(t) = u_{\max}, \quad t \in [0, \tau_1'].$$

We proceed now exactly as in the second part of the proof of Theorem 3, staying on β_2, then increasing again along the solution of (3.12) until reaching x_T. Hence (note that $\sigma < \tau_1' < \tau_2$)

$$\begin{aligned} x_*(t) &= \beta_2, \quad t \in [\tau_1', \tau_2], \\ u_*(t) &= 1 - \beta_2, \quad t \in [\tau_1', \tau_2], \\ x_*(t) &= \frac{1}{1 + \gamma_T e^{T-t}}, \quad t \in [\tau_2, T], \\ u_*(t) &= 0, \quad t \in [\tau_2, T]. \end{aligned}$$

This completes the proof. □

Lemma 7. *The unique solution of* (3.17) *is*

$$x(t) = \frac{u_{\max} - 1}{(\gamma_2 + 1)\, e^{(u_{\max}-1)t} - 1}.$$

Proof. For convenience, we introduce the notation

$$\alpha := u_{\max} - 1 > 0.$$

Suppose x solves (3.17). For

$$\tilde{x} = \frac{1}{x},$$

we get

$$\tilde{x}' = -\frac{x'}{x^2} = \frac{\alpha x + x^2}{x^2} = \frac{\alpha}{x} + 1 = \alpha \tilde{x} + 1,$$

i.e.,

$$e^{-\alpha t} = e^{-\alpha t} \tilde{x}'(t) - \alpha e^{-\alpha t} \tilde{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t} \left(e^{-\alpha t} \tilde{x}(t) \right).$$

Hence, by integrating, we obtain

$$e^{-\alpha t}\tilde{x}(t) = -\frac{1}{\alpha}e^{-\alpha t} + C,$$

i.e.,

$$\tilde{x}(t) = Ce^{\alpha t} - \frac{1}{\alpha}. \tag{3.18}$$

Next, using the condition $x(0) = x_0$ from (3.17), we find

$$\frac{1}{x_0} = \tilde{x}(0) = C - \frac{1}{\alpha},$$

so

$$C = \frac{1}{x_0} + \frac{1}{\alpha}.$$

Furthermore, substituting this C into (3.18), we get

$$x(t) = \frac{1}{\tilde{x}(t)} = \frac{1}{Ce^{\alpha t} - \frac{1}{\alpha}} = \frac{1}{\left(\frac{1}{x_0} + \frac{1}{\alpha}\right) e^{\alpha t} - \frac{1}{\alpha}} = \frac{\alpha}{(\gamma_2 + 1)\, e^{\alpha t} - 1}.$$

Conversely, it can be checked that this x solves (3.17) as

$$\begin{aligned} x(0) &= \frac{\alpha}{\gamma_2 + 1 - 1} = \frac{\alpha}{\gamma_2} = x_0, \\ -\alpha x(t) - x^2(t) &= -\frac{\alpha^2}{(\gamma_2 + 1)\, e^{\alpha t} - 1} - \frac{\alpha^2}{((\gamma_2 + 1)e^{\gamma t} - 1)^2} \\ &= -\frac{\alpha^2\, (\gamma_2 + 1)\, e^{\alpha t}}{((\gamma_2 + 1)e^{\alpha t} - 1)^2} = x'(t). \end{aligned}$$

This completes the proof. □

4. THE BEVERTON–HOLT MODEL (DISCRETE CASE)

Beverton and Holt were interested in studying a discrete population model in 1957, see [3, 32, 34]. Many authors discussed the discrete case, and many published about the Beverton–Holt equation, see [7, 12, 17], which turned out to be very beneficial in different fields such as economics and social science, see [2, 3, 19, 34]. The Beverton–Holt model is given by

$$x_{n+1} = \frac{vK_n x_n}{K_n + (v-1)x_n}, \quad n \in \mathbb{N}. \tag{4.1}$$

As was the case for the logistic equation and for the Schaefer model, the initial point x_0 is assumed to be positive. The carrying capacity is denoted by $K_n > 0$. The inherent growth rate is denoted by $v > 1$, see [11]. Some studies consider the carrying capacity to be periodic, i.e., $K_{n+\rho} = K_n$ for some $\rho \in \mathbb{N}$. Many authors have studied this equation by different metods, sometimes by considering the equation as a special case of so-called "rational difference equations", see [11, 16, 17].

We introduce

$$\alpha := \frac{v-1}{v}, \quad \text{i.e.,} \quad v = \frac{1}{1-\alpha}$$

so that

$$v > 1 \quad \text{if and only if} \quad 0 < \alpha < 1.$$

Assuming x solves (4.1), we obtain

$$x_{n+1} = \frac{K_n x_n}{(1-\alpha)K_n + \alpha x_n},$$

which can be rearranged as

$$x_n(K_n - \alpha x_{n+1}) = (1-\alpha)K_n x_{n+1},$$

i.e.,

$$\begin{aligned} x_n &= \frac{(1-\alpha)K_n x_{n+1}}{K_n - \alpha x_{n+1}} \\ &= \frac{(1-\alpha)K_n x_{n+1}}{K_n - \alpha x_{n+1}} + x_{n+1} - x_{n+1} \\ &= \frac{\alpha x_{n+1}(x_{n+1} - K_n)}{K_n - \alpha x_{n+1}} + x_{n+1}, \end{aligned}$$

i.e.,

$$\Delta x_n = \frac{\alpha x_{n+1}(K_n - x_{n+1})}{K_n - \alpha x_{n+1}},$$

where

$$\Delta x_n = x_{n+1} - x_n.$$

We next get

$$K_n \Delta x_n - \alpha x_{n+1}(x_{n+1} - x_n) = \alpha x_{n+1}(K_n - x_{n+1}),$$

i.e.,

$$\Delta x_n K_n = \alpha x_{n+1}(K_n - x_n).$$

Dividing by K_n, we reach to

$$\Delta x_n = \alpha x_{n+1}\left(1 - \frac{x_n}{K_n}\right). \tag{4.2}$$

The time scales analogue [5] to (4.2) is obtained by replacing x_n by $x(t)$, x_{n+1} by $x^\sigma(t)$, and Δx_n by $x^\Delta(t)$, see [7, 13]. The dynamic equation is then appearing as

$$x^\Delta = \alpha x^\sigma \left(1 - \frac{x}{K(t)}\right). \tag{4.3}$$

The differential equation (1.1) is accessible from (4.3) by replacing x^Δ by x', while both x^σ and x have to be replaced by x, i.e.,

$$x' = \alpha x \left(1 - \frac{x}{K(t)}\right). \tag{4.4}$$

4.1. CASE STATEMENT

We now consider the Beverton–Holt model with harvesting

$$x_{n+1} = \frac{vK_n x_n}{K_n + (v-1)x_n} - h_n x_{n+1}, \tag{4.5}$$

see [6, (11)]. By again introducing

$$\alpha := \frac{v-1}{v},$$

(4.5) turns into

$$x_{n+1} = \frac{K_n x_n}{(1-\alpha)K_n + \alpha x_n} - h_n x_{n+1}, \tag{4.6}$$

see [6, (12)], or equivalently, by performing the same calculations as in Section 4.1,

$$\Delta x_n = \alpha x_{n+1}\left(1 - \frac{x_n}{K_n}\right) - E_n x_n, \tag{4.7}$$

see [6, (14)], where

$$E_n = \frac{h_n}{1+h_n}.$$

To avoid extinction, we assume

$$0 < h_n \leq \frac{\alpha}{1-\alpha},$$

i.e.,

$$0 < E_n < \alpha.$$

We now consider (4.6) with constant K and constraints, i.e.,

$$\begin{cases} X_{n+1} = \frac{KX_n}{(1-\alpha)K+\alpha X_n} - qU_nX_{n+1}, \\ X_0 > 0, \quad X_T > 0, \\ 0 < X_{\min} \le X_n \le K, \quad 0 \le U_n \le U_{\max}, \\ \bar{J}(X,U) = \sum_{n=0}^{T-1} \frac{1}{(1+\delta)^n} \left[\bar{p}_n qU_nX_{n+1} - c(U_n)\right] \longrightarrow \max, \end{cases} \tag{4.8}$$

where $\bar{J}(X,U)$ is the discounted profit, $\delta > 0$ is the discount factor, $\bar{p}$ is the price function, and c is the harvesting cost. Suppose c is a linear function, i.e., $c(U) = cU$.

4.2. GEOMETRIC DISTRIBUTION OF THE PRICE FUNCTION

We now present the discrete analogue of the continuous price function used in Chapter 3, see Section 3.3. Assume the price function is given by

$$\bar{p}_n = \begin{cases} p_0, & n < \tau, \\ p_0 + \theta, & n \ge \tau, \end{cases}$$

where $p_0 > 0$ and $\theta > 0$. Suppose τ is a geometrically distributed random variable with parameter $\gamma > 0$, i.e.,

$$P(\tau \le n) = F(n) = 1 - (1-\gamma)^n .$$

The expected value of the price function is calculated as

$$\begin{aligned} \mathbb{E}(\bar{p}_n) &= p_0 P(n < \tau) + (p_0+\theta)P(n \ge \tau) \\ &= p_0(1 - F(n)) + (p_0+\theta)F(n) \\ &= p_0(1-\gamma)^n + (p_0+\theta)(1-(1-\gamma)^n) \\ &= p_0 + \theta - \theta(1-\gamma)^n \\ &= p_0 + \theta(1-(1-\gamma)^n). \end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{E}(\bar{J}(X,U)) &= \sum_{n=0}^{T-1} \frac{1}{(1+\delta)^n} \left[\mathbb{E}(\bar{p}_n) q U_n X_{n+1} - c U_n\right] \\
&= \sum_{n=0}^{T-1} \frac{1}{(1+\delta)^n} \left[\left(p_0 + \theta(1-(1-\gamma)^n)\right) q U_n X_{n+1} - c U_n\right] \\
&= \sum_{n=0}^{T-1} \frac{1}{(1+\delta)^n} \left[p_0 \left(1 + \beta_1 \left(1-(1-\gamma)^n\right)\right) q U_n X_{n+1} - c U_n\right],
\end{aligned}$$

where

$$\beta_1 := \frac{\theta}{p_0}.$$

4.3. CHANGE OF VARIABLES

Assume (X, U) solves (4.8). If we perform the change of variables

$$x_n = \frac{X_n}{K}, \quad u_n = q U_n,$$

then x satisfies

$$\begin{aligned}
x_{n+1} &= \frac{X_{n+1}}{K} = \frac{\frac{K X_n}{(1-\alpha)K + \alpha X_n} - q U_n X_{n+1}}{K} \\
&= \frac{\frac{X_n}{K}}{1-\alpha+\alpha\frac{X_n}{K}} - q U_n \frac{X_{n+1}}{K} \\
&= \frac{x_n}{1-\alpha+\alpha x_n} - u_n x_{n+1},
\end{aligned}$$

so that

$$x_{n+1} = \frac{x_n}{1-\alpha+\alpha x_n} - u_n x_{n+1} \tag{4.9}$$

and

$$x_0 = \frac{X_0}{K}, \quad x_T = \frac{X_T}{K} \in [0,1],$$

$$0 < x_{\min} \le x_n \le 1, \quad x_{\min} := \frac{X_{\min}}{K},$$

$$0 \le u_n \le u_{\max} := qU_{\max}.$$

After the new variables are set up, we substitute them into the objective function of the optimal control problem as

$$\begin{aligned} J(x,u) = \mathbb{E}\left(\bar{J}(X,U)\right) &= \sum_{n=0}^{T-1} \frac{1}{(1+\delta)^n} \left[p_0 \left(1 + \beta_1 (1 - (1-\gamma)^n)\right) u_n K x_{n+1} - \frac{cU_n}{q} \right] \\ &= \sum_{n=0}^{T-1} \frac{1}{(1+\delta)^n} \left(p_n x_{n+1} - B\right) u_n, \end{aligned}$$

where

$$p_n := A\left[1 + \beta_1 \left(1 - (1-\gamma)^n\right)\right], \quad A := p_0 K, \quad B = \frac{c}{q}.$$

Summarizing, we are concerned with the problem

$$\begin{cases} x_{n+1} = \frac{x_n}{1-\alpha+\alpha x_n} - u_n x_{n+1}, \\ x_0 > 0, \quad x_T > 0, \\ 0 < x_{\min} \le x_n \le 1, \quad 0 \le u_n \le u_{\max}, \\ J(x,u) = \sum_{n=0}^{T-1} \frac{1}{(1+\delta)^n} \left(p_n x_{n+1} - B\right) u_n \longrightarrow \max. \end{cases} \tag{4.10}$$

4.4. EULER–LAGRANGE EQUATION

Solving (4.9) for u, we obtain

$$\begin{aligned} u_n &= \frac{x_n}{(1-\alpha+\alpha x_n)\, x_{n+1}} - 1 = \frac{x_n - x_{n+1} + \alpha x_{n+1} - \alpha x_n x_{n+1}}{x_{n+1} - \alpha x_{n+1} + \alpha x_n x_{n+1}} \\ &= \frac{\alpha x_{n+1}\,(1-x_n) - \Delta x_n}{x_{n+1} - \alpha x_{n+1}\,(1-x_n)} \\ &= \frac{\alpha x_{n+1}\,(1-(x_{n+1}-\Delta x_n)) - \Delta x_n}{x_{n+1} - \alpha x_{n+1}\,(1-(x_{n+1}-\Delta x_n))}, \end{aligned}$$

and therefore,

$$\begin{aligned} J(x,u) &= \sum_{n=0}^{T-1} \frac{1}{(1+\delta)^n} (p_n x_{n+1} - B) \frac{\alpha x_{n+1}(1-(x_{n+1}-\Delta x_n)) - \Delta x_n}{x_{n+1} - \alpha x_{n+1}(1-(x_{n+1}-\Delta x_n))} \\ &= \sum_{n=0}^{T-1} L(n, x_{n+1}, \Delta x_n), \end{aligned}$$

where

$$L(n,x,v) = \frac{1}{(1+\delta)^n} (p_n x - B) \frac{\alpha x(1-(x-v)) - v}{x - \alpha x(1-(x-v))}.$$

To solve (4.10), we must solve the Euler–Lagrange equation (see [1, Theorem 4.3])

$$\Delta L_v(n, x_{n+1}, \Delta x_n) = L_x(n, x_{n+1}, \Delta x_n). \tag{4.11}$$

The rest of this thesis will show work on solving (4.11). For convenience, let us introduce

$$\tilde{H} = \alpha x(1-(x-v)) = \alpha x - \alpha x^2 + \alpha x v$$

and

$$H = \frac{\tilde{H} - v}{x - \tilde{H}}.$$

Thus,

$$\tilde{H}_v = \alpha x, \quad \tilde{H}_x = \alpha - 2\alpha x v + \alpha v,$$

and

$$\begin{aligned} H_v &= \frac{\left(\tilde{H}_v - 1\right)\left(x - \tilde{H}\right) - \left(-\tilde{H}_v\right)\left(\tilde{H} - v\right)}{\left(x - \tilde{H}\right)^2} = \frac{\tilde{H}_v x - x - \tilde{H}_v \tilde{H} + \tilde{H} + \tilde{H}_v \tilde{H} - \tilde{H}_v v}{(x-\tilde{H})^2} \\ &= \frac{\tilde{H}_v(x-v) - x + \tilde{H}}{(x-\tilde{H})^2} = \frac{\alpha x(x-v) - x + \alpha x - \alpha x^2 + \alpha x v}{(x-\tilde{H})^2} \\ &= \frac{x(\alpha-1)}{(x-\tilde{H})^2}, \\ H_x &= \frac{\tilde{H}_x(x-\tilde{H}) - (1-\tilde{H}_x)(\tilde{H}-v)}{(x-\tilde{H})^2} = \frac{x\tilde{H}_x - \tilde{H}_x\tilde{H} - \tilde{H} + v + \tilde{H}\tilde{H}_x - v\tilde{H}_x}{(x-\tilde{H})^2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\tilde{H}_x(x-v)-\tilde{H}+v}{(x-\tilde{H})^2} = \frac{(\alpha-2\alpha xv+\alpha v)(x-v)-(\alpha x-\alpha x^2+\alpha xv)+v}{(x-\tilde{H})^2} \\
&= \frac{\alpha x-2\alpha x^2 v+\alpha vx-\alpha v+2\alpha xv^2-\alpha v^2-\alpha x+\alpha x^2-\alpha xv+v}{(x-\tilde{H})^2} \\
&= \frac{(1-\alpha)v-\alpha(x-v)^2}{(x-\tilde{H})^2}.
\end{aligned}
$$

We note that

$$L(n,x,v) = \frac{1}{(1+\delta)^n}(p_n x - B)H.$$

First, we calculate

$$L_v(n,x,v) = \frac{1}{(1+\delta)^n}(p_n x - B)H_v.$$

For the calculation of L_x, we first establish the formula

$$
\begin{aligned}
&(\tilde{H}-v)(x-\tilde{H})+(1-\alpha)xv-\alpha x(x-v)^2+x^2(\alpha-1) \\
&= (\tilde{H}-v)(x-\tilde{H})+(1-\alpha)x(v-x)-\alpha x(x-v)^2 \\
&= (\tilde{H}-v)(x-\tilde{H})+(v-x)\{(1-\alpha)x+\alpha x(x-v)\} \\
&= (\tilde{H}-v)(x-\tilde{H})+(v-x)\{x-\alpha x+\alpha x(x-v)\} \\
&= (\tilde{H}-v)(x-\tilde{H})+(v-x)(x-\tilde{H}) \\
&= (\tilde{H}-v+v-x)(x-\tilde{H}) \\
&= (\tilde{H}-x)(x-\tilde{H}),
\end{aligned}
$$

i.e.,

$$(\tilde{H}-v)(x-\tilde{H})+(1-\alpha)xv-\alpha x(x-v)^2+x^2(\alpha-1) = -(x-\tilde{H})^2. \qquad (4.12)$$

Now we calculate

$$L_x(n,x,v) = \frac{1}{(1+\delta)^n}\{p_n H + (p_n x - B)\,H_x\}$$

$$
\begin{aligned}
&= \frac{1}{(1+\delta)^n}\left\{p_n\frac{\tilde{H}-v}{x-\tilde{H}}+(p_nx-B)\frac{(1-\alpha)v-\alpha(x-v)^2}{(x-\tilde{H})^2}\right\} \\
&= \frac{1}{(1+\delta)^n}\left\{p_n\frac{(\tilde{H}-v)(x-\tilde{H})+(1-\alpha)vx-\alpha x(x-v)^2}{(x-\tilde{H})^2}\right. \\
&\quad \left.-B\frac{(1-\alpha)v-\alpha(x-v)^2}{(x-\tilde{H})^2}\right\} \\
&= -L_v+\frac{1}{(1+\delta)^n}\left\{p_n\frac{(\tilde{H}-v)(x-\tilde{H})+(1-\alpha)vx-\alpha x(x-v)^2+x^2(\alpha-1)}{(x-\tilde{H})^2}\right. \\
&\quad \left.-B\frac{(1-\alpha)(v-x)-\alpha(x-v)^2}{(x-\tilde{H})^2}\right\} \\
&\overset{(4.12)}{=} -L_v+\frac{1}{(1+\delta)^n}\left\{p_n\frac{-(x-\tilde{H})^2}{(x-\tilde{H})^2}-B\frac{(1-\alpha)(v-x)-\alpha(x-v)^2}{(x-\tilde{H})^2}\right\} \\
&= -L_v+\frac{1}{(1+\delta)^n}\left\{-p_n-B\frac{(1-\alpha)(v-x)-\alpha(x-v)^2}{(x-\tilde{H})^2}\right\} \\
&= -L_v-\frac{1}{(1+\delta)^n}\left\{p_n+B\frac{(1-\alpha)(v-x)-\alpha(x-v)^2}{(x-\tilde{H})^2}\right\}.
\end{aligned}
$$

In summary, since

$$x-\tilde{H}=x-\alpha x(1-(x-v))=x\{1-\alpha+\alpha(x-v)\},$$

we have

$$L_v(n,x,v)=\frac{1}{(1+\delta)^n}\frac{(\alpha-1)(p_nx-B)}{x\{1-\alpha+\alpha(x-v)\}^2}$$

and

$$L_x(n,x,v)=-L_v(n,x,v)-\frac{1}{(1+\delta)^n}\left(p_n-B\frac{x-v}{x^2\{1-\alpha+\alpha(x-v)\}}\right).$$

Now we use the Euler–Lagrange equation (4.11), i.e., we require

$$
\begin{aligned}
0 &= L_x(n,x_{n+1},\Delta x_n)-\Delta L_v(n,x_{n+1},\Delta x_n) \\
&= -L_v(n,x_{n+1},\Delta x_n)-\frac{1}{(1+\delta)^n}\left(p_n-B\frac{x_n}{x_{n+1}^2(1-\alpha+\alpha x_n)}\right) \\
&\quad -L_v(n+1,x_{n+2},\Delta x_{n+1})+L_v(n,x_{n+1},\Delta x_n) \\
&= -\frac{1}{(1+\delta)^n}\left(p_n-B\frac{x_n}{x_{n+1}^2\,(1-\alpha+\alpha x_n)}\right)+\frac{1-\alpha}{(1+\delta)^{n+1}}\frac{p_{n+1}-\frac{B}{x_{n+2}}}{(1-\alpha+\alpha x_{n+1})^2}
\end{aligned}
$$

$$= \frac{1}{(1+\delta)^{n+1}} \left\{ (1-\alpha) \frac{p_{n+1} - \frac{B}{x_{n+2}}}{(1-\alpha+\alpha x_{n+1})^2} - (1+\delta) \left(p_n - B \frac{x_n}{x_{n+1}^2 (1-\alpha+\alpha x_n)} \right) \right\},$$

i.e.,

$$p_{n+1} - \frac{B}{x_{n+2}} = \frac{1+\delta}{1-\alpha} \left(p_n - B \frac{x_n}{x_{n+1}^2 (1-\alpha+\alpha x_n)} \right) (1-\alpha+\alpha x_{n+1})^2,$$

i.e., the solution x of (4.11) is given recursively by

$$x_{n+2} = \frac{B}{p_{n+1} - \frac{1+\delta}{1-\alpha} \left(p_n - B \frac{x_n}{x_{n+1}^2 (1-\alpha+\alpha x_n)} \right) (1-\alpha+\alpha x_{n+1})^2}.$$

5. CONCLUSION AND FUTURE ACHIEVEMENT

In this thesis, we explain the logistic equation by Pierre François Verhulst carefully. Its historical part was significant. Malthus's essay inspired Pierre François Verhulst, and he made an excellent transformation of the model without the environment limitation respective to the limited model. Gordon–Schaefer adapted the logistic growth model economically. A harvesting term is involved, which means the logistic model is not just a biological model anymore but it becomes bioeconomic. After that, we discuss four strategies to explain the optimal harvesting level in the so-called Schaefer model. The procedure requests solving the optimal control equation; we solve it by the Euler–Lagrange equation. We find the equation has two roots, and we must exclude one since it does not satisfy one of the conditions. The second root $x_2(t)$ is considered to administrate the four optimal strategies. Furthermore, obtaining the optimal harvesting level is complicated. Each process needs to be explained accurately. In other words, we need to consider each detail within a specific period since the Schaefer model considers the continuous-time case of the logistic model. After we prove the four strategies in the Schaefer model (the continuous-time case), we study the Beverton–Holt model (the discrete case). We discuss the geometric distribution of the price function. After that, we use the discrete version of the Euler–Lagrange equation to find the solution of the Beverton–Holt model.

Future work on the discrete case contains confirming the four strategies as we confirmed them in the continuous case. In addition, it can be interesting to consider other distributions of the price function in both the continuous and discrete cases, for example, the Weibull distribution in the continuous case. Also, it would be interesting to consider the case when q, r, and K are time varying instead of constant.

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