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The spinning top

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THE SPINNING TOP

BY

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INTRODUCTION

Several mathematicians have solved the problems of motion of the top and gyroscope most completely, but none of them have considered in their solutions the effects of the supporting gimbal rings upon the motion or the effects of a variable rotor speed.

It is the purpose of this paper to investigate the top equations by two well known methods; namely, by the method of Lagrange and by the method of Jacobi; considering in both the dynamics of the gimbal rings and the varying rotor speed.

Fig.1. THE GYROPENDULUM.

APPARATUS

The apparatus considered in this paper is the gyropendulum; a heavy symetrical top mounted diametrically on pivots in a ring. This ring is mounted concentrically in another ring on pivots at right angles to the axis of the top, and this second ring is mounted on pivots at right angles to the pivots of the first ring: the last named pivots being vertical. The center of gravity of the top is on its axis some distance from the center of the rings.

A gyropendulum was built in the summer of 1930 in order to observe better its movements and to determine the friction of its pivots, and with the idea in mind of possibly using it in geophysical prospecting. Its rotor consists of a brass sphere 11.431 cm. in diameter on an axis 1.270 x 14.920 cm. The rings were made of .635 x 1.270 cm. stock, their mean diameter being 17.78 cm. and 20.32 cm. The pivots were small ball bearings.

The Symbols:

$A =$ The physical moment of inertia of the sphere and axle about the center line of the axle.

$B =$ The physical moment of inertia of the sphere and axle about the line through the center of the instrument perpendicular to the axle of the sphere.

$C =$ The physical moment of inertia of the outer ring about a diameter.

$D =$ The physical moment of inertia of the inner ring about a diameter.

$\mathcal{E} =$ The physical moment of inertia of the inner ring about its axis.

$\mathring{\zeta} \equiv \frac{d\zeta}{dt} =$ The angular velocity of the sphere about its axis.

$\mathring{\phi} \equiv \frac{d\phi}{dt} =$ The rate of precession of the sphere about the vertical.

$\mathring{\theta} \equiv \frac{d\theta}{dt} =$ The angular velocity of the sphere about the horizontal and through the axle and perpendicular to it.

$\phi =$ The azmuth distance of the axle of the sphere measured from the north toward the east.

$\theta =$ The zenith distance of the axis of the sphere.

$t =$ The time in seconds.

$m =$ Mass of the rotor and axle in grams.

$T =$ The kinetic energy of the moving parts in ergs.

g = The acceleration of gravity in cm/sec.

g' = The local value of the acceleration of gravity.

$\mathcal{D} = \mathcal{B} + \mathcal{D}$,

$F = \mathcal{B} + \mathcal{C} + \mathcal{E}$,

$K = \mathcal{C} + \mathcal{D}$,

W = The work in ergs.

Θ = The moment tending to increase θ.

Φ = The moment tending to increase ϕ.

Z = The moment tending to increase z.

$\ddot{\theta} = \dfrac{d^2\theta}{dt^2}$,

$\ddot{\phi} = \dfrac{d^2\phi}{dt^2}$,

$\ddot{z} = \dfrac{d^2 z}{dt^2}$,

h = A constant.

$\bar{w}_r$ = Weight of rotor and axle in dynes.

R = A moment.

l = Length of string wound on rotor in cm.

$\dot{u}$ = Radians per second of the rotor at any instant during the fall of the weight.

A = 82,978.4610 gr. cm., physical moment of inertia of rotor about its axis.

I = 262.2195 gr. cm., physical moment of inertia of the pulley.

a = .724 cm., radius of rotor drum to center of string.

b = 2.469 cm. radius of the pulley.

u = The angle of the rotor turns through counted from the beginning of the fall of the weight up to any instant before the string is free from the rotor.

ζ = The angle of the rotor turns through counted from the fall of the weight.

$\tilde{w}$ = Weight in dynes of the falling weight.

v = Velocity in cm/sec.

h = The distance from the center of the gimbal rings to the center of gravity of the rotor.

p = The momenta.

$-U$ = The potential.

$$S = \int_{t_0}^{t} (T+U)\,dt.$$

$N, K, \beta, \lambda, a_0\ a_1\ a_2\ J.$ constants.

The Top Equations by the Method of Lagrange.

From figure one we have the kinetic energy of all the moving parts is

$$T = \tfrac{1}{2}\Big[A(\dot{z}+\dot{\phi}\cos\theta)^2 + B(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta) + C\dot{\phi}^2 + D(\dot{\theta}^2+\dot{\phi}^2\cos^2\theta)$$
$$+ E\dot{\phi}^2\sin^2\theta\Big] \quad \text{-----------} \quad 1.$$
$$= \tfrac{1}{2}\Big[A(\dot{z}+\dot{\phi}\cos^2\theta) + (B+D)\dot{\theta}^2 + (B+C+E)\dot{\phi}^2\sin^2\theta$$
$$+ (C+D)\dot{\phi}^2\cos^2\theta\Big] \quad \text{-----------} \quad 2.$$
$$= \tfrac{1}{2}\Big[A(\dot{z}+\dot{\phi}\cos\theta) + G\dot{\theta}^2 + F\dot{\phi}^2\sin^2\theta + K\dot{\phi}^2\cos^2\theta\Big] \quad \text{---} \quad 3.$$

By the equations of Lagrange the work done by the effective forces when the coordinate θ is changed by an infinitesimal amount $\delta\theta$ without changing the others, z and ϕ is

$$\delta_\theta W = \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta}\right]\delta\theta = \Theta\,\delta\theta \quad \text{------} \quad 4.$$

and similarly

$$\delta_\phi W = \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi}\right]\delta\phi = \Phi\,\delta\phi \quad \text{------} \quad 5.$$

$$\delta_z W = \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z}\right]\delta z = Z\,\delta z \quad \text{------} \quad 6.$$

From the three above equations it follows that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) - \frac{\partial T}{\partial \theta} = \Theta. \quad\text{-----}\quad 7.$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\phi}\right) - \frac{\partial T}{\partial \phi} = \Phi. \quad\text{-----}\quad 8.$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot z}\right) - \frac{\partial T}{\partial z} = Z. \quad\text{-----}\quad 9.$$

Since θ, ϕ and z are angles and their differentials multiplied by Θ, Φ and Z respectively give the differential of work, Θ, Φ and Z must be the moments tending to increase these angles.

From equation (3) we have

$$\frac{\partial T}{\partial \theta} = -A(\dot z + \dot\phi\cos\theta)\dot\phi\sin\theta + F\dot\phi^2\sin\theta\cos\theta - K\dot\phi^2\sin\theta\cos\theta$$

$$= -A(\dot z + \dot\phi\cos\theta)\dot\phi\sin\theta + (F-K)\dot\phi^2\sin\theta\cos\theta \quad\text{----}\quad 10.$$

$$\frac{\partial T}{\partial \dot\theta} = \mathcal{J}\dot\theta. \quad\text{-----}\quad 11.$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) = \mathcal{J}\ddot\theta. \quad\text{-----}\quad 12$$

Substituting (10) and (12) in (7) we have

$$\mathcal{J}\ddot\theta + A(\dot z + \dot\phi\cos\theta)\dot\phi\sin\theta + (K-C)\dot\phi^2\sin\theta\cos\theta = \Theta \quad\text{---}\quad 13$$

From equation (3) we have

$$\frac{\partial T}{\partial \phi} = 0. \quad\text{-----}\quad 14.$$

$$\frac{\partial T}{\partial \dot\phi} = A(\dot\zeta + \dot\phi\cos\theta)\cos\theta + F\dot\phi\sin^2\theta + K\dot\phi\cos^2\theta$$

$$= A\dot\zeta\cos\theta + (A - F + K)\dot\phi\cos^2\theta + F\dot\phi \text{ - - - - } 15.$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\phi}\right) = A\ddot\zeta\cos\theta - A\dot\zeta\dot\theta\sin\theta + (A - F + K)\ddot\phi\cos^2\theta$$

$$-2\dot\phi\dot\theta(A - F + K)\sin\theta\cos\theta + F\ddot\phi \text{ - - - - } 16.$$

Substituting (14) and (16) in (8) we have

$$A\ddot\zeta\cos\theta - A\dot\zeta\dot\theta\sin\theta + (A - F + K)\ddot\phi\cos^2\theta$$

$$-2\dot\phi\dot\theta(A - F + K)\sin\theta\cos\theta + F\ddot\phi = \Phi \text{ - - - - } 17.$$

Also from (3) we have

$$\frac{\partial T}{\partial \zeta} = 0. \text{ - - - - - - - - - - } 18$$

$$\frac{\partial T}{\partial \dot\zeta} = A(\dot\zeta + \dot\phi\cos\theta) \text{ - - - - - - - - - - } 19.$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\zeta}\right) = A\ddot\zeta + A\ddot\phi\cos\theta - A\dot\phi\dot\theta\sin\theta \text{ - - - - - - } 20.$$

Substituting (18) and (20) in (9) we have

$$A\ddot\zeta + A\ddot\phi\cos\theta - A\dot\zeta\dot\phi\sin\theta = Z \text{ - - - - - } 22.$$

Multiplying equation (13) by $\dot\theta$

Multiplying equation (17) by $\dot\phi$ and

Multiplying equation (21) by $\dot z$ and adding we have

$$\Big[\mathcal{J}\ddot\theta\dot\theta + A(\dot z+\dot\phi\cos\theta)\dot\phi\dot\theta\sin\theta + (K-F)\dot\phi^2\dot\theta\sin\theta\cos\theta$$

$$+A\ddot z\dot\phi\cos\theta - A\dot z\dot\theta\dot\phi\sin\theta + (A-F+K)\ddot\phi\dot\phi\cos^2\theta$$

$$-2\dot\phi^2\dot\theta(A-F+K)\sin\theta\cos\theta + F\ddot\phi\dot\phi + A\ddot z\dot z$$

$$+A\ddot\phi\dot z\cos\theta - A\dot\phi\dot\theta\dot z\sin\theta\Big] = \Theta\dot\theta + \Phi\dot\phi + Z\dot z \quad - 22$$

Rearranging (22) we have

$$\Big[\mathcal{J}\dot\theta\frac{d\dot\theta}{dt} + F\dot\phi\frac{d\dot\phi}{dt} + A\dot z\frac{d\dot z}{dt} - (A-F+K)\dot\phi^2\sin\theta\cos\theta\frac{d\theta}{dt}$$

$$+(A-F+K)\cos^2\theta\dot\phi\frac{d\dot\phi}{dt} + A\Big(\dot\phi\cos\theta\frac{d\dot z}{dt} + \dot z\cos\theta\frac{d\dot\phi}{dt}$$

$$-\dot\phi\dot z\sin\theta\frac{d\theta}{dt}\Big)\Big] = \Theta\frac{d\theta}{dt} + \Phi\frac{d\phi}{dt} + Z\frac{dz}{dt} \quad ---- 23.$$

or

$$\Big[d\Big(\frac{\mathcal{J}\dot\theta^2+F\dot\phi^2+A\dot z^2}{2}\Big) + (A-F+K)\frac{\dot\phi^2 d(\cos^2\theta)+\cos^2\theta\, d(\dot\phi^2)}{2}$$

$$+A\Big(\dot\phi\cos\theta\, d\dot z + \dot z\cos\theta\, d\dot\phi + \dot\phi\dot z\, d(\cos\theta)\Big)\Big] =$$

$$= \Theta\, d\theta + \Phi\, d\phi + Z\, dz. \quad ------------ 24$$

$$\tfrac{1}{2}d\Big[\mathcal{J}\dot\theta^2 + F\dot\phi^2 + A\dot z^2 + (A-F+K)\dot\phi^2\cos^2\theta + 2A\dot\phi\dot z\cos\theta\Big]$$

$$= \Theta\, d\theta + \Phi\, d\phi + Z\, dz \quad ------------ 25$$

Integrating (25)

$$\tfrac{1}{2}\left[\mathcal{G}\dot{\theta}^2 + F\dot{\phi}^2 + A\dot{z}^2 + (A-F+K)\dot{\phi}^2\cos^2\theta + 2A\dot{\phi}\dot{z}\cos\theta\right.$$

$$= \int \Theta\, d\theta + \int \Phi\, d\phi + \int Z\, dz + h \text{ - - - - - - } 26$$

or

$$\tfrac{1}{2}\left[A(\dot{z}^2 + 2\dot{\phi}\dot{z}\cos\theta + \dot{\phi}^2\cos^2\theta) + \mathcal{G}\dot{\theta}^2 + F\dot{\phi}^2\sin^2\theta\right.$$

$$\left. + K\dot{\phi}^2\cos^2\theta\right] =$$

$$= \tfrac{1}{2}\left[A(\dot{z} + \dot{\phi}\cos\theta)^2 + \mathcal{G}\dot{\theta}^2 + F\dot{\phi}^2\sin^2\theta + K\dot{\phi}^2\cos^2\theta\right]$$

$$= T \text{ - - - - - - - - - - } 27.$$

Hence

$$T - \int \Theta\, d\theta - \int \Phi\, d\phi - \int Z\, dz = h \text{ - - - - - } 28.$$

Calculation of the moments and the potential.

The forces acting on the gyropendulum at any given set-up are constant. By leveling the base of the instrument the axis of the outer ring is set normal to the level surface at the set-up.

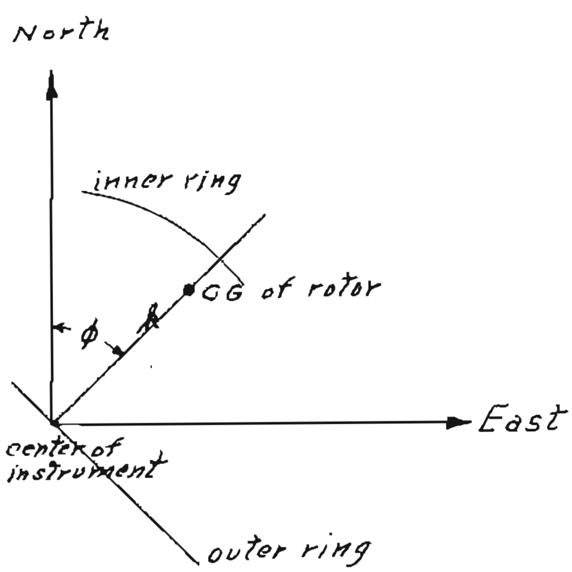

Fig. 2

Line Diagram Of Instrument

Plan View

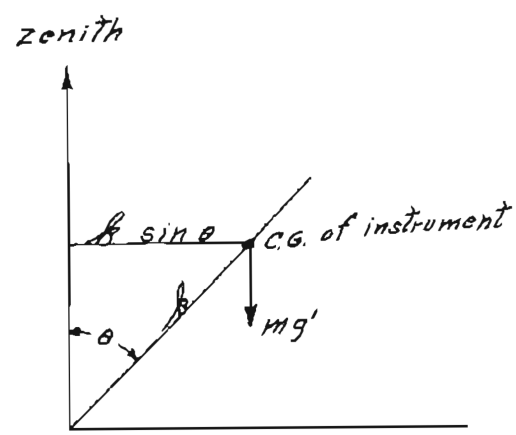

Fig. 3

Line Diagram Of Instrument

Section through vertical and C.G. of rotor

$$\Theta = mgR\sin\theta - R_\theta \text{ ——— } 29$$

Where R_θ is the resisting moment due to the friction in the bearings supporting the inner ring. It depends on the radius of the bearing and on the coeffecient of friction of the bearing.

$$\Phi = -R_\phi \text{ ——— } 30.$$

Where R_ϕ is the resisting moment due to the friction in the bearings supporting the inner ring. It depends on the radius of the bearing and on the coeffecient of friction of the bearing.

$$Z = A\frac{d\dot{z}}{dt} \text{ ——— } 31.$$

This resisting moment cannot be taken constant as could the other two because of the windage at the high speed of the rotor. By experiment $\dot{z}$ was found to be equal to a second degree function of the time, in seconds from the start of the rotor, or

$$\frac{dz}{dt} = \dot{z} = \dot{z}_0 + ct + et^2 \text{ ——— } 32.$$

$$dz = (\dot{z}_0 + ct + et^2)dt \text{ ——— } 33$$

$$\frac{d\dot{z}}{dt} = (c + 2et) \text{ ——— } 34.$$

$$\therefore Z = A(c + 2et) \text{ ——— } 35$$

Substituting these values of Θ, Φ and Z in equation (28).

$$T = \int (mg'k\sin\theta - R_\theta)\,d\theta - \int (-R_\varphi)\,d\varphi - \int A(c+2\epsilon t)(\dot{z}_0 + ct + \epsilon t^2)^2\,dt$$

$$= h \qquad\qquad 36$$

$$T + mg'k\cos\theta + R_\theta\theta + R_\varphi\varphi - \frac{A}{2}(\dot{z}_0 + ct + \epsilon t^2)^2 = h \qquad 37.$$

If we start from the vertical position when

$$\theta = \varphi = z = t = 0 \qquad\qquad 38$$

$$T_0 + mg'k - \frac{A\dot{z}_0^2}{2} = h \qquad\qquad 39$$

$$\text{but}\quad T_0 = \frac{A\dot{z}_0^2}{2}$$

(39) becomes

$$mg'k = h \qquad\qquad 40.$$

Thus giving the constant of integration of equation (26) and (37) becomes

$$T + mg'k(\cos\theta - 1) + R_\theta\theta + R_\varphi\varphi - \frac{A}{2}(\dot{z}_0 + ct + \epsilon t^2)^2 = 0 \qquad 41.$$

Substituting the values of Θ, Φ and Z from equations (29), (30) and (31) in equations (13), (17) and (21) we have

$$K\ddot{\theta} + A(\dot{z} + \dot{\varphi}\cos\theta)\dot{\varphi}\sin\theta + (K - F)\dot{\varphi}^2\sin\theta\cos\theta = mg'k\sin\theta - R_\theta \qquad 42$$

$$A\ddot{z}\cos\theta - A\dot{z}\dot{\theta}\sin\theta + (A - F + K)\ddot{\varphi}\cos^2\theta - 2\dot{\varphi}\dot{\theta}(A - F + K)\sin\theta\cos\theta$$

$$+ F\ddot{\varphi} = -R_\varphi \qquad\qquad 43$$

$$A(\ddot{z} - \ddot{\varphi}\cos\theta - \dot{\varphi}\dot{\theta}\sin\theta) = A\frac{d\dot{z}}{dt} \qquad\qquad 44$$

These equations cannot be solved.

The Top Equations by the Method of Jacobi.

We have for the kinetic energy.

$$T=\frac{1}{2}\left[A(\dot{z}+\dot{\phi}\cos\theta)^2+B(\dot{\theta}^2+\dot{\phi}^2\cos^2\theta)+D(\dot{\theta}^2+\dot{\phi}^2\cos^2\theta)+E\dot{\phi}^2\sin^2\theta\right] \quad \text{--- 46.}$$

The moments about the respective axis of θ, ϕ and z are

$$p_\theta=\frac{\partial T}{\partial\dot{\theta}}=B\dot{\theta}+D\dot{\theta}=(B+D)\dot{\theta} \quad \text{--- 47}$$

$$p_\phi=\frac{\partial T}{\partial\dot{\phi}}=A(\dot{z}+\dot{\phi}\cos\theta)\cos\theta+B\dot{\phi}\sin^2\theta+C\dot{\phi}+D\dot{\phi}\cos^2\theta+E\dot{\phi}\sin^2\theta \quad \text{--- 48}$$

$$p_z=\frac{\partial T}{\partial\dot{z}}=A(\dot{z}+\dot{\phi}\cos\theta) \quad \text{--- 49}$$

$$\dot{\theta}=\frac{p_\theta}{B+D} \quad \text{--- 50,}$$

$$\dot{\phi}=\frac{p_\phi-p_z\cos\theta}{B\sin^2\theta+C+D\cos^2\theta+E\sin^2\theta} \quad \text{--- 51}$$

$$\dot{z}=\frac{p_z}{A}-\frac{\cos\theta(p_\phi-p_z\cos\theta)}{B\sin^2\theta+C+D\cos^2\theta+E\sin^2\theta} \quad \text{--- 52}$$

The energy then becomes the Hamiltonian Function.

$$H=T-U \quad \text{--- 53}$$

$$H=\frac{1}{2}\left[\frac{p_z^2}{A}+\frac{(p_\phi-p_z\cos\theta)^2}{B\sin^2\theta+C+D\cos^2\theta+E\sin^2\theta}+\frac{p_\theta^2}{B+D}\right]+mgR\cos\theta \quad \text{-- 54}$$

Let

$$S = \int_{t_0}^{t} (T+U)\,dt \quad \text{------} \quad 55$$

$$\frac{dS}{dt} = T+U \quad \text{------} \quad 56$$

$$= \frac{\partial S}{\partial t} + \frac{\partial S}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial S}{\partial \varphi}\frac{d\varphi}{dt} + \frac{\partial S}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial S}{\partial t} + 2T \quad \text{by Euler's Theorem.} \quad \text{---} \quad 57$$

Since U does not contain $\dot\theta, \dot\varphi$ or $\dot z$ and

$$P_\theta = \frac{\partial T}{\partial \dot\theta} = \frac{\partial}{\partial \dot\theta}\left(\frac{\partial S}{\partial t}\right) = \frac{\partial S}{\partial \theta} \quad \text{------} \quad 58,$$

$$\text{and } P_\varphi = \frac{\partial S}{\partial \varphi} \quad \text{------} \quad 59$$

$$P_z = \frac{\partial S}{\partial z} \quad \text{------} \quad 60$$

Since $H = T - U$ the sum of the energies and from (11)

$$T + U = \frac{\partial S}{\partial t} + 2T$$

$$\frac{\partial S}{\partial t} + T - U = 0 = \frac{\partial S}{\partial t} + H = 0 \quad \text{------} \quad 61$$

Substituting the results of (58),(59),(60) in (54),(61) becomes

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left[\frac{1}{A}\left(\frac{\partial S}{\partial z}\right)^2 + \frac{\left(\frac{\partial S}{\partial \varphi} - \frac{\partial S}{\partial z}\cos\theta\right)^2}{B\sin^2\theta + C + D\cos^2\theta + B\sin^2\theta} + \frac{\left(\frac{\partial S}{\partial \theta}\right)}{B + D}\right]$$

$$+ Mg'k\cos\theta = 0. \quad \text{------} \quad 62$$

Try

$$S = -\alpha t + (B + D)\left[\lambda z + \beta\varphi + f(\theta)\right] \qquad 63$$

as a solution where $f(\theta)$ is a function of θ, where α, λ and β are constant.

$$\left.\begin{aligned}
\frac{\partial S}{\partial t} &= -\alpha \\
\frac{\partial S}{\partial z} &= \lambda(B + D) \\
\frac{\partial S}{\partial \varphi} &= \beta(B + D) \\
\frac{\partial S}{\partial \theta} &= (B + D) f'(\theta)
\end{aligned}\right\} \qquad 64$$

Inserting these values of the derivities of S in (62)

$$-\alpha + \frac{1}{2}\left[\frac{\lambda^2(B+D)^2}{A} + \frac{\{\beta(B+D) - \lambda\cos\theta(B+D)\}^2}{B\sin^2\theta + C + D\cos^2\theta + E\sin^2\theta} + \frac{(B+D)^2\{f'(\theta)\}^2}{B+D}\right] + mg'h\cos\theta = 0 \qquad 65$$

or

$$-\alpha + \frac{(B+D)^2}{2}\left[\frac{\lambda^2}{A} + \frac{(\beta - \lambda\cos\theta)^2}{B\sin^2\theta + C + D\cos^2\theta + E\sin^2\theta} + \frac{\{f'(\theta)\}^2}{B+D}\right] + mg'h\cos\theta = 0. \qquad 66$$

Solving this for

$$\{f'(\theta)\}^2 = \frac{2\alpha}{B+D} - \frac{(B+D)\lambda^2}{A} - \frac{(B+D)(\beta - \lambda\cos\theta)^2}{(B - D + E)\sin^2\theta + C + D} - \frac{2mg'h\cos\theta}{B+D} = F(\theta) \qquad 67$$

$$f(\theta) = \int \sqrt{F\theta}\, d\theta \quad \text{------------------} \quad 68$$

Differentiating S with respect to the constants α, β and λ

$$\frac{\partial S}{\partial \alpha} = -t + \int \frac{d\theta}{\sqrt{F(\theta)}} = c_1 \quad \text{------------------} \quad 69$$

$$\frac{\partial S}{\partial \beta} = (B+D)\left[\varphi - \int \frac{(\beta - \lambda\cos\theta)\, d\theta}{\sqrt{F(\theta)}\left[(B-D+E)\sin^2\theta + (C+D)\right]}\right]$$

$$= c_2 \quad \text{------------------} \quad 70$$

$$\frac{\partial S}{\partial \lambda} = (B+D)\left[\bar{z} + (B+D)\int\left[\frac{(\beta-\lambda\cos\theta)\cos\theta}{(B-D+E)\sin^2\theta + (C+D)} - \frac{\lambda}{A}\right]\frac{d\theta}{\sqrt{F(\theta)}}\right]$$

$$= c_3 \quad \text{------------------} \quad 71.$$

From (58), (59) and (60)

$$\frac{\partial S}{\partial \theta} = P_\theta = (B+D)f'(\theta) = (B+D)\sqrt{F\theta} \quad \text{------------------} \quad 72.$$

$$\frac{\partial S}{\partial z} = P_z = A(\dot{z} + \dot{\varphi}\cos\theta) = (B+D)\lambda \quad \text{or}$$

$$\lambda = \frac{A(\dot{z} + \dot{\varphi}\cos\theta)}{B+D} \quad \text{------------------} \quad 73$$

If there is no friction then by the Lagrange's Equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z} = 0 \quad \text{------------------} \quad 74$$

$$\frac{d}{dt}\left[A(\dot{\zeta}+\dot{\varphi}\cos\theta)\right] = 0$$

$$\dot{\zeta}+\dot{\varphi}\cos\theta = N \quad \text{a constant} \qquad (75)$$

hence

$$\lambda = \frac{AN}{B+b} \qquad (76)$$

From (62)

$$\frac{\partial S}{\partial t} + H = 0$$

$$\lambda = H = T - U \quad \text{the total energy} \qquad (77)$$

$$\frac{\partial S}{\partial \varphi} = P_\varphi = AN\cos\theta + (B - D + E)\dot{\varphi}\sin^2\theta + (C+D)\dot{\varphi}$$

$$= (B+b)\beta$$

$$\beta = \frac{1}{B+b}\left\{AN\cos\theta + \left[(B+E-D)\sin^2\theta + (C+D)\right]\dot{\varphi}\right\}$$

$$= \frac{P_\varphi}{B+b} \qquad (78)$$

Equations (76), (77) and (78) give the constants α, λ, and β when observed data is substituted in the equations. Equations (70), (71) and (72) do not lend themselves to any method of solution. Jacobi was able to reduce them to elliptic integrals by neglecting the gimbal rings.

The Speed-Time Curve of the Rotor Determined by Experiment as a Function of Time.

For this experiment the gimbal rings were clamped in a fixed position. 130 cm. of string were wound on the shaft of the rotor and run over a pulley and a weight was suspended from the other end of the string. The time t_1 was observed from the beginning of the fall of the weight to the instant the string came free from the rotor, then the time t_2 was recorded, the time taken by friction to consume the energy of the rotor.

$$\left(A + I\frac{a^2}{b^2} + \frac{\bar{w}a^2}{g}\right)\frac{d^2u}{dt^2} = \bar{w}g = J\frac{d^2u}{dt^2}$$

$$\frac{d\tilde{u}}{dt^2} = \bar{w}a$$

$$\frac{du}{dt} = \bar{w}at + c \qquad t = 0 \text{ when } \frac{du}{dt} = 0 \quad \therefore c = 0$$

c/130

$$\tfrac{1}{2}\left(A\frac{v^2}{a^2} + I\frac{v^2}{b^2} + \frac{\bar{w}v^2}{g}\right) = \bar{w}l. \quad \text{If there is no friction.}$$

$$v = \sqrt{\frac{2\bar{w}l}{A/a^2 + I/b^2 + \frac{\bar{w}}{g}}}$$

$$u_0 = \dot{\zeta}_0 = \frac{v}{a} = \sqrt{\frac{2wl}{A + I\frac{a^2}{b^2} + \frac{\bar{w}a^2}{g}}}$$

$$\dot{u}_0 = \sqrt{2lg}\sqrt{\frac{\bar{w}/g}{A + I\frac{a^2}{b^2} + \frac{\bar{w}a^2}{g}}}$$

$$= \sqrt{2\times 130 \times 979.8}\sqrt{\frac{\bar{w}/g}{82,978 + 22 + .525\frac{\bar{w}}{g}}}$$

$$= 503\sqrt{\frac{\bar{w}/g}{83,000 + .525\frac{\bar{w}}{g}}}$$

$$\dot{u}_0 = 503\sqrt{\frac{935}{83490}} = 53.2 \text{ rad/sec} \quad \text{when } \frac{\bar{w}}{g} = 935 \text{ gr.}$$

Other values of $\dot{u}_0$ were found for different values of $\bar{w}/g$ to be

$$\dot{u}_0 = 76.3 \text{ rad/sec} \quad \text{when } \frac{\bar{w}}{g} = 1935 \text{ gr.}$$

$$\dot{u}_0 = 93.5 \quad \text{''} \quad \text{''} \quad \frac{w}{g} = 2935 \text{ gr.}$$

$$\dot{u}_0 = 132.0 \quad \text{''} \quad \text{''} \quad \frac{w}{g} = 5935.$$

Since $\dot{u}_0 = \dot{\bar{z}}_0$ when $t = 0$ assume

$$\dot{z} = a_0 + a_1 t + a_2 t^2$$

If we make $\dot{z}_0 = 0$ when $t = 900$ and from the data

$$132.0 = a_0 + 11a_1 + 121a_2$$

$$76.3 = a_0 + 352a_1 + 123904a_2$$

$$0 = a_0 + 900a_1 + 810000a_2$$

Solving simultaneously which gives

$$\dot{z} = 132.908 - .17306t + .0000027t^2$$

Determination of the resisting torque on the rotor.

If $\mathcal{R}$ is a resisting torque

$$\mathcal{R} = \frac{1}{\dot{z}}\frac{d}{dt}\left(\frac{1}{2}A\dot{z}^2\right) = \frac{1}{\dot{z}}\cdot\frac{1}{2}A\,2\dot{z}\frac{d\dot{z}}{dt} = A\frac{d\dot{z}}{dt}$$

$$= A\frac{d}{dt}\left(133.908 - .17306t + .000027t^2\right)$$

$$= A(-.17306 + .000054t) \text{ dynes centemeters}$$

resisting torque.

$\mathcal{R}_{100} = 82,978(-.17306 + .0054) = 13,900$ dyne-cm, when $t = 100$

$\mathcal{R}_0 = 82,978(-.17306) = 14,306$ " " " $t = 0$

$\mathcal{R}_{200} = 82,978(-.17306 + .0108) = 13,360$ " " " $t = 200$

$\mathcal{R}_{300} = 82,978(-.17306 + .0162) = 13,010$ " " $t = 300$

$\mathcal{R}_{400} = 82,978(-.17306 + .0216) = 12,540$ " " $t = 400$

The negative sign indicates that the torque opposes the motion.

Assuming the worst condition possible the energy dissipated in the bearings is

In the rotor $\dfrac{14,360 \times 130}{.724} = 2,580,000$ ergs

In the pulley $\dfrac{14,306 \times 130 \times \sqrt{2}}{2.469} = 1,067000$ "

Total 3,657,000 ergs

$$\frac{1}{2}\left(A\frac{v^2}{a^2} + I\frac{v^2}{b^2} + \frac{\bar{w}v^2}{g}\right) = \bar{w}l - 3,657,000 \text{ ergs}$$

Thus accounting for friction in starting

$$\dot{u}_0 = \sqrt{\frac{2(\bar{w}l - 3657000)}{A + I\frac{a^2}{b^2} + \frac{\bar{w}a^2}{g}}}$$

$= 131.8$ rad/sec.

Against 132 with no friction in stating.

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