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Several problems in nonlinear Schrödinger equations

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SEVERAL PROBLEMS IN NONLINEAR SCHRÖDINGER EQUATIONS

by

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Paper II: Pages 26-49, Well-Posedness and Blowup for the Dispersion-Managed Nonlinear Schrödinger Equation has been submitted to Proceedings of the American Mathematical Society for review. This paper is joint work with Dr. Jason Murphy.

ABSTRACT

We study several different problems related to nonlinear Schrödinger equations, specifically the dispersion-managed equation

$$\begin{cases} (i\partial_t + \gamma(t)\Delta)u = \pm|u|^p u \\ u(0,x) = u_0(x) \end{cases}$$

where $u = u(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is a complex valued function of space and time. The function γ is a 1-periodic map that adds a time anisotropy to the equation. We also study the $1d$ periodic NLS

$$\begin{cases} (i\partial_t + \Delta)u = \pm|u|^p u \\ u(0,x) = u_0(x) \end{cases}$$

where $u = u(t,x) : \mathbb{R} \times \mathbb{T} \to \mathbb{C}$ is a *periodic* complex-valued function of space and time.

We prove several new results for the first equation: a modified scattering result for both an averaged version of the equation and the full equation, as well as a set of Strichartz estimates and a blowup result for the $3d$ cubic problem.

We also present an exposition of the classical work of Bourgain on invariant measures for the second equation in the mass-subcritical regime.

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SECTION

1. INTRODUCTION

In this work, we study several problems involving power-type nonlinear Schrödinger equations. We begin with the study of the so-called 'dispersion-managed' model, or DMNLS:

$$\begin{cases} i\partial_t u + \gamma(t)\Delta u = |u|^p u \\ u(0,x) = u_0(x) \end{cases} \quad \text{on } \mathbb{R}_t \times \mathbb{R}_x^d. \tag{1.1}$$

Here, $\gamma(t)$ is a one-periodic function called the *dispersion map*, and it obeys certain admissibility conditions which will be outlined later.

Physically speaking, this model is related to the dispersion-free nonlinear Schrödinger equation: it serves as a model for light traveling in a fiber optic cable, where in contrast to the standard 'isotropic' model of a fiber optic cable, one alternates the composition of the fiber optic cable along the length. This anisotropy is modeled by the dispersion map γ.

We discuss several results for this model. First, we prove small-data modified scattering for an averaged version of this model using techniques developed by Hayashi-Naumkin and Kato-Pusateri ([16], [21]). Next, we prove a set of global-in-time Strichartz estimates for the unaveraged model, and use these to demonstrate a well-posedness and blowup result for the $3d$ cubic DMNLS. Finally, using these Strichartz estimates, we prove a modified scattering result for the unaveraged $1d$ cubic DMNLS, the natural next step from the first result.

The final section contains an exposition of Bourgain's result ([4]) on invariance of the Gibbs measure for the $1d$ NLS on the torus.

1.1. DISCUSSION OF MAIN RESULTS

In this thesis, we present several different problems. The first involves well-posedness and modified scattering for the averaged dispersion-managed NLS given by

$$i\partial_t + d_{\text{av}}\partial_{xx}u = c\int_0^1 e^{-iD(\tau)\Delta}\{|e^{iD(\tau)\Delta}u|^2 e^{iD(\tau)\Delta}u\}d\tau \tag{1.2}$$

In short, we show that small-data solutions to this equation (in a particular Sobolev norm) give rise to global solutions of the equation above, and the solution obeys certain L^∞ decay estimates. Further, we are able to show that up to a logarithmic phase correction, the solution scatters in L^∞.

Next, we discuss the more general dispersion-managed NLS, where the dispersion coefficient in front of the Laplacian is time-varying and 1-periodic. We establish a collection of Strichartz estimates for this model, and use these to prove a well-posedness and blowup, as well as a modified scattering result in 1d analogous to the above result for the averaged equation.

Finally, we conclude with an exposition of Bourgain's 1994 result on invariant measures for the 1d periodic mass-subcritical NLS.

1.2. NOTATION AND BASIC ESTIMATES

In this section, we introduce some notation that will be used throughout the rest of the paper. First, we write $A \lesssim B$ or $B \gtrsim A$ to denote the inequality $A \leq CB$ for some $C > 0$. If $A \lesssim B$ and $B \lesssim A$, we write $A \sim B$. We denote explicit dependence of the constant C on various parameters by subscripts, e.g. $A \lesssim_u B$ denotes $A \leq CB$ for some $C = C(u)$. We also utilize the standard 'big oh' notation $\mathcal{O}$. Finally, we use the Japanese bracket notation $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$.

For a time interval I, we write $L_t^q L_x^r(I \times \mathbb{R}^d)$ for the Banach space of functions $u : I \times \mathbb{R}^d \to \mathbb{C}$ equipped with the norm

$$\|u\|_{L_t^q L_x^R(I \times \mathbb{R}^d} = \left(\int_I \|u(t)\|_{L_x^r}^q \, \mathrm{d}t \right)^{1/q},$$

making the usual adjustments when q or r is infinite. We will often abbreviate by removing the explicit space over which the norm is taken when clear from context. Given $r \in [1, \infty]$, we write r' for the Hölder conjugate to r, i.e. the solution to $\frac{1}{r} + \frac{1}{r'} = 1$.

Since we exclusively deal with the power-type nonlinearity $F(u) = |u|^p u$ ($p > 0$) in this work, we record the following estimate, as it will be utilized in many of our well-posedness results.

Lemma 1. *Let $F(u)$ be as described above. Then we have the following estimate:*

$$|F(u) - F(v)| \lesssim_p (|u|^p + |v|^p)|u - v| \tag{1.3}$$

1.3. TOOLS FROM HARMONIC ANALYSIS

We define the Fourier transform on $\mathbb{R}^d$ by

$$(\mathcal{F} f)(\xi)\widehat{f}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) \, \mathrm{d}x \tag{1.4}$$

and on the torus $\mathbb{T}^d = \mathbb{R} \,/\, \mathbb{Z}$ by

$$\widehat{f}(k) := \int_{\mathbb{T}^d} e^{-2\pi i k\cdot x} f(x) \, \mathrm{d}x \tag{1.5}$$

The fractional differentiation operators $\langle\nabla\rangle^s$ ($s > 0$) are then defined by $\widehat{\langle\nabla\rangle^s f}(\cdot) := \langle\cdot\rangle^s \widehat{f}(\cdot)$.

Using this, we define the inhomogeneous Sobolev norm by

$$\|u\|_{H^s_x(\mathbb{R}^d)} := \|\langle\nabla\rangle^s f\|_{L^2_x(\mathbb{R}^d)}$$

and the weighted Sobolev norm by

$$\|u\|_{H^{s,p}_x(\mathbb{R}^d)} := \|u\|_{H^s_x(\mathbb{R}^d)} + \|\langle x\rangle^p u\|_{L^2_x(\mathbb{R}^d)}$$

1.4. LINEAR THEORY FOR THE SCHRÖDINGER EQUATION

In this section, we discuss the structure and estimates for the linear Schrödinger equation on $\mathbb{R}^d$.

1.4.1. Representation of Solutions and Related Objects.

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u(0,x) = u_0(x) \end{cases} \quad \text{on} \quad \mathbb{R}\times\mathbb{R}^d \tag{1.6}$$

First, recall that we can write the solution $u(t,x)$ to this equation as follows:

$$u(t,x) = e^{it\Delta}u_0(x)$$

where the solution operator $e^{it\Delta}$ is defined as a Fourier multiplier: $e^{it\Delta}f(x) = \mathcal{F}^{-1}\, e^{-it\xi^2}\,\widehat{f}(\xi)$. By explicitly evaluating the Gaussian integral and using analytic continuation, we arrive at the following physical-space representation of $e^{it\Delta}$:

$$e^{it\Delta}f(x) = (4\pi it)^{-d/2}\int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}}\, f(y)\,\mathrm{d}y$$

This readily implies the factorization identity

$$e^{it\Delta} = \mathcal{M}(t)\,\mathcal{D}(t)\,\mathcal{F}\,\mathcal{M}(t) \tag{1.7}$$

where

$$[\mathcal{M}(t)f](x) = e^{i|x|^2/4t}f(x) \quad \text{and} \quad [\mathcal{D}(t)f(x)] = (2it)^{-\frac{1}{2}}f\left(\frac{x}{2t}\right)$$

We will also make use of the Galilean operator $J(t) = x + 2it\nabla$. On the one hand, one can directly compute and show that

$$J(t) = \mathcal{M}(t)[2it\nabla]\,\mathcal{M}(-t)$$

On the other hand, an ODE argument leads to the identity

$$J(t) = e^{it\Delta}xe^{-it\Delta} \tag{1.8}$$

We will also make use of the following chain rule for $J(t)$, which follows from a direct computation: for any $p > 0$,

$$J(t)[|z|^p z] = \frac{p+2}{2}|z|^p[J(t)z] - \frac{p}{2}|z|^{p-2}z^2[\overline{J(t)z}] \tag{1.9}$$

Note that if we consider the linear dispersion-managed Schrödinger equation

$$\begin{cases} i\partial_t u + \gamma(t)\Delta u = 0 \\ u(0,x) = u_0(x) \end{cases} \quad \text{on} \quad \mathbb{R}\times\mathbb{R}^d, \tag{1.10}$$

all of the above identities hold; we merely make the replacement $t \mapsto \Gamma(t)$, where $\Gamma(t) := \int_0^t \gamma(s)\,\mathrm{d}s$. Such modifications will be used in Section 3 to prove modified scattering for the full cubic DMNLS. In this context, we will either infer the change from t into $\Gamma(t)$, or we will explicitly denote it by writing the argument $\Gamma(t)$ or a subscript Γ on the operator.

1.4.2. Estimates for the Linear Schrödinger Equation. In this section, we collect some standard estimates for the linear Schrödinger equation - see [32] for more detailed exposition.

First, we establish the 'dispersive estimates' for the linear Schrödinger equation.

Lemma 2 (Dispersive estimate)**.** *Let $e^{it\Delta}$ be the solution operator for the linear Schrödinger equation given above. Then for $1 \leq p \leq 2$,*

$$\left\|e^{it\Delta}u\right\|_{L^{p'}} \lesssim t^{-d(\frac{1}{p}-\frac{1}{2})}\|u\|_{L^p}$$

Proof. Let u be a Schwartz function on $\mathbb{R}^d$. First note that by Plancherel, $\|e^{it\Delta}u\|_{L^2} = \|u\|_{L^2}$, so $e^{it\Delta} : L^2 \to L^2$. Next, we use the convolution representation of $e^{it\Delta}$ to conclude that $\|e^{it\Delta}u\|_{L^\infty} \lesssim t^{-\frac{d}{2}}\|u\|_{L^1}$. Using Marcinkiewicz interpolation, we conclude that $\|e^{it\Delta}u\|_{L^{p'}} \lesssim t^{-d\left(\frac{1}{p}-\frac{1}{2}\right)}\|u\|_{L^p}$ for every $1 \leq p \leq 2$. □

Using this, one can prove the following Strichartz estimates for the linear Schrödinger equation:

Theorem 1 (Strichartz estimates)**.** *Fix $d \geq 1$. Call a pair (q, r)* admissible *if $q, r \in [2, \infty]$, $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, and $(q, r, d) \neq (2, \infty, 2)$. Then for any admissible exponents (q, r) and $(\tilde{q}, \tilde{r})$, we have the homogeneous Strichartz estimate*

$$\|e^{it\Delta}u_0\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim_{d,q,r} \|u_0\|_{L^2} \tag{1.11}$$

the dual homogeneous Strichartz estimate

$$\left\| \int_{\mathbb{R}} e^{-is\Delta} F(s) \, \mathrm{d}s \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \tag{1.12}$$

and the retarded Strichartz estimate ([8])

$$\left\| \int_s^t e^{i(t-s)\Delta} F(s) \, \mathrm{d}s \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \tag{1.13}$$

for any $s < t$.

Remark. The endpoint cases of this theorem require a different treatment - this is done in [22].

Proof. Fix a (nonendpoint) admissible pair (q, r). The problem amounts to proving that the operator $T = e^{it\Delta} : L^2 \to L_t^q L_x^r$. To prove this, we use the method of TT^*; this means we need to compute the adjoint of T. Using that T is unitary on L^2, we find that

$$\begin{aligned} \langle Tf, G \rangle &= \iint e^{it\Delta} f(x) G(t,x) \, \mathrm{d}x \, \mathrm{d}t \\ &= \int f(x) \int \overline{e^{-it\Delta} G(t,x)} \, \mathrm{d}t \, \mathrm{d}x, \end{aligned}$$

so that in particular

$$T^* G(x) = \int_{\mathbb{R}} e^{-is\Delta} G(s,x) \, \mathrm{d}s$$

and

$$TT^* F(t,x) = \int_{\mathbb{R}} e^{i(t-s)\Delta} F(s,x) \, \mathrm{d}s$$

Using the dispersive estimate (Lemma 2), Hardy-Littlewood-Sobolev, and the scaling relation defining admissibility, we estimate

$$\begin{aligned}\|TT^*F\|_{L_t^q L_x^r} &\lesssim \left\| \int |t-s|^{-(\frac{d}{2}-\frac{d}{r})} \|F(s)\|_{L_x^{r'}} \,\mathrm{d}s \right\|_{L_t^q} \\ &\lesssim \| |t|^{-\frac{2}{q}} * \|F(t)\|_{L_x^{r'}} \|_{L_t^q} \\ &\lesssim \| |t|^{-\frac{2}{q}} \|_{L_t^{\frac{q}{2},\infty}} \|F\|_{L_t^{q'} L_x^{r'}} \\ &\lesssim \|F\|_{L_t^{q'} L_x^{r'}}.\end{aligned}$$

By the method of TT^*, we find that T maps $L^2 \to L_t^q L_x^r$ boundedly, and its adjoint T^* maps $L_t^{q'} L_x^{r'} \to L^2$ boundedly; if we compose these estimates, we conclude the inhomogeneous Strichartz estimate. Finally, the retarded Strichartz estimate follows from applying the Christ-Kiselev lemma [8]. □

These Strichartz estimates will be paramount in the well-posedness results we will state later. Finally, we state an elementary estimate that will be used several times later: for $0 \leq \alpha \leq 1$,

$$|e^{ix} - 1| \lesssim |x|^\alpha \tag{1.14}$$

PAPER

I. MODIFIED SCATTERING FOR A DISPERSION-MANAGED NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT

We prove sharp L^∞ decay and modified scattering for a one-dimensional dispersion-managed cubic nonlinear Schrödinger equation with small initial data chosen from a weighted Sobolev space. Specifically, we work with an averaged version of the dispersion-managed NLS in the strong dispersion management regime. The proof adapts techniques from Hayashi-Naumkin and Kato-Pusateri, which established small-data modified scattering for the standard $1d$ cubic NLS.

1. INTRODUCTION

We study the long-time behavior of small solutions for a 'dispersion-managed' nonlinear Schrödinger equation (NLS) in one space dimension. Such models arise as the envelope equation for electromagnetic wave propagation in fiber-optics communication systems in which the dispersion is varied periodically along the optical fiber (see e.g. [1, 6, 24]). This may be modeled via an equation of the form

$$i\partial_t w + d(t)\partial_{xx} w = c|w|^2 w \tag{1}$$

for some 1-periodic function $d(t)$ and some $c \in \mathbb{R}\setminus\{0\}$. One may then decompose

$$d(t) = d_{\text{av}} + d_0(t),$$

with d_0 1-periodic and mean zero, and d_{av} giving the average dispersion.

In the present paper, we will not work directly with the model (1), but rather with an 'averaged' version in the so-called strong dispersion management regime (as introduced in [13]). In particular, we consider the equation

$$i\partial_t u + d_{\text{av}}\partial_{xx} u = c\int_0^1 e^{-iD(\tau)\Delta}\big\{|e^{iD(\tau)\Delta}u|^2 e^{iD(\tau)\Delta}u\big\}\,d\tau, \tag{2}$$

where

$$D(\tau) := \int_0^\tau d_0(\sigma)d\sigma$$

(see e.g. [6, Section 1.2] for a derivation of this model). Here we work with the specific choice

$$d(t) = \begin{cases} d_+, & 0 \le t < t_+, \\ -d_-, & t_+ < t \le 1, \end{cases} \tag{3}$$

with $d_\pm$ and t_+ chosen so that $d_{\text{av}} \neq 0$. As a matter of fact, because we consider only small initial conditions, the sign of d_{av} and c play no essential role, and so we assume without loss of generality that $d_{\text{av}} = c = 1$.

Dispersion-managed nonlinear Schrödinger equations have been the subject of a great deal of recent research, due largely to their connection with applications in fiber-optics based communications. This includes both numerical investigations and rigorous mathematical studies (see e.g. [2, 6, 10, 12, 13, 18, 19, 30, 33]). Much of the work to date has centered on questions of well-posedness and the existence and properties of soliton solutions. In this work, we will show that for small initial data chosen from a weighted Sobolev space, the corresponding solutions to (2) are global in time and decay as $|t| \to \infty$. Moreover, such solutions exhibit modified scattering, that is, asymptotically linear behavior up to a logarithmic phase correction (as in the case of the standard $1d$ cubic NLS). In particular, our result demonstrates the absence of small coherent structures for (2).

Our main result is the following.

Theorem 2. *Let $u_0 \in H^{1,1}$ satisfy $\|u_0\|_{H^{1,1}} = \varepsilon > 0$. If ε is sufficiently small, then there exists a unique solution $u \in C_t H_x^{1,1}([0,\infty) \times \mathbb{R})$ to (2) with $u|_{t=0} = u_0$. Furthermore, the solution obeys*

$$\|u(t)\|_{L^\infty} \lesssim \varepsilon(1+|t|)^{-\frac{1}{2}} \tag{4}$$

for all $t \geq 0$, and there exists $W \in L^\infty(\mathbb{R})$ such that

$$u(t,x) = (2it)^{-1/2} e^{ix^2/4t} \big[\exp\{-\tfrac{i}{2}|W(\tfrac{x}{2t})|^2 \log t\} W(\tfrac{x}{2t})\big] + \mathcal{O}(t^{-\frac{1}{2}-\frac{1}{20}})$$

in L^∞ as $t \to \infty$.

Theorem 2 fits in the general context of modified scattering for long-range nonlinear Schrödinger equations. In particular, many previous works have considered the standard $1d$ cubic NLS

$$i\partial_t u + \partial_{xx} u = \pm |u|^2 u, \tag{5}$$

for which one also obtains sharp L^∞ and modified scattering for small data in $H^{1,1}$. In fact, in the defocusing case, one can capitalize on the completely integrable structure of (5) to obtain this result without any size restriction on the initial data [9]. Several different approaches have been utilized in order to establish small-data modified scattering for (5) (see e.g. [16, 20, 21, 26], as well as [27] for a review). Essentially, each of these approaches are based off of a bootstrap argument involving some 'dispersive' type norm and some 'energy' type norm, using an ODE argument to obtain estimates for the dispersive part and a chain-rule type estimate and Grönwall to control the energy part. We follow the same general strategy, adapting techniques particularly from [21] and [16]. In particular, we use the Fourier representation and a 'space-time non-resonance' type approach as in [21] to control the dispersive norm, while the energy-type estimate relies on the chain-rule type identity satisfied by the Galilean operator $J(t) = x + 2it\partial_x$. The estimate for the dispersive norm follows largely as in [21], while the estimate of energy norm requires some modifications. In particular, while the energy estimate for the standard NLS relies directly on the L^∞-decay for solutions, we must instead rely on the factorization of the free propagator (see (1.7)) in order to exhibit suitable decay in the nonlinearity. For the details, see (19) and (20) below.

The problem for (2) (as opposed to the original model (1)) is simplified by the fact that the underlying linear part of the equation is still given by the standard Schrödinger equation. In the setting of (1), the situation is more complicated due to the time-dependence in the linear part of the equation, which affects the underlying linear dispersion (see e.g. [2]). We plan to study the model (1) in a future work.

The rest of the paper is organized as follows: In Section 2, we discuss the H^1 and $H^{1,1}$ well-posedness for (2). In Section 3, we prove the main result, Theorem 2. In particular, in Section 3.1, we establish global existence and L^∞ decay, while in Section 3.2 we establish the long-time asymptotic behavior of solutions.

2. LOCAL WELL-POSEDNESS

In this section we discuss the H^1 and $H^{1,1}$ well-posedness for (2) (see e.g. [2, 6] for other well-posedness results for dispersion-managed NLS). As much of what follows is standard, we focus only on the main points and the new estimates needed to treat the specific model (2); we refer the reader to [5] for a general introduction to well-posedness for nonlinear Schrödinger equations. We construct solutions to (2) as solutions to the following Duhamel formula:

$$u(t) = e^{it\Delta}u_0 - i\int_1^t \int_0^1 e^{i(t-s)\Delta}\big[e^{-iD(\tau)\Delta}\{|e^{iD(\tau)\Delta}u(s)|^2 e^{iD(\tau)\Delta}u(s)\}\big]\,d\tau\,ds.$$

We apply the standard contraction mapping argument, with the key nonlinear estimate given as follows: By the chain rule, the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, and the fact that $e^{i\cdot\Delta}$ is unitary on H^1, we have

$$\begin{aligned}
&\left\|\int_0^1 \langle\partial_x\rangle\left[e^{-iD(\tau)\Delta}\left\{|e^{iD(\tau)\Delta}u|^2 e^{iD(\tau)\Delta}u\right\}\right]\,d\tau\right\|_{L_t^1L_x^2([0,T]\times\mathbb{R})}\\
&\lesssim \sup_{\tau\in[0,1]} T\left\|\langle\partial_x\rangle\big[e^{-iD(\tau)\Delta}\{|e^{iD(\tau)\Delta}u|^2 e^{iD(\tau)\Delta}u\}\big]\right\|_{L_t^\infty L_x^2}\\
&\lesssim \sup_{\tau\in[0,1]} T\left\|e^{iD(\tau)\Delta}u\right\|_{L_{t,x}^\infty}^2\|\langle\partial_x\rangle u\|_{L_t^\infty L_x^2} \lesssim T\,\|\langle\partial_x\rangle u\|_{L_t^\infty L_x^2}^3.
\end{aligned}$$

This allows us to establish local existence for $u_0\in H^1$ for times $T\sim\|u_0\|_{H^1}^{-2}$. Similarly, by utilizing the chain rule for $J(t)$ (see (1.9)), we may establish local existence for $u_0\in H^{1,1}$, again with $T\sim\|u_0\|_{H^1}^{-2}$, albeit only with the crude bound

$$\|xu(t)\|_{L^2}\lesssim\|J(t)u(t)\|_{L^2}+t\|\partial_x u(t)\|_{L^2}\lesssim(1+t)\|u_0\|_{H^{1,1}}.$$

This H^1-subcritical well-posedness also includes the usual blowup alternative, that is, either u is forward-global or there exists $T_* < \infty$ such that

$$\lim_{t \to T_*} \|u(t)\|_{H^1} = \infty.$$

We will eventually obtain explicit bounds on the growth of the H^1-norm of solutions, which in particular imply that the solutions may be extended to be forward-global in time.

3. PROOF OF MAIN RESULT

We let u be a solution to (2) with $\|u_0\|_{H^{1,1}} = \varepsilon$. By the local theory and Sobolev embedding, we may assume that

$$\|u(t)\|_{H^{1,1}} \leq 2\varepsilon. \tag{6}$$

for $t \in [0,1]$, and that the solution exists on some maximal interval $(0, T_*)$ with $T_* > 1$.

3.1. GLOBAL EXISTENCE AND DECAY

The first part of the proof of Theorem 2 is based on a bootstrap argument for times $t \geq 1$ involving the following 'dispersive' and 'energy' norms:

$$\begin{aligned} \|u(t)\|_{X_D} &:= \|\hat{f}(t)\|_{L^\infty}, \quad \text{where} \quad f(t) = e^{-it\Delta}u(t), \\ \|u(t)\|_{X_E} &:= t^{-\frac{1}{20}}\big\{\|\langle\partial_x\rangle u(t)\|_{L^2} + \|J(t)u(t)\|_{L^2}\big\}. \end{aligned} \tag{7}$$

Note that by (1.8), we may also write

$$\|J(t)u(t)\|_{L^2} = \|xf(t)\|_{L^2}.$$

We then define

$$\|u(t)\|_X = \sup_{s\in[1,T]} \{\|u(s)\|_{X_D} + \|u(s)\|_{X_E}\}.$$

Control over these norms will imply the desired L^∞ decay.

Lemma 3. *For any* $t \geq 1$*,*

$$\|u(t)\|_{L^\infty} \lesssim t^{-\frac{1}{2}}\{\|u(t)\|_{X_D} + \|u(t)\|_{X_E}\}.$$

Proof. We write $f(t) = e^{-it\Delta}u(t)$ as above. By (1.7), Hausdorff–Young, and Cauchy–Schwarz, we have

$$\begin{aligned}
\|u(t)\|_{L^\infty} &= \|M(t)D(t)\,\mathcal{F}\,M(t)f(t)\|_{L^\infty} \\
&\lesssim t^{-\frac{1}{2}}\{\|\hat{f}(t)\|_{L^\infty} + \|\,\mathcal{F}[M(t)-1]f(t)\|_{L^\infty}\} \\
&\lesssim t^{-\frac{1}{2}}\{\|u(t)\|_{X_D} + t^{-\frac{1}{20}}\||x|^{\frac{1}{10}}f(t)\|_{L^1}\} \\
&\lesssim t^{-\frac{1}{2}}\{\|u(t)\|_{X_D} + t^{-\frac{1}{20}}\|\langle x\rangle f(t)\|_{L^2}\} \\
&\lesssim t^{-\frac{1}{2}}\{\|u(t)\|_{X_D} + \|u(t)\|_{X_E}\}.
\end{aligned}$$

□

The first part of Theorem 2 will follow from the following bootstrap estimate.

Proposition 3. *Let* $u : [1,T]\times\mathbb{R} \to \mathbb{C}$ *be a solution to* (2) *satisfying* (6)*. Then there exists* $C > 0$ *(independent of* T*) so that*

$$\|u(t)\|_X \leq 8\varepsilon + C\|u(t)\|_X^3$$

for all $t \in [1,T]$*.*

We split this proposition into two lemmas. We begin by estimating the dispersive norm.

Lemma 4 (Dispersive bound). *For any $t \geq 1$,*

$$\|u(t)\|_{X_D} \leq 2\varepsilon + C \int_1^t s^{-1-\frac{1}{20}} \|u(s)\|_X^3 \, ds. \tag{8}$$

Proof of Lemma 4. We begin with the Duhamel formula for the profile $f(t) = e^{-it\Delta}u(t)$:

$$f(t) = f(1) - i \int_1^t \int_0^1 e^{-is\Delta} e^{iD(\tau)\Delta} F(e^{iD(\tau)\Delta} u(s)) \, d\tau \, ds, \tag{9}$$

where $F(z) = |z|^2 z$. Taking the Fourier transform yields the following:

$$\begin{aligned} \hat{f}(t) = \hat{f}(1) - i(2\pi)^{-1} \int_1^t \int_0^1 \iint & \Big[e^{i(s+D(\tau))(\xi^2 - (\xi-\eta)^2 + (\eta-\sigma)^2 - \sigma^2)} \\ & \times \hat{f}(s, \xi - \eta) \hat{\bar{f}}(s, \eta - \sigma) \hat{f}(s, \sigma) \Big] \, d\eta \, d\sigma \, d\tau \, ds \end{aligned} \tag{10}$$

Changing variables via $\xi - \sigma \mapsto \sigma$, we find that

$$\hat{f}(t) = \hat{f}(1) - i(2\pi)^{-1} \int_1^t \int_0^1 \iint e^{2i(s+D(\tau))\eta\sigma} G(s, \xi, \eta, \sigma) \, d\sigma \, d\eta \, d\tau \, ds, \tag{11}$$

where

$$G(s, \xi, \eta, \sigma) := \hat{f}(\xi - \eta) \hat{\bar{f}}(\eta - \xi + \sigma) \hat{f}(\xi - \sigma).$$

By Plancherel, we may write

$$\begin{aligned} \hat{f}(t) = \hat{f}(1) - i(2\pi)^{-1} \int_1^t \int_0^1 \iint & \mathcal{F}_{\eta,\sigma} \left[e^{2i(s+D(\tau))\eta\sigma} \right] \\ & \times \mathcal{F}^{-1}_{\eta,\sigma} \left[G(s, \xi, \eta, \sigma) \right] \, d\sigma \, d\eta \, d\tau \, ds \end{aligned} \tag{12}$$

Noting the identity

$$\mathcal{F}_{\eta,\sigma}\left[e^{2i(s+D(\tau))\eta\sigma}\right] = \frac{1}{2(s+D(\tau))}e^{\frac{-i\eta\sigma}{2(s+D(\tau))}} \tag{13}$$

and the fact that

$$G(s,\xi,0,0) = |\hat{f}(s,\xi)|^2\hat{f}(s,\xi),$$

we may therefore write

$$\begin{aligned}\hat{f}(t,\xi) = \hat{f}(1,\xi) - i(2\pi)^{-1}\int_1^t\int_0^1 \frac{1}{2(s+D(\tau))}|\hat{f}(s,\xi)|^2\hat{f}(s,\xi)\,d\tau\,ds \\ + \int_1^t\int_0^1 \frac{1}{2(s+D(\tau))}\left[\iint \left[e^{\frac{-i\eta\sigma}{2(s+D(\tau))}} - 1\right]\mathcal{F}^{-1}_{\eta,\sigma}[G]\,d\sigma\,d\eta\right]d\tau\,ds.\end{aligned}$$

In particular, this implies

$$i\partial_t\hat{f}(t,\xi) = \int_0^1 \frac{1}{2(t+D(\tau))}|\hat{f}(t,\xi)|^2\hat{f}(t,\xi)\,d\tau + \int_0^1 \frac{1}{2(t+D(\tau))}\mathcal{R}(t,\tau,\xi)\,d\tau, \tag{14}$$

where

$$\mathcal{R}(t,\tau,\xi) = (2\pi)^{-1}\iint\left[e^{-\frac{i\eta\sigma}{2(t+D(\tau))}} - 1\right]\mathcal{F}^{-1}_{\eta,\sigma}[G]\,d\sigma\,d\eta.$$

We now employ an integrating factor to remove the first term on the right-hand side of (14). With

$$\Theta(t) = \int_1^t\int_0^1 \frac{1}{2(s+D(\tau))}|\hat{f}(s,\xi)|^2\,d\tau\,ds \tag{15}$$

and $g = e^{i\Theta(t)}\hat{f}$, we obtain

$$i\partial_t g = e^{i\Theta(t)}\int_0^1 \frac{1}{2(t+D(\tau))}\mathcal{R}(t,\xi)\,d\tau.$$

We want to estimate this quantity in L^∞_ξ. Using the definition of $\mathcal{R}$ from above, Cauchy–Schwarz and the bound (1.14), we find that

$$|\partial_t g| \lesssim \int_0^1 \iint |t + D(\tau)|^{-1-\frac{1}{5}} |\eta|^{\frac{1}{5}} |\sigma|^{\frac{1}{5}} |\mathcal{F}^{-1}_{\eta,\sigma}[G](s,\xi,\eta,\sigma)| \, d\sigma \, d\eta \, d\tau \tag{16}$$

To proceed, we now need to invert the Fourier transform appearing in (16). Writing

$$\mathcal{F}^{-1}_{\eta,\sigma}[G] = \mathcal{F}^{-1}_{\sigma}\left[\mathcal{F}^{-1}_{\eta}\left[\hat{f}(\xi-\eta)\hat{f}(\eta-\xi+\sigma)\right]\hat{f}(\xi-\sigma)\right],$$

a direct computation leads to the estimate

$$|\mathcal{F}^{-1}_{\eta,\sigma}[G](\eta,\sigma)| \lesssim \int |f(z-\eta)|\,|f(z)|\,|f(z-\sigma)| \, dz. \tag{17}$$

Thus, by the triangle inequality and Cauchy–Schwarz, and the fact that $D(\tau) \geq 0$[1],

$$\begin{aligned}
|(16)| &\lesssim \int_0^1 \iiint |t + D(\tau)|^{-1-\frac{1}{5}} \left[|z-\eta|^{\frac{1}{5}} + |z|^{\frac{1}{5}}\right] \left[|z-\sigma|^{\frac{1}{5}} + |z|^{\frac{1}{5}}\right] \\
&\qquad\qquad \times |f(z-\eta)||f(z)||f(z-\eta)| \, dz \, d\sigma \, d\eta \, d\tau \\
&\lesssim t^{-1-\frac{1}{5}} \|f\|_{L^1} \|\langle x\rangle^{\frac{2}{5}} f\|_{L^1}^2 \\
&\lesssim t^{-1-\frac{1}{5}} \|\langle x\rangle f\|_{L^2}^3 \lesssim t^{-1-\frac{1}{20}} \|u(t)\|_X^3.
\end{aligned}$$

The desired estimate then follows from the fundamental theorem of calculus and the triangle inequality:

$$\begin{aligned}
|g(t)| = |\hat{f}(t)| &\leq |\hat{f}(1)| + \int_1^t |\partial_s g(s)| \, ds \\
\implies \|u(t)\|_{X_D} &\leq 2\varepsilon + C \int_1^t s^{-1-\frac{1}{20}} \|u(s)\|_X^3 \, ds.
\end{aligned}$$

□

[1]The fact that $D(\tau) \geq 0$ is a convenient consequence of our particular choice of parameters above. To treat situations in which $D(\cdot)$ may take negative values, one only needs to observe that $\sup_{\tau\in[0,1]} |D(\tau)| \leq T_0$ for some $T_0 > 0$ and then begin the bootstrap estimate at times $t \geq 2T_0$, for example.

As this is the desired estimate, we can move on to the second component of the X-norm. This leads us to the following lemma:

Lemma 5 (Energy bound)**.** *For u as above, we have the following estimate:*

$$\|u(t)\|_{X_E} \leq 2t^{-\frac{1}{20}}\varepsilon + Ct^{-\frac{1}{20}} \int_1^t s^{-1+\frac{1}{20}} \|u(s)\|_X^3 \, ds. \tag{18}$$

Proof of Lemma 5. The starting point is the Duhamel formula

$$u(t) = e^{i(t-1)\Delta}u(1) - i \int_1^t \int_0^1 e^{i(t-s)\Delta} e^{-iD(\tau)\Delta} F(e^{iD(\tau)\Delta}u(s)) \, d\tau \, ds$$

with $F(z) = |z|^2 z$.

We first estimate the weighted component of the X_E-norm. Using (1.8), we deduce

$$\begin{aligned} J(t)u(t) &= e^{it\Delta}xf(1) \\ &\quad - i \int_0^t \int_0^1 e^{i(t-s)\Delta} e^{-iD(\tau)\Delta} \left[J(s + D(\tau)) F(e^{iD(\tau)\Delta}u(s)) \right] \, d\tau \, ds. \end{aligned}$$

Thus, using (1.9), (6), (1.8), Sobolev embedding, and the unitarity of $e^{i\cdot\Delta}$, we obtain

$$\begin{aligned} \|J(t)u(t)\|_{L^2} &\leq 2\varepsilon + C \int_1^t \int_0^1 \left\| e^{iD(\tau)\Delta}u(s) \right\|_{L^\infty}^2 \left\| J(s + D(\tau)) e^{iD(\tau)\Delta}u(s) \right\|_{L^2} d\tau \, ds \\ &\leq 2\varepsilon + C \int_1^t \int_0^1 \left\| e^{iD(\tau)\Delta}u(s) \right\|_{L_x^\infty}^2 \left\| e^{iD(\tau)\Delta} J(s)u(s) \right\|_{L^2} d\tau \, ds \\ &\leq 2\varepsilon + C \int_1^t s^{\frac{1}{20}} \left[\int_0^1 \left\| e^{i(s+D(\tau))\Delta} f(s) \right\|_{L_x^\infty}^2 d\tau \right] \|u(s)\|_{X_E} \, ds. \end{aligned} \tag{19}$$

This leaves us at an impasse, as we do not currently have any estimates for the quantity in square brackets.

To proceed, we use the factorization identity (1.7) and estimate as we did in the proof of Lemma 3. This yields

$$\begin{aligned}
&\|e^{i(s+D(\tau))\Delta}f(s)\|_{L_x^\infty}\\
&\lesssim |s+D(\tau)|^{-\frac{1}{2}}\big\{\|\hat{f}\|_{L^\infty}+\|\,\mathcal{F}[\mathcal{M}(s+D(\tau))-1]f\|_{L^\infty}\big\}\\
&\lesssim |s+D(\tau)|^{-\frac{1}{2}}\big\{\|\hat{f}\|_{L^\infty}+\|[\mathcal{M}(s+D(\tau))-1]f\|_{L^1}\big\}\\
&\lesssim |s+D(\tau)|^{-\frac{1}{2}}\big\{\|\hat{f}\|_{L^\infty}+|s+D(\tau)|^{-\frac{1}{5}}\||x|^{\frac{2}{5}}f\|_{L^1}\big\}\\
&\lesssim |s+D(\tau)|^{-\frac{1}{2}}\|\hat{f}\|_{L^\infty}+|s+D(\tau)|^{-\frac{1}{2}-\frac{1}{5}}\|\langle x\rangle f\|_{L^2}\\
&\lesssim \big\{|s+D(\tau)|^{-\frac{1}{2}}+|s+D(\tau)|^{-\frac{1}{2}-\frac{3}{20}}\big\}\|u(s)\|_X\lesssim |s|^{-\frac{1}{2}}\|u(s)\|_X,
\end{aligned}\tag{20}$$

where we have again used the fact that $D(\tau)\geq 0$ (cf. the footnote on page 18). Inserting this into (19), we obtain the desired estimate for the weighted component of the X_E-norm.

For the H^1 component of the X_E-norm, we estimate in much the same way, using the chain rule directly in place of (1.9). □

With Lemma 4 and Lemma 5 in place, we readily obtain the estimate appearing in Proposition 3. Using a standard continuity argument, the well-posedness theory discussed in Section 2, and Lemma 3, we deduce the the first part of Theorem 2. In particular, we have the following:

Corollary 4. *For $\|u_0\|_{H^{1,1}}=\varepsilon$ sufficiently small, there exists a unique, forward-global solution $u\in C_tH_x^{1,1}([0,\infty)\times\mathbb{R})$ to (2) with $u|_{t=0}=u_0$ obeying*

$$\|u(t)\|_X\lesssim\varepsilon\quad\textit{for all}\quad t\geq 1.$$

In particular,

$$\|u(t)\|_{L^\infty}\lesssim\varepsilon(1+|t|)^{-\frac{1}{2}}\quad\textit{for all}\quad t\geq 0.$$

3.2. ASYMPTOTIC BEHAVIOR

We now turn to the second part of Theorem 2, namely, the asymptotic behavior of small solutions to (2).

Proof of (4). To begin, we return to the setting of the proof of Lemma 4, this time with the bounds provided by Corollary 4 in hand. In particular, recalling

$$g(t) = e^{i\Theta(t)}\hat{f}(t), \quad \text{with} \quad \Theta(t) = \int_1^t \int_0^1 \frac{1}{2(s+D(\tau))} |\hat{f}(s,\xi)|^2 \, d\tau \, ds,$$

the estimate of (16) now yields the bound

$$\|\partial_t g\|_{L^\infty_\xi} \lesssim t^{-1-\frac{1}{20}} \varepsilon^3.$$

It follows that

$$\|g(t) - W_0\|_{L^\infty} \lesssim \varepsilon^3 t^{-\frac{1}{20}}$$

for some $W_0 \in L^\infty$. In particular, we have $|\hat{f}(t)| \to |W_0|$ in L^∞, with the same rate of convergence.

We next observe that

$$\left| \int_0^1 \frac{1}{2(s+D(\tau))} \, d\tau - \frac{1}{2s} \right| \lesssim s^{-2}$$

for all $s \geq 1$. Using this, we deduce that

$$\Theta(t) = \tfrac{1}{2}|W_0|^2 \log t + \Phi(t),$$

where $\Phi(t)$ converges to a real-valued limit Φ_∞ in L^∞ (with a rate of $t^{-\frac{1}{10}}$).

Thus, setting $W = e^{-i\Phi_\infty} W_0$, we may obtain by direct substitution into the formula for $\hat{f}$, the following new formula for $\hat{f}$:

$$\hat{f}(t) = e^{-i\frac{1}{2}|W_0|^2 \log t} e^{-i\Phi_\infty} W_0 + \mathcal{O}(t^{-\frac{1}{20}}) = e^{-\frac{1}{2}|W|^2 \log t} W + \mathcal{O}(t^{-\frac{1}{20}}) \tag{21}$$

in L^∞ as $t \to \infty$. Finally, using (1.7) and estimating as we did for Lemma 3, we obtain

$$u(t) = e^{it\Delta} f(t) = \mathcal{M}(t)\mathcal{D}(t)\hat{f}(t) + \mathcal{O}(t^{-\frac{1}{20}}).$$

Inserting the asymptotic behavior for $\hat{f}$ obtained in (21), we obtain the desired asymptotic behavior for $u(t)$. □

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II. WELL-POSEDNESS AND BLOWUP FOR THE DISPERSION-MANAGED NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT

We consider the nonlinear Schrödinger equation with periodic dispersion management. We first establish global-in-time Strichartz estimates for the underlying linear equation with suitable dispersion maps. As an application, we establish a small-data scattering result for the $3d$ cubic equation. Finally, we use a virial argument to demonstrate the existence of blowup solutions for the $3d$ cubic equation with piecewise constant dispersion map.

1. INTRODUCTION

We study the initial-value problem for a certain class of *dispersion-managed* nonlinear Schrödinger equations (DMNLS). In general, these equations take the form

$$i\partial_t u + \gamma(t)\Delta u + |u|^p u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \tag{1}$$

where the dispersion map γ is some time-periodic real-valued function. Such equations arise naturally in the context of nonlinear fiber optics, where one often encounters the cubic case ($p = 2$) in one dimension ($d = 1$) with piecewise constant dispersion map γ as in (2) below (see e.g. [1, 24]). For some representative mathematical results in this case (as well as some other cases), we refer the reader to [6, 7, 10, 12, 13, 18, 19, 28, 30, 33]. We note that in many of these works, one does not study (1) directly, but instead averages over one period of the dispersion map and studies the resulting autonomous equation.

The authors of [2] initiated the study of the initial-value problem for (1) for general dimensions and powers $p > 0$ with piecewise constant 1-periodic dispersion map of the form

$$\gamma(t) = \begin{cases} \gamma_+ & 0 \leq t < t_+ \\ \gamma_- & t_+ \leq t < 1, \end{cases} \tag{2}$$

with $\gamma_\pm > 0$. Their results included a local well-posedness theory in H^1 for energy-subcritical equations (i.e. $p < \frac{4}{d-2}$); a global well-posedness result in L^2 for the mass-subcritical case (i.e. $p < \frac{4}{d}$); and a sharp global well-posedness result in H^1 for the mass-critical case (i.e. $p = \frac{4}{d}$), including the existence of finite-time blowup solutions at the sharp threshold. The local theory appearing in [2] relies on the use of local-in-time Strichartz estimates for the usual linear Schrödinger equation. Combining this with mass conservation yields the global result in the mass-subcritical case. In the mass-critical case, the authors rely on the sharp Gagliardo–Nirenberg inequality to establish the global well-posedness result, while their blowup result relies on the use of the pseudoconformal symmetry.

Our goal in this work is to initiate the study of the global behavior of solutions in the intercritical setting (that is, for powers between the mass- and energy-critical exponents), where so far only the local behavior has been understood. We focus on the model case of the $3d$ cubic equation, namely

$$\begin{cases} i\partial_t u + \gamma(t)\Delta u + |u|^2 u = 0, \\ u|_{t=0} = u_0 \in H^1(\mathbb{R}^3), \end{cases} \tag{3}$$

with γ as in (2), although most of what we do carries over to the general intercritical case (i.e. $\frac{4}{d} < p < \frac{4}{d-2}$) in a straightforward way.

Our first result is a global-in-time Strichartz estimate for dispersion-managed Schrödinger equations (see Theorem 6), which may be of more general interest than the specific applications given here. The proof applies to a class of dispersion maps that is somewhat more general than the piecewise constant case (see Definition 5). The most essential restrictions are (i) non-vanishing average dispersion, i.e. $\langle\gamma\rangle := \int_0^1 \gamma(t)\,dt \neq 0$, and (ii) non-vanishing of the dispersion map itself (e.g. $\gamma^{-1} \in L_t^\infty$). A related NLS model in which the dispersion map itself vanishes at some points was considered in [11]; it is possible that in this case, one could recover some Lorentz-space modified Strichartz estimates, although we did not pursue that direction here.

As an application of the global Strichartz estimates, we establish a global well-posedness and scattering result for solutions to (3) with non-zero average dispersion and *small* initial data in H^1 (see Theorem 9). On the other hand, we can also demonstrate the existence of solutions to (3) that blow up in finite time (see Theorem 10). In particular, we may obtain finite-time blowup regardless of the sign of the average dispersion. To prove the blowup result, we combine the virial identity with a scaling argument to demonstrate the existence of blowup solutions in any 'focusing' step. We briefly summarize our main results as follows (with the statements given in the main body of the paper:

Theorem.

1. *The standard Strichartz estimates hold for the linear equation*

$$i\partial_t u + \gamma(t)\Delta u = 0$$

   for admissible dispersion maps γ. See Theorem 6 and Theorem 8.

2. *For admissible dispersion maps γ, small initial data in H^1 lead to global solutions to* (3) *that scatter. See Theorem 9.*

3. *For piecewise constant dispersion maps γ, equation* (3) *admits solutions that blow up in finite time. See Theorem 10.*

This combination of results, namely small-data scattering together with the existence of finite-time blowup solutions, leads to the interesting problem of identifying sharp conditions on the initial data that guarantee global well-posedness and scattering. In the setting of [2], the authors considered the mass-critical dispersion-managed NLS and were able to describe a sharp condition for global-wellposedness purely in terms of the conserved mass. In the case of the standard intercritical NLS (e.g. (3) with $\gamma \equiv 1$), the sharp condition for scattering versus blowup is described in terms of a combination of the mass and energy (see e.g. [17]). In the dispersion-managed setting, one does not have a conserved energy; instead, a different energy is conserved on each interval on which γ is constant. This makes it challenging to adapt any type of 'energy trapping' argument in order to propagate bounds even over one full period of the dispersion map. We plan to revisit this problem in a future work.

The rest of the paper is organized as follows: In Section 1 we establish the global-in-time Strichartz estimates for the underlying linear model. In Section 3, we establish global well-posedness and scattering for sufficiently small data. Finally, in Section 4, we prove the blowup result.

STRICHARTZ ESTIMATES FOR THE DISPERSION-MANAGED EQUATION

In this section we establish global-in-time Strichartz estimates for dispersion-managed Schrödinger equations of the form

$$\begin{cases} i\partial_t u + \gamma(t)\Delta u = 0, \\ u|_{t=s} = \varphi. \end{cases} \tag{4}$$

The equation (4) has the solution

$$u(t,s) = e^{i\Gamma(t,s)\Delta}\varphi, \quad \text{where} \quad \Gamma(t,s) = \int_s^t \gamma(\tau)\,d\tau. \tag{5}$$

Our main result addresses the following class of dispersion maps. A typical example is shown in Figure 1.

Definition 5 (Admissible)**.** We call $\gamma : \mathbb{R} \to \mathbb{R}$ *admissible* if it satisfies the following conditions:

- γ is one-periodic: $\gamma(t+1) = \gamma(t)$ for all $t \in \mathbb{R}$.
- γ and γ^{-1} are bounded: $\|\gamma\|_{L^\infty} + \|\gamma^{-1}\|_{L^\infty} < \infty$.
- γ has at most finitely many discontinuities on $[0,1]$.
- γ has nonzero average over its period: $\langle\gamma\rangle := \int_0^1 \gamma(t)\,dt \neq 0$.

The class of admissible functions includes the important piecewise constant case (2) in the case of nonzero average dispersion. Indeed, Definition 5 is basically a slight generalization of this special case. On the other hand, Definition 5 does *not* permit the case that $\gamma \to 0$ along some sequence of times. In particular, the proof of the global Strichartz estimates given below requires that γ stay bounded away from zero. A model of NLS in which γ vanishes was considered in [11]. It is possible

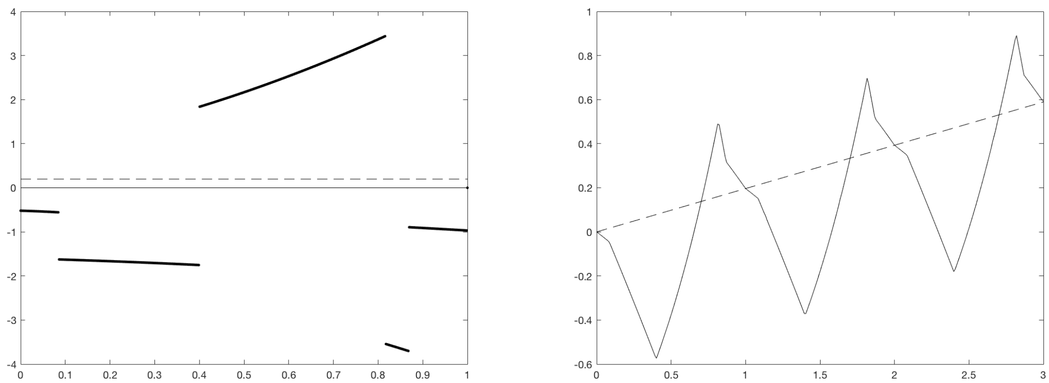

Figure 1. On the left is plotted one period of γ. The solid line is the line $y = 0$; the dotted line is $y = \langle \gamma \rangle$, which in this case is small and positive. On the right is plotted $t \mapsto \Gamma(t, 0)$ for $t \in [0, 3]$; the dotted line corresponds to the line $y = \langle \gamma \rangle t$.

that some Lorentz-modified Strichartz estimates could be established in this setting (e.g. by modifying the estimate in (10)), although we did not pursue that direction here. Instead, our main result shows that for admissible functions γ, we can derive global Strichartz estimates for (4) directly from those known to hold for the usual Schrödinger equation (cf. [14, 22, 31]).

Theorem 6 (Strichartz estimates)**.** *For any dimension $d \geq 1$, any $2 \leq q, r \leq \infty$ satisfying*

$$\tfrac{2}{q} + \tfrac{d}{r} = \tfrac{d}{2} \quad \textit{and} \quad (d, q, r) \neq (2, 2, \infty), \tag{6}$$

and any admissible γ (in the sense of Definition 5), there exists $C = C(\gamma) > 0$ so that for any $\varphi \in L^2$ and $s \in \mathbb{R}$, we have

$$\| e^{i\Gamma(t,s)\Delta} \varphi \|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} \leq C \| \varphi \|_{L^2(\mathbb{R}^d)}, \quad \textit{where} \quad \Gamma(t, s) = \int_s^t \gamma(\tau)\, d\tau.$$

Remark. The proof below will show that the constant may be taken to be

$$C(\gamma) = C_{\mathrm{Str}} \cdot \| \gamma^{-1} \|_{L^\infty}^{\frac{1}{q}} \cdot \left[1 + N_\gamma \left\{ 1 + \frac{[1 + 4\|\gamma\|_{L^\infty}]}{|\langle \gamma \rangle|} \right\} \right]^{\frac{1}{q}}, \tag{7}$$

where $\langle\gamma\rangle = \int_0^1 \gamma(s)\,ds$ is the average dispersion; N_γ is the number of discontinuities of γ in $[0,1]$; and C_{Str} is the constant for the standard $L_x^2(\mathbb{R}^d) \to L_t^q L_x^r(\mathbb{R}^d)$ Strichartz estimate. In particular, our estimate breaks down when γ becomes unbounded or approaches zero; when the average dispersion tends to zero; or when the number of discontinuities in one period becomes unbounded.

The proof relies on a few lemmas. We begin with the following:

Lemma 6. *Let γ be a one-periodic function on $\mathbb{R}$. Define*

$$\Gamma(t) = \int_0^t \gamma(s)\,ds \quad \textit{and} \quad \langle\gamma\rangle = \int_0^1 \gamma(s)\,ds.$$

Then

$$|\Gamma(t) - t\langle\gamma\rangle| \le 2\|\gamma\|_{L^\infty} \quad \textit{for all} \quad t \in \mathbb{R}\,. \tag{8}$$

Proof. Writing $\lfloor t \rfloor$ for the floor of t and using 1-periodicity of γ, we have that

$$\begin{aligned}\Gamma(t) &= \lfloor t \rfloor \langle\gamma\rangle + \int_{\lfloor t \rfloor}^t \gamma(s)\,ds \\ &= t\langle\gamma\rangle + (t - \lfloor t \rfloor)\langle\gamma\rangle + \int_{\lfloor t \rfloor}^t \gamma(s)\,ds.\end{aligned}$$

As $|t - \lfloor t \rfloor| \le 1$ for all $t \in \mathbb{R}$, this yields

$$\big|\Gamma(t) - t\langle\gamma\rangle\big| \le |\langle\gamma\rangle| + \|\gamma\|_{L^\infty},$$

which implies the result. □

In particular, Lemma 6 implies (via the intermediate value theorem) that for $\langle\gamma\rangle \neq 0$, the range of $\Gamma(\cdot, 0)$ equals $\mathbb{R}$. We also have the following corollary, which plays a role when we later partition the range of $\Gamma(\cdot, 0)$.

Corollary 7. *Let γ be a one-periodic function on $\mathbb{R}$, with*

$$\Gamma(t) = \int_0^t \gamma(s)\,ds, \quad \textit{and} \quad \langle\gamma\rangle = \int_0^1 \gamma(s)\,ds \neq 0.$$

Then for any $\delta > 0$ and any $t_1, t_2 \in \mathbb{R}$, we have that

$$|t_2 - t_1| > \frac{\delta + 4\|\gamma\|_{L^\infty}}{|\langle\gamma\rangle|} \implies |\Gamma(t_2) - \Gamma(t_1)| > \delta.$$

Proof. Without loss of generality, suppose that $\langle\gamma\rangle > 0$ and $t_2 > t_1$. Then by Lemma 6,

$$\Gamma(t_2) - \Gamma(t_1) \geq (t_2 - t_1)\langle\gamma\rangle - 4\|\gamma\|_{L^\infty},$$

which implies the result. □

We turn to the proof of Theorem 6.

Proof of Theorem 6. We fix the dimension $d \geq 1$ and (q, r) obeying (6). We let γ be admissible in the sense of Definition 5 and suppose $\langle\gamma\rangle = \int_0^1 \gamma(s)\,ds > 0$. It will suffice to prove that

$$\|e^{i\Gamma(t)\Delta}\varphi\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|\varphi\|_{L^2},$$

where $\varphi \in L^2$ and

$$\Gamma(t) = \int_0^t \gamma(s)\,ds.$$

For convenience, we split $\mathbb{R}$ (viewed as the domain of Γ) into a disjoint union $\mathbb{R} = \mathbb{I} \cup \mathbb{D}$ so that Γ is increasing on $\mathbb{I}$ and decreasing on $\mathbb{D}$ (equivalently, γ is positive on $\mathbb{I}$ and negative on $\mathbb{D}$). It then suffices to establish $L_t^q L_x^r$ bounds on $\mathbb{I} \times \mathbb{R}^d$, say. Note that $\mathbb{I}$ and $\mathbb{D}$ are themselves disjoint unions of intervals; moreover, if we write N_γ for the number of discontinuities of γ in one period, then the number of such intervals comprising $\mathbb{I}$ (or $\mathbb{D}$) within any unit interval is at most $N_\gamma + 1$.

For each $n \in \mathbb{Z}$, we may now decompose

$$\Gamma^{-1}([n, n+1)) \cap \mathbb{I} = \bigcup_{k=1}^{K_n} U_k^n$$

for some disjoint collection of intervals $\{U_k^n\}_{k=1}^{K_n}$, where each U_k^n is contained in a distinct interval of $\mathbb{I}$. In particular, Γ is injective when restricted to each U_k^n. The key in what follows is to bound K_n uniformly in n. In particular, we claim that

$$K_n \leq K_\gamma := 1 + N_\gamma \left\{ 1 + \frac{[1 + 4\|\gamma\|_{L^\infty}]}{\langle \gamma \rangle} \right\} \quad \text{for all} \quad n \in \mathbb{Z}. \tag{9}$$

Indeed, by Corollary 7, we have that

$$|\Gamma^{-1}([n, n+1))| \leq \frac{1 + 4\|\gamma\|_{L^\infty}}{\langle \gamma \rangle} \quad \text{for each} \quad n \in \mathbb{Z},$$

where $|\cdot|$ denotes Lebesgue measure. We now observe that each U_k^n corresponds to a distinct interval in $\mathbb{I}$, and hence to at least one discontinuity of γ (at least, if $K_n > 1$). In particular, if (9) were to fail, then we could find more than $N_\gamma(1 + C)$ discontinuities of γ in a subset of $\mathbb{R}$ of total length less than C, a contradiction.

Now, using the change of variables $s = \Gamma(t)$ on each U_k^n and the standard $L_x^2 \to L_t^q L_x^r$ Strichartz estimates, we estimate

$$\begin{aligned} \int_{\mathbb{I}} \|e^{i\Gamma(t)\Delta}\varphi\|_{L_x^r}^q \, dt &\leq \sum_{n\in\mathbb{Z}} \sum_{k=1}^{K_n} \int_{U_k^n} \|e^{i\Gamma(t)\Delta}\varphi\|_{L_x^r}^q \, dt \\ &\leq \sum_{n\in\mathbb{Z}} K_\gamma \|\gamma^{-1}\|_{L^\infty} \int_n^{n+1} \|e^{is\Delta}\varphi\|_{L_x^r}^q \, ds \\ &\leq K_\gamma \|\gamma^{-1}\|_{L^\infty} \|e^{it\Delta}\varphi\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)}^q \\ &\leq C_{\text{Str}}^q K_\gamma \|\gamma^{-1}\|_{L^\infty} \|\varphi\|_{L^2}^q, \end{aligned} \tag{10}$$

where C_{Str} is the constant in the usual $L^2(\mathbb{R}^d) \to L_t^q L_x^r(\mathbb{R}^d)$ Strichartz estimate. □

With Theorem 6 in place, we may obtain the full range of Strichartz estimates (other than the double L_t^2 endpoint, which we did not pursue here) via the method of TT^* and the Christ–Kiselev lemma (see [8]). In particular, we obtain the following:

Theorem 8 (Inhomogeneous Strichartz estimates). *Let $d \geq 1$ and let $q, \tilde{q}, r, \tilde{r}$ obey (6) with $(q, \tilde{q}) \neq (2, 2)$. Let γ be admissible in the sense of Definition 5. Then for any $t_0 \in \mathbb{R}$ we have the estimates*

$$\left\| \int_{\mathbb{R}} e^{i\Gamma(t_0,t)\Delta} F(s)\, ds \right\|_{L^2(\mathbb{R}^d)} \leq C \|F\|_{L_t^{q'} L_x^{r'}(\mathbb{R}\times\mathbb{R}^d)},$$
$$\left\| \int_{t_0}^{t} e^{i\Gamma(t,s)\Delta} F(s)\, ds \right\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \leq C \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R}\times\mathbb{R}^d)}$$

for suitable $C = C(\gamma)$, where $\Gamma(t,s) = \int_s^t \gamma(\tau)\, d\tau$.

3. WELL-POSEDNESS

In this section, we consider the initial-value problem

$$\begin{cases} i\partial_t u + \gamma(t)\Delta u + |u|^2 u = 0 \\ u|_{t=t_0} = u_0 \in H^1(\mathbb{R}^3) \end{cases} \tag{11}$$

on $\mathbb{R}\times\mathbb{R}^3$, where γ is an admissible function in the sense of Definition 5. This includes the important special case

$$\gamma(t) = \begin{cases} \gamma_+, & 0 < t < t_+ \\ -\gamma_- & t_+ < t < 1 \end{cases} \tag{12}$$

for $\gamma_\pm > 0$, extended periodically to $\mathbb{R}$, provided the average dispersion

$$\langle \gamma \rangle := \int_0^1 \gamma(s)\, ds \tag{13}$$

is nonzero. As in the previous section, we define

$$\Gamma(t,s) = \int_s^t \gamma(\tau)\, d\tau, \tag{14}$$

so that the Duhamel formula for the solution to (11) is given by

$$u(t) = e^{i\Gamma(t,t_0)\Delta}u_0 + i\int_{t_0}^t e^{i\Gamma(t,s)\Delta}(|u|^2u)(s)\, ds.$$

The local H^1 theory (i.e. local existence for H^1 initial data) for (11)–(12) was previously considered in [2]. In this section, we apply the Strichartz estimates obtained in the previous section to establish a global result, namely, global well-posedness and scattering for sufficiently small initial data in H^1. In fact, with the global Strichartz estimates in hand, the proof follows from a fairly standard contraction mapping argument.

Theorem 9. *Let γ be an admissible function in the sense of Definition 5 and define*

$$\Gamma(t,s) = \int_s^t \gamma(\tau)\, d\tau.$$

Then there exists $\eta_0 = \eta_0(\gamma) > 0$ such that the following holds: Given $t_0 \in \mathbb{R}$ and $u_0 \in H^1(\mathbb{R}^3)$, if

$$\big\| |\nabla|^{\frac{1}{2}} e^{i\Gamma(t,t_0)\Delta} u_0 \big\|_{L_t^5 L_x^{\frac{30}{11}}([t_0,\infty)\times\mathbb{R}^3)} < \eta < \eta_0,$$

then there exists a unique forward-global solution $u : [t_0,\infty)\times\mathbb{R}^3 \to \mathbb{C}$ to (11), which scatters in the sense that

$$\lim_{t\to\pm\infty} \|u(t) - e^{i\Gamma(t,t_0)\Delta}u_+\|_{H^1} = 0$$

for some $u_+ \in H^1$. Analogous statements hold backward in time.

Proof. The proof is based on a contraction mapping argument using the Strichartz estimates established in the previous section. We define

$$\Phi(u)(t) := e^{i\Gamma(t,t_0)\Delta}u_0 + i\int_{t_0}^{t} e^{i\Gamma(t,s)\Delta}|u|^2u(s)\,ds. \tag{15}$$

We let $u_0 \in H^1$ and set $A = \|u_0\|_{H^1}$. Given $\eta > 0$ to be determined below, we define the complete metric space

$$X = \{u : \|u\|_{L_t^\infty H_x^1} \leq 2CA,\ \|u\|_{L_t^5 H_x^{1,\frac{30}{11}}} \leq 2CA,\ \||\nabla|^{\frac{1}{2}}u\|_{L_t^5 L_x^{\frac{30}{11}}} \leq 2\eta\},$$

with

$$d(u,v) = \|u - v\|_{L_t^5 L_x^{\frac{30}{11}}} \quad \text{for} \quad u, v \in X.$$

Here and below space-time norms are taken over $[t_0,\infty)\times\mathbb{R}^3$, and $C > 0$ is chosen to encode the implicit constants arising in Sobolev embedding and Strichartz estimates; in particular, the dependence of η on γ arises through the implicit constants in the Strichartz estimates (see e.g. (7) above).

To show that $\Phi : X \to X$, we first let $u \in X$ and use Strichartz and Sobolev embedding to estimate

$$\begin{aligned}
\|\Phi(u)\|_{L_t^\infty H_x^1} &\lesssim \|u_0\|_{H_x^1} + \left\| \int_{t_0}^{t} e^{i\Gamma(t,s)\Delta}|u|^2u(s)\,ds \right\|_{L_t^\infty H_x^1} \\
&\lesssim A + \left\| \langle\nabla\rangle \left[|u|^2u\right] \right\|_{L_t^{\frac{5}{3}} L_t^{\frac{30}{23}}} \\
&\lesssim A + \|u\|_{L_{t,x}^5}^2 \|\langle\nabla\rangle u\|_{L_t^5 L_x^{\frac{30}{11}}} \\
&\lesssim A + 8C\eta^2 A \leq 2CA
\end{aligned}$$

for suitable $C > 0$ and $\eta = \eta(C)$ sufficiently small. We then obtain the same estimate for the $L_t^5 H_x^{1,\frac{30}{11}}$-norm of $\Phi(u)$, as well.

For the $L_t^5 \dot{H}_x^{\frac{1}{2},\frac{30}{11}}$ estimate, we begin with an application of Sobolev embedding. Using the fractional chain rule, we obtain

$$\begin{aligned}
\||\nabla|^{\frac{1}{2}}\Phi(u)\|_{L_t^5 L_x^{\frac{30}{11}}} &\leq \||\nabla|^{\frac{1}{2}} e^{i\Gamma(t,t_0)\Delta}u_0\|_{L_t^5 L_x^{\frac{30}{11}}} + \left\||\nabla|^{\frac{1}{2}} \int_{t_0}^t e^{i\Gamma(t,s)\Delta}|u|^2 u(s)\, ds\right\|_{L_t^5 L_x^{\frac{30}{11}}} \\
&\leq \eta + C\|u\|_{L_{t,x}^5}^2 \||\nabla|^{\frac{1}{2}} u\|_{L_t^5 L_x^{\frac{30}{11}}} \\
&\leq \eta + 8C^4\eta^3 \leq 2\eta
\end{aligned}$$

provided η is sufficiently small.

To show that Φ is a contraction, we let $u, v \in X$ and estimate as above to deduce

$$\|\Phi(u) - \Phi(v)\|_{L_{t,x}^5} \lesssim \||u|^2 u - |v|^2 v\|_{L_t^{\frac{5}{3}} L_x^{\frac{30}{23}}} \tag{16}$$

$$\lesssim \{\|u\|_{L_{t,x}^5}^2 + \|v\|_{L_{t,x}^5}^2\}\|u - v\|_{L_t^5 L_x^{\frac{30}{11}}} \tag{17}$$

$$\lesssim \eta^2 \|u - v\|_{L_t^5 L_x^{\frac{30}{11}}}, \tag{18}$$

which implies that Φ is a contraction for η sufficiently small.

It follows that Φ has a unique fixed point u, yielding the desired solution. To prove the scattering result, it suffices to show that $\{e^{i\Gamma(t_0,t)\Delta}u(t)\}$ is Cauchy in H^1 as $t \to \infty$. To this end, we observe that by the Duhamel formula (15),

$$e^{i\Gamma(t_0,t)\Delta}u(t) - e^{i\Gamma(t_0,s)\Delta}u(s) = i\int_s^t e^{i\Gamma(t_0,\tau)\Delta}|u|^2 u(\tau)\, d\tau.$$

(This follows from the fact that both pieces on the left-hand side have the same initial datum u_0, and thus their free evolutions are the same).

Thus, applying the Strichartz estimates and estimating as above, we obtain

$$\begin{aligned}\|e^{i\Gamma(t_0,t)\Delta}u(t) - e^{i\Gamma(t_0,s)\Delta}u(s)\|_{H^1} &\lesssim \|\langle\nabla\rangle[|u|^2u]\|_{L_t^{\frac{5}{3}}L_x^{\frac{30}{23}}((s,t)\times\mathbb{R}^3)} \\ &\lesssim \|u\|^2_{L^5_{t,x}((s,t)\times\mathbb{R}^d)}\|u\|_{L_t^5H_x^{1,\frac{30}{11}}((s,t)\times\mathbb{R}^d)} \\ &\to 0 \quad \text{as} \quad s,t\to\infty,\end{aligned}$$

which yields the result. □

4. FINITE-TIME BLOWUP

In this section, we continue to consider the equation (11) but restrict attention only to the piecewise constant case (12). We will adapt the virial argument to demonstrate the possibility of finite-time blowup solutions. In particular, we will exhibit solutions that blow up on the first interval $[0,t_+)$, although the same argument would apply on any 'focusing' step. The existence of local-in-time solutions follows from [2] (or from suitable modifications of Theorem 9 above), and so we will take the existence of solutions for granted and focus on the issue of blowup.

To state our main result, we firstly introduce the *ground state* solution Q for the standard cubic NLS, which is the unique positive, radial, decreasing solution to

$$-Q + \Delta Q + Q^3 = 0$$

and plays a key role in the determination of sharp scattering/blowup results in that setting (see e.g. [17]). Fixing $\gamma_+ > 0$, we then define

$$R_+(x) = Q(\tfrac{x}{\sqrt{\gamma_+}}),$$

which solves

$$-R_+ + \gamma_+ \Delta R_+ + R_+^3 = 0. \tag{19}$$

In particular, $u(t,x) = e^{it} R_+(x)$ solves (11)–(12) on $[0, t_+)$.

We next define the *mass*

$$M(u) = \int_{\mathbb{R}^3} |u|^2 \, dx$$

and the *energies*

$$E_\pm(u) = \int_{\mathbb{R}^3} \frac{\gamma_\pm}{2} |\nabla u|^2 \mp \frac{1}{4} |u|^4 \, dx.$$

We observe that solutions to (11)–(12) conserve the mass, while neither of $E_\pm(u)$ is globally conserved. Instead, we have conservation of E_+ on intervals $[n, n+t_+)$ and conservation of E_- on intervals $[n+t_+, n+1)$, where $n \in \mathbb{Z}$.

Our result is the following:

Theorem 10. *Suppose that* $u_0 \in H^1(\mathbb{R}^3)$ *satisfies*

$$\begin{aligned} M(u_0)E_+(u_0) &< M(R_+)E_+(R_+), \\ \|u_0\|_{L^2} \|\nabla u_0\|_{L^2} &\geq \|R_+\|_{L^2} \|\nabla R_+\|_{L^2}. \end{aligned} \tag{20}$$

Suppose further that either $xu_0 \in L^2$ *or* u_0 *is radial. Then there exists* $\lambda > 0$ *sufficiently large that the solution to* (11)–(12) *with* $u|_{t=0} = \lambda u_0(\lambda \cdot)$ *blows up at some time* $T \in (0, t_+)$.

The condition (20) would be a typical blowup condition for the cubic NLS

$$i\partial_t u + \gamma_+ \Delta u + |u|^2 u = 0, \tag{21}$$

with the proof following the standard virial argument (see e.g. [15, 17]). In the present setting, the basic idea is to combine this argument with scaling to make the blowup happen before the end of the first focusing step (that is, before time t_+).

The key identity is the following standard virial identity:

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^3}|x|^2|u(t,x)|^2\,dx = \frac{d}{dt}\left[4\Im\int_{\mathbb{R}^3}\bar{u}\nabla u\cdot x\,dx\right]$$
$$= 8\int_{\mathbb{R}^3}\gamma_+|\nabla u|^2 - \tfrac{3}{4}|u|^4\,dx \tag{22}$$

for solutions to (21) (see e.g. [17] or [5, Section 6.5]). The role of the condition (20) is to guarantee that the right-hand side of (22) is quantitatively negative throughout the lifespan of the solution:

Lemma 7. *Suppose $u_0 \in H^1$ satisfies*

$$M(u_0)E_+(u_0) \leq (1-\delta)M(R_+)E_+(R_+), \tag{23}$$

$$\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} \geq \|R_+\|_{L^2}\|\nabla R_+\|_{L^2} \tag{24}$$

for some $\delta > 0$. Let u be the solution to (21) with $u|_{t=0} = u_0$. Then there exists $\delta' = \delta'(\delta) > 0$, $c = c(\delta) > 0$, and $\varepsilon = \varepsilon(\delta) > 0$ so that

$$\|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2} \geq (1+\delta')\|R_+\|_{L^2}\|\nabla R_+\|_{L^2}, \tag{25}$$

and

$$\int_{\mathbb{R}^3}\gamma_+(1+\varepsilon)|\nabla u(t,x)|^2 - \tfrac{3}{4}|u(t,x)|^4\,dx < -c \tag{26}$$

uniformly for t in the lifespan of u.

Proof. This is the standard 'energy trapping' argument (see e.g. [17, Theorem 4.2]). The key observation is that the ground state Q (and hence the rescaled ground state $R_+(\cdot) = Q(\frac{\cdot}{\sqrt{\gamma_+}})$) is an optimizer for the sharp Gagliardo–Nirenberg inequality

$$\|u\|_{L^4(\mathbb{R}^3)}^4 \leq C_0 \|u\|_{L^2(\mathbb{R}^3)} \|\nabla u\|_{L^2(\mathbb{R}^3)}^3,$$

so that in particular

$$C_0 = \frac{\|R_+\|_{L^4}^4}{\|R_+\|_{L^2} \|\nabla R_+\|_{L^2}^3}. \tag{27}$$

We can connect the various norms of R_+ to one another via the following Pohozaev identities (obtained by multiplying (19) by R_+ and $x \cdot \nabla R_+$ and integrating):

$$-\|R_+\|_{L^2}^2 - \gamma_+ \|\nabla R_+\|_{L^2}^2 + \|R_+\|_{L^4}^4 = 0,$$
$$\tfrac{3}{2}\|R_+\|_{L^2}^2 + \tfrac{\gamma_+}{2}\|\nabla R_+\|_{L^2}^2 - \tfrac{3}{4}\|R_+\|_{L^4}^4 = 0.$$

In particular,

$$\tfrac{\gamma_+}{3}\|\nabla R_+\|_{L^2}^2 = \tfrac{1}{4}\|R_+\|_{L^4}^4, \quad \text{so that} \quad E_+(R_+) = \tfrac{\gamma_+}{6}\|\nabla R_+\|_{L^2}^2. \tag{28}$$

We can therefore use the Gagliardo–Nirenberg inequality to obtain

$$\begin{aligned}(1-\delta)M(R_+)E_+(R_+) &\geq M(u)E_+(u) \\ &\geq \tfrac{\gamma_+}{2}\|u(t)\|_{L^2}^2 \|\nabla u(t)\|_{L^2}^2 - \tfrac{C_0}{4}\|u(t)\|_{L^2}^3 \|\nabla u(t)\|_{L^2}^3,\end{aligned}$$

which, using (27) and (28), implies

$$(1-\delta) \geq 3\left[\frac{\|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2}}{\|R_+\|_{L^2}\|\nabla R_+\|_{L^2}}\right]^2 - 2\left[\frac{\|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2}}{\|R_+\|_{L^2}\|\nabla R_+\|_{L^2}}\right]^3$$

Thus, by (24) and a continuity argument, we deduce that (25) holds.

For (26), we use (28), (23), and (25) to write

$$\begin{aligned}\int \gamma_+(1+\varepsilon)|\nabla u|^2 - \tfrac{3}{4}|u|^4\,dx &= \tfrac{1}{M(u)}\big[3M(u)E_+(u) - \gamma_+(\tfrac{1}{2}-\varepsilon)\|u(t)\|_{L^2}^2\|\nabla u(t)\|_{L^2}^2\big] \\ &\leq \frac{\gamma_+\|R_+\|_{L^2}^2\|\nabla R_+\|_{L^2}^2}{2M(u)}\big[(1-\delta)-(1+\delta')^2(1-2\varepsilon)\big] \\ &= -\frac{\gamma_+\|R_+\|_{L^2}^2\|\nabla R_+\|_{L^2}^2}{2M(u)}\big[\delta+2\delta'+(\delta')^2-\varepsilon(1+\delta')^2\big],\end{aligned}$$

which, choosing ε sufficiently small, yields the result. □

We turn to the proof of Theorem 10.

Proof of Theorem 10. We take $u_0 \in H^1$ such that (20) holds.

We first consider the case $xu_0 \in L^2$. We let u denote the maximal-lifespan solution to (11)–(12) with initial data u_0, and for $\lambda \geq 1$ we let u^λ denote the maximal-lifespan solution with initial data $u_0^\lambda := \lambda u_0(\lambda\cdot)$. We let I_λ denote the intersection of the lifespan of u^λ with $[0, \lambda^{-2}t_+]$. In particular,

$$u^\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$$

for all $t \in I_\lambda$. Thus, by Lemma 7 and scaling, there exists $c > 0$ so that

$$\int_{\mathbb{R}^3} \gamma_+|\nabla u^\lambda(t,x)|^2 - \tfrac{3}{4}|u^\lambda(t,x)|^4\,dx < -\lambda c \tag{29}$$

uniformly for $t \in I_\lambda$ (here the ε improvement in Lemma 7 is not needed).

We now set

$$f_\lambda(t) = \int_{\mathbb{R}^3} |x|^2|u^\lambda(t,x)|^2\,dx \quad \text{for} \quad t \in I_\lambda$$

and write

$$f_\lambda(t) = f_\lambda(0) + tf_\lambda'(0) + \int_0^t\int_0^s f_\lambda''(\tau)\,d\tau\,ds. \tag{30}$$

We observe (cf. (22) and (29)) that

$$\begin{aligned} &|f_\lambda(0)| = C_1\lambda^{-3}, \\ &|f_\lambda'(0)| = \left|4\gamma_+\Im\int_{\mathbb{R}^3} \bar{u}_0^\lambda \nabla u_0^\lambda \cdot x\,dx\right| \le C_2\lambda^{-1}, \\ &f_\lambda''(\tau) < -c\lambda, \end{aligned} \tag{31}$$

uniformly for $\tau \in I_\lambda$, where $C_1 := \|xu_0\|_{L^2}^2$ and $C_2 := 4\|xu_0\|_{L^2}\|\nabla u_0\|_{L^2}$. We therefore obtain

$$0 \le f_\lambda(t) \le -\tfrac{1}{2}c\lambda t^2 + C_2\lambda^{-1}t + C_1\lambda^{-3} \quad \text{for} \quad t \in I_\lambda.$$

We now observe that the quadratic polynomial on the right-hand side equals zero at time

$$T_\lambda := \frac{C_2 + \sqrt{C_2^2 + 2cC_1}}{c\lambda^2}.$$

Choosing λ large enough that $T_\lambda < t_+$, we therefore obtain that the solution u^λ blows up at or before time T_λ.

We next consider the case that u_0 (and hence the solution u) is radial. In this case, we use a localized version of the virial identity. As before, we consider the rescaled solution u^λ on I_λ. We then introduce a weight $w_R(x) = R^2\phi(\frac{x}{R})$, where ϕ is a smooth, nonnegative, radial function satisfying

$$\phi(x) = \begin{cases} |x|^2 & |x| \le 1 \\ \text{constant} & |x| \ge 3 \end{cases}$$

and obeying the bounds

$$|\nabla\phi(x)| \le 2|x| \quad \text{and} \quad |\partial_{jk}\phi(x)| \le 2 \quad \text{for all} \quad x \in \mathbb{R}^3.$$

We will later specialize to $R \sim \lambda^{-1}$.

Proceeding as above, we define

$$f_\lambda(t) = \int w_R(x)|u^\lambda(t,x)|^2\,dx > 0$$

and use the expansion (30). Using the fact that

$$f_\lambda'(t) = 4\gamma_+\Im\int_{\mathbb{R}^3} \bar{u}^\lambda \nabla u^\lambda \cdot \nabla(w_R)\,dx,$$

we first derive the bounds

$$|f_\lambda(0)| \lesssim R^2\lambda^{-1} \quad \text{and} \quad |f_\lambda'(0)| \lesssim R. \tag{32}$$

Computing the second derivative of f_λ, we derive the following analogue of (22):

$$f_\lambda''(t) = 8\int \gamma_+|\nabla u^\lambda|^2 - \tfrac{3}{4}|u^\lambda|^4\,dx \tag{33}$$

$$+ 4\gamma_+\Re\int_{|x|>R}(\partial_j\bar{u}^\lambda)(\partial_k u^\lambda)\partial_{jk}[w_R]\,dx - 8\int_{|x|>R}|\nabla u^\lambda|^2\,dx \tag{34}$$

$$+ \mathcal{O}\Big[\int_{|x|>R} R^{-2}|u^\lambda|^2 + |u^\lambda|^4\,dx\Big] \tag{35}$$

(see e.g. [17, Section 4]).

Now, (34)≤ 0 by the Cauchy–Schwarz inequality and the assumptions on w_R, while for the first term in (35), we use conservation of mass to obtain

$$R^{-2}\|u^\lambda\|_{L^2}^2 \lesssim R^{-2}\lambda^{-1} \quad \text{uniformly on} \quad I_\lambda.$$

For the remaining term, we use the radial Sobolev embedding estimate, Young's inequality, and the conservation of mass to obtain

$$\begin{aligned}\|u^\lambda\|_{L^4(|x|>R)}^4 &\leq R^{-2}\|u^\lambda\|_{L^2}^2\||x|u^\lambda\|_{L^\infty}^2 \\ &\leq CR^{-2}\|u^\lambda\|_{L^2}^3\|\nabla u^\lambda\|_{L^2} \\ &\leq 8\varepsilon\gamma_+\|\nabla u^\lambda\|_{L^2}^2 + \tfrac{C}{\varepsilon\gamma_+}R^{-4}\|u^\lambda\|_{L^2}^6 \\ &\leq 8\varepsilon\gamma_+\|\nabla u^\lambda\|_{L^2}^2 + \tfrac{C}{\varepsilon\gamma_+}R^{-4}\lambda^{-3},\end{aligned}$$

where we have allowed the constant C to change in each line and ε is as in Lemma 7. Thus, continuing from above and using Lemma 7, we obtain

$$\begin{aligned}f_\lambda''(t) &\leq 8\int (1+\varepsilon)\gamma_+|\nabla u^\lambda|^2 - \tfrac{3}{4}|u^\lambda|^4\,dx + CR^{-2}\lambda^{-1} + \tfrac{C}{\varepsilon\gamma_+}R^{-4}\lambda^{-3} \\ &\leq -2c\lambda + CR^{-2}\lambda^{-1} + \tfrac{C}{\varepsilon\gamma_+}R^{-4}\lambda^{-3}\end{aligned}$$

for some $c > 0$. In particular, choosing

$$R = C_0\lambda^{-1} \quad \text{for sufficiently large} \quad C_0 = C_0(c, \varepsilon, \gamma_+)$$

and recalling (32), we derive

$$|f_\lambda(0)| \leq C_1\lambda^{-3}, \quad |f_\lambda'(0)| \leq C_2\lambda^{-1}, \quad \text{and} \quad f_\lambda''(t) \leq -c\lambda$$

for some $c, C_1, C_2 > 0$, uniformly on I_λ. As this puts us in exactly the same situation as (31) above, we deduce that for sufficiently large λ, we obtain blowup before time t_+. □

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SECTION

2. UNPUBLISHED CONTENT

In this section, we present a proof of modified scattering for the full dispersion-managed cubic NLS, and a result of Bourgain on invariant measures for NLS on the $1d$ torus.

2.1. MODIFIED SCATTERING FOR THE FULL DMNLS

In this section, we present a proof of modified scattering for the full $1d$ cubic DMNLS, as mentioned at the beginning of the first paper. Note that in the previous case, we were able to obtain a modified scattering result, as all dispersive behavior was pushed into the nonlinearity $F(u) = |u|^2 u$, and the underlying linear equation remained the ordinary dispersion-free Schrödinger equation.

However, the techniques from the first paper [28] are inadequate to handle the full dispersion-managed problem. This is for the simple reason that in this regime, one does not immediately have a global-in-time Strichartz estimate! However, with the techniques developed in [29], we are able to adapt the arguments in [16] and obtain a modified scattering result for the $1d$ cubic DMNLS.

2.1.1. Problem Statement. As in [28], we focus primarily on the following Cauchy problem:

$$\begin{cases} i\partial_t u + \frac{1}{2}\gamma(t)\partial_x^2 u = |u|^2 u \\ u|_{t=0} = u_0 \end{cases} \quad \text{on} \quad \mathbb{R}\times\mathbb{R} \tag{2.1}$$

where the dispersion map $\gamma(t)$ satisfies several admissibility conditions:

- γ is one-periodic: $\gamma(t+1) = \gamma(t)$ for all $t \in \mathbb{R}$.

- γ and γ^{-1} are bounded: $\|\gamma\|_{L^\infty} + \|\gamma^{-1}\|_{L^\infty} < \infty$.
- γ has at most finitely many discontinuities on $[0, 1]$.
- γ has nonzero average over its period: $\langle\gamma\rangle := \int_0^1 \gamma(t)\,dt \neq 0$.

Our main result is the following.

Theorem 11. *Let $u_0 \in H^{1,1}(\mathbb{R})$ satisfy $\|u_0\|_{H^{1,1}} = \varepsilon = \varepsilon(\langle\gamma\rangle) > 0$. If ε is sufficiently small, then there exists a unique solution $u \in C_t H_x^{1,1}([0,\infty) \times \mathbb{R})$ to (2.1) with $u(0, x) = u_0$. Furthermore, the solution obeys*

$$\|u(t)\|_{L^\infty} \lesssim \varepsilon(1 + |t|)^{-\frac{1}{2}} \tag{2.2}$$

for all $t \geq 0$ and there exists $W \in L^\infty(\mathbb{R})$ such that

$$u(t, x) = (i\Gamma(t))^{-1/2} e^{ix^2/2\Gamma(t)} \big[\exp\{-\tfrac{i}{\langle\gamma\rangle}|W(\tfrac{x}{\Gamma(t)})|^2 \log(t)\} W(\tfrac{x}{\Gamma(t)})\big] + \mathcal{O}(t^{-\frac{1}{2}-\frac{1}{20}})$$

in L^∞ as $t \to \infty$.

2.1.2. Notational Changes. In a departure from our previous notational conventions, we will set several new conventions for the remainder of this chapter.

First, our main equation will be the following:

$$\begin{cases} i\partial_t + \frac{1}{2}\gamma(t)\Delta u = |u|^2 u \\ u(0, x) = u_0(x) \end{cases} \quad \text{on} \quad \mathbb{R} \times \mathbb{R} \tag{2.3}$$

Solutions to the linear DMNLS with $u|_{t=0} = u_0$ are given by

$$u(t, x) = e^{i\frac{\Gamma(t)}{2}\Delta} u_0 \quad \text{where} \quad e^{i\frac{\Gamma(t)}{2}\Delta} = \mathcal{F}^{-1}\, e^{-i\frac{\Gamma(t)}{2}|\xi|^2}\, \mathcal{F}$$

and

$$\Gamma(t) = \int_0^t \gamma(\tau)\,\mathrm{d}\tau, \quad \text{and} \quad \Gamma(t,s) = \int_s^t \gamma(\tau)\,\mathrm{d}\tau$$

as in the case of the linear (constant-dispersion) Schrödinger equation, one can express the action of $e^{i\Gamma(t)\Delta}$ as a convolution kernel:

$$e^{i\frac{\Gamma(t)}{2}\Delta}(x,y) = (2\pi i\Gamma(t))^{-\frac{1}{2}} e^{i\frac{(x-y)^2}{2\Gamma(t)}}$$

which in turn implies the factorization identity (as in the constant-dispersion case)

$$e^{i\frac{\Gamma(t)}{2}\Delta} = \mathcal{M}(\Gamma(t))\mathcal{D}(\Gamma(t))\,\mathcal{F}\,\mathcal{M}(\Gamma(t)) \tag{2.4}$$

where $\mathcal{M}(t)$ and $\mathcal{D}(t)$ are defined in the following fashion:

$$[\mathcal{M}(t)f](x) = e^{i\frac{|x|^2}{2t}} f(x), \;\; [\mathcal{D}(t)f](x) = (it)^{-\frac{1}{2}} f\left(\frac{x}{t}\right).$$

One can think of $\mathcal{M}$ as being the "modulation" operator and $\mathcal{D}$ as being the "dilation" operator.

We will make use of the adapted Galilean operator $J_\Gamma(t) = x + i\Gamma(t)\partial_x$. As in the case of the dispersion-free Schrödinger equation, one may directly compute and show that

$$J_\Gamma(t) = \mathcal{M}(\Gamma(t))[i\Gamma(t)\partial_x]\,\mathcal{M}(-\Gamma(t))$$

On the other hand, though, an argument by ODEs shows that

$$J_\Gamma(t) = e^{i\frac{\Gamma(t)}{2}\Delta} x e^{-i\frac{\Gamma(t)}{2}\Delta} \tag{2.5}$$

We will also need the following chain rule identity for $J_\Gamma(t)$, which one can prove by direct computation: for any $p > 0$,

$$J_\Gamma(t)[|z|^p z] = \frac{p+2}{2}|z|^p[J_\Gamma(t)z] - \frac{p}{2}|z|^{p-2}z^2[\overline{J_\Gamma(t)z}] \tag{2.6}$$

2.1.3. Well-posedness. In this section, we discuss the $H^1(\mathbb{R})$ and $H^{1,1}(\mathbb{R})$ well-posedness for (11). As in our previous work, much of what follows is very standard, but is impossible without the introduction of our new global-in-time Strichartz estimates. With these in hand, much of what follows is standard, so we focus on the main ideas.

As usual, we construct solutions to (11) as fixed points of the map given by the Duhamel formula:

$$\Phi[u] = e^{i\frac{\Gamma(t)}{2}\Delta}u_0 - i\int_0^t e^{i\frac{\Gamma(t,s)}{2}\Delta}|u|^2u(s)\,\mathrm{d}s \tag{2.7}$$

We apply the standard contraction mapping scheme; for brevity, we only show the estimate of the nonlinear integral term here. The estimate proceeds as follows: by Strichartz, Hölder, and Sobolev embedding, we have

$$\begin{aligned}\|\langle\partial_x\rangle|u|^2u\|_{L^1_TL^2_x(\mathbb{R})} &\lesssim T\||u|^2\|_{L^\infty_{T,x}}\|\langle\partial_x\rangle u\|_{L^\infty_TL^2_x}\\ &\lesssim T\|u\|^3_{L^\infty_TH^1_x}\end{aligned}$$

which is an acceptable estimate. Thus, we're able to get local existence for times $T \sim \|u_0\|^2_{H^1_x}$. Similarly, by commuting $J_\Gamma(t)$ with the equation and using (2.6), one finds a similar local existence result for $u_0 \in H^{1,1}$. However, in this case, one has only a crude estimate on $\|xu\|_{L^2}$:

$$\|xu\|_{L^2} \lesssim (1+\Gamma(t))\|u_0\|_{H^{1,1}}$$

2.1.4. Proof of Main Result. In this section, we prove the main result Theorem 11. To begin, we fix an admissible dispersion map $\gamma(t)$. By the admissibility conditions, this yields a $T_* = T_*(\langle\gamma\rangle) > 0$ such that $t \geq T_*$ implies that $\Gamma(t) \gtrsim t$.

In particular, by the $H^{1,1}$ local theory described above, for any $\varepsilon > 0$, we can find $\delta = \delta(\langle\gamma\rangle)$ sufficiently small such that if $\|u_0\|_{H^{1,1}} < \delta$, then

$$\|\langle\partial_x\rangle u(T_*)\|_{L^2} + \|J_\Gamma(T_*)u(T_*)\|_{L^2} < \varepsilon. \tag{2.8}$$

2.1.5. Global Existence and Decay. With times $t < T_*$ handled by the local theory, we can now begin to prove the first half of Theorem 11. This is accomplished through a bootstrap argument for $t \geq T_*$, using the following 'dispersive' and 'energy' norms:

$$\begin{aligned} \|u(t)\|_{X_D} &:= \|\hat{f}(t)\|_{L^\infty}, \quad \text{where} \quad f(t) = e^{-i\frac{\Gamma(t)}{2}\Delta}u(t), \\ \|u(t)\|_{X_E} &:= t^{-\frac{1}{20}}\{\|\langle\partial_x\rangle u(t)\|_{L^2} + \|J_\Gamma(t)u(t)\|_{L^2}\} \end{aligned} \tag{2.9}$$

In particular, by (2.5), we may write

$$\|J_\Gamma(t)u(t)\|_{L^2} = \|xf(t)\|_{L^2}$$

We then define

$$\|u(t)\|_X = \sup_{s\in[T_*,T]}\{\|u(s)\|_{X_D} + \|u(s)\|_{X_E}\}$$

Lemma 8. *For any $t \geq T_*$,*

$$\|u(t)\|_{L^\infty} \lesssim t^{-\frac{1}{2}}\{\|u(t)\|_{X_D} + \|u(t)\|_{X_E}\}$$

Proof. Write $f(t) = e^{-i\frac{\Gamma(t)}{2}\Delta}u(t)$ as above. By (2.4), Hausdorff-Young, and Cauchy-Schwarz, we have the estimate

$$\begin{aligned}
\|u(t)\|_{L^\infty} &= \| \mathcal{M}(\Gamma(t))\, \mathcal{D}(\Gamma(t))\, \mathcal{F}\, \mathcal{M}(\Gamma(t)) f(t)\|_{L^\infty} \\
&\lesssim \Gamma(t)^{-\frac{1}{2}}\{\|\hat{f}(t)\|_{L^\infty} + \| \mathcal{F}[\mathcal{M}(\Gamma(t)) - 1] f(t)\|_{L^\infty}\} \\
&\lesssim \Gamma(t)^{-\frac{1}{2}}\{\|u(t)\|_{X_D} + \Gamma(t)^{-\frac{1}{20}} \||x|^{\frac{1}{10}} f(t)\|_{L^1}\} \\
&\lesssim t^{-\frac{1}{2}}\{\|u(t)\|_{X_D} + \|\langle x\rangle f\|_{L^2}\} \\
&\lesssim t^{-\frac{1}{2}}\{\|u(t)\|_{X_D} + \|u(t)\|_{X_E}\}
\end{aligned}$$

□

The first part of Theorem 11 will follow from the following bootstrap estimate on $[T_*, T]$:

Proposition 12. *Let $u : [T_*, T] \times \mathbb{R} \to \mathbb{C}$ be a solution to* (11) *satisfying* (2.8)*. Then there exists $C > 0$ independent of T so that*

$$\|u(t)\|_X \leq 4\varepsilon + C(\langle\gamma\rangle)\|u(t)\|_X^3$$

for every $t \in [T_*, T]$.

We divide the proof of this estimate into two separate lemmas, as we may clearly handle the energy norm and dispersive norm separately.

Lemma 9 (Energy estimate)**.** *For u solving* (11) *as above, we have the following estimate:*

$$\|u(t)\|_{X_E} \leq 2\varepsilon t^{-\frac{1}{20}} + Ct^{-\frac{1}{20}} \int_{T_*}^{t} s^{-1+\frac{1}{20}} \|u(s)\|_X^3 \,\mathrm{d}x \tag{2.10}$$

Proof. We begin with the Duhamel formula from time $t = T_*$:

$$u(t) = e^{i\frac{\Gamma(t)}{2}\Delta} e^{-i\frac{\Gamma(T_*)}{2}\Delta} u(T_*) - i \int_{T_*}^{t} e^{i\frac{\Gamma(t)}{2}\Delta} e^{-i\frac{\Gamma(s)}{2}\Delta} |u|^2 u(s)\,\mathrm{d}s. \tag{2.11}$$

To estimate the X_E-norm, we need to consider both the action of $\langle\partial_x\rangle$ and that of $J_\Gamma(t)$ on $u(t)$. In view of the identity (2.5) and the fact that Fourier multipliers commute, (2.11) will be the appropriate way to estimate $\|u(t)\|_{X_E}$.

We first estimate $\|J_\Gamma(t)u(t)\|_{L^2}$. Using the identity (2.5) and (2.11), we find

$$J_\Gamma(t)u(t) = e^{i\frac{\Gamma(t)}{2}\Delta}e^{-i\frac{\Gamma(T_*)}{2}\Delta}J_\Gamma(T_*)u(T_*) - i\int_{T_*}^{t} e^{i\frac{\Gamma(t)}{2}\Delta}e^{-i\frac{\Gamma(s)}{2}\Delta}J_\Gamma(s)|u|^2u(s)\,\mathrm{d}s \quad (2.12)$$

Taking the L^2 norm, we have

$$\|J_\Gamma(t)u(t)\|_{L^2} \lesssim \|J_\Gamma(T_*)u(T_*)\|_{L^2} + \int_{T_*}^{t} \|J_\Gamma(s)|u|^2u(s)\|_{L^2}\,\mathrm{d}s \quad (2.13)$$

$$\lesssim \varepsilon + \int_{T_*}^{t} \|u(s)\|_{L^\infty}^2\|J_\Gamma(s)u(s)\|_{L^2}\,\mathrm{d}s \quad (2.14)$$

where we have used the chain rule for $J_\Gamma(t)$ in the last inequality. Similarly, one finds the estimate (using the ordinary fractional chain rule)

$$\|\langle\partial_x\rangle u(t)\|_{L^2} \lesssim \varepsilon + \int_{T_*}^{t} \|u(s)\|_{L^\infty}^2\|\langle\partial_x\rangle u(s)\|_{L^2}\,\mathrm{d}s \quad (2.15)$$

Upon adding (2.13) and (2.15) together, using Lemma 8 and the definitions of the dispersive and energy norms one readily concludes the estimate (2.10). □

Next, we establish a corresponding lemma for the dispersive norm:

Lemma 10 (Dispersive estimate)**.** *For u solving* (11) *as above, we have the following estimate:*

$$\|u\|_{X_D} \leq \varepsilon + C\int_{T_*}^{t} s^{-1-\frac{1}{20}}\|u\|_X^3\,\mathrm{d}s$$

Proof. We begin by examining the equation satisfied by the 'profile' $f = e^{-i\frac{\Gamma(t)}{2}\Delta}u$; a straightforward computation reveals that f satisfies the equation

$$i\partial_t f = e^{-i\frac{\Gamma(t)}{2}\Delta}|u|^2 u. \tag{2.16}$$

Now recalling the factorization identity (2.4), we rewrite (2.16) as

$$i\partial_t f = [\mathcal{M}\,\mathcal{D}\,\mathcal{F}\,\mathcal{M}]^{-1}|u|^2 u. \tag{2.17}$$

Directly computing, we find that

$$\begin{aligned}
[\mathcal{M}\,\mathcal{D}\,\mathcal{F}\,\mathcal{M}]^{-1}|u|^2 u &= \mathcal{M}^{-1}\,\mathcal{F}^{-1}\,\mathcal{D}^{-1}\,|\,\mathcal{M}^{-1}\,u|^2\,\mathcal{M}^{-1}\,u \\
&= \Gamma(t)^{-1}\,\mathcal{M}^{-1}\,\mathcal{F}^{-1}\,|\,\mathcal{D}^{-1}\,\mathcal{M}^{-1}\,u|^2\,\mathcal{D}^{-1}\,\mathcal{M}^{-1}\,u \\
&= \Gamma(t)^{-1}\,\mathcal{M}^{-1}\,\mathcal{F}^{-1}\,|\,\mathcal{F}\,\mathcal{M}\,f|^2\,\mathcal{F}\,\mathcal{M}\,f.
\end{aligned} \tag{2.18}$$

Thus the PDE for f now reads

$$i\partial_t f = \Gamma(t)^{-1}\,\mathcal{M}^{-1}\,\mathcal{F}^{-1}\,|\,\mathcal{F}\,\mathcal{M}\,f|^2\,\mathcal{F}\,\mathcal{M}\,f. \tag{2.19}$$

Recall that since we are working with times $t \geq T_*$, we need not worry about the factor of $\Gamma(t)^{-1}$ appearing in (2.19); in fact, by the discussion in [MVH-2], we may freely replace this by (a constant multiple of) t which we will later do; this is why our estimates will depend in some way on the average dispersion $\langle\gamma\rangle$.

Making this replacement, we creatively rearrange the equation (2.19) as the following equation:

$$\begin{aligned} i\partial_t f &= \Gamma(t)^{-1}\,\mathcal{F}^{-1}\,|\hat{f}|^2\hat{f} \\ &\quad + \Gamma(t)^{-1}(\mathcal{M}^{-1}-1)\,\mathcal{F}^{-1}\,|\,\mathcal{F}\,\mathcal{M}\,f|^2\,\mathcal{F}\,\mathcal{M}\,f \\ &\quad + \Gamma(t)^{-1}\,\mathcal{F}^{-1}\left(|\,\mathcal{F}\,\mathcal{M}\,f|^2\,\mathcal{F}\,\mathcal{M}\,f - |\hat{f}|^2\hat{f}\right) \qquad (2.20) \\ &= \Gamma(t)^{-1}\,\mathcal{F}^{-1}\,|\hat{f}|^2\hat{f} + \Gamma(t)^{-1}\{I_1(t) + I_2(t)\} \qquad (2.21) \end{aligned}$$

where we define

$$I_1(t) = (\mathcal{M}^{-1}-1)\,\mathcal{F}^{-1}\,|\,\mathcal{F}\,\mathcal{M}\,f|^2\,\mathcal{F}\,\mathcal{M}\,f$$

and

$$I_2(t) = \mathcal{F}^{-1}\left(|\,\mathcal{F}\,\mathcal{M}\,f|^2\,\mathcal{F}\,\mathcal{M}\,f - |\hat{f}|^2\hat{f}\right)$$

Now we rearrange (2.20) and take the Fourier transform, yielding the following equation:

$$i\partial_t\hat{f} - \Gamma(t)^{-1}|\hat{f}|^2\hat{f} = \Gamma(t)^{-1}\left\{\widehat{I_1} + \widehat{I_2}\right\} \qquad (2.22)$$

To remove the second term on the left–hand side of (2.22), we employ an integating factor:

$$B(t) = \exp\left(i\int_{T_*}^{t}\frac{|\hat{f}(s)|^2}{\Gamma(s)}\,\mathrm{d}s\right) \qquad (2.23)$$

Now set $g = \hat{f}B$. Then g solves the PDE

$$i\partial_t g(t) = B(t)\Gamma(t)^{-1}(\widehat{I_1} + \widehat{I_2}) \qquad (2.24)$$

so that by the fundamental theorem of calculus,

$$g(t) = g(T_*) - i\int_{T_*}^{t} B(s)\Gamma(s)^{-1}(\widehat{I_1}(s) + \widehat{I_2}(s))\,\mathrm{d}s$$

All that remains now is to estimate $\|\widehat{I_1}\|_{L^\infty}$ and $\|\widehat{I_2}\|_{L^\infty}$. To this end, note that by Hausdorff-Young, (1.14) and Hölder, we have the following estimate on $\|\widehat{I_1}\|_{L^\infty}$

$$\begin{aligned}
\|\widehat{I_1}\|_{L^\infty} &= \|\mathcal{F}(\mathcal{M}^{-1}-1)\mathcal{F}^{-1}|\mathcal{F}\mathcal{M}f|^2\mathcal{F}\mathcal{M}f\|_{L^\infty} \\
&\lesssim \|(\mathcal{M}^{-1}-1)\mathcal{F}^{-1}|\mathcal{F}\mathcal{M}f|^2\mathcal{F}\mathcal{M}f\|_{L^1} \\
&\lesssim \Gamma(t)^{-\frac{1}{5}}\||x|^{\frac{1}{10}}\mathcal{F}^{-1}|\mathcal{F}\mathcal{M}f|^2\mathcal{F}\mathcal{M}f\|_{L^1} \\
&\lesssim \Gamma(t)^{-\frac{1}{5}}\|\langle x\rangle f\|_{L^2}\|f\|_{L^1}^2 \\
&\lesssim \Gamma(t)^{-\frac{1}{5}}\|\langle x\rangle f\|_{L^2}^3 \lesssim t^{-\frac{1}{20}}\|u\|_X^3
\end{aligned}$$

Similarly we have (using the same estimates as before)

$$\begin{aligned}
\|\widehat{I_2}\|_{L^\infty} &= \||\mathcal{F}\mathcal{M}f|^2\mathcal{F}\mathcal{M}f - |\hat{f}|^2\hat{f}\|_{L^\infty} \\
&\lesssim \||\mathcal{F}\mathcal{M}f|^2|\hat{f}|^2\|_{L^\infty}\|\mathcal{F}\mathcal{M}f-\hat{f}\|_{L^\infty} \\
&\lesssim \|f\|_{L^1}^2\|\mathcal{F}(\mathcal{M}-1)f\|_{L^\infty} \\
&\lesssim \Gamma(t)^{-\frac{1}{5}}\|\langle x\rangle f\|_{L^2}^3 \lesssim t^{-\frac{1}{20}}\|u\|_X^3
\end{aligned}$$

Putting everything together, we have

$$|g(t)| = |\hat{f}(t)| \leq \varepsilon + \int_{T_*}^{t} s^{-1-\frac{1}{20}}\|u(s)\|_X^3\,\mathrm{d}s \tag{2.25}$$

which establishes the desired estimate. □

With these two lemmata in hand, one readily deduces the estimate appearing in Proposition 12. A standard bootstrap argument, the well-posedness theory from earlier and Lemma 8, we are able to conclude the first part of Theorem 11. In particular, the following holds:

Corollary 13. *For $\|u_0\|_{H^{1,1}} = \varepsilon$ sufficiently small (depending on our choice of dispersion map γ), there exists a unique forward-global solution $u \in C_t H^{1,1}_x([0,\infty)\times\mathbb{R})$ to* (11) *with $u(0,x) = u_0$ obeying*

$$\|u(t)\|_X \lesssim \varepsilon \quad \textit{for all} \quad t \geq T_*$$

and in particular, we have the L^∞ decay estimate

$$\|u(t)\|_{L^\infty} \lesssim \varepsilon(1+|t|)^{-\frac{1}{2}} \quad \textit{for all} \quad t \geq 0.$$

2.1.6. Asymptotic Behavior. In this section, we use the estimates obtained from closing the bootstrap (Proposition 12) to study the asymptotic behavior of solutions to (11).

Recall the setting of Lemma 10; in particular, recall that we set

$$g(t) = B(t)\hat{f}(t) \quad \text{with} \quad B(t) = \exp\left(i\int_{T_*}^{t} \frac{|\hat{f}(s)|^2}{\Gamma(s)^2}\,\mathrm{d}s\right).$$

Having closed the bootstrap, the estimates in Lemma 10 on $\partial_t g$ furnish the following estimate:

$$\|\partial_t g\|_{L^\infty} \lesssim \varepsilon^3 t^{-1-\frac{1}{20}}$$

In particular, by the fundamental theorem of calculus, we can find $W_0 \in L^\infty$ such that

$$\|g(t) - W_0\|_{L^\infty} \lesssim \varepsilon^3 t^{-\frac{1}{20}}$$

In particular, we know that $|\hat{f}| \to |W_0|$ in L^∞ with the same rate of convergence.

Next, using Lemma 6, we observe that

$$\frac{\langle\gamma\rangle^2}{2\|\gamma\|_{L^\infty}}\left|\frac{1}{\Gamma(s)} - \frac{1}{\langle\gamma\rangle s}\right| \leq \frac{1}{s^2}.$$

From this we deduce that

$$B(t) = \exp\left(i\frac{1}{\langle\gamma\rangle}|W_0|^2\log(t)\right)\exp(i\Phi(t))$$

where $\Phi(t) \to \Phi_\infty$ in L^∞ with a rate of $t^{-\frac{1}{10}}$. Thus, rearranging the relationship between g and $\hat{f}$ and setting $e^{-i\Phi_\infty}W_0 = W$, we find

$$\hat{f}(t) = e^{-i\frac{1}{\langle\gamma\rangle}|W|^2\log(t)}W + \mathcal{O}(t^{-\frac{1}{20}}) \tag{2.26}$$

as $t \to \infty$. Finally, using the representation (2.4) and estimating as in Lemma 8 we obtain

$$u(t) = e^{i\Gamma(t)\Delta}f(t) = \mathcal{M}(\Gamma(t))\,\mathcal{D}(\Gamma(t))\hat{f}(t) + O(t^{-\frac{1}{20}})$$

Plugging in the asymptotic behavior for $\hat{f}$ obtained in (2.26), we obtain the asymptotic behavior for $u(t)$.

2.2. THE PERIODIC NLS AND INVARIANT MEASURES

In this section, we present an exposition of the seminal paper of Bourgain [4] on invariant measures for the cubic NLS on the $1d$ torus. We try to add more details to the presentation in the original paper, which tends to run light in that regard.

However, in the spirit of the remainder of the paper, we begin with a motivation for the study of Gibbs measures. Recall from the study of ordinary differential equations that for a Hamiltonian H on $\mathbb{R}^{2n}$, we may associate the following system of $2n$ ODEs:

$$\begin{cases} \dot{p}_j = \partial_{q_j}H \\ \dot{q}_j = -\partial_{p_j}H \end{cases} \tag{2.27}$$

where the Hamiltonian function $H = H(q_1, \cdots, q_n, p_1, \cdots p_n) : \mathbb{R}^{2n} \to \mathbb{R}$ is sufficiently regular. By Liouville's theorem, the Lebesgue measure $\mathrm{d}\vec{p}\mathrm{d}\vec{q}$ on $\mathbb{R}^{2n}$ is conserved under the flow, and since H is conserved under its flow, we expect the **Gibbs measure** $e^{-H(\vec{q},\vec{p})}\mathrm{d}\vec{p}\mathrm{d}\vec{q}$ to be conserved under the flow.

It turns out that one may consider 'Hamiltonian PDE', and ask a similar question: if we have a Hamiltonian PDE, is its associated Gibbs measure an invariant of the dynamics of the PDE? Formally, of course, this follows from Liouville's theorem, but this is far from an actual proof. As a byproduct of Bourgain's proof, he also constructs global-in-time dynamics for the NLS on the support of the measure. This adds another use for such problems: invariance of the Gibbs measure goes hand-in-hand with well-posedness of the NLS on the support of the measure. These measures can be supported on highly irregular function spaces (i.e. negative order Sobolev spaces), which makes these a good tool to study the low-regularity dynamics of dispersive equations.

2.2.1. Problem Statement. Consider the following nonlinear Schrödinger equation in the space-periodic setting:

$$\begin{cases} (i\partial_t + \partial_x^2)u(t,x) = \pm|u|^{p-2}u \\ u|_{t=0} = u_0(x) \end{cases} \quad \text{on} \quad \mathbb{R}\times\mathbb{T} \tag{NLS}$$

Recall that in the flat case (i.e. where $\mathbb{T}$ is replaced with $\mathbb{R}$ in the above), we have a robust theory of local well-posedness. In particular, we have local-in-time solutions for $u_0 \in H^s(\mathbb{R})$ for $s > 0$ and $p - 2 \le \frac{4}{1-2s}$. If $p > 6$, we have blowup even for smooth data. However, in the periodic case, the local theory is different and somewhat ill-understood. However, for our purposes, we have the following results:

Theorem 14. *($p = 4$). The Cauchy problem*

$$\begin{cases} (i\partial_t + \partial_x^2)u = \pm|u|^2 u \\ u(0,x) = u_0(x) \end{cases} \tag{2.28}$$

is globally well-posed for $u_0 \in H^s(\mathbb{T})$, $s \geq 0$, and the solution u lies in the space $C_t H_x^s(\mathbb{R} \times \mathbb{T})$. We also have that the data-to-solution map is continuous in H^s.

Theorem 15. *($p > 4$). The Cauchy problem*

$$\begin{cases} (i\partial_t + \partial_x^2)u = -|u|^{p-2} u \\ u(0,x) = u_0(x) \end{cases} \tag{2.29}$$

is locally well-posed on a time interval $I = [0,T]$ for $u_0 \in H^s$, where we impose the following conditions on the regularity exponent s:

$$\begin{cases} s > 0 \text{ for } p \leq 6, \text{ and} \\ s > s_c, \text{ for } p > 6. \end{cases} \tag{2.30}$$

We have that $T = T\left(\|u_0\|_{H^s}\right)$ and $u \in C_t H_x^s(I \times \mathbb{T})$, and the data-to-solution map is Lipschitz in u_0.

Remark. One small bit of notation that will crop up later - the number s_c is the Schrödinger scaling critical equation for the NLS above; it is defined by (in d dimensions)

$$s_c = \frac{d}{2} - \frac{2}{p-2}.$$

This comes from noting that the NLS above is invariant under the rescaling $u \mapsto \lambda^{\frac{2}{p-2}} u(\lambda^2 t, \lambda x)$, so that the only L^2-based Sobolev space whose norm is invariant under this rescaling is precisely H^{s_c}.

The motivation for studying such a periodic NLS as above originates in the '80s paper of Lebowitz-Rose-Speer ([25]), who had motivations stemming from statistical physics. We will prove the invariance of the measure they constructed on a certain canonical ensemble. To do this, in the $p = 4$ case, we just combine the result of LRS with the global well-posedness result from Theorem 14. In the case $4 < p \leq 6$, we have problems, as here we have only a local-in-time solution. Also, we don't have any *a priori* bounds on the H^s norm of the solution for $s \in (0, 1)$ in this range.

The workaround will be the following procedure: we combine the local well-posedness result from Theorem 15 and some invariant measures ideas to construct *both* the flow on the canonical ensemble and the measure therein. This will give global solutions to NLS for $p \in (4, 6]$ and data $u_0 \in H^s(\mathbb{T}), s < 1$.

More precisely, if $(\Omega, \mathcal{U}, \mathbb{P})$ is an ambient probability space, the random Fourier series

$$\phi_{a,\omega}(x) := a + \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{g_k(\omega)}{k} e^{2\pi i k x}$$

for $a \in \mathbb{C}$ and $\{g_k(\omega)\}_{k \in \mathbb{Z} \setminus \{0\}}$ iid L^2-normalized Gaussians gives (for $p < 6$) a "good datum" in the sense that (NLS) with $\phi_{a,\omega}$-data is globally well-posed. However, for $p = 6$, we need an L^2-cutoff on the data: $\|\phi_{a,\omega}\|_{L^2} \leq C_1$ for some specific $C_1 > 0$.

Recall that the natural conserved quantities for (NLS) are the following:

1. L^2-norm (or mass):
$$N(u) = \|u\|_{L^2}^2$$

2. Hamiltonian:
$$H(u) = \frac{1}{2} \int_{\mathbb{T}} |\partial_x u|^2 - \frac{1}{p} \int_{\mathbb{T}} |u|^p \, \mathrm{d}x$$

The measure given in Lebowitz-Rose-Speer is formally given by

$$\mu \propto \exp\left[-\beta H(\phi)\right] \prod_{x\in\mathbb{T}} d\phi(x),$$

for $\beta > 0$ a parameter. To make this normalizable, we impose a cutoff $\|\phi\|_{L^2}^2 < B$ for some $B > 0$. For $p < 6$, this is an arbitrary imposition; but for $p = 6$, the choice is specific. Note that by the conservation of mass, this is an acceptable decision. By a (formal!) application of the Liouville theorem from Hamiltonian ODE theory, conservation of H implies invariance of μ (under the flow generated by H). This gives us the following

Main Problem:. Make the previously stated invariance of μ rigorous.

In Bourgain's paper (setting aside contemporary results that no doubt trivialize the old theory), there are two main approaches mentioned:

1. Replacing (NLS) by a discrete system of ODEs:

$$i\dot{q}_n + \frac{q_{n+1} + q_{n-1} - 2q_n}{h^2} = -|q_n|^{p-2} q_n$$

where $n \in \mathbb{Z}/N\mathbb{Z}$ and $h = \frac{2\pi}{N}$. Note that this is Hamiltonian (with the Hamiltonian naturally following from the form given above). We define a statistical ensemble by normalizing the density $e^{-\beta H(q)}$ on a suitable ball $\left(\sum_n |q_n|^2\right)^{1/2}$ in $\mathbb{C}^N$. However, there are some technical difficulties associated with replacing the circle group $\mathbb{T}$ with the cyclic group $\mathbb{Z}/N\mathbb{Z}$, so we avoid this method.

2. The alternative to the above method is that of Zhidkov in [34]. Fix N and define the Dirichlet projection by

$$P_N\phi = \sum_{|n|\le N} \hat{\phi}(n) e^{2\pi i n x}$$

(i.e. the Nth partial sum operator).

Consider the truncated equation

$$i\partial_t u^N + \partial_x^2 u^N + P_N(|u^N|^{p-2}u^N) = 0 \tag{NLS$_N$}$$

where u^N takes the form

$$u^N(t,x) = \sum_{|n|\leq N} a_n(t)e^{2\pi i n x}. \tag{2.31}$$

We can identify u^N with the sequence $(a_n)_{|n|\leq N}$ through the formula (2.31). Note that (NLS$_N$) is Hamiltonian with Hamiltonian given (in terms of $a = (a_n)$) by

$$H(a) = 2\pi^2 \sum_{|n|\leq N} n^2|a_n|^2 - \frac{1}{p}\int_{\mathbb{T}} \left| \sum_{|n|\leq N} a_n e^{2\pi i n x} \right|^p \mathrm{d}x. \tag{2.32}$$

This is equivalent to saying that

$$\frac{da}{dt} = i\frac{\partial}{\partial \overline{a}} H.$$

Claim. (2.32) *is the Hamiltonian for* (NLS$_N$).

Proof. By Plancherel and the representation (2.31), we have that

$$H(u) = \frac{1}{2}\int_{\mathbb{T}} |\partial_x P_N u|^2 - \frac{1}{p}\int_T |P_N u|^p \tag{2.33}$$

If we consider the standard L^2 symplectic form given by $\omega(u,v) = \langle u, v\rangle_{L^2}$, and we place the complex conjugate on the second element in the L^2-inner product. To compute the $\overline{a}$-derivative of the Hamiltonian, we need to compute the following quantity:

$$\frac{d}{ds} H(u+sv)\bigg|_{s=0}$$

where s is a real parameter. One readily verifies the following identity:

$$\frac{d}{ds}H(u+sv)\bigg|_{s=0} = \operatorname{Re}\int_{\mathbb{T}}(-\partial_x^2 P_N u)\overline{v}\,\mathrm{d}x - \operatorname{Re}\int_{\mathbb{T}}|P_N u|^{p-2}P_N u\;\overline{P_N v}\,\mathrm{d}x \tag{2.34}$$

which gives (after comparison with the L^2-inner product)

$$\frac{d}{ds}H(u+sv)\bigg|_{s=0} = \left\langle -\partial_x^2 P_N u - P_N\left[|P_N u|^{p-2}P_N u\right], v\right\rangle_{L^2(\mathbb{T})} \tag{2.35}$$

where we used self-adjointness of the Dirichlet projection to move the P_N over from the $\overline{v}$ in the second integral term. The entry in the first slot of the inner product is precisely the derivative we're looking for. A bit of algebra yields the claim. □

This tells us that we should use (NLS_N) as our finite-dimensional models. It also turns out (due to Zhidkov) that (NLS) can be well-understood by these finite-dimensional models, provided we have sufficient knowledge of the Cauchy problem. For $p = 4$, this is provided by Theorem 14. For $p \in (4, 6]$, we have to replace L^2 by H^s for $s > 0$, and we only get local-in-time solutions. In what remains, we need to accomplish the following:

Step 1 Understand the Cauchy problem for (NLS_N) with estimates independent of N, and describe how (NLS_N) approximates (NLS).

Step 2 Construct invariant measures for (NLS_N), and use the invariant measure to stick local solutions together into global ones on large subsets of the canonical ensemble for (NLS_N).

Step 3 Global well-posedness of (NLS) for a.e. datum u_0 in the Wiener space of suitable mass-truncated functions.

Step 4 Establish invariance of the limit measure.

2.2.2. Estimates for the Cauchy Problem. In this section, we recall/establish some estimates for the Cauchy problem for the periodic NLS. Consider the equation

$$\begin{cases} (i\partial_t + \partial_x^2)u + \Gamma(u) = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{2.36}$$

where Γ is an arbitrary (and potentially nonlocal) nonlinearity. We may rewrite (2.36) using the Duhamel formula:

$$u(t,x) = e^{it\Delta}u_0 + i\int_0^t e^{i(t-s)\Delta}\Gamma(u(s))\,\mathrm{d}s \tag{2.37}$$

where (as usual)

$$e^{it\Delta}\psi(x) = \sum_{k\in\mathbb{Z}} e^{i(kx-k^2t)}\hat{\psi}(k)$$

is the free Schrödinger propagator.

The method for proving local well-posedness for this equation is virtually identical to the real-line case: the Banach fixed-point theorem still applies, and we try and compensate the loss of regularity from the nonlinearity by regularizing effects from the free propagator.

Recall the L^4 estimate

$$\|e^{it\Delta}f\|_{L^4_{t,x}([0,1]\times\mathbb{T})} \lesssim \|f\|_{L^2(\mathbb{T})} \tag{2.38}$$

and the so-called 'multiplier' estimate (which can be found in Bourgain's original paper on Strichartz estimates on tori, [3]):

$$\left\|\sum_{k\in\mathbb{Z}}\int a(k,\lambda)e^{i(kx+\lambda t)}\,\mathrm{d}\lambda\right\|_{L^4_{t,x}([0,1]\times\mathbb{T})} \lesssim \left(\sum_k \int (1+|\lambda+k^2|)^{\frac{3}{4}}|a(k,\lambda)|^2\,\mathrm{d}\lambda\right)^{\frac{1}{2}} \tag{2.39}$$

Recall also the L^6 Strichartz estimate:

$$\left\|e^{it\Delta}P_N f\right\|_{L^6_{t,x}([0,1]\times\mathbb{T})} \lesssim N^{\varepsilon}\|f\|_{L^2(\mathbb{T})} \tag{2.40}$$

for any $\varepsilon > 0$ (see [23]).

Next, in the Duhamel integral, rewrite the integral term using the space-time Fourier transform. For the time being, we denote the integral by $V(t,x)$. Note that $V(0,x) \equiv 0$ - this will crop up later.

Claim. *The space-time Fourier transform* $\widetilde{V}(\lambda, n)$ *of* $V(t,x)$ *is given by the following formula:*

$$\widetilde{V}(\lambda, n) = \frac{\widetilde{\Gamma}(\lambda, n)}{\lambda + n^2} \tag{2.41}$$

Proof. We take the space-time Fourier transform $\mathcal{F}_{t,x}$ of $V(t,x)$. Note that we have (since characters are multiplicative),

$$\mathcal{F}_{t,x} = \mathcal{F}_t\,\mathcal{F}_x$$

so that

$$\mathcal{F}_{t,x}\left\{V(t,x)\right\}(\lambda, n) = i\int_{\mathbb{R}} e^{-i\lambda t}\left\{\int_0^t e^{-i(t-s)n^2}\widehat{\Gamma}(u)(s,n)\,\mathrm{d}s\right\}\mathrm{d}t \tag{2.42}$$

(since we know exactly what the spatial Fourier transform of the free Schrödinger propagator applied to Γ is). If we change the order of integration, (noting that we need to split into $t \geq 0$ and $t < 0$), we have

$$\text{RHS}(2.42) =$$
$$i\int_0^{\infty} e^{-i\lambda t}\int_s^{\infty} e^{-i(t-s)k^2}\widehat{\Gamma(s)}\,\mathrm{d}t\,\mathrm{d}s + i\int_{-\infty}^{0} e^{-i\lambda t}\int_{-\infty}^{s} e^{-i(t-s)k^2}\widehat{\Gamma(s)}\,\mathrm{d}t\,\mathrm{d}s \tag{2.43}$$

If we change variables $u = t - s$ in the t integral and note that that the integrals are then over the first and third quadrants in $\mathbb{R}^2$, we can write

$$\mathrm{RHS}(2.43) = i \iint_{\mathbb{R}^2} e^{-i(\lambda+k^2,\lambda)\cdot(t,s)} \left\{ \widehat{\Gamma(s)} \Theta(ts) \right\} \mathrm{d}t\, \mathrm{d}s \tag{2.44}$$

where Θ is the standard Heaviside step function. Since morally speaking, RHS(2.44) is the t, s Fourier transform of $\widehat{\Gamma(s)}\Theta(ts)$, we can compute directly using the commutativity properties of the space-time Fourier transform and the fact that $\widehat{\Gamma}$ does not depend on t,

$$\begin{aligned} \mathrm{RHS}(2.44) &= i \int \mathcal{F}_s \widehat{\Gamma}(\lambda - \sigma) \left(\mathcal{F}_{t,s}\{\Theta(ts)\}\right)(\sigma, \lambda + n^2)\, \mathrm{d}\sigma \\ &= i \int \mathcal{F}_s \{\widehat{\Gamma}(\lambda - \sigma)\} \frac{\delta(\sigma)}{i(\lambda + n^2)}\, \mathrm{d}\sigma \\ &= \frac{\widetilde{\Gamma}(\lambda, n)}{\lambda + n^2} \end{aligned} \tag{2.45}$$

This establishes the identity

$$i \int_0^t e^{i(t-s)\Delta} \Gamma(u)(s)\, \mathrm{d}s = \sum_{n \in \mathbb{Z}} \int \widetilde{\Gamma}(u)(\lambda, n) \frac{e^{i(kx + \lambda t)}}{\lambda + n^2}\, \mathrm{d}\lambda \tag{2.46}$$

as desired. □

Now, note that we also have the identity

$$\sum_k \int \widetilde{\Gamma}(\lambda, k) \frac{e^{i(kx - k^2 t)}}{\lambda + k^2}\, \mathrm{d}\lambda = \sum_k e^{i(kx - k^2 t)} \{\mathcal{F}_x V(0, x)\} \tag{2.47}$$

But RHS(2.47) $\equiv 0(!)$. This means that we can freely write the identity (2.46) as

$$i \int_0^t e^{i(t-s)\Delta} \Gamma(u)(s)\, \mathrm{d}s = \sum_{n \in \mathbb{Z}} \int \widetilde{\Gamma}(u)(\lambda, n) e^{ikx} \left[\frac{e^{i\lambda t} - e^{-in^2 t}}{\lambda + n^2} \right] \mathrm{d}\lambda \tag{2.48}$$

To the naked eye, this may appear an inconsequential change (since all we did was add zero). However, note that in magnitude the term in brackets above obeys the following estimate:

$$\left|\frac{e^{i\lambda t}-e^{-in^2t}}{\lambda+n^2}\right| \lesssim \left|\frac{(\lambda+n^2)t}{\lambda+n^2}\right| = |t|$$

and since we're only concerned about local theory, $|t| \lesssim 1$. Thus, this ingenious addition of zero has saved us at the low space-time frequencies, but only because we're working locally in time.

Having derived the identity (2.48), we now specialize to the case of a cubic nonlinearity $\Gamma(u) = |u|^2u$ or $\Gamma(u) = P_N|u|^2u$. To perform a contraction mapping argument, we use the space $L^4_{t,x}([0,1]\times\mathbb{T})$, and take the $L^4_{t,x}$ norm of (2.37). Note that by (2.38)

$$\|e^{it\Delta}u_0\|_{L^4_{t,x}} \lesssim \|u_0\|_{L^2}$$

so that term is safe. For the nonlinear term, there are two essentially two contributions:

$$\sum_n \int_{|\lambda+n^2|>1} \frac{\widetilde{\Gamma}(u)}{\lambda+n^2} e^{i(nx+\lambda t)}\,\mathrm{d}\lambda \tag{I}$$

and

$$\sum_n \int_{|\lambda+n^2|>1} \frac{\widetilde{\Gamma}(u)}{\lambda+n^2} e^{i(nx-n^2 t)}\,\mathrm{d}\lambda. \tag{II}$$

To estimate these, we use the multiplier estimate above. For the first term, we have

$$\begin{aligned}
\|(\mathrm{I})\|_{L^4_{t,x}} &\lesssim \left(\sum_n \int (1+|\lambda+n^2|)^{\frac{3}{4}} \frac{|\widetilde{\Gamma}(u)|^2}{|\lambda+n^2|^2}\right)^{\frac{1}{2}} \\
&\sim \left(\sum_n \int \frac{|\widetilde{\Gamma}(u)|^2}{1+|\lambda+n^2|^{\frac{5}{4}}}\right)^{\frac{1}{2}} \\
&\lesssim \left(\sum_n \int \frac{|\widetilde{\Gamma}(u)|^2}{1+|\lambda+n^2|^{\frac{3}{4}}}\right) \\
&\lesssim \|\Gamma(u)\|_{L^{4/3}_{t,x}} \qquad (2.49)
\end{aligned}$$

where the last inequality follows from the dual estimate to (2.39).

To estimate (II), note that by Hölder's inequality, we have the estimate

$$\begin{aligned}
\|(\mathrm{II})\|_{L^4_{t,x}} &\lesssim \sum_n \left(\int \frac{\widetilde{\Gamma}(u)}{1+|\lambda+n^2|} \right)^2 \mathrm{d}\lambda \\
&\lesssim \sum_n \int \frac{|\widetilde{\Gamma}(u)|^2}{1+|\lambda+n^2|^2} \mathrm{d}\lambda \\
&\lesssim \sum_n \int \frac{|\widetilde{\Gamma}(u)|^2}{1+|\lambda+n^2|^{\frac{3}{4}}} \\
&\lesssim \|\Gamma(u)\|_{L^{4/3}_{t,x}} \qquad (2.50)
\end{aligned}$$

again using the dual estimate, as above. If we specialize to the cases $\Gamma(u) = |u|^2 u$ or $\Gamma(u) = P_N |u|^2 u$, we have (by Hölder),

$$\|\Gamma(u)\|_{L^{4/3}_{t,x}} \lesssim \|u\|^3_{L^4_{t,x}} \qquad (2.51)$$

in the former case, and (by boundedness of Littlewood-Paley projections on L^p for $p \in (1, \infty)$),

$$\|\Gamma(u)\|_{L^{4/3}_{t,x}} \lesssim \|u\|^3_{L^4_{t,x}} \qquad (2.52)$$

in the latter case, as well.

One can readily verify continuity of the data-to-solution map in the L^4-norm in the usual way. Further, if we combine all the estimates from above, what we essentially have is the following estimate:

$$\|u\|_{L^\infty_t L^2_x} \lesssim \|u_0\|_{L^2_x} + C\|u\|^3_{L^4_{t,x}} \qquad (2.53)$$

and from the above discussion of continuity of the data-to-solution map, $u \in C_t L_x^2([0,T] \times \mathbb{T})$. Note that if we want H^s regularity for $s > 0$, note that all we have to do is pass around a copy of $\langle \partial_x \rangle^s$. This readily implies continuity of the data-to-solution map in H^s.

On the other hand, if we want to repeat the above argument for $p > 4$, we have to work a bit harder, as our Strichartz estimates are at best frequency-localized, and we also lose a bit in the frequency scale N. In particular, we can't do general L^2 data, and the local theory requires H^s regularity for $p \in (4,6]$. The norm we'll use for the contraction mapping is the following Besov modified $X^{s,b}$-norm:

$$\|u\|_{\mathcal{B}^s} = \|\langle n \rangle^s \langle \lambda - n^2 \rangle \tilde{u}\|_{B^{1/2}_{2,\infty;\lambda} \ell^2_n} \tag{2.54}$$

for posterity, we record the definition as it appears in Bourgain:

$$\|u\|_{\mathcal{B}^s} = \sup_{K \in 2^{\mathbb{Z}}} (1+K)^{1/2} \left(\sum_{n \in Z} \langle n \rangle^{2s} \int_{K \leq |\lambda + n^2| \leq 2K+1} |\tilde{u}(\lambda, n)|^2 \right)^{1/2}. \tag{2.55}$$

To compute $\|u\|_{\mathcal{B}^s}$, we take the infimum over all u satisfying the equation

$$u(t,x) = \sum_n \int \tilde{u}(\lambda, n) e^{i(nx + \lambda t)} \, \mathrm{d}\lambda \tag{2.56}$$

To estimate the integral term, write a Littlewood-Paley decomposition of u in the spatial variable x, i.e.

$$u = \sum_{M \in 2^{\mathbb{Z}}} P_M u - P_{\frac{M}{2}} u \tag{2.57}$$

and writing $u^+ = u$ and $u^- = \overline{u}$, write $|u|^{p-2} u$ as

$$|u|^{p-2} u = \sum_{\substack{M_1 > M_2 \text{ dyadic}, \\ \sigma \in \{\pm\}}} u^{\sigma}_{M_1} u^{\sigma}_{M_2} F(P_{10 M_1} u) \tag{2.58}$$

where F absorbs the pieces of $|u|^{p-2}u$ left over from taking the Littlewood-Paley decomposition. As an aside, note that if p is even, this decomposition is purely algebraic. Otherwise, we justify this by arguing that $z \mapsto |z|^{p-2}z$ is differentiable "enough".

The idea of the proof of the necessary estimates for the nonlinear term in the Duhamel formula is as follows: the Littlewood-Paley projected u_{M_1} eats the factor M^s appearing in the definition of $\|u\|_Y$, and the factor M_2^{-s} that comes from u_{M_2} eats the N^ε factor coming from estimating the $\|\cdot\|_Y$ norm of the summands in (2.58) by the $(6,6)$ Strichartz norm. This is very nebulous, but the details are found in Bourgain's restriction estimates paper.

The point is that at the end of the day, for u_0 in $H^s(\mathbb{T})$, for $s > 0$ we can get a local solution for $p \leq 6$. The time existence T depends on s and $\|u_0\|_{H^s}$, but on $[0,T]$ the solution is $C_t H_x^s$ and we have all the standard regularity estimates. Namely, if u, v are solutions with data u_0, v_0, then $\|u - v\|_{H^s} \leq 2\|u_0 - v_0\|_{H^s}$.

It is of extreme importance to understand the size T of the interval of existence depends on the size of the H^s norm of the data as a power. To be quantitative, if $p < 6$, we have

$$T \gtrsim_{p,s} \|u_0\|_{H^s}^{-C(p)} \tag{2.59}$$

and if $p = 6$,

$$T \gtrsim_{s} \|\phi\|_{H^s}^{-C(s)} \tag{2.60}$$

The problem now is to build global solutions from these local ones. However, the main obstacle is that $\|u(t)\|_{H^s}$ is not a conserved quantity under the NLS flow for $s \in (0,1)$. The argument herein will show how the invariant measure can yield a substitute for this conservation law, where we consider the Cauchy problem for a *set* of data, rather than a singular datum.

Recall that (NLS_N) is actually a vector-valued ODE in the coefficients $(a_n)_{|n|\leq N}$. From the a priori bound arising from the Hamiltonian nature of the equation, we know that the solution $(a_n)_{|n|\leq N}$ is bounded and thus exists for all time (from the standard ODE existence theory). On the other hand, we can easily reproduce all the well-posedness theory arguments for (NLS) above and get a well-posedness result for (NLS_N) with bounds independent of N, as long as the data is in $H^s(\mathbb{T})$ for $s > 0$. The time of existence will be as above. This completes the first part of Step 1 as enumerated above. To complete step 1, we need a quantitative comparison of (NLS_N) and (NLS). This is accomplished in the following

Lemma 11 (Comparing (NLS_N) and (NLS))**.** *Let $p \leq 6$, $s > 0$, $\phi \in H^s(\mathbb{T})$ with $\|\phi\|_{H^s} < A$ and $N \gg 1$ a large integer. For notational convenience, write $v = u^N$.*

Then if the solution to

$$\begin{cases} i\partial_t v + \partial_x^2 v + P_N\left(|v|^{p-2}v\right) = 0 \\ v(0,x) = P_N\phi \end{cases} \tag{2.61}$$

obeys the estimate

$$\|v(t)\|_{H^s} \leq A \quad \textit{for} \;\; t \leq T \tag{2.62}$$

then the Cauchy problem (NLS) *is well-posed on* $[0,T]$ *and we have the approximation for $t < T$ and $0 < s_1 < s$*

$$\|u(t) - v(t)\|_{H^s} \lesssim_{p,s} \exp\left\{(1+A)^{C(p,s)}T\right\} N^{s_1 - s} \tag{2.63}$$

provided RHS(2.63) *remains bounded above by* 1*. If $p < 6$, $C(p,s)$ does not depend on s.*

Proof. Fix $0 < s_1 < s$ and denote by τ the interval given by the local theory for (NLS) for an initial datum bounded by $A+1$ in H^{s_1}-norm. We will then get information on a time interval $[0, T]$ by piecing together estimates on intervals of length τ.

First note that we have the estimate

$$\begin{aligned} \|u(0) - v(0)\|_{H^{s_1}} &= \left(\sum_{|n|>N} \langle n \rangle^{2s_1} |\phi(n)|^2 \right)^{1/2} \\ &= \left(\sum_{|n|>N} \langle n \rangle^{2(s_1 - s)} \langle n \rangle^{2s} |\phi(n)|^2 \right)^{1/2} \\ &\lesssim N^{s_1 - s} A \end{aligned} \tag{2.64}$$

by Hölder's inequality.

To get started, assume that for $t \leq t_0$, we obtained the estimate $\|u(t) - v(t)\|_{H^{s_1}} < \delta < 1$. Then by the triangle inequality, we have

$$\|u(t_0)\|_{H^{s_1}} \leq \|v(t_0)\|_{H^{s_1}} + \delta < A + 1 \tag{2.65}$$

where the last inequality follows from the assumption. Using the local regularity theory for (NLS), the two IVPS

$$\begin{cases} (i\partial_t + \partial_x^2) u = -|u|^{p-2} u \\ u(t_0, x) = u(t_0) \end{cases} \quad \begin{cases} (i\partial_t u' + \partial_x^2 u') = -|u'|^{p-2} u' \\ u'(t_0, x) = v(t_0) \end{cases} \tag{2.66}$$

are well-posed on $[t_0, t_0 + \tau]$, by the H^s local theory. Using the well-posedness estimates from the local theory above,

$$\|u(t) - u'(t)\|_{H^{s_1}} \leq 2 \|u(t_0) - u'(t_0)\|_{H^{s_1}} < 2\delta \tag{2.67}$$

Next, we need to compare $u'(t)$ and $v(t)$ on $[t_0, t_0+\tau]$, since u' is basically the solution of (NLS) with $v(t_0$ as data). Recall that v solves (NLS$_N$), so that by Duhamel (since u' and v have the same free solution),

$$u'(t) - v(t) = i \int_0^t e^{i(t-s)\Delta} \mathcal{F}(u(s)) \, \mathrm{d}s \tag{2.68}$$

where the nonlinear term F is given by

$$F(u(s)) = |u'|^{p-2}u' - P_N(|v|^{p-2}v) \tag{2.69}$$

Creatively add zero, and rewrite F as

$$F(u) = \underbrace{\left[|u'|^{p-2}u' - P_N(|u'|^{p-2}u')\right]}_{\text{term I}} + \underbrace{\left[P_N\left(|u'|^{p-2}u' - |v|^{p-2}v\right)\right]}_{\text{term II}} \tag{2.70}$$

We then seek to estimate $\|u' - v\|_{\mathcal{B}^{s_1}}$. We estimate terms (I) and (II) separately. For the first term, we have the estimate

$$\|(\mathrm{I})\|_{\mathcal{B}^{s_1}} \lesssim C\tau^\delta \|u' - P_{N_1}u'\|_{\mathcal{B}^{s_1}} \|u'\|_{s_1}^{p-2} \tag{2.71}$$

for $N_1 \sim N$ (i.e. $\frac{N}{2} \lesssim N_1 \lesssim 2N$). For the second term, we have the estimate

$$\|(\mathrm{II})\|_{\mathcal{B}^{s_1}} \lesssim C\tau^\delta \|u' - v\|_{\mathcal{B}^{s_1}} (\|u'\|_{\mathcal{B}^{s_1}} + \|v\|_{\mathcal{B}^{s_1}})^{p-2} \tag{2.72}$$

From the local theory we have the estimates

$$\begin{aligned} \|u'\|_{\mathcal{B}^{s_1}} &\le C\|u'(t_0)\|_{H^{s_1}} = C\|v(t_0)\|_{H^{s_1}} < CA \\ \|v\|_{\mathcal{B}^{s_1}} &\le C\|v(t_0)\|_{H^{s_1}} < CA \end{aligned} \tag{2.73}$$

Then by choosing $\tau = \tau(A)$ sufficiently small, we can ensure that

$$\|u' - v\|_{\mathcal{B}^{s_1}} \lesssim \|u' - P_{N_1} u'\|_{\mathcal{B}^{s_1}} \tag{2.74}$$

Now consider an $s > s_1$. Then by the local theory, we have

$$\|u'\|_{\mathcal{B}^s(t_0, t_0+\tau)} \lesssim C\|u'(t_0)\|_{H^s} = C\|v(t_0)\|_{H^s} < CA \tag{2.75}$$

by the assumptions above and the local H^s theory. From the above and the definition of the $\mathcal{B}^s$-norm, we have

$$\|u' - P_{N_1} u'\|_{\mathcal{B}^{s_1}} \lesssim CAN_1^{s_1 - s} \sim CAN^{s_1 - s} \tag{2.76}$$

Then for $t \in [t_0, t_0 + \tau]$, we estimate

$$\|u'(t) - v(t)\|_{\mathcal{B}^{s_1}} \lesssim \|u' - v\|_{\mathcal{B}^{s_1}} + \|u' - P_{N_1} u'\|_{\mathcal{B}^{s_1}} < CAN^{s_1 - s} \tag{2.77}$$

this implies by the triangle inequality that

$$\|u - v\|_{H^{s_1}} < 2\delta + CAN^{s_1 - s} \quad \text{on } [t_0, t_0 + \tau] \tag{2.78}$$

Now break up the interval $[0, T]$ into subintervals of length τ, and define $t_j = j\tau$, for $j \in \{0, \dots, \frac{T}{\tau}\}$. Then we have (iterating the results and estimates above),

$$\begin{cases} \|u(t_j) - v(t_j)\|_{H^{s_1}} \equiv \delta_{j+1} < 2\delta_j + CAN^{s_1 - s} \\ \delta_0 < N^{s_1 - s} A \end{cases} \tag{2.79}$$

By induction, we find that $\delta_j \lesssim C^{j+1} AN^{s_1 - s}$. Since $j < \frac{T}{\tau}$ and $\tau \gtrsim_{p,s} \frac{C}{(1+A)^{C_1}}$, we conclude the estimate (2.63). $\square$

Remark. In the application of this estimate, the parameters p, s, A and T are fixed and we send $N \to \infty$, so the sharp inequality is actually irrelevant.

Note that this completes Step 1 above! We now move onto step 2, where we actually construct the invariant measures for the equation (NLS_N)

2.2.3. Invariant Measures for the Truncated Equation. In this section, we construct invariant measures for the system (NLS_N). Recall that (NLS_N) is an equation for $u^N(t,x)$, where

$$u(t,x) = \sum_{|n|\leq N} a_n(t) e^{2\pi i n x} \tag{2.80}$$

Recall that we have a conservation of mass

$$\left(\sum_{|n|\leq N} |a_n|^2 \right)^{1/2} \tag{2.81}$$

and of the Hamiltonian

$$H(a) = 2\pi^2 \sum_{|n|\leq N} n^2 |a_n|^2 - \frac{1}{p} \int_{\mathbb{T}} \left| \sum_{|n|\leq N} a_n e^{2\pi i n x} \right|^p \mathrm{d}x \tag{2.82}$$

Note that (NLS_N) has a distinct advantage over the original equation (NLS): its phase space is $\mathbb{C}^{2N+1}$ (i.e. finite dimensional!). Further, the flow of (NLS_N) is global, by the standard ODE existence theory. Finally, since (NLS_N) is Hamiltonian, **the flow is measure-preserving** on $\mathbb{C}^{2N+1}$.

Remark. For notational convenience, define the set N_0 by

$$N_0 = \{n \in \mathbb{Z} \mid |n| \leq N \text{ and } n \neq 0\},$$

so for example

$$1_0 = \{-1, 1\}$$

Let $B > 0$ be positive (to be specified) and consider the ball in $\mathbb{C}^{2N+1}$ defined by

$$\Omega_{N,B} = \left\{ (a_n)_{|n|\leq N} \;\middle|\; \left(\sum_{|n|\leq N} |a_n|^2 \right)^{1/2} \leq B \right\} \tag{2.83}$$

which by conservation of mass is invariant under the flow.

Define ρ_N to be the measure on $\mathbb{C}^{2N} = [(a_n)_{n\in N_0}]$ defined by

$$\mathrm{d}\rho_N = \frac{e^{-2\pi^2 \sum_{n\in N_0} n^2|a_n|^2}}{\displaystyle\int_{\mathbb{C}^{2N}} e^{-2\pi^2 \sum_{n\in N_0} n^2|a_n|^2}\mathrm{d}(a_n)} \mathrm{d}(a_n) \tag{2.84}$$

Claim. *ρ_N is the image measure on $\mathbb{C}^{2N}$ under the map*

$$\omega \overset{F}{\mapsto} \left[\frac{g_n(\omega)}{2\pi n} \right]_{n\in N_0} \tag{2.85}$$

where ω is an element in some ambient probability space $(\mathcal{O}, \mathcal{U}, \mathbb{P})$ and $\{g_n\}$ are iid Gaussian random variables with mean zero and variance one.

Proof. We need to check that

$$\int_E \mathrm{d}\rho_N = \int_{F^{-1}(E)} \mathrm{d}\mathbb{P}(\omega) \tag{2.86}$$

where by Radon-Nikodym E is a Lebesgue-measurable Borel set in $\mathbb{C}^{2N}$. By inner/outer regularity of Lebesgue measure, we can write $E = \prod_{n \in N_0} I_n$, where each I_n is open in $\mathbb{C}$. Then by direct computation,

$$\begin{aligned}
\int_E \mathrm{d}\rho_N &= \int_{F^{-1}(E)} \mathrm{d}\mathbb{P}(\omega) \\
&= \mathbb{E}(\chi_{F^{-1}(E)}) \\
&= \mathbb{P}(\omega \in F^{-1}(E)) \\
&= \mathbb{P}(F(\omega) \in E) \\
&= \mathbb{P}\left(\left[\frac{g_n(\omega)}{2\pi n} \right]_{n \in N_0} \in E \right) \\
&= \prod_{n \in N_0} \mathbb{P}(g_n(\omega) \in 2\pi n I_n) \\
&= \prod_{n \in N_0} \int_{2\pi n I_n} (2\pi)^{-\frac{1}{2}} e^{-|a_n|^2/2} \mathrm{d}a_n \\
&= \prod_{n \in N_0} \int_{I_n} e^{-2\pi^2 n^2 |a_n|^2} (2\pi)^{\frac{1}{2}} n \mathrm{d}a_n \\
&= \int_E \prod_{n \in N_0} (2\pi)^{\frac{1}{2}} n e^{-2\pi^2 n^2 |a_n|^2} \mathrm{d}a_n
\end{aligned} \tag{2.87}$$

If we compute the product, what we end up with is the following:

$$(2.87) = (2\pi)^N \left(\prod_{n \in N_0} n \right) \int_E \exp\left\{ \sum_{n \in N_0} -2\pi^2 n^2 |a_n|^2 \right\} \mathrm{d}(a_n). \tag{2.88}$$

We then note that the first two terms appearing in (2.88) comprise precisely the value of

$$\int_{\mathbb{C}^{2N}} \exp\left\{ -2\pi^2 \sum_{n \in N_0} n^2 |a_n|^2 \right\} \mathrm{d}(a_n) \tag{2.89}$$

which can be shown by direct computation. This shows the result. □

The statistical ensemble will be the measure space obtained by endowing $\Omega_{N,B}$ with the measure

$$\mathrm{d}\mu_N = \exp\left\{\frac{1}{p}\int_{\mathbb{T}}\left|\sum_{|n|\leq N} a_n e^{2\pi inx}\right|^p \mathrm{d}x\right\}(\mathrm{d}a_0 \otimes \mathrm{d}\rho_N) \tag{2.90}$$

which by construction is invariant under the flow of (NLS_N). Thus, μ_N is a Weiner measure restricted to $\Omega_{N,B}$. We claim that $L^1(\mathrm{d}\rho_N)$ estimates for $p < 6$ and if $p = 6$ for sufficiently small B:

Lemma 12. *The random variable*

$$G(\omega) = \exp\left\{\left\|\sum_{n\neq 0}\frac{g_n(\omega)}{n}e^{2\pi inx}\right\|_{L^p}^p\right\}\mathbb{1}_{\left\{\left(\sum_{n\neq 0}\frac{|g_n(\omega)|^2}{n^2}\right)^{1/2}<B\right\}} \tag{2.91}$$

is an element of $L^1(\mathrm{d}\mathbb{P}(\omega))$ for $p < 6$ and B arbitrary, and for $p = 6$, B sufficiently small.

Proof. We need to estimate the following probability:

$$\mathbb{P}\left\{\left\|\sum_{n\neq 0}\frac{g_n(\omega)}{n}e^{2\pi inx}\right\|_{L^p} > \lambda,\ \left(\sum_{n\neq 0}\frac{|g_n(\omega)|^2}{n^2}\right)^{1/2} < B\right\}. \tag{2.92}$$

To this end, suppose $\exists\omega$ so that both conditions in (2.92) hold. Then for *any* $M_0 \in 2^{\mathbb{Z}}$, we have the trivial inequality

$$\left\|\sum_{n>M_0}F(n)\right\|_{L^p} \geq \left\|\sum_{n}F(n)\right\|_{L^p} - \left\|\sum_{n\leq M_0}F(n)\right\|_{L^p} \tag{2.93}$$

If we specialize to the case $F(n) = \frac{g_n(\omega)}{n}e^{2\pi inx}$, then the first term on the right-hand side above is bounded below by λ by assumption. We then seek $M_0 \in 2^{\mathbb{Z}}$ so that the remaining term on the right-hand is bounded below by (say) $\frac{\lambda}{2}$.

To this end, note that by Bernstein,

$$\left\| \sum_{n \le M_0} F(n) \right\|_{L^p} \le C_{\text{Ber}} M_0^{\frac{1}{2}-\frac{1}{p}} \left\| \sum_{n \le M_0} F(n) \right\|_{L^2}$$
$$\le C_{\text{Ber}} M_0^{\frac{1}{2}-\frac{1}{p}} B \tag{2.94}$$

Bounding from below by $\frac{\lambda}{2}$, we find that $M_0 \ge \left(\frac{\lambda}{2CB}\right)^{\frac{2p}{p-2}}$

We now want to look for a single 'scale of concentration', i.e. some $M > M_0$ so that most of the L^p-norm of F is concentrated at scale larger or equal to M. This will play a crucial role in reducing the dimension of the problem (to dimension less than or equal to M).

Claim. *Let* $\ell^1_{\{M>M_0\}} \ni \sigma_M = c\left[M^{-\frac{1}{p}} + \left(\frac{M_0}{M}\right)^{\frac{1}{2}}\right]$*, where* c *is chosen to make* $\|\sigma_M\|_{\ell^1} < 1$*. Then* $\exists M \ge M_0$ *so that*

$$\left\| \sum_{n \sim M} \frac{g_n(\omega)}{n} e^{2\pi i n x} \right\|_{L^p} > \frac{1}{2}\sigma_M \lambda$$

Proof. If not, then

$$\left\| \sum_{n > M_0} \frac{g_n(\omega)}{n} e^{2\pi i n x} \right\|_{L^p} \le \sum_{M > M_0} \left\| \sum_{n \sim M} \frac{g_n(\omega)}{n} e^{2\pi i n x} \right\|_{L^p} \tag{2.95}$$
$$\le \sum_{M > M_0} \frac{1}{2}\sigma_M \lambda < \frac{1}{2}\lambda \tag{2.96}$$

which contradicts the assumption that the high frequencies capture half the L^p-norm (from earlier). □

Hence M is the scale for concentration. This allows us to reduce to a finite-dimensional problem.

By a theorem of Banach space geometry (probably due to Bourgain), we can find a 'norming set' $\xi = \{\phi\}$ obeying the following properties:

$$\log(\#\xi) \lesssim M, \tag{2.97}$$

$$\forall f \in \operatorname{span}\{e^{inx} | n \sim M\}\ \exists \phi \in \xi \text{ with } |\langle f, \phi \rangle| \gtrsim \|f\|_{L^p} \tag{2.98}$$

$$\|\phi\|_{L^2} \lesssim M^{\frac{1}{2}-\frac{1}{p}}. \tag{2.99}$$

Using this, note the following: if

$$\left\| \sum_{n \sim M} g_n(\omega) e^{2\pi i n x} \right\|_{L^p} \gtrsim \sigma_M M \lambda$$

then by the norming set property above, we can find $\phi = \phi(\omega) \in \xi$ so that

$$\left| \left\langle \sum_{n \sim M} g_n(\omega) e^{2\pi i n x}, \phi \right\rangle \right| \gtrsim \sigma_M M \lambda$$

so by definition of the Fourier transform, we have

$$\left| \sum_{n \sim M} g_n(\omega) \hat{\phi}(n) \right| \gtrsim \sigma_M M \lambda$$

$$\implies \left| \sum_{n \sim M} g_n(\omega) \frac{\hat{\phi}(n)}{\|\phi\|_{L^2}} \right| \gtrsim \sigma_M M^{\frac{1}{2}+\frac{1}{p}} \lambda \tag{2.100}$$

Define (2.100) to be the statement $S(\omega) = S(\omega; \phi, M)$, so that we have the containment

$$\left\{ \omega \left| \left\| \sum \frac{g_n(\omega)}{n} e^{2\pi i n x} \right\|_{L^p} > \lambda \right. \right\} \subset \bigcup_{M \geq M_0} \bigcup_{\phi \in \xi} \{\omega | S(\omega) \text{ holds for that choice of } \phi\} \tag{2.101}$$

Taking probabilities, we see

$$\mathbb{P}\left\{\left\|\sum \frac{g_n(\omega)}{n} e^{2\pi inx}\right\|_{L^p} > \lambda\right\} \leq \sum_{M \geq M_0} (\#\xi) \sup_{\phi \in \xi} \mathbb{P}\{S(\omega)\}$$
$$\lesssim \sum_{M \geq M_0} e^{CM} e^{-C_1 M_0^{1+\frac{2}{p}} \lambda^2} \tag{2.102}$$

where the estimate on $\mathbb{P}\{S\}$ follows from classical estimates about sums of iid random variables against unit vectors (the proof follows from Chebyshev's inequality).

Next, define Ω_1 to be the intersection of the two events

$$\left\{\left\|\sum \frac{g_n(\omega)}{n} e^{2\pi inx}\right\|_{L^p} > \lambda\right\}$$

and

$$\left\{\left\|\sum \frac{g_n(\omega)}{n} e^{2\pi inx}\right\|_{L^2} \leq B\right\}$$

By the above results, if $\omega \in \Omega_1$, $\exists M \geq M_0 = M_0(\lambda, B)$ and $\phi = \phi(\omega, M) \in \xi_M$ so that ω belongs to the event

$$\left\{\left|\sum g_n(\omega) \frac{\hat{\phi}(n)}{\|\phi\|_{L^2}}\right| > \sigma_M M^{\frac{1}{2}+\frac{1}{p}} \lambda\right\}$$

This implies (as before)

$$\mathbb{P}[\Omega_1] \leq \sum_{M \geq M_0} e^{CM} e^{-C_1 \sigma_M^2 \lambda^2 M^{1+\frac{2}{p}}} \tag{2.103}$$

Now make the specific choice

$$\sigma_M = M^{-\frac{1}{p}} + \left(\frac{M_0}{M}\right)^{\frac{1}{2}}$$

Then RHS(2.103) becomes

$$\sum_{M \geq M_0} e^{CM - C_1(M^{-\frac{1}{p}} + (\frac{M_0}{M})^{\frac{1}{2}})^2 \lambda^2 M^{1+\frac{2}{p}}}$$

If one simplifies this (multiply out the quadratic expression and combine like terms), we find that the highest overall power of M_0 and M in the exponential is $1 + \frac{2}{p}$, which comes with a negative sign. We thus pull this out in ℓ^∞ and using the exponential decay in λ we can sum the remaining terms, which gives us the estimate

$$\mathbb{P}[\Omega_1] \leq e^{-C\lambda^2 M_0^{1+\frac{2}{p}}} \tag{2.104}$$

Finally, recall that at the outset, we wanted to prove an integrability estimate; specifically, we wanted $e^{F(\omega)^p} \in L^1(\mathrm{d}\omega)$. To prove such an estimate, we need to analyze the following integral:

$$\int_{\lambda_0}^{\infty} e^{\lambda^p} \mathbb{P}[F(\omega) > \lambda] \mathrm{d}\lambda \tag{2.105}$$

However, we have an estimate on $\mathbb{P}[F > \lambda]$ given in (2.104). If we plug this in to our integral, use the definition of M_0 and simplify, we arrive at the following:

$$\int_{\lambda_0}^{\infty} e^{\lambda^p} \left[e^{-C\lambda^{2+\frac{2(p+2)}{p-2}} B^{-\frac{2(p+2)}{p-2}}} \right] \mathrm{d}\lambda \tag{2.106}$$

Comparing the exponentials, we see that for this function to be integrable with respect to λ, we need

$$p - \left[2 + \frac{2(p+2)}{p-2} \right] \tag{2.107}$$

$$\Leftrightarrow p - \frac{4p}{p-2} < 0 \tag{2.108}$$

which is true precisely when $p < 6$. For $p = 6$, note that $p = \frac{4p}{p-2}$, so the powers actually match one another. However, if we choose B sufficiently small, we can guarantee integrability. This establishes the result. □

Remark. Note that this argument actually allows us to establish $L^r(\mathrm{d}\omega)$ estimates on the density for $r \in [1, \infty)$, by simply adding a factor of λ^r and changing the integration measure to $\frac{\mathrm{d}\lambda}{\lambda}$ in the usual way.

Morally speaking, what this lemma tells us is that sets of small ρ_N-measure are sets of small μ_N-measure. Essentially, we're proving a reverse absolute continuity result.

Next, let $s \in (0, 1)$ and $K > 1$. Consider the following subset of $\Omega_{N,B}$ (given by (2.83)):

$$\Omega^{s,K} = \left\{ (a_n)_{|n| \leq N} \in \Omega_{N,B} \,\middle|\, \left\| \sum_{|n| \leq N} a_n e^{2\pi i n x} \right\|_{H^s(\mathbb{T})} \leq K \right\} \tag{2.109}$$

Pushing forward by ρ_N as in the lemma above, we have the trivial estimate

$$\mathbb{P}\left\{ \left(\sum_{n \neq 0} \langle n \rangle^{2s} \frac{g_n(\omega)}{n^2} \right)^{\frac{1}{2}} > K \right\} \lesssim e^{-CK^2} \tag{2.110}$$

In particular, by the remark above, we have

$$\rho_N(\Omega \setminus \Omega^{s,K}) \lesssim e^{-CK^2} \quad \text{and} \quad \mu_N(\Omega \setminus \Omega^{s,K}) \lesssim e^{-CK^2}$$

Next, recall the equation (NLS_N), where $u^N(t, x)$ is defined to be

$$u^N(t, x) = \sum_{|n| \leq N} a_n(t) e^{2\pi i n x}$$

For $p = 4$, there is a regularity (i.e. global well-posedness) result for L^2 initial data. For $4 < p \leq 6$, we only get a local result, and our data needs to come from $H^s, s > 0$. The interval of existence is of length τ, where $\tau = \|u_0\|_{H^s}^{-C}$. To move forward, the main idea is the following: Combine the local theory with invariance of μ_N to get well-posedness on large subsets of $\Omega_{N,B}$.

To do this, first note that the flow map associated to (NLS_N) is a measure-preserving transformation on $(\Omega_{N,B}, \mu_N)$. Denote the flow map [given by flowing by the free evolution to time τ] by S. Consider $\overline{\Omega} \subset \Omega_{N,B}$ defined by (for fixed $T > 0$)

$$\overline{\Omega} = \Omega^{s,K} \cap S^{-1}(\Omega^{s,K}) \cap \cdots \cap S^{-\lfloor \frac{T}{\tau} \rfloor}(\Omega^{s,K}) \tag{2.111}$$

By construction, for any $\overline{a} = (a_n) \in \overline{\Omega}$, there is N-independent local well-posedness on each subinterval $[j\tau, (j+1)\tau] \subset [0, T]$ with corresponding data $u(j\tau)$, as by construction $\|u(j\tau)\|_{H^s} \leq K$. In particular, the regularity estimates from earlier give us that

$$\|u(t)\|_{H^s} < 2K \tag{2.112}$$

for any $t < T$. Furthermore, since S is a measure-preserving transformation and $\overline{\Omega} \subset \Omega^{s,K}$, we deduce the following:

$$\mu_N(\Omega_{N,B} \setminus \overline{\Omega}) \leq \frac{T}{\tau} \mu_N(\Omega_{N,B} \setminus \Omega^{s,K}) < T K^{C(p,s)} e^{-CK^2}.$$

Fixing T and sending $K \to \infty$, we deduce the following lemma:

Lemma 13. *Fix $s \in (0, \frac{1}{2}), p \leq 6, T < \infty$, and $\delta > 0$. Then there exists $\overline{\Omega} \subset \Omega_{N,B}$ such that $\mu_N(\Omega_{N,B} \setminus \overline{\Omega}) < \delta$ and for every $\overline{a} = (a_n)_{|n| \leq N} \in \overline{\Omega}$, the solution u of the initial-value problem*

$$\begin{cases} i\partial_t u + \partial_x^2 u + P_N(|u|^{p-2}u) = 0 \\ u(0,x) = \sum_{|n| \leq N} a_n e^{2\pi i n x} \end{cases} \tag{2.113}$$

satisfies for $|t| < T$

$$\|u(t)\|_{H^s} \lesssim \left(\log\left(\frac{T}{\delta}\right)\right)^{1/2}$$

Remark. Note that by interpolation between the estimates $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$ and the estimate $\|u(t)\|_{H^{\frac{1}{2}-}} \lesssim \left(\log\left(\frac{T}{\delta}\right)\right)^{\frac{1}{2}}$, we can actually improve the estimate to

$$\|u(t)\|_{H^s} \lesssim \left(\log\frac{T}{\delta}\right)^{s+}$$

where $s+$ is any number greater than s.

Our next goal is to globalize this result using the previous lemma. To do this, we prove the following:

Lemma 14. *Fix* $s \in (0, \frac{1}{2})$*,* $p \leq 6$*, and* $\delta > 0$*. Then there exists a subset* $\overline{\Omega} \subset \Omega_{N,B}$ *such that* $\mu_N(\Omega_{N,B} \setminus \overline{\Omega}) < \delta$ *and for any* $\overline{a} = (a_n)_{|n|\leq N} \in \overline{\Omega}$*, the solution of the initial-value problem*

$$\begin{cases} i\partial_t u + \partial_x^2 u + P_N(|u|^{p-2}u) = 0 \\ u(0,x) = \sum_{|n|\leq N} a_n e^{2\pi i n x} \end{cases} \tag{2.114}$$

satisfies for every $t \in \mathbb{R}$

$$\|u(t)\|_{H^s} \lesssim \left(\log\left(\frac{1+|t|}{\delta}\right)\right)^{s+}$$

where $s+$ *denotes any real number greater than* s*.*

Proof. Fix s, p and δ satisfying the constraints above. For $j \in \mathbb{N}$, define a sequence of times $\{T_j\}_{j\in\mathbb{N}}$ by $T_j = 2^j$ for every j. Then for each j, by Lemma 13 we can find a set $\overline{\Omega}_j$ such that $\mu_N(\Omega_{N,B} \setminus \overline{\Omega}_j) < 2^{-j}\delta$. Define $\overline{\Omega}$ to be simply the intersection of all these sets. The result immediately follows from the the definition of $\overline{\Omega}$ and the second half of the argument for Lemma 13. □

Recall what we set out to show: we know that for any $s > 0$, we can find a local solution to (NLS$_N$) on a time interval of size τ where the time of existence τ is (up to constants) an inverse power of the size of the initial data (smaller data gives longer times of existence). This result combined with invariance of the measure μ_N gives us well-posedness for large time intervals for data taken from large subsets of the canonical ensemble $\Omega_{N,B}$.

With these lemmas in hand, we can now move on to prove the invariance of the limiting measures μ and ρ.

2.2.4. Flow and Invariant Measures for the Nonlinear Schrödinger Equation. In this section, we consider the limits ρ and μ of the measures ρ_N and μ_N defined earlier as $N \to \infty$. In particular, ρ is defined to be the image measure under the map

$$\omega \mapsto \sum_{n \neq 0} \frac{g_n(\omega)}{2\pi n} e^{2\pi i n x}$$

Since a straightforward check shows that the random Fourier series above is an element of $H^s(\mathbb{T})$ almost surely for every $s < \frac{1}{2}$, we can view ρ as a measure on $H_0^{\frac{1}{2}-}(\mathbb{T})$, where we define

$$H^{\frac{1}{2}-} = \bigcap_{0<\sigma<\frac{1}{2}} H^{\sigma}$$

Next, define the measure μ by

$$\mathrm{d}\mu = \left(e^{\frac{1}{p}\|\phi\|_{L^p}^p} \chi_{\{\|\phi\|_{L^2} \leq B\}} \right) \cdot (\mathrm{d}a_0 \otimes \mathrm{d}\rho) \tag{2.115}$$

where B is the L^2 cutoff from above: an arbitrary positive number if $p < 6$ and a specific small constant for $p = 6$, and $a_0 = \hat{\phi}(0)$. Note that by Lemma 12, the density $\frac{\mathrm{d}\mu}{\mathrm{d}\rho}$ is an element of $L^1(\rho)$. In what follows, denote $E_N = \mathrm{span}\{e^{2\pi i n x} \mid |n| \in N_0\}$.

Claim. *Let $U \subset H^{\frac{1}{2}-}$ be open (in the norm topology). Then the following hold:*

$$\rho(U) = \lim_{N\to\infty} \rho_N(U \cap E_N) \tag{2.116}$$

$$\mu(U) = \lim_{N\to\infty} \mu_N(U \cap E_N) \tag{2.117}$$

Proof. Fix $s \in (0, \frac{1}{2})$ and $N \in \mathbb{N}$. Since $H^s(\mathbb{T})$ is a topological vector space, without loss of generality we may take the open set U to be the open ball of radius 1 around 0. By using the definitions of ρ_N and μ_N as pushforward measures and the structure of the pushforward maps F_N and F, the problem reduces to proving that

$$\mathbb{P}[\|F_N(\omega)\|_{H^s(\mathbb{T})} < 1] \to \mathbb{P}[\|F(\omega)\|_{H^s(\mathbb{T})} < 1]$$

However, recall that the Fourier series for $F(\omega)$ converges in $H^s(\mathbb{T})$, so that we have (potentially after taking the Fourier transform),

$$\mathbb{E}[\|F_N(\omega) - \mathcal{F}(\omega)\|_{H^s}] \to 0$$

which then implies

$$\mathbb{E}[\|F_N(\omega)\|_{H^s}] \to \mathbb{E}[\|F(\omega)\|_{H^s}]$$

Using dominated convergence, we conclude the desired result for ρ_N and ρ. To see that μ_N converges to μ in the sense desired, recall that the densities of μ_N and μ are integrable with respect to ρ_N and ρ, respectively. Appealing to dominated convergence yet again, we deduce the desired convergence. □

Next, we want the full equation (NLS) to be well-posed μ almost everywhere. Thankfully, this is the case, as the following lemma shows:

Lemma 15 (Well-posedness of (NLS) on supp(μ))**.** *For any $p \le 6$ and $\delta > 0$, the initial-value problem*

$$\begin{cases} i\partial_t u + \partial_x^2 u + |u|^{p-2}u = 0 \\ u(0,x) = \phi(x) \end{cases} \tag{2.118}$$

is globally well posed and the solution u obeys the bound

$$\|u(t)\|_{H^s} \lesssim \left(\log \frac{1+|t|}{2}\right)^{s+}$$

for any $s < \frac{1}{2}$ and a set of data $\Phi \ni \phi$ where $\mu(H^s(\mathbb{T}) \setminus \Phi) < \delta$.

Proof. We will combine the well-posedness result in Lemma 14 and the approximation result in Lemma 11. To this end, first fix $0 < s_1 < s < \frac{1}{2}$ and $\delta_1 > 0$.

Claim. *The set*

$$\Phi_1 := \left\{ \phi \mid \|\phi\|_{H^s} \lesssim_s \log^{1/2}\left(\frac{1}{\delta_1}\right) \right\}$$

satisfies $\mu(\Phi_1^{\mathsf{c}}) \lesssim \delta_1$.

Proof. (of Claim) First note that Φ_1^{c} is given by $\{\phi \mid \|\phi\|_s \gtrsim_s \log^{1/2}(1/\delta_1)\}$. Using the identity 2.110 (and the following conclusion about μ_N) and the convergence of $\mu_N \to \mu$ proven above, we conclude that

$$\mu(\Phi_1^{\mathsf{c}}) \lesssim e^{-C(\log(\delta_1^{-1})} \lesssim_s \delta_1$$

for a constant $C = C(s)$ chosen appropriately. □

Next, choose $1 \ll N \in \mathbb{N}$ and apply Lemma 14 with s, δ_1, p as above). This gives us a set $\mathcal{O}_N$ such that $\mu_N(\Omega_{N,B} \setminus \mathcal{O}_N) < \delta_1^2$. Further, for $(a_n)_{|n|\le N} \in \mathcal{O}_N$, we know that the solution $v = P_N u$ of (NLS$_N$) satisfies

$$\|v(t)\|_{H^s} \lesssim \log^{\frac{1}{2}}\left(\frac{1+|t|}{2}\right)$$

Next, define the set $\Phi_2 = \{\phi \in H^s(\mathbb{T}) \mid (\widehat{\phi}(n))_{|n|\leq N} \in \mathcal{O}_N\}$. We then have the inclusion

$$\{\phi \mid \|\phi\|_{L^2} \leq B \text{ and } \phi \in \mathcal{O}_N^{\mathsf{c}}\} \subset \{a \in \mathbb{C}^{2N+1} \,|\, \|a\|_{L^2} \leq B \text{ and } a \in \mathcal{O}_N^{\mathsf{c}}\}$$

In particular, by subadditivity of measure and the fact that $\frac{\mathrm{d}\mu_N}{\mathrm{d}\rho_N}$ is positive,

$$\begin{aligned}
&(\mathrm{d}a_0 \otimes \rho)\left(\{\phi \mid \|\phi\|_{L^2} \leq B \text{ and } \phi \in \mathcal{O}_N^{\mathsf{c}}\}\right) \\
&\leq (\mathrm{d}a_0 \otimes \rho_N)\left(\{a \in \mathbb{C}^{2N+1} \,|\, \|a\|_{L^2} \leq B \text{ and } a \in \mathcal{O}_N^{\mathsf{c}}\}\right) \\
&\leq \mu_N(\Omega_{N,B} \setminus \mathcal{O}_N) \lesssim \delta_1^2
\end{aligned}$$

This in turn implies

$$\mu(\Phi_2^{\mathsf{c}}) < \delta_1$$

by the convergence of $\mu_N \to \mu$ and the fact that the Radon-Nikodym derivative $\frac{\mathrm{d}\mu}{\mathrm{d}\rho} \in L^2(\mathrm{d}a_0 \otimes \mathrm{d}\rho)$. In particular, this tells that for all but an exceptional set of μ-measure at most δ_1, we can ensure that

$$\|\phi\|_{H^s} \lesssim \log^{\frac{1}{2}}\left(\frac{1+|t|}{2}\right)$$

and

$$\|v\|_{H^s} \lesssim \log^{\frac{1}{2}}\left(\frac{1+|t|}{2}\right)$$

for v a solution to (NLS_N) with datum

$$v(0,x) = \sum_{|n|\leq N} \widehat{\phi}(n) e^{2\pi i n x}.$$

With this in hand, we apply Lemma 11 with $A = C\log^{\frac{1}{2}}\left(\frac{T}{\delta_1}\right)$. Then there exists a set $\overline{\Phi}_{s_1,T} \subset \Phi_2$ such that for $\phi \in \overline{\Phi}_{s_1,T}$, we have that the equation

$$\begin{cases} i\partial_t u + \partial_x^2 u + |u|^{p-2}u = 0 \\ u(0,x) = \phi \end{cases} \tag{2.119}$$

is well-posed on $[0,T]$, and $\|u(t) - v(t)\|_{H^{s_1}} \lesssim N^{s_1 - s}\exp 1 + A)_*^C T$. In particular,

$$\|u(t)\|_{H^{s_1}} \lesssim 2A \lesssim \log^{\frac{1}{2}}\left(\frac{T}{\delta_1}\right)$$

for $|t| < T$. Note that the set $\overline{\Phi}$ above then depends on both T and $s_1 < s$. Intersecting these sets $\overline{\Phi}$ for an increasing sequence of times T_j, we are able to globalize the above result in H^{s_1}:

$$\|u(t)\|_{H^{s_1}} \lesssim_{s_1} \log^{1/2}\left(\frac{1+|t|}{2}\right)$$

for every time $t \in \mathbb{R}$. Finally, intersecting along a sequence of regularities $\{s_i < s\}$, we obtain the estimate above at regularity $s = \frac{1}{2}-$, as desired. □

Remark. Note that one can replace the power of 1/2 in the logarithm by $s+$ for any $s < 1/2$ by interpolation with the conservation of mass estimate, as in the previous results of this type. Further, we can ensure that Φ is compact in H^s by the local theory and dominated convergence.

To recap, this lemma shows that up to an exceptional set of μ-measure zero, (NLS) is globally well-posed on the support of μ. We can now turn to the final result: invariance of the measure μ under the dynamics of the equation.

Theorem 16 (Invariance of μ under NLS dynamics)**.** *The measure μ is invariant under the dynamics of* (NLS) *for $p \in [4,6]$.*

Proof. We know from the previous lemma that (NLS) is globally well-posed on a compact set Φ in $H^s(\mathbb{T})$. This set is simply the union of the compact sets

$$K_{\delta,A} = \left\{\phi \in H^s \mid \|\phi\|_{L^2} \leq B \text{ and } \|u(t)\|_{H^s} \leq_s A \log^{s+}\left(\frac{1+|t|}{2}\right),\ \forall t \in \mathbb{R}, s < \frac{1}{2}\right\}$$

where the union runs over all $\delta > 0$ and $A < \infty$, and $u(t)$ is the solution of (NLS) with datum ϕ.

First note that the flow of (NLS) maps Φ into itself, and that we want to show that μ is invariant on Φ. Set S_t to be the map $u_0 = \phi \mapsto u(t)$ for some $t \in \mathbb{R}$ and let $K \subset \Phi$ be compact. Let $s \in (0, 1/2)$, and denote by B_r the ball of radius r in H^s-norm. Further let $S_{N,t}$ to be the finite-dimensional flow of (NLS_N) on the ensemble $\Omega_{N,B} \subset E_N$, which we considered earlier.

Due to the H^s-regularity of the map $S_{N,t}$ and the approximation lemma Lemma 11 for $S_{N,t}$ and S_t, we have for any $\varepsilon > 0$ and $N > N_0$ sufficiently large,

$$S_{N,t}((K + B_\varepsilon) \cap E_N) \subset S_{N,t}(P_N(K)) + B_{\varepsilon/2} \subset S_t(K) + B_\varepsilon.$$

Using the limit characterization of μ in terms of μ and the above result,

$$\begin{aligned}\mu(S_t(K) + B_\varepsilon) &= \lim_{N\to\infty} \mu_N((S_K + B_\varepsilon) \cap E_N) \\ &\geq \liminf_{N\to\infty} \mu_N(S_{N,t}((K + B_\varepsilon) \cap E_N)).\end{aligned}$$

However, since μ_N is invariant under $S_{N,t}$ by Liouville's theorem,

$$\begin{aligned}\mu(S_t(K) + B_\varepsilon) &\geq \lim_{N\to\infty} \mu_N(K + (B_\varepsilon) \cap E_N) \\ &= \mu(K + B_\varepsilon) \geq \mu(K)\end{aligned}$$

In particular, $\mu(S_t(K)) \geq \mu(K)$. By reversibility of the NLS flow, the other direction follows immediately, and thus $\mu(S_t(K)) = \mu(K)$, establishing the invariance of μ under the flow. $\square$

3. SUMMARY AND CONCLUSIONS

To conclude the thesis, we present a simple overview of the main results in each section.

In the first paper, we study an averaged dispersion-managed nonlinear Schrödinger equation. The averaging allows us to shift the dipersion-management from the underlying linear equation into the nonlinear term, which in our case is the cubic nonlinearity $F(u) = |u|^2u$. In particular, this averaging allows us to leverage many standard tools from the study of power-type NLS, and we thus establish a well-posedness and modified scattering result using techniques adapted from [16] and [21].

In the second paper, we consider the full dispersion-managed model without averaging. In this regime, the dispersion-managment is built into the underlying linear Schrödinger equation, which at face value disallows many of the standard techniques for establishing well-posedness; in particular, Strichartz estimates no longer apply 'out of the box'. Imposing certain physically relevant conditions on the dispersion map $\gamma(t)$, we are able to establish a set of nonendpoint Strichartz estimates for this model. Using these, we establish a global well-posedness and scattering result for the $3d$ cubic DMNLS with $H^1(\mathbb{R}^3)$ data. However, we also exhibit solutions that blow up in finite time.

The remainder of the thesis concludes with two unrelated results. The first result is a modified scattering result for the dispersion-managed NLS in the same spirit as in the first paper. This is established in much the same way as [16], but was previously impossible without the global-in-time Strichartz estimates introduced in the second paper.

The final section is an exposition of Bourgain's 1994 result establishing invariance of the Gibbs measure associated to the $1d$ mass-subcritical NLS on the torus $\mathbb{T}$. We fill in many of the details that are omitted in Bourgain's original work, and refactor some of the notation for the contemporary reader.

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VITA

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