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Methods for synthesizing a four-bar mechanism so it will have a low value of jerk.

Ronald D. Brenner

METHODS FOR SYNTHESIZING A FOUR-BAR MECHANISM

SO IT WILL HAVE A LOW VALUE OF JERK

BY

RONALD D. BRENNER

A

THESIS

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Approved by

Charles L. Edwards (Advisor) B. L. Atchley

Archie R. Sholan Ralph E. Schowalter

ABSTRACT

The purpose of this thesis is to find a method of synthesis, for a four-bar mechanism, so that the mechanism that is synthesized will have a low value of jerk. In the field of synthesis there has been little or no effort made to consider jerk when designing mechanisms.

Jerk is defined as the time rate of change of acceleration. Jerk can also be expressed as being proportional to the time rate of change of the inertia forces. When the jerk is large the effect on the link of a mechanism is the same as that of an impact load. Another effect of jerk is the excessive wear at the pin connections.

The model used to study jerk is the four-bar mechanism and in particular the crank-lever mechanism. A crank-lever mechanism has one fixed link (the ground link), one link that rotates (the crank), one link that oscillates with plane motion (the coupler), and one link that oscillates with rotary motion (the lever). As the crank rotates the lever oscillates through an angle θ. The advance swing of the lever take place during a larger angle of crank rotation than the return swing of the lever so we have a quick-return mechanism. The time ratio determines an angle α which along with θ specifies a family of mechanisms. A family of mechanisms is a group of mechanisms having the same angles θ and α, but having different proportions. It is assumed that some criterion can be found that will predict or deter-

mine which mechanism in a family will have the lowest value of maximum jerk. For the analysis four different families were chosen, each family contains ten mechanisms.

The velocities, accelerations, and jerks for a four-bar mechanism can most readily be determined by graphical means. However the graphical methods become quite tedious when they must be repeated many times. For this reason analytical equations were derived that could be used on the digital computer. These equations were derived by writing position vector equations, for a four-bar, and differentiating once to obtain the angular velocity, differentiating twice to determine the angular acceleration, and differentiating a third time to determine the angular jerk. A program can be written for the IBM 1620 that will calculate the values of jerk for a mechanism. These calculated values were then plotted against the crank angle. From the trend of these curves the methods of synthesis were determined.

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I. INTRODUCTION

The purpose of this thesis is to find a method of synthesis for a four-bar mechanism, so that the mechanism that is synthesized will have a low value of jerk. In the field of synthesis there has been little or no effort made to consider jerk when designing mechanisms. The reason for this is first, little is known about jerk and secondly, the knowledge of synthesis and analytical methods, in mechanism design, has been rather limited. The process of synthesizing a mechanical linkage has long been by trial and error. The engineer could, by constantly varying the lengths of the links in a mechanism, design a mechanism that is workable but this would not neccesarily be the best one. In recent years the increasing availability and use of modern high speed digital computers has made it possible to solve complicated mechanism problems by analytical methods and has led to the increased interest in the methods of synthesis. However none of these methods of synthesis take into account the possibility of a mechanism having a large value of jerk.

Jerk is defined as the time rate of change of acceleration i.e. jerk = $\frac{dA}{dt}$. To think of jerk only as the time rate of change of acceleration of a link one does not get a clear picture of what is actually happening. The forces due to the accelerations are the important considerations. From Newton's second law of motion one knows that $F = MA$. Considering the mass to be constant and differentiating

both sides with respect to time the result is

$$\frac{dF}{dt} = M \frac{dA}{dt}$$

Rearranging this equation an expression for jerk is obtained.

$$\text{jerk} = \frac{dA}{dt} = \frac{1}{M} \frac{dF}{dt}$$

One can now think of jerk as being proportional to the time rate of change of the inertia forces. In most mechanisms the acceleration is made to change from a positive value to a zero or negative value in a small time interval. As this time interval becomes infinitely small the jerk becomes infinitely large. Infinite jerk is rarely ever achieved in actual practice because of the clearances at the pins and the elasticity of the links. When the jerk is large the effect on the link is that of an impact load which induces vibrations that affect the resulting stress. Another effect of jerk is excessive wear at the pin connections.

The model used to study jerk will be the four-bar mechanism. The four-bar mechanism was choosen for several reasons.

(1) The four bar mechanism is the most widely used of all mechanisms.

(2) The four-bar mechanism is used as the basis for more complex mechanisms. Therefore theory developed for the four-bar is applicable to these other mechanisms.

(3) Many mechanisms which are not four-bars are equivalent to four-bars in some aspects of their motion. As

far as these motions are concerned, four-bar theory applies to them.

The four-bar and its equivalent mechanisms appear in some of the oldest and in some of the most modern machinery known. The functions performed by the four-bar defy classification except in the most general manner such as guiding, converting or transforming motion, and providing mechanical advantage.* (1)

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* Numbers in parentheses refer to sources listed in the Bibliography.

II. DISCUSSION

There are many types of four-bar mechanisms but the most widely used is the crank-lever mechanism so it will be used for the analysis. A crank-lever mechanism, as shown in Figure 1, has one link that is fixed (the ground link r_1), one link that rotates (the crank r_2), one

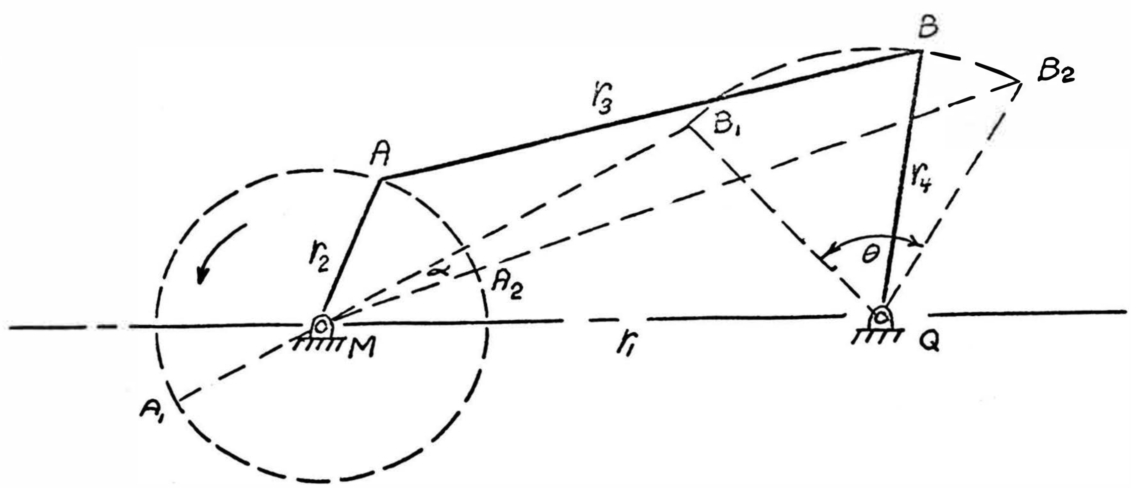

FIGURE 1

CRANK-LEVER MECHANISM

link that oscillates with plane motion (the coupler r_3), and one link that oscillates with rotary motion (the lever r_4). Thus as the crank rotates the lever oscillates through some angle θ. In reference to Figure 1, B_1 and B_2 are the extreme positions of the lever pin and A_1 and A_2 are the corresponding positions of the crank pin. The two swings of the lever do not take place during the same angle of crank rotation.

Starting at A_2 the forward swing occurs as the crank rotates through the angle $180 + \alpha$ and the return swing takes place as the crank rotates through an angle of $180 - \alpha$. This then is a quick-return mechanism. The time ratio for this mechanism is the time of advance divided by the time of return. The time ratio can also be expressed in terms of the angle through which the crank rotates for each swing of the lever.

$$T. R. = \frac{\text{time for advance}}{\text{time for return}} = \frac{180 + \alpha}{180 - \alpha}$$

The most common design problem will be one in which the angle through which the lever oscillates θ, and the angle α (or the time ratio which determines α) are specified. It can be shown[1] that for a specified θ and α a family of mechanisms can be designed. A family of mechanisms is a group of mechanisms all of which have the same values for θ and α . The mechanisms in a family however will have different proportions.

To analyze the jerk in a crank-lever mechanism it will be assumed that one knows the lengths of the links, the angular velocity of the driving crank, and the angular position of the driving crank. These are the neccesary conditions to determine the jerk in the mechanism.

The problem is to find some criterion that will predict or determine which mechanism in a family will have the lowest value of maximum jerk. The maximum jerk being the largest

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1 Appendix A

value occuring in a link during one revolution of the crank. For the analysis four different families of mechanisms were choosen each family containing 10 mechanisms. The families will be designated as families one, two, three, and four. For family one $\theta = 80°$ and $\alpha = 36°$, for family two $\theta = 80$ and $\alpha = 25$, $\theta = 60$ and $\alpha = 36$ for family three, and for family four $\theta = 60$ and $\alpha = 25$. The dimensions for the mechanisms used were determined by the first method as shown in Appendix A. The actual calculations were done on the Royal McBee LGP-30 digital computer. The second method as shown in Appendix A was not used because for a specified θ and α the dimensions are nearly proportional to each other and thus the velocities, accelerations, and jerks would also be proportional.

The velocities, accelerations, and jerks for a four-bar mechanism in a given position can be most readily determined by graphical methods. When the velocities, accelerations, and jerks must be determined for many positions of a mechanism the graphical methods become quite tedious. Most analytical methods are too complex and unwieldy for hand computation but they lend themselves readily to the digital computer. The best analytical method applicable to the digital computer is Raven's (2) method. The four-bar in Figure 2 will be used to illustrate

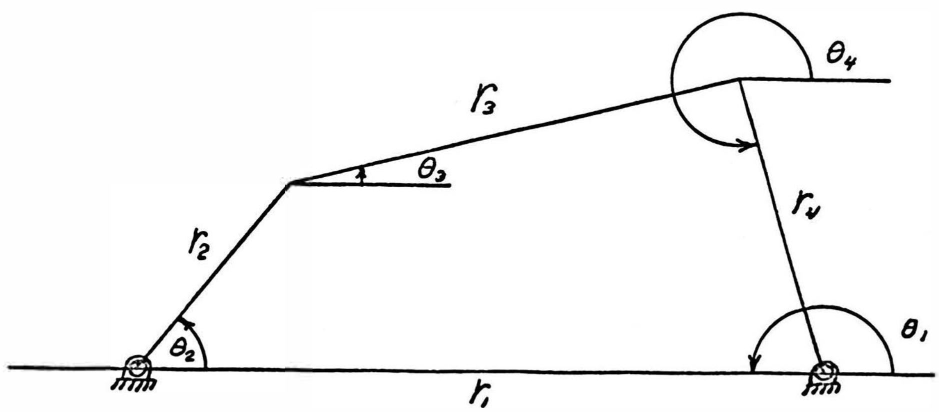

FIGURE 2

FOUR-BAR MECHANISM

Ravens method. The lengths of the links and the angle θ_2 are known quantities. The angles θ_3 and θ_4 can be determined from the geometry of the mechanism[2] The first step is to replace the links of the four-bar by position vectors as shown in Figure 3.

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2 Appendix B

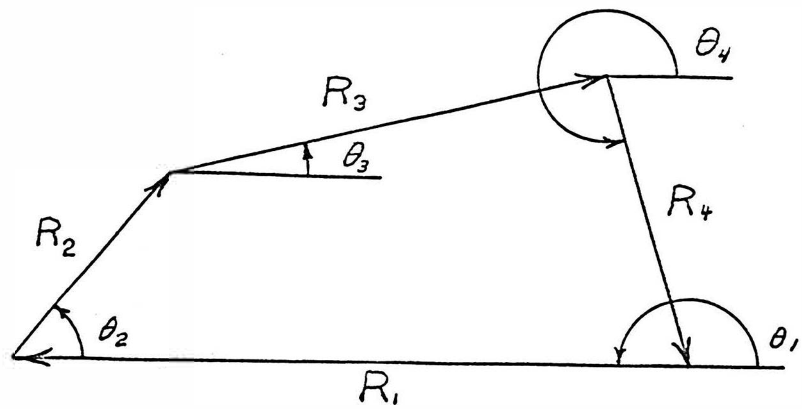

FIGURE 3

POSITION VECTORS

The position vector equation is

$$R1 \mathrel{+\!\!\!\rightarrow} R2 \mathrel{+\!\!\!\rightarrow} R3 \mathrel{+\!\!\!\rightarrow} R4 = 0$$

The position vector equation can be differentiated to give a velocity vector equation . The velocity equation can be differentiated to give an equation that will determine the accelerations. Carrying Raven's method one step further and differentiating the acceleration equation, an equation for jerk results. To perform the differentiation the position vector equation can be rewritten in polar form.

$$r_1 e^{i\theta_1} + r_2 e^{i\theta_2} + r_3 e^{i\theta_3} + r_4 e^{i\theta_4} = 0$$

Differentiating with respect to time the first term is zero because θ_1 is a constant. The resulting equation is

$$i r_2 \dot{\theta}_2 e^{i\theta_2} + i r_3 \dot{\theta}_3 e^{i\theta_3} + i r_4 \dot{\theta}_4 e^{i\theta_4} = 0$$

Knowing that $\dot{\theta} = \omega$ and using Euler's notation, $e^{i\theta} = COS\,\theta + i\,SIN\,\theta$ the equation becomes

$$i r_2 \omega_2 (COS\,\theta_2 + i\,SIN\,\theta_2) + i r_3 \omega_3 (COS\,\theta_3 + i\,SIN\,\theta_3) + i r_4 \omega_4 (COS\,\theta_4 + i\,SIN\,\theta_4) = 0$$

The equation can now be separated into real and imaginary parts.

$$i\,[r_2\,\omega_2 \cos\theta_2 + r_3\,\omega_3 \cos\theta_3 + r_4\,\omega_4 \cos\theta_4] = 0$$

$$[-r_2\,\omega_2 \sin\theta_2 - r_3\,\omega_3 \sin\theta_3 - r_4\,\omega_4 \sin\theta_4] = 0$$

The unknown quantities in the above equations are ω_3 and ω_4. These two quantities can best be found by using determinants. Rearranging the equations for the determinants we have

$$r_3\,\omega_3 \cos\theta_3 + r_4\,\omega_4 \cos\theta_4 = -r_2\,\omega_2 \cos\theta_2$$

$$r_3\,\omega_3 \sin\theta_3 + r_4\,\omega_4 \sin\theta_4 = -r_2\,\omega_2 \sin\theta_2$$

$$\omega_3 = \frac{\begin{vmatrix} -r_2\,\omega_2 \cos\theta_2 & r_4 \cos\theta_4 \\ -r_2\,\omega_2 \sin\theta_2 & r_4 \sin\theta_4 \end{vmatrix}}{\begin{vmatrix} r_3 \cos\theta_3 & r_4 \cos\theta_4 \\ r_3 \sin\theta_3 & r_4 \sin\theta_4 \end{vmatrix}}$$

$$\omega_3 = \frac{-r_2\,\omega_2 \cos\theta_2\, r_4 \sin\theta_4 + r_4 \cos\theta_4\, r_2\,\omega_2 \sin\theta_2}{r_3 \cos\theta_3\, r_4 \sin\theta_4 - r_4 \cos\theta_4\, r_3 \sin\theta_3}$$

$$\omega_3 = \frac{r_2\,\omega_2\,(\cos\theta_4 \sin\theta_2 - \cos\theta_2 \sin\theta_4)}{r_3\,(\cos\theta_3 \sin\theta_4 - \cos\theta_4 \sin\theta_3}$$

$$\omega_3 = \frac{r_2\,\omega_2 \sin(\theta_4 - \theta_2)}{r_3 \sin(\theta_3 - \theta_4)}$$

ω_4 is determined in the same manner

$$\omega_4 = \frac{r_2 \omega_2 \, SIN(\theta_2 - \theta_3)}{r_4 \, SIN(\theta_3 - \theta_4)}$$

Differentiating the velocity equation

$$i r_2 \dot{\theta}_2 e^{i\theta_2} + i r_3 \dot{\theta}_3 e^{i\theta_3} + i r_4 \dot{\theta}_4 e^{i\theta_4} = 0$$

the acceleration equation is obtained

$$(i)^2 r_2 (\dot{\theta}_2)^2 e^{i\theta_2} + \overset{0}{\cancel{i r_2 \ddot{\theta}_2 e^{i\theta_2}}} + (i)^2 r_3 (\dot{\theta}_3)^2 e^{i\theta_3}$$

$$+ i r_3 \ddot{\theta}_3 e^{i\theta_2} + (i)^2 r_4 (\dot{\theta}_4)^2 e^{i\theta_4} + i r_4 \ddot{\theta}_4 e^{i\theta_4} = 0$$

The second term in the equation is zero because ω_2 is constant. Using Euler's notation and substituting ω for $\dot{\theta}$ and α for $\ddot{\theta}$ the equation is written

$$-r_2 \omega_2^2 (COS\theta_2 + i\,SIN\theta_2) - r_3 \omega_3^2 (COS\theta_3 + i\,SIN\,\theta_3)$$

$$+ i r_3 \alpha_3 (COS\theta_3 + i\,SIN\theta_3) - r_4 \omega_4^2 (COS\theta_4 + i\,SIN\theta_4)$$

$$+ i r_4 \alpha_4 (COS\,\theta_4 + i\,SIN\theta_4) = 0$$

The equation can now be broken up into its real and imaginary parts.

$$[-r_2 \omega_2^2 \cos\theta_2 - r_3 \omega_3^2 \cos\theta_3 - r_3 \alpha_3 \sin\theta_3$$

$$-r_4 \omega_4^2 \cos\theta_4 - r_4 \alpha_4 \sin\theta_4] = 0$$

$$i[-r_2 \omega_2^2 \sin\theta_2 - r_3 \omega_3^2 \sin\theta_3 + r_3 \alpha_3 \cos\theta_3$$

$$-r_4 \omega_4^2 \sin\theta_4 + r_4 \alpha_4 \cos\theta_4] = 0$$

The above equations can be rearranged so that determinants can be used to solve for α_3 and α_4 .

$$r_3 \alpha_3 \sin\theta_3 + r_4 \alpha_4 \sin\theta_4 = -(r_4 \omega_4^2 \cos\theta_4 + r_2 \omega_2^2 \cos\theta_2 + r_3 \omega_3^2 \cos\theta_3)$$

$$r_3 \alpha_3 \cos\theta_3 + r_4 \alpha_4 \cos\theta_4 = (r_4 \omega_4^2 \sin\theta_4 + r_2 \omega_2^2 \sin\theta_2 + r_3 \omega_3^2 \sin\theta_3)$$

For ease in evaluating the determinants the right hand members can be denoted by A and B respectively.

$$r_3 \alpha_3 \sin\theta_3 + r_4 \alpha_4 \sin\theta_4 = -A$$

$$r_3 \alpha_3 \cos\theta_3 + r_4 \alpha_4 \cos\theta_4 = B$$

$$\alpha_3 = \frac{\begin{vmatrix} -A & r_4 \sin\theta_4 \\ B & r_4 \cos\theta_4 \end{vmatrix}}{\begin{vmatrix} r_3 \sin\theta_3 & r_4 \sin\theta_4 \\ r_3 \cos\theta_3 & r_4 \cos\theta_4 \end{vmatrix}}$$

$$\alpha_3 = \frac{-A\cos\theta_4 - B\sin\theta_4}{r_3\sin\theta_3\, r_4\cos\theta_4 - r_4\sin\theta_4\, r_3\cos\theta_3}$$

$$\alpha_3 = \frac{A\cos\theta_4 + B\sin\theta_4}{-r_3\sin(\theta_3 - \theta_4)}$$

Substituting in the values for A and B

$$\alpha_3 = \frac{r_4\omega_4^2 + r_2\omega_2^2\cos(\theta_2 - \theta_4) + r_3\omega_3^2\cos(\theta_3 - \theta_4)}{-r_3\sin(\theta_3 - \theta_4)}$$

By the same manner α_4 is found.

$$\alpha_4 = \frac{r_3\omega_3^2 + r_4\omega_4^2\cos(\theta_4 - \theta_3) + r_2\omega_2^2\cos(\theta_2 - \theta_3)}{r_4\sin(\theta_3 + \theta_4)}$$

Since jerk is a vector quantity it can be determined by carrying Raven's method one step farther and differentiating the acceleration equation. The result of this differentiation is:

$$-i r_2(\dot\theta_2)^3 e^{i\theta_2} - i r_3(\dot\theta_3)^3 e^{i\theta_3} - 2 r_3\dot\theta_3\ddot\theta_3 e^{i\theta_3}$$
$$+ (i)^2 r_3\dot\theta_3\ddot\theta_3 e^{i\theta_3} + i r_3\dddot\theta_3 e^{i\theta_3} - i r_4(\dot\theta_4)^3 e^{i\theta_4}$$
$$-2 r_4\dot\theta_4\ddot\theta_4 e^{i\theta_4} + (i)^2 r_4\dot\theta_4\ddot\theta_4 e^{i\theta_4} + i r_4\dddot\theta_4 e^{i\theta_4} = 0$$

The quantity $\dddot{\theta}$ is the angular jerk and is denoted by β . Now by changing from polar form to trigonometric form and substituting ω and α for $\dot{\theta}$ and $\ddot{\theta}$ the equation becomes

$$-i r_2 \omega_2^3 (COS\theta_2 + i\, SIN\theta_2) - i r_3 \omega_3^3 (COS\theta_3 + i\, SIN\theta_3)$$
$$-3 r_3 \omega_3 \alpha_3 (COS\theta_3 + i\, SIN\theta_3) + i r_3 \beta_3 (COS\theta_3 + i\, SIN\theta_3)$$
$$-i r_4 \omega_4^3 (COS\theta_4 + i\, SIN\theta_3) - 3 r_4 \omega_4 \alpha_4 (COS\theta_4 + i\, SIN\theta_4)$$
$$+ i r_4 \beta_4 (COS\theta_4 + i\, SIN\theta_4) = 0$$

The equation can now be broken up into its real and imaginary parts in order to solve for β_3 and β_4 by determinants.

$$i\Big[-r_2 \omega_2^3 COS\theta_2 - r_3 \omega_3^3 COS\theta_3 - 3 r_3 \omega_3 \alpha_3 SIN\theta_3 + r_3 \beta_3 COS\theta_3$$
$$- r_4 \omega_4^3 COS\theta_4 - 3 r_4 \omega_4 \alpha_4 SIN\theta_4 + r_4 \beta_4 COS\theta_4\Big] = 0$$

$$\Big[r_2 \omega_2^3 SIN\theta_2 + r_3 \omega_3^3 SIN\theta_3 - 3 r_3 \omega_3 \alpha_3 COS\theta_3 - r_3 \beta_3 SIN\theta_3$$
$$+ r_4 \omega_4^3 SIN\theta_4 - 3 r_4 \omega_4 \alpha_4 COS\theta_4 - r_4 \beta_4 SIN\theta_4\Big] = 0.$$

These equations can be rewritten as

$$r_3 \beta_3 COS\theta_3 + r_4 \beta_4 COS\theta_4 = A$$
$$r_3 \beta_3 SIN\theta_3 + r_4 \beta_4 SIN\theta_4 = -B$$

Where the quantities A and B are

$$A = \left[r_2\,\omega_2^3 \cos\theta_2 + r_3\,\omega_3^3 \cos\theta_3 + 3 r_3\,\omega_3\,\alpha_3 \sin\theta_3 + r_4\,\omega_4^3 \cos\theta_4 + 3 r_4\,\omega_4\,\alpha_4 \sin\theta_4\right]$$

$$B = \left[r_2\,\omega_2^3 \cos\theta_2 + r_3\,\omega_3 \sin\theta_3 - 3 r_3\,\omega_3\,\alpha_3 \cos\theta_3 + r_4\,\omega_4^3 \sin\theta_4 - 3 r_4\,\omega_4\,\alpha_4 \cos\theta_4\right]$$

Solving for β_3 and β_4 by determinants we get.

$$\beta_3 = \frac{\begin{vmatrix} A & r_4 \cos\theta_4 \\ -B & r_4 \sin\theta_4 \end{vmatrix}}{\begin{vmatrix} r_3 \cos\theta_3 & r_4 \cos\theta_4 \\ r_3 \sin\theta_3 & r_4 \sin\theta_4 \end{vmatrix}}$$

$$\beta_3 = \frac{A\, r_4 \sin\theta_4 + B\, r_4 \cos\theta_4}{r_3 \cos\theta_3\, r_4 \sin\theta_4 - r_4 \cos\theta_4\, r_3 \sin\theta_3}$$

$$\beta_3 = \frac{A \sin\theta_4 + B \cos\theta_4}{-r_3 \sin(\theta_3 - \theta_4)}$$

Upon substitution of the values for A and B the equation becomes

$$\beta_3 = \frac{r_2 \omega_2^3 SIN(\theta_2 + \theta_4) + r_3 \omega_3^3 SIN(\theta_3 + \theta_4) + r_4 \omega_4^3 SIN\, 2\theta_4}{-r_3 SIN(\theta_3 - \theta_4)}$$

$$+ \frac{-r_3 \omega_3 \alpha_3 COS(\theta_3 + \theta_4) + 3 r_4 \omega_4 \alpha_4 (SIN^2 \theta_4 - COS^2 \theta_4)}{-r_3 SIN(\theta_3 - \theta_4)}$$

By the same manner it is found that

$$\beta_4 = \frac{r_2 \omega_2^3 SIN(\theta_2 + \theta_3) + r_3 \omega_3^3 SIN\, 2\theta_3 + r_4 \omega_4^3 SIN(\theta_3 + \theta_4)}{r_4 SIN(\theta_3 - \theta_4)}$$

$$+ \frac{3 r_3 \omega_3 \alpha_3 (SIN^2 \theta_3 - COS^2 \theta_3) - 3 r_4 \omega_4 \alpha_4 COS(\theta_3 + \theta_4)}{r_4 SIN(\theta_3 - \theta_4)}$$

The equations that were just derived for the angular velocities, angular accelerations, and the angular jerks and the equations for determining the angles θ_3 and θ_4 are sufficient to calculate the angular jerks in the lever and coupler of a crank-lever mechanism when the lengths of the links, angular velocity of the crank, and the angular position of the crank are known. With the equations and given conditions known a computer program was written in the "Fortran Language" for the IBM 1620 digital computer.[3]

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3 Appendix C

The program was designed to compute and print the values of θ_2, β_3, and β_4. The input data for the program was ω_2, delta θ_2, and the dimensions of the links which were computed on the LGP-30 digital computer. The quantity delta θ_2 was choosen to be 3°. This allowed the jerk to be calculated for every 3° of crank rotation, from an initial value of 0° to a final value of 360°. The value for ω_2 was choosen to be 100 radians per second. This value was choosen because a realistic mechanism, in which jerk must be considered, will operate at least at this speed or higher speeds. At speeds much lower than this jerk would probably not be a problem. The values of jerk calculated by the computer were plotted against the crank angles for all of the forty mechanisms. It should be understood that the jerk that was computed is the angular jerk. The total jerk has three components that are $r\omega^3$, $r\omega\alpha$, and $r\beta$. The final equations used in the derivation of β were equations for the total jerk so therefore the value of angular jerk is a reflection of the total jerk. Angular jerk is the largest single component of the total jerk in a crank-lever mechanism. The curves, Figure 4 through Figure 44, show that a similar pattern exists for each family of mechanisms. The first thing that is noticed is that the maximum jerk, in both the lever and coupler, occurs when θ_2 is in the range of plus or minus 15°. Also the jerk in the lever and the coupler are the same when θ_2 equals 0° and when θ_2 equals 180°. Another similarity

TABLE I

VALUES OF JERK FOR MECHANISM ONE FAMILY ONE

θ_2	β_3 (10^{-6} RAD/SEC^3)	β_4 (10^{-6} RAD/SEC^3)
1	17.077425	17.077426
3	18.815544	18.934864
6	18.544113	18.556592
9	15.951276	15.985609
12	11.761066	12.157441
15	7.171959	8.241724
18	3.161991	5.039079
21	.187480	2.818648
24	-1.734142	1.479881
27	-2.808861	.777398
30	-3.293511	.468752
33	-3.408728	.372804
36	-3.314412	.374573
39	-3.113651	.409746
42	-2.869785	.446741
45	-2.617645	.472773
48	-2.375422	.484758
51	-2.151436	.494013
54	-1.948554	.473413
57	-1.766788	.465023
60	-1.604804	.435192
63	-1.460273	.411022
66	-1.332562	.397103
69	-1.218437	.363871
72	-1.116652	.342071
75	-1.025731	.322179
78	-.944408	.304477
81	-.871617	.289110
84	-.806464	.276139
87	-.748208	.265566
90	-.696241	.257362
93	-.650068	.251481
96	-.609293	.247871
99	-.573604	.246478
102	-.542763	.247254
105	-.516595	.250150
108	-.494976	.255121
111	-.477823	.262113
114	-.465082	.271065
117	-.456711	.281892
120	-.452664	.294477
123	-.452867	.308651
126	-.457192	.324177
129	-.465418	.340726

TABLE I (continued)

VALUES OF JERK FOR MECHANISM ONE FAMILY ONE

θ_2	β_3 (10^{-6} RAD/SEC^3)	β_4 (10^{-6} RAD/SEC^3)
132	-.477194	.357851
135	-.491988	.374963
138	-.509041	.391297
141	-.527307	.405903
144	-.545417	.417627
147	-.561648	.425125
150	-.573925	.426899
153	-.579881	.421372
156	-.576961	.407007
159	-.562617	.382473
162	-.534559	.346844
165	-.491069	.299814
168	-.431323	.241891
171	-.355669	.174521
174	-.265790	.100099
177	-.164695	.021837
180	-.056508	-.056614
183	.053925	-.131052
186	.161560	-.198151
189	.261700	-.254864
192	.350463	-.299199
195	.425107	-.330248
198	.484153	-.348158
201	.527333	-.353962
204	.555400	-.349333
207	.569856	-.336316
210	.572673	-.317070
213	.566023	-.293681
216	.552080	-.268011
219	.532864	-.241630
222	.510164	-.215783
225	.485491	-.191403
228	.460076	-.169144
231	.434891	-.149419
234	.410684	-.132454
237	.388011	-.119325
240	.367277	-.107002
243	.348776	-.098382
246	.332717	-.092309
249	.319257	-.088600
252	.308524	-.087054
255	.300634	-.874638

TABLE I (continued)

VALUES OF JERK FOR MECHANISM ONE FAMILY ONE

θ_2	β_3 (10^{-6} RAD/SEC^3)	β_4 (10^{-6} RAD/SEC^3)
258	.295716	-.896204
261	.293919	-.093315
264	.295436	-.098339
267	.300513	-.104479
270	.309471	-.111514
273	.322718	-.119208
276	.340778	-.127298
279	.364312	-.135481
282	.395153	-.143398
285	.431345	-.150614
288	.477196	-.156587
291	.533336	-.160632
294	.601800	-.161879
297	.685124	-.159211
300	.786461	-.151187
303	.909722	-.135953
306	1.059736	-.111114
309	1.242428	-.073596
312	1.464977	-.019468
315	1.735928	.056245
318	2.065159	.159763
321	2.463556	.298724
324	2.942115	.482276
327	3.510393	.721060
330	4.173124	1.027100
333	4.926200	1.413853
336	5.751568	1.897482
339	6.615229	2.501492
342	7.476032	3.268740
345	8.317487	4.283972
348	9.208052	5.699370
351	10.357770	7.725322
354	12.068512	10.511526
357	14.461088	13.879905
360	17.065348	17.046309

TABLE II

DIMENSIONS OF THE MECHANISMS

FAMILY ONE

	r_1	r_2	r_3	r_4
10	10.147332	6.221388	8.127615	10.000000
20	10.319135	5.965550	9.765499	
30	10.509127	5.668297	11.329061	
40	10.710731	5.325904	12.806402	
50	10.917351	4.942977	14.186278	
60	11.122606	4.522432	15.458189	
70	11.320501	4.067468	16.612453	
80	11.505552	3.581549	17.640286	
90	11.672873	3.068371	18.533866	
100	11.818234	2.531842	19.286392	

FAMILY TWO

	r_1	r_2	r_3	r_4
10	10.781305	6.279217	8.930434	10.000000
20	11.709390	6.082769	11.365025	
30	12.727357	5.840028	13.713123	
40	13.787970	5.552840	15.956854	
50	14.853377	5.2233919	18.079145	
60	15.893474	4.854191	20.063842	
70	16.884181	4.448046	21.895841	
80	17.806047	4.008050	23.561200	
90	18.643229	3.537549	25.047244	
100	19.382788	3.040126	26.342663	

TABLE II (continued)

DIMENSIONS OF THE MECHANISMS

FAMILY THREE

	r_1	r_2	r_3	r_4
12	9.786099	4.802793	6.581137	10.000000
24	9.521066	4.552965	8.090170	
36	9.212628	4.253254	9.510566	
48	8.870496	3.906944	10.826761	
60	8.506509	3.517828	12.024336	
72	8.134734	3.090170	13.090171	
84	7.771467	2.629656	14.012586	
96	7.434961	2.138341	14.781477	
108	7.144676	1.624598	15.388418	
120	6.919818	1.093056	15.826761	

FAMILY FOUR

	r_1	r_2	r_3	r_4
12	10.259345	4.856743	7.330097	10.000000
24	10.592996	4.660274	9.579884	
36	10.980353	4.412746	11.724711	
48	11.399943	4.116870	13.741080	
60	11.831009	3.775890	15.606899	
72	12.254498	3.393541	17.301726	
84	12.653567	3.393541	18.806991	
96	13.013780	2.521897	20.016230	
108	13.323128	2.042153	21.185127	
120	13.571975	1.540034	22.031942	

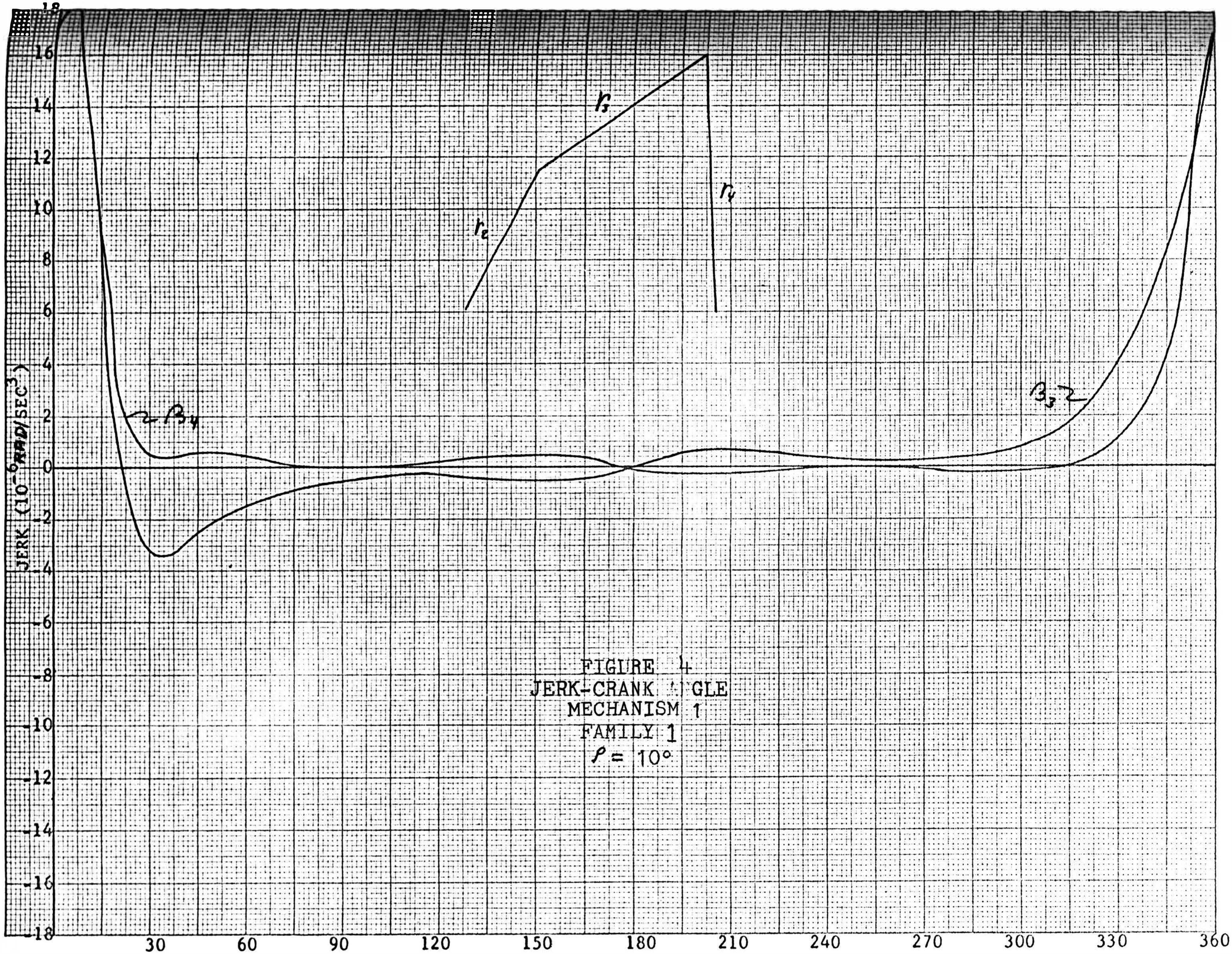


FIGURE 4
JERK-CRANK ANGLE
MECHANISM 1
FAMILY 1
$\rho = 10°$

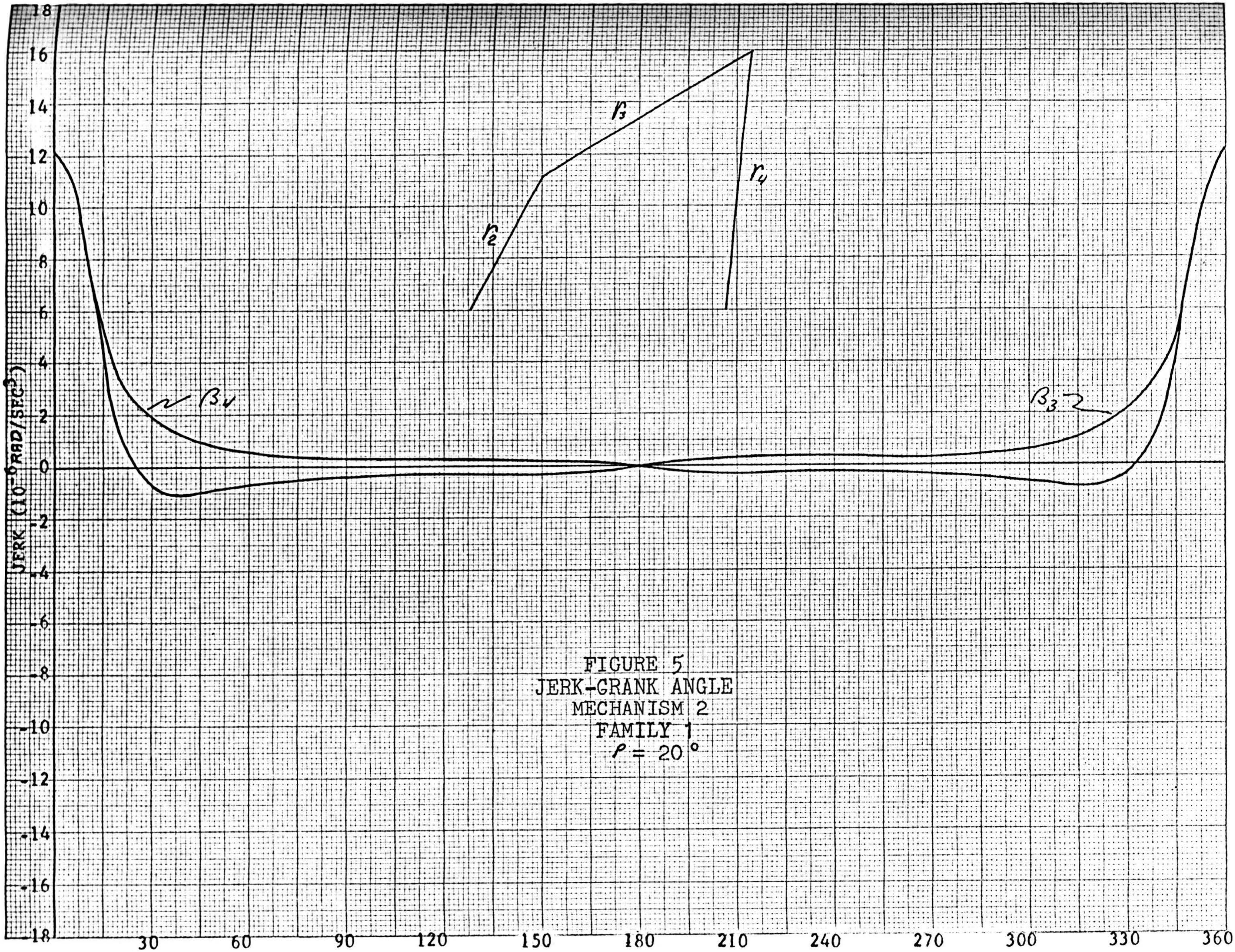


FIGURE 5
JERK-GRANK ANGLE
MECHANISM 2
FAMILY 1
$\rho = 20°$

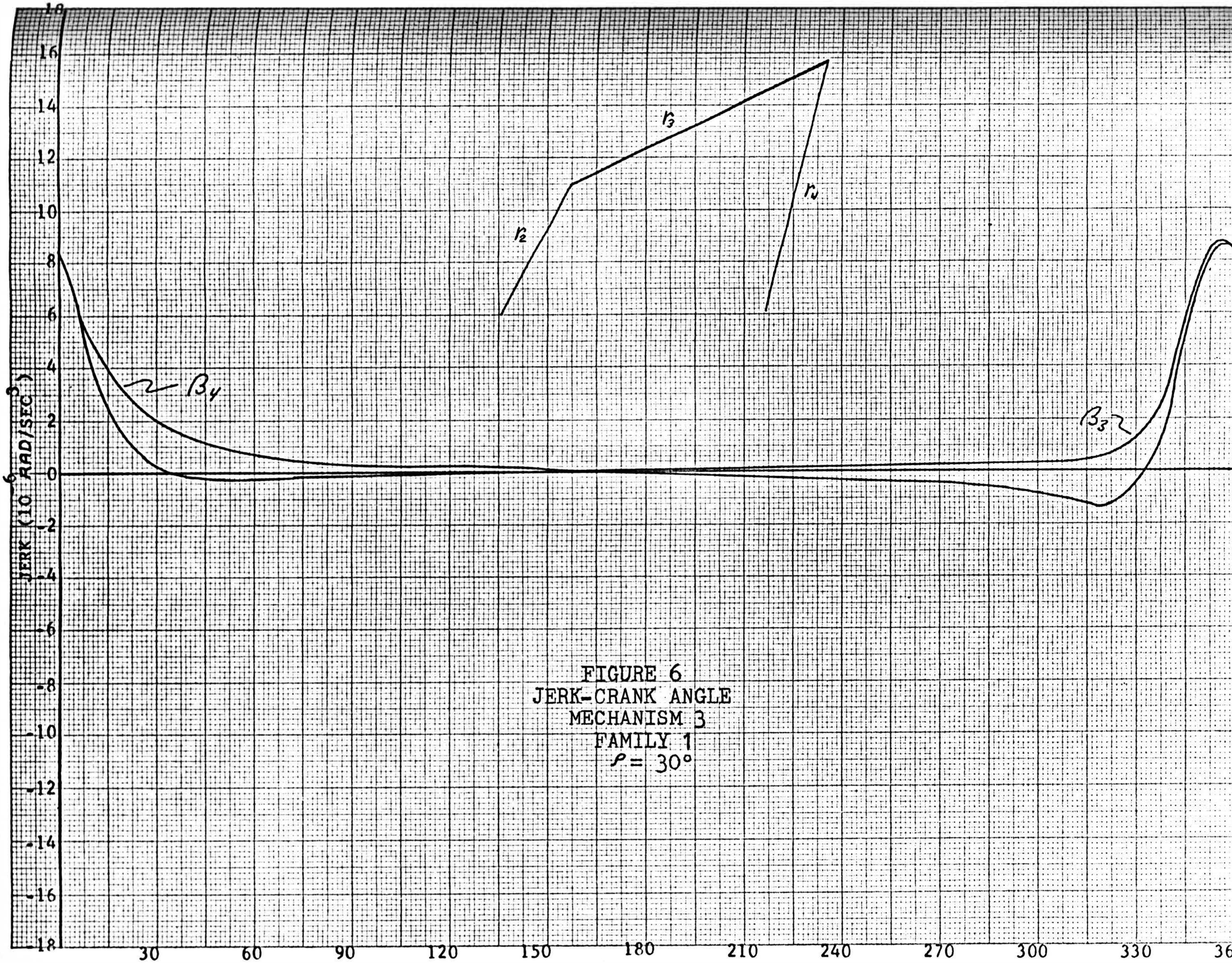
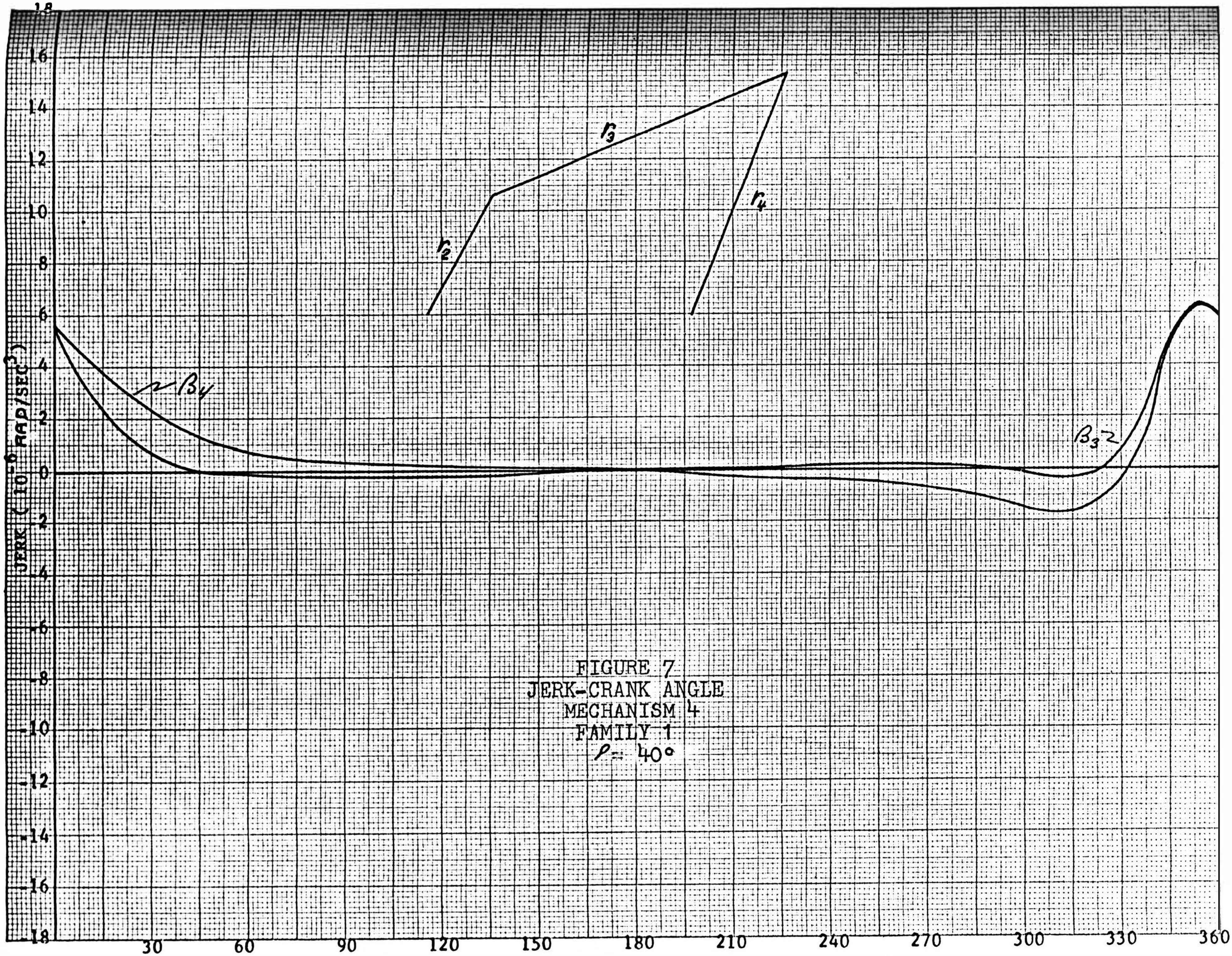


FIGURE 6
JERK-CRANK ANGLE
MECHANISM 3
FAMILY 1
$\rho = 30°$


JERK (10^{-6} RAD/SEC^3)
r_2
r_3
r_4
β_4
β_3
FIGURE 7
JERK-CRANK ANGLE
MECHANISM 4
FAMILY 1
ρ = 40°

25

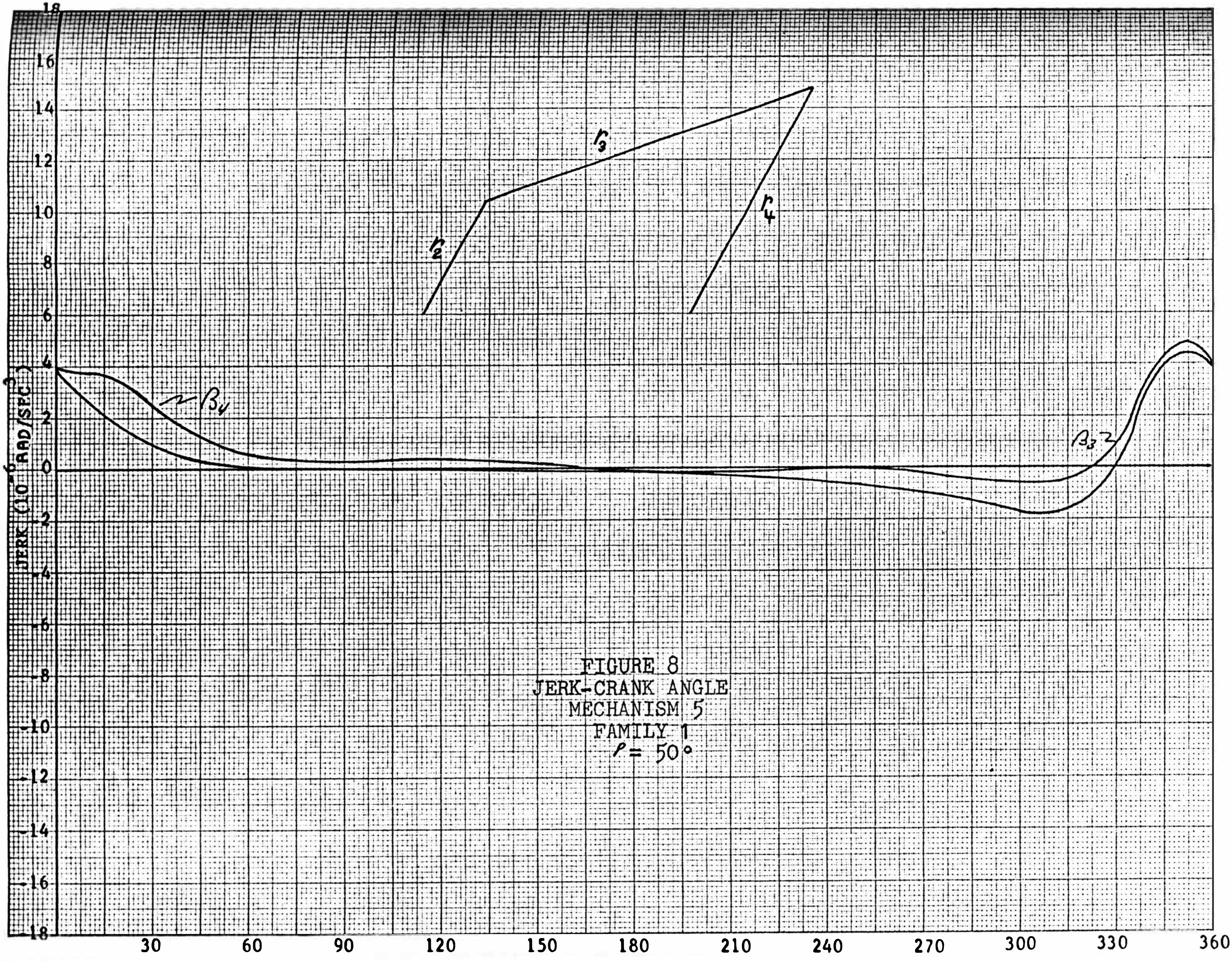
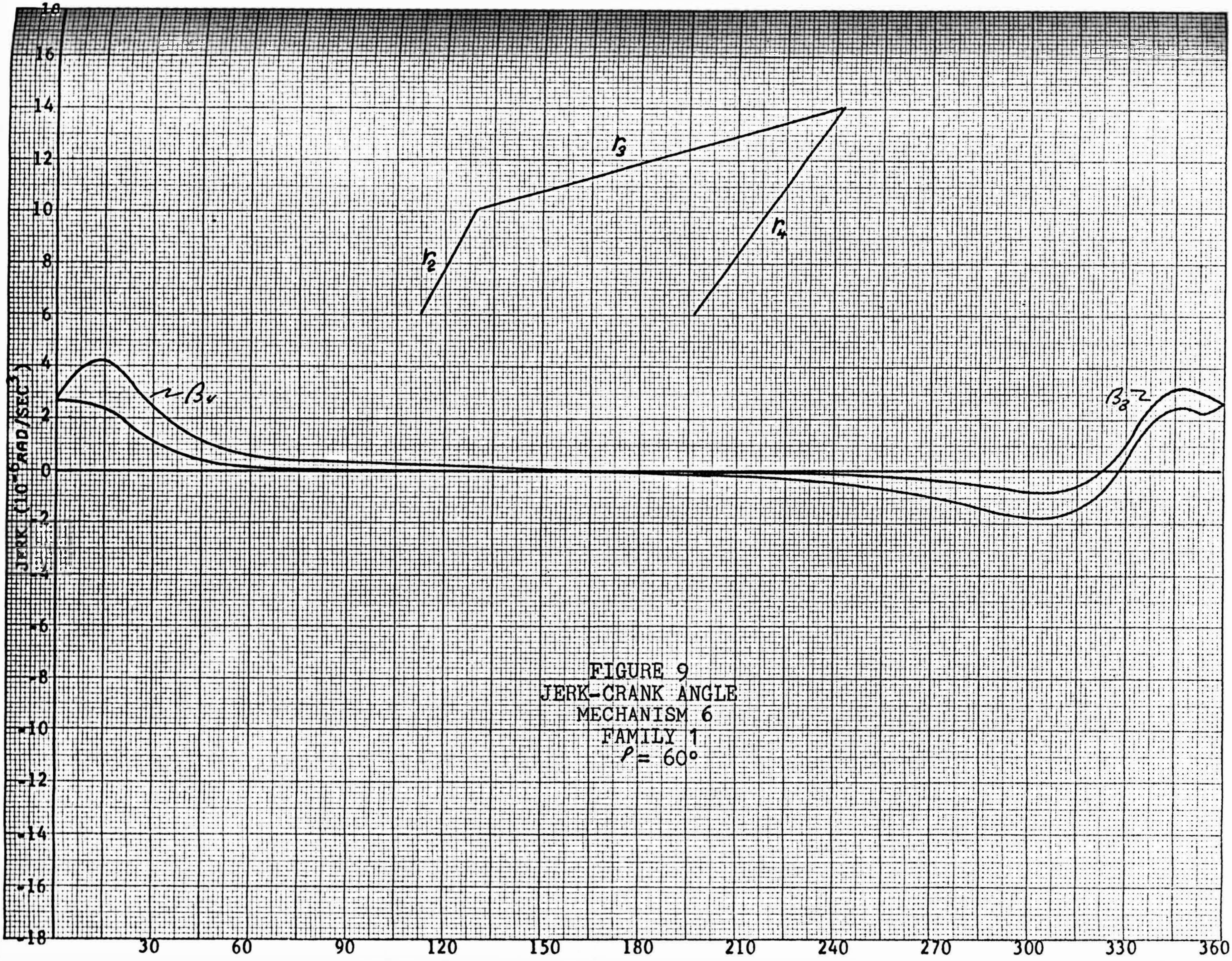


FIGURE 8
JERK-CRANK ANGLE
MECHANISM 5
FAMILY 1
$\rho = 50°$


JERK (10^{-6} RAD/SEC^3)
r_2
r_3
r_4
β_4
β_3
FIGURE 9
JERK-CRANK ANGLE
MECHANISM 6
FAMILY 1
ρ = 60°

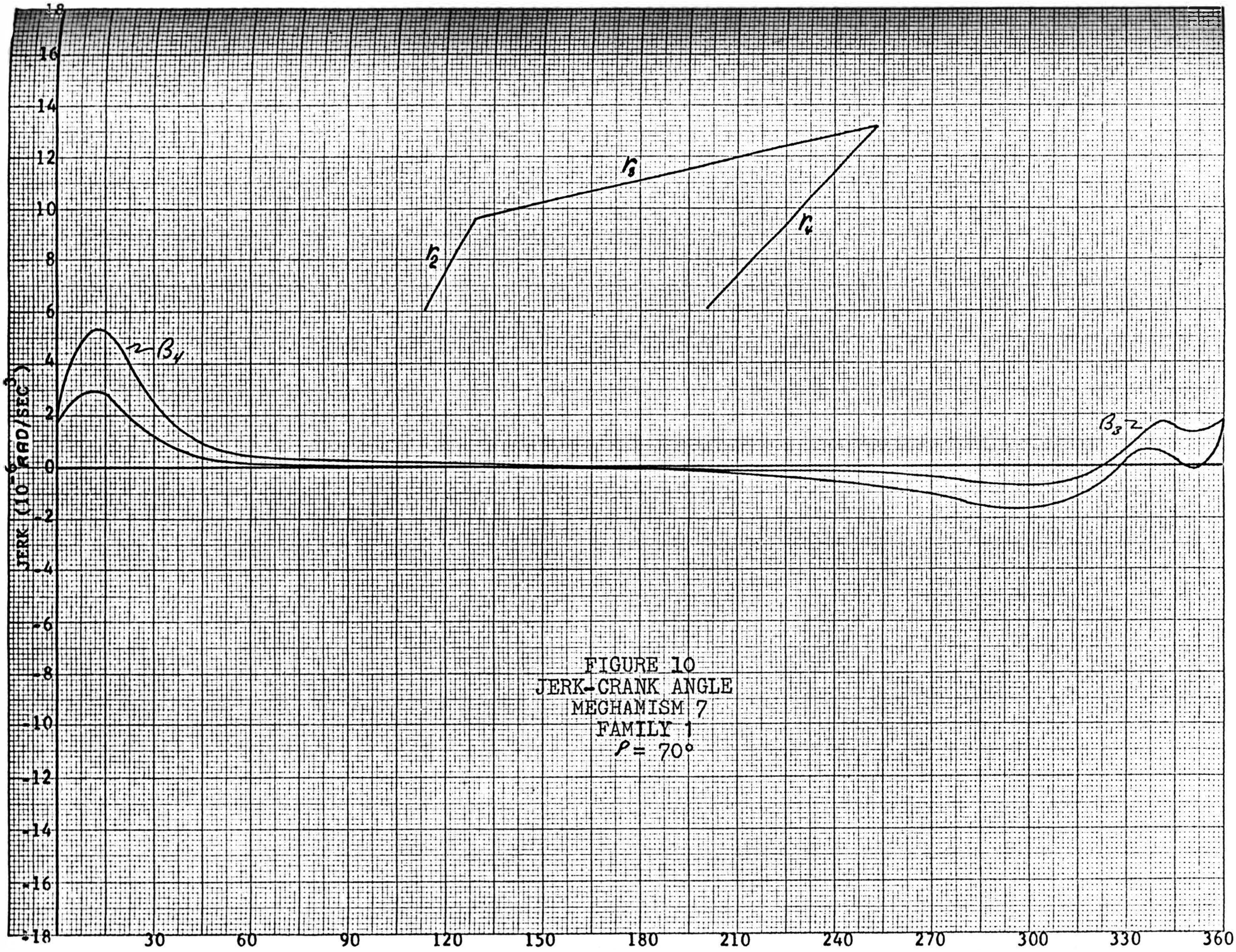


FIGURE 10
JERK-CRANK ANGLE
MECHAMISM 7
FAMILY 1
$\rho = 70°$

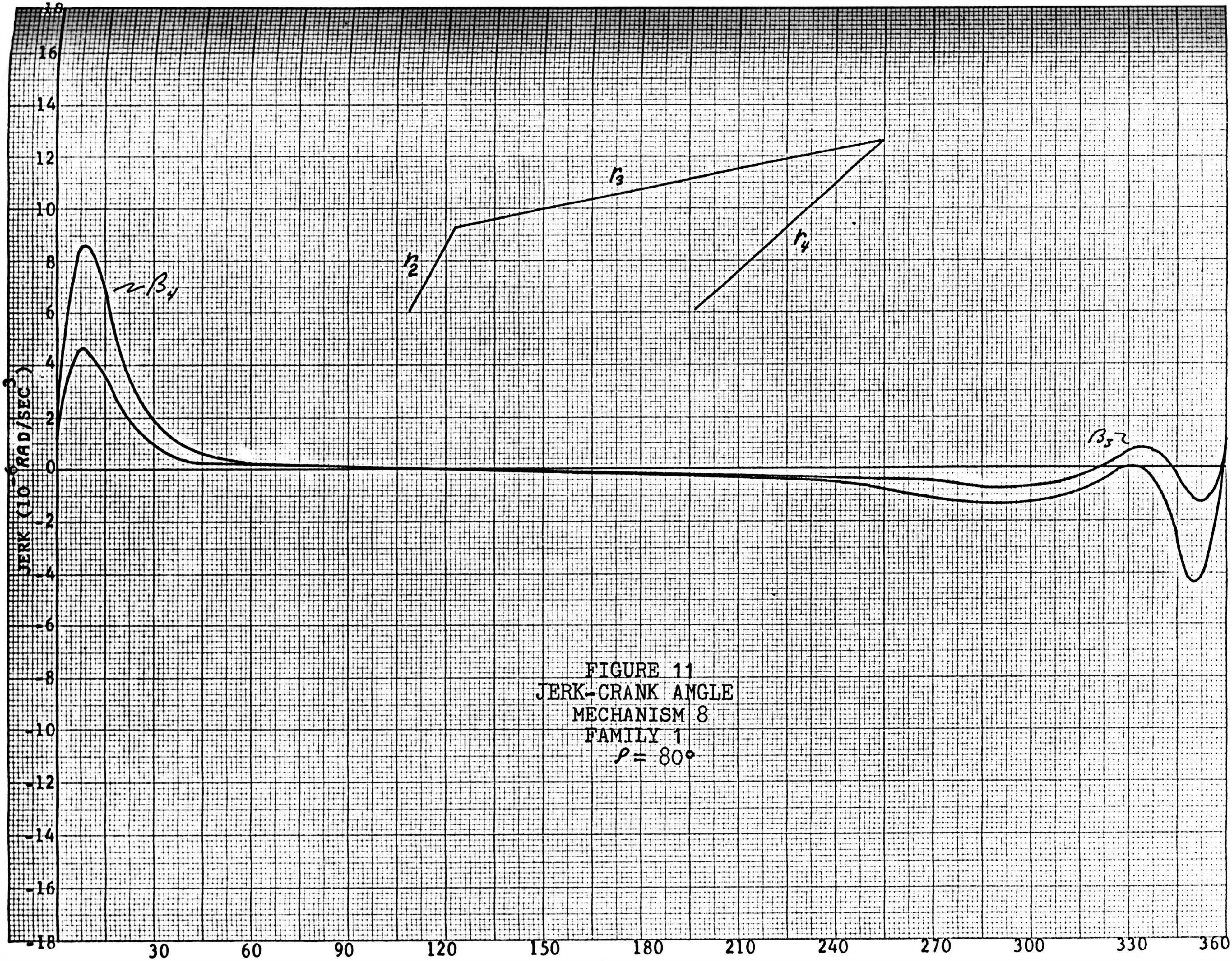


FIGURE 11
JERK-CRANK AMGLE
MECHANISM 8
FAMILY 1
$\rho = 80°$

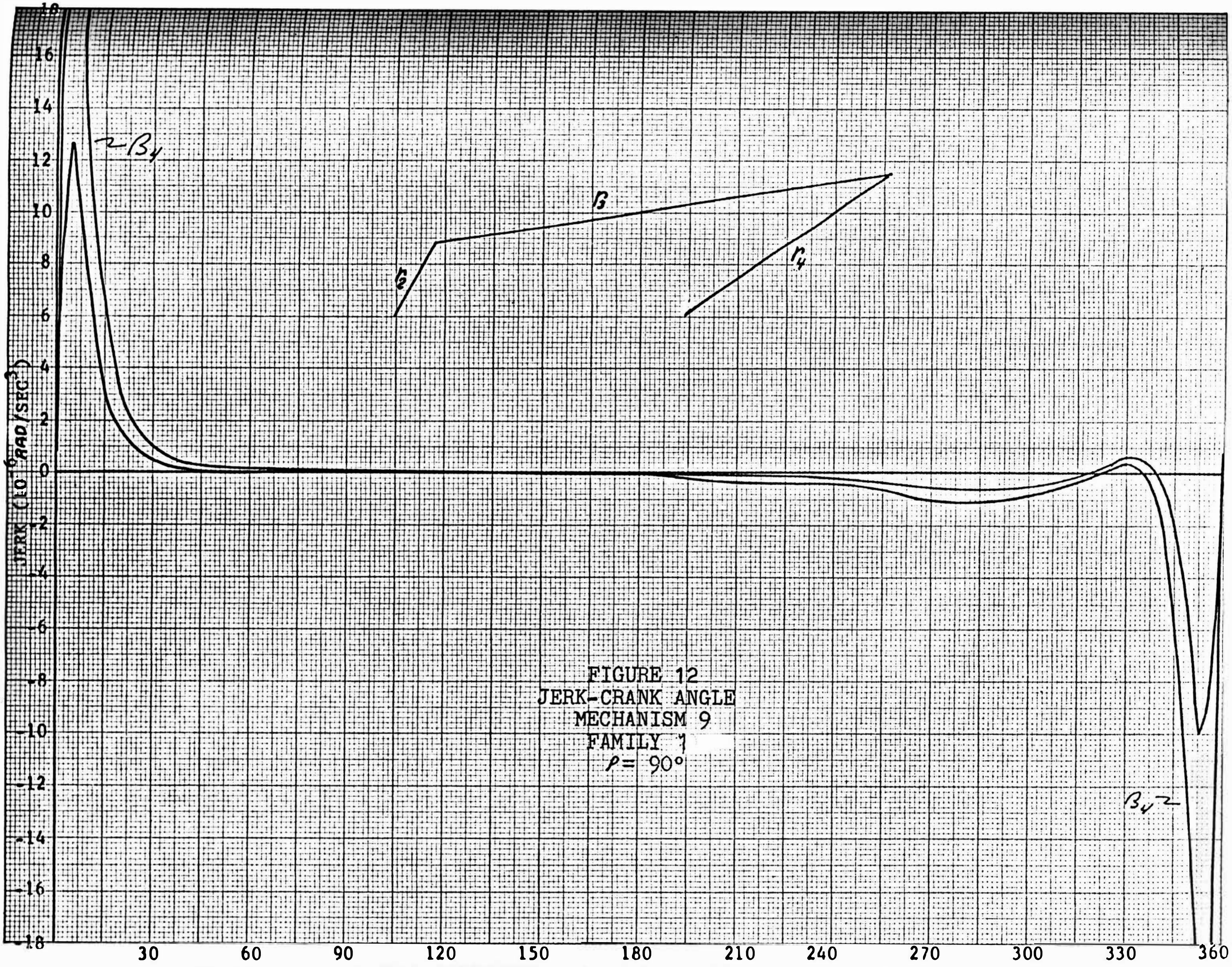


FIGURE 12
JERK-CRANK ANGLE
MECHANISM 9
FAMILY 1
$\rho = 90°$

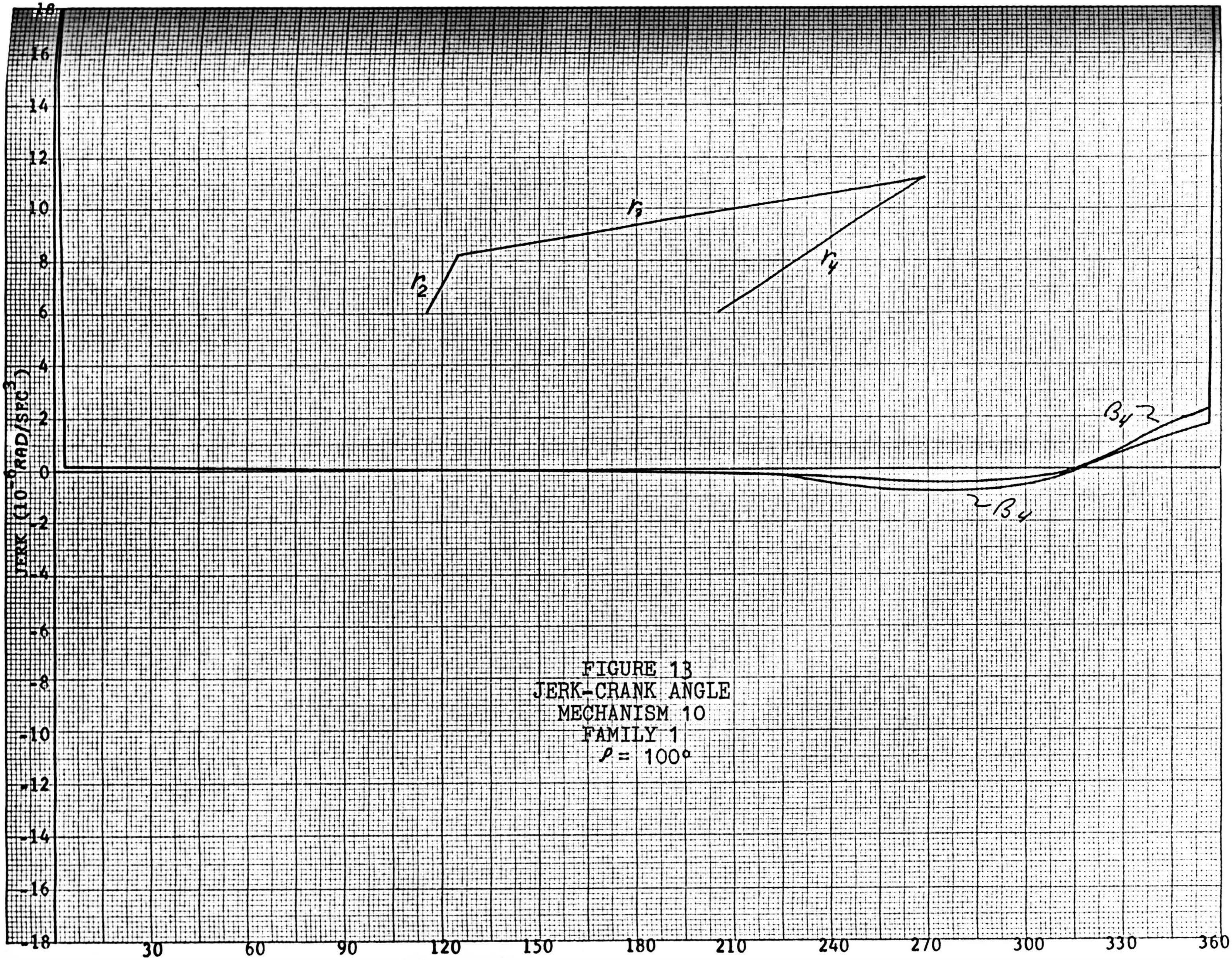


FIGURE 13
JERK-CRANK ANGLE
MECHANISM 10
FAMILY 1
$\rho = 100°$

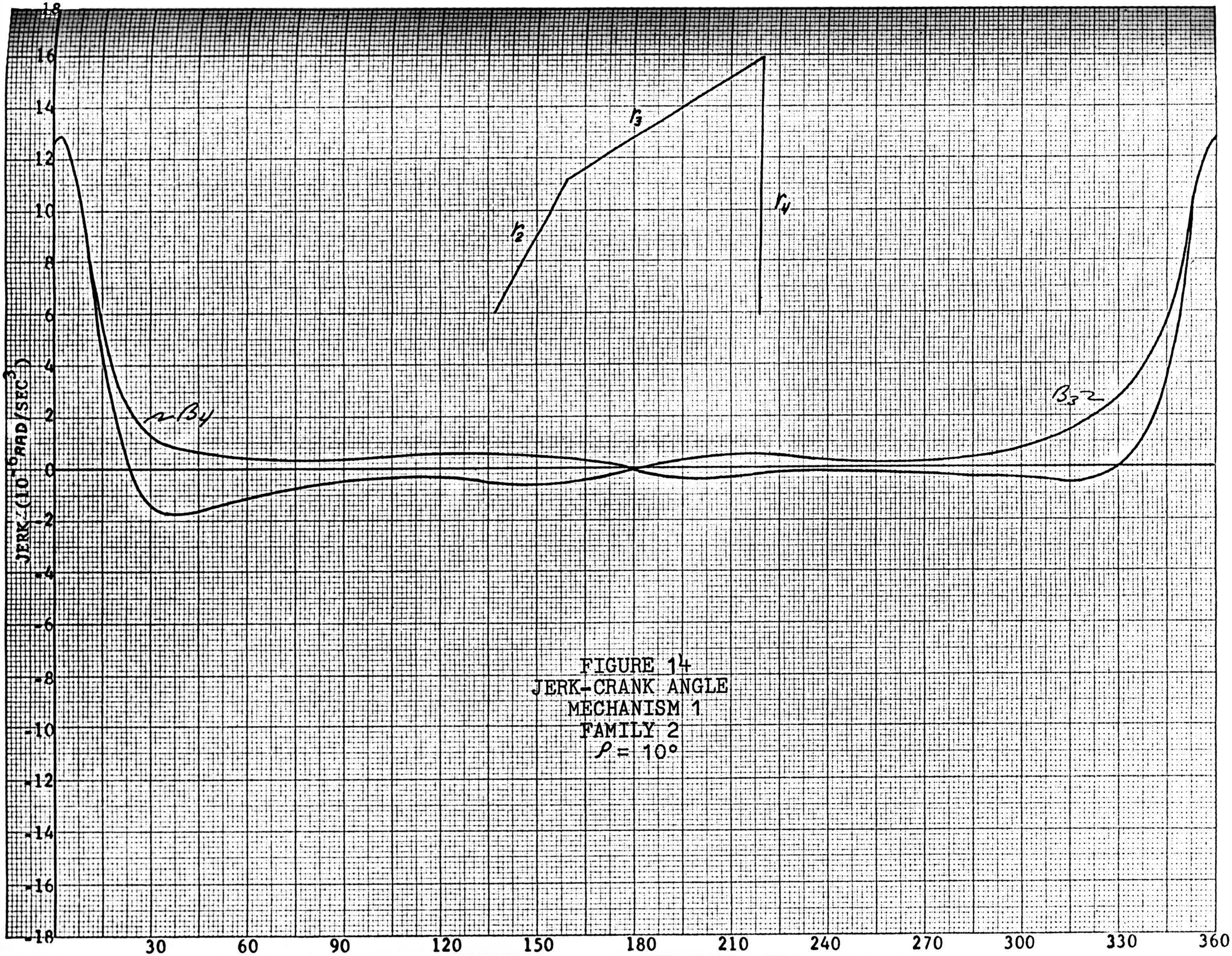


FIGURE 14
JERK-CRANK ANGLE
MECHANISM 1
FAMILY 2
$\rho = 10°$

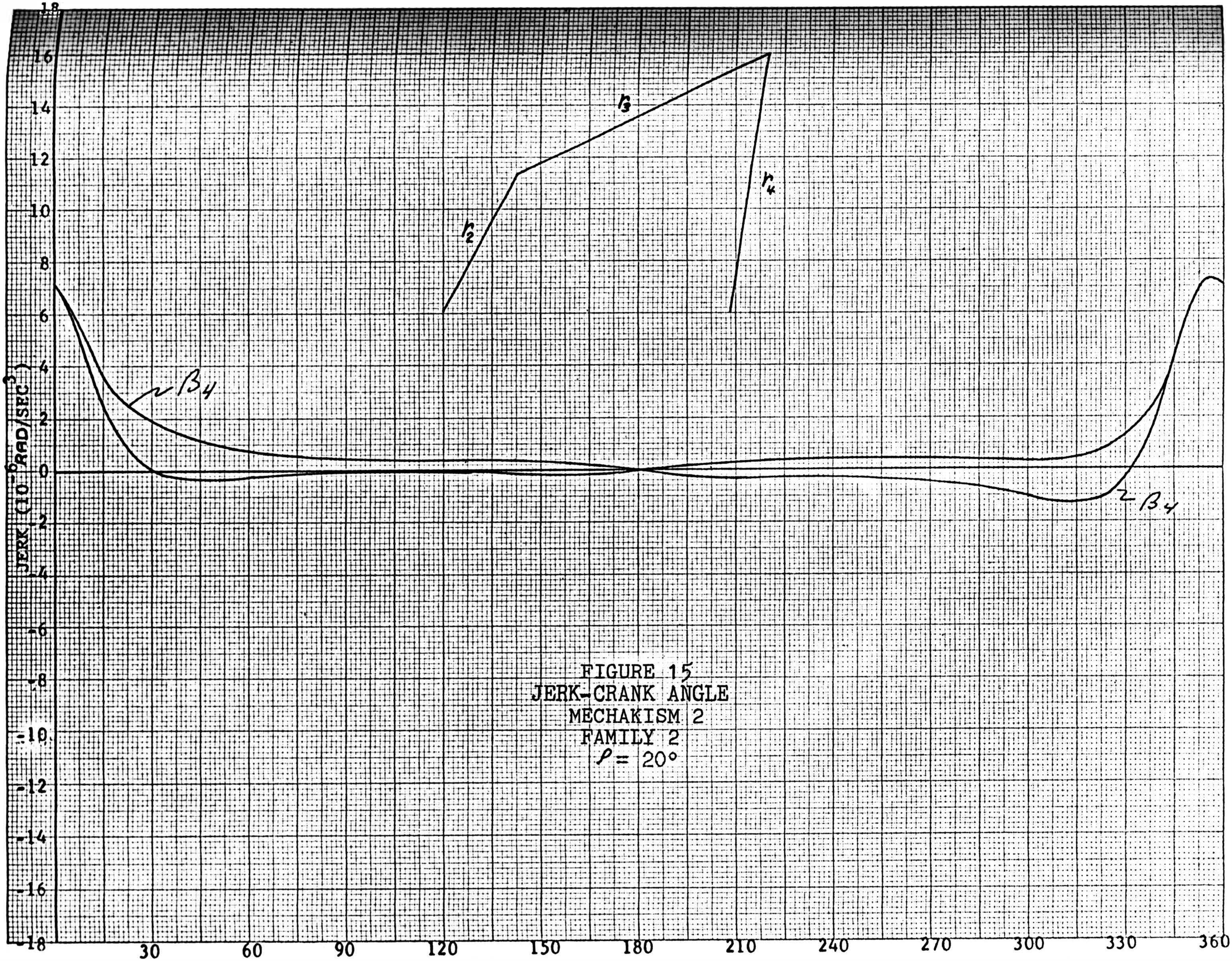


FIGURE 15
JERK-CRANK ANGLE
MECHAKISM 2
FAMILY 2
$\rho = 20°$

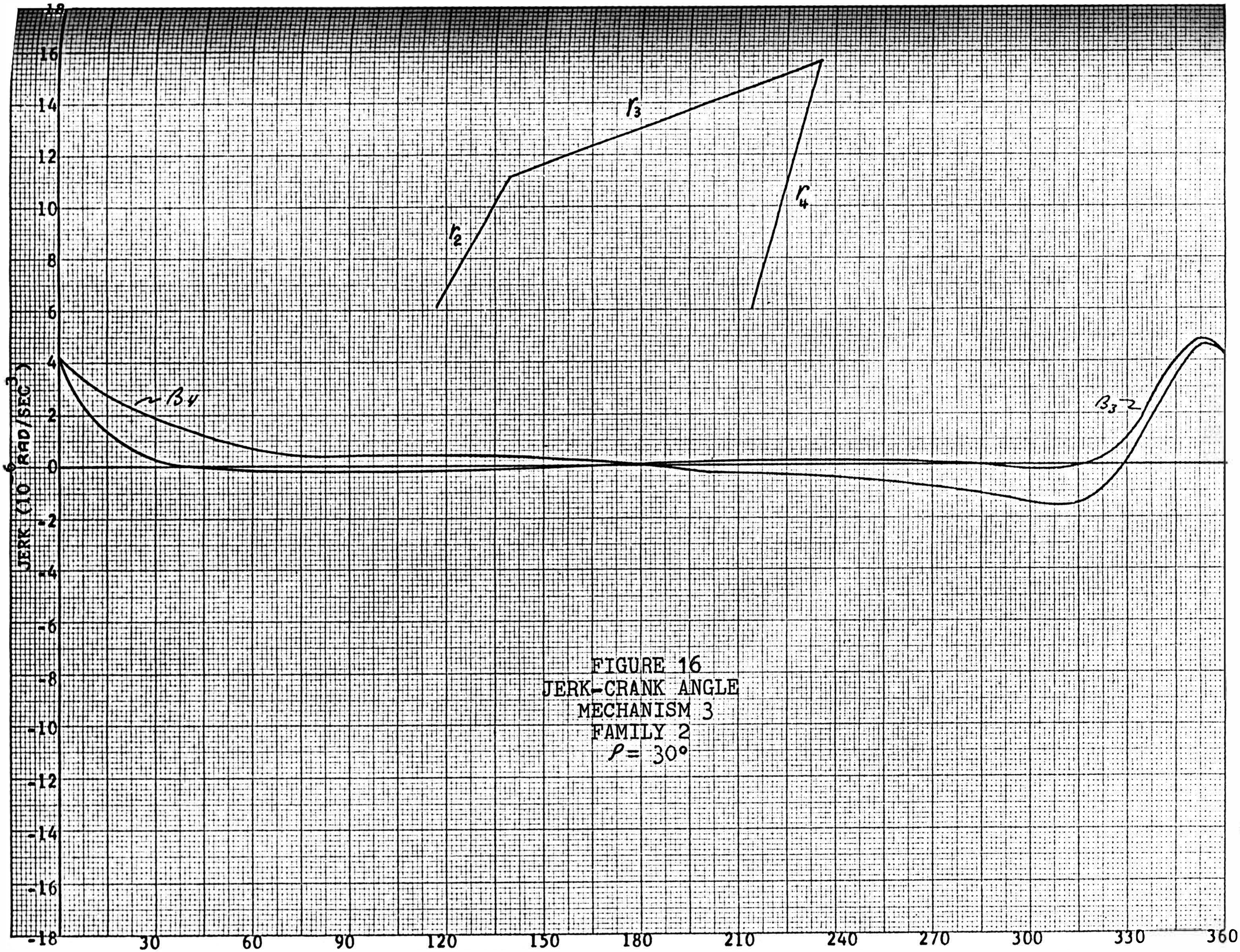


FIGURE 16
JERK-CRANK ANGLE
MECHANISM 3
FAMILY 2
$\rho = 30°$

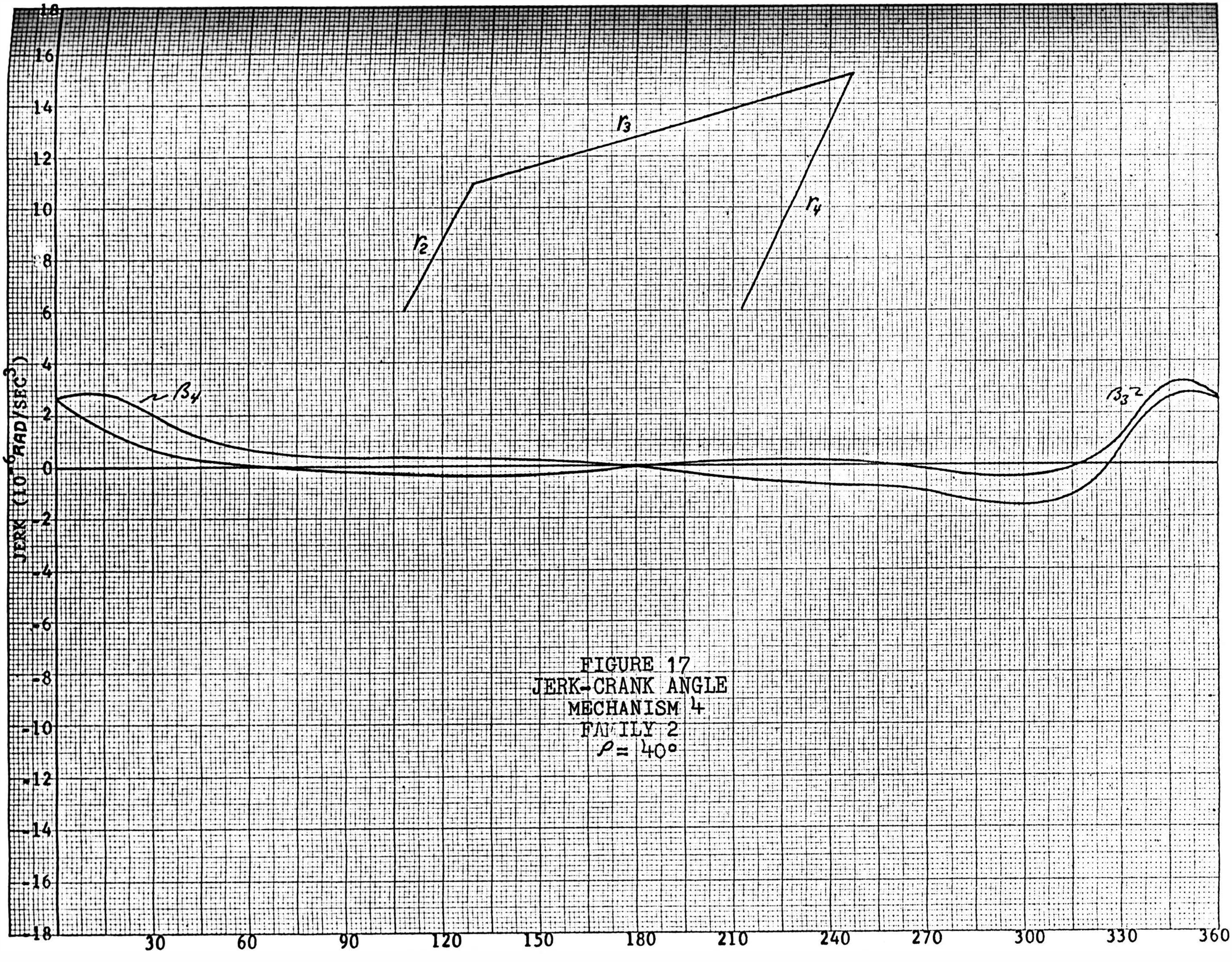


FIGURE 17
JERK-CRANK ANGLE
MECHANISM 4
FAMILY 2
$\rho = 40°$

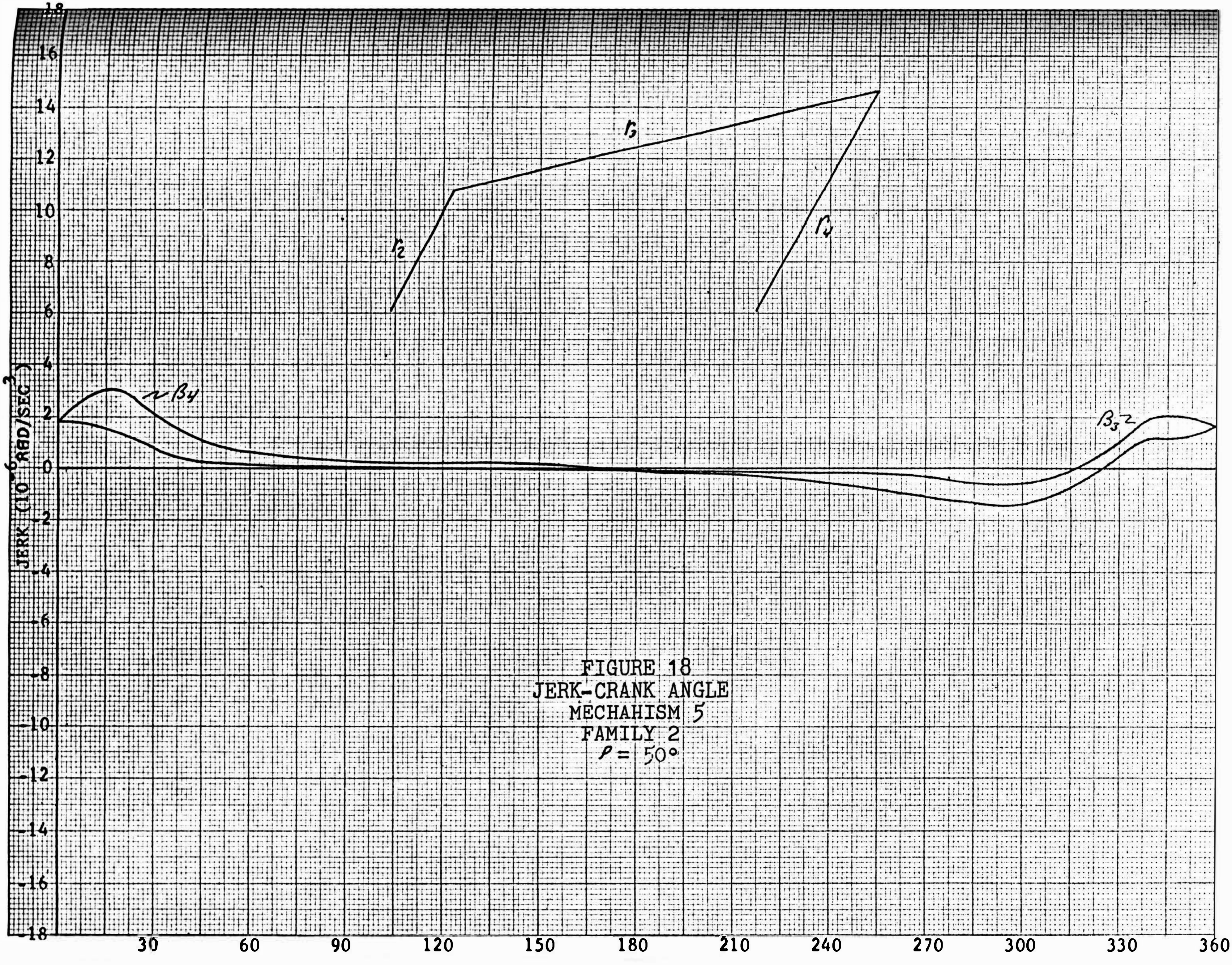


FIGURE 18
JERK-CRANK ANGLE
MECHAHISM 5
FAMILY 2
$\rho = 50°$

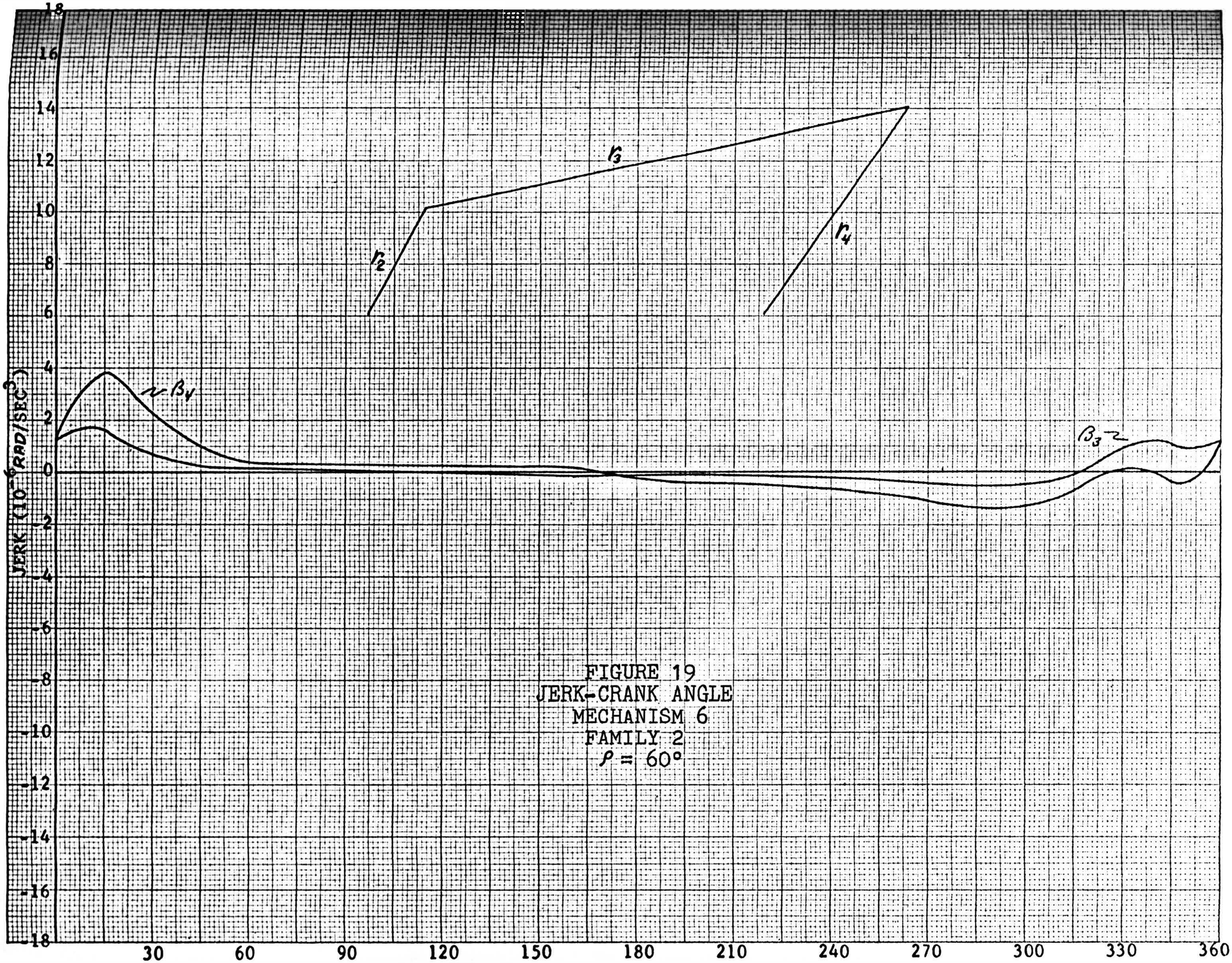


FIGURE 19
JERK-CRANK ANGLE
MECHANISM 6
FAMILY 2
$\rho = 60°$

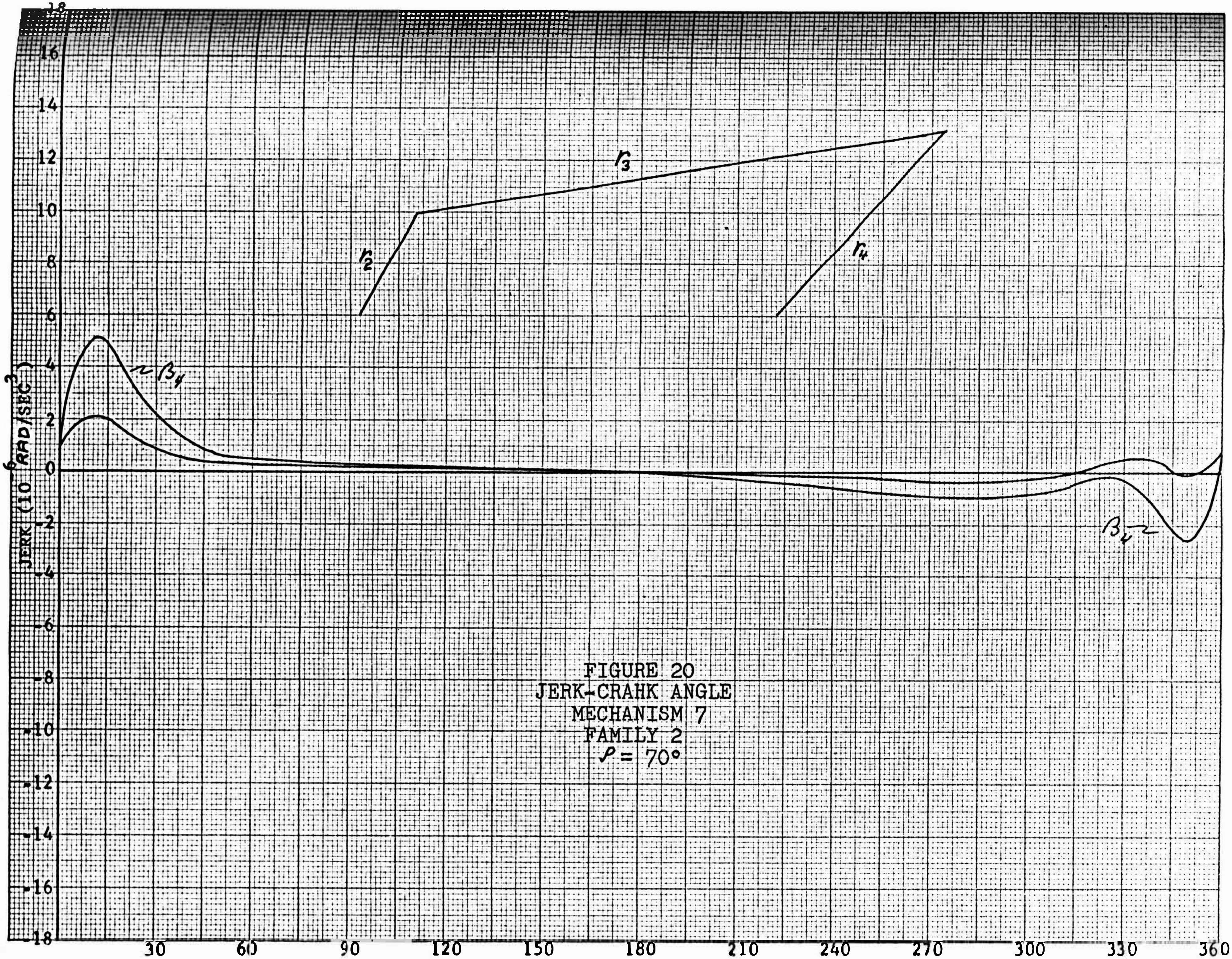


FIGURE 20
JERK-CRAHK ANGLE
MECHANISM 7
FAMILY 2
$\rho = 70°$

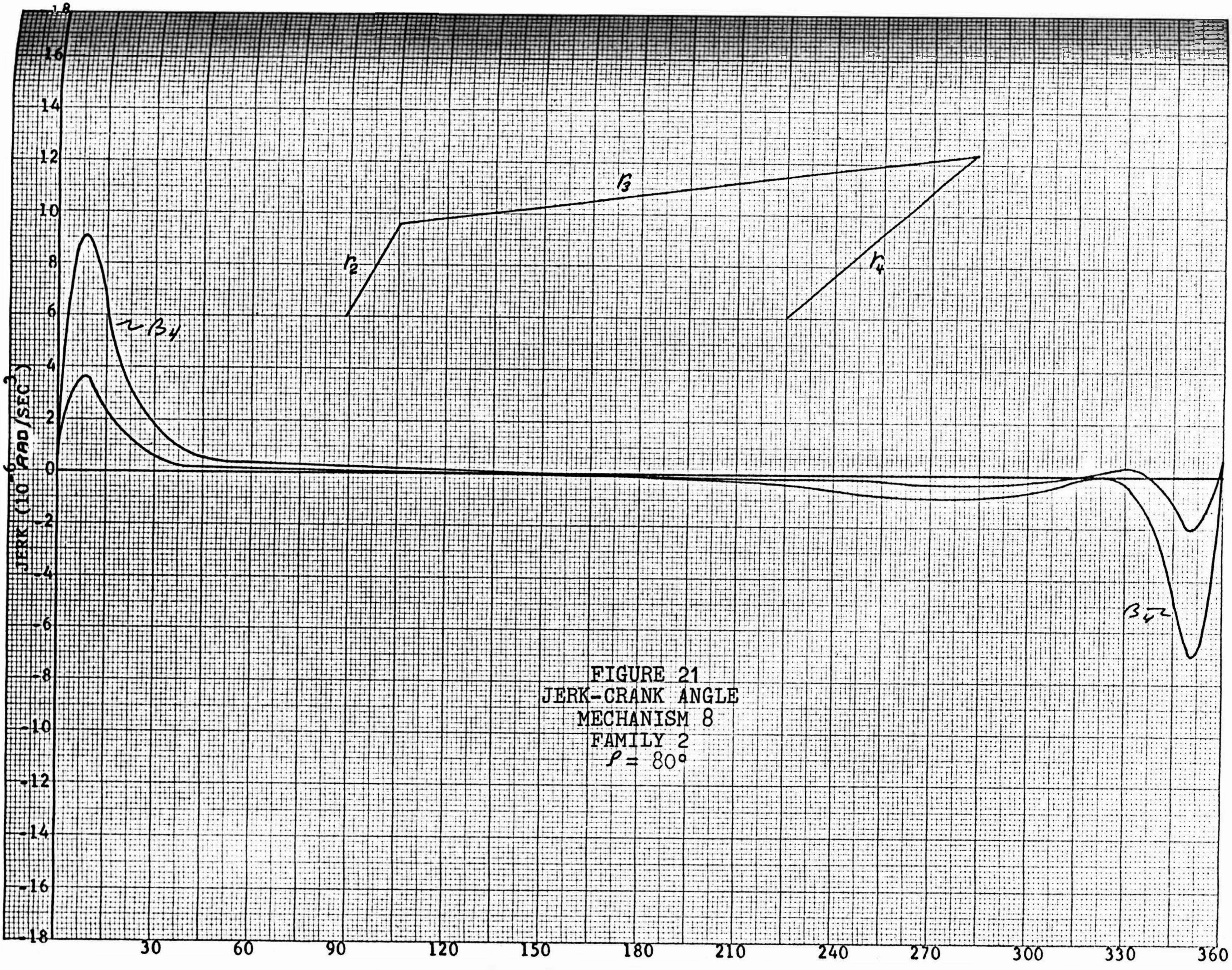


FIGURE 21
JERK-CRANK ANGLE
MECHANISM 8
FAMILY 2
$\rho = 80°$

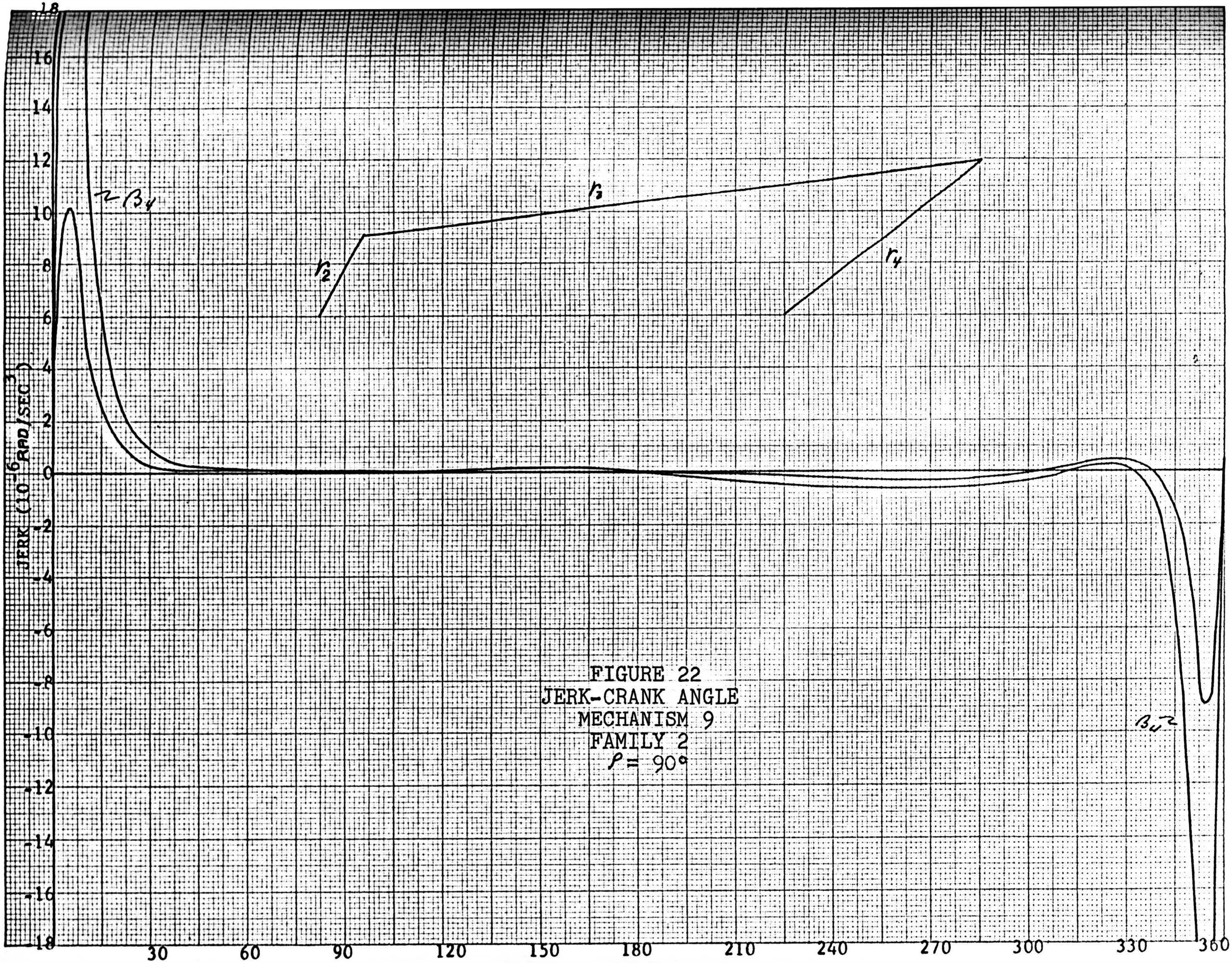


FIGURE 22
JERK-CRANK ANGLE
MECHANISM 9
FAMILY 2
$\rho = 90°$

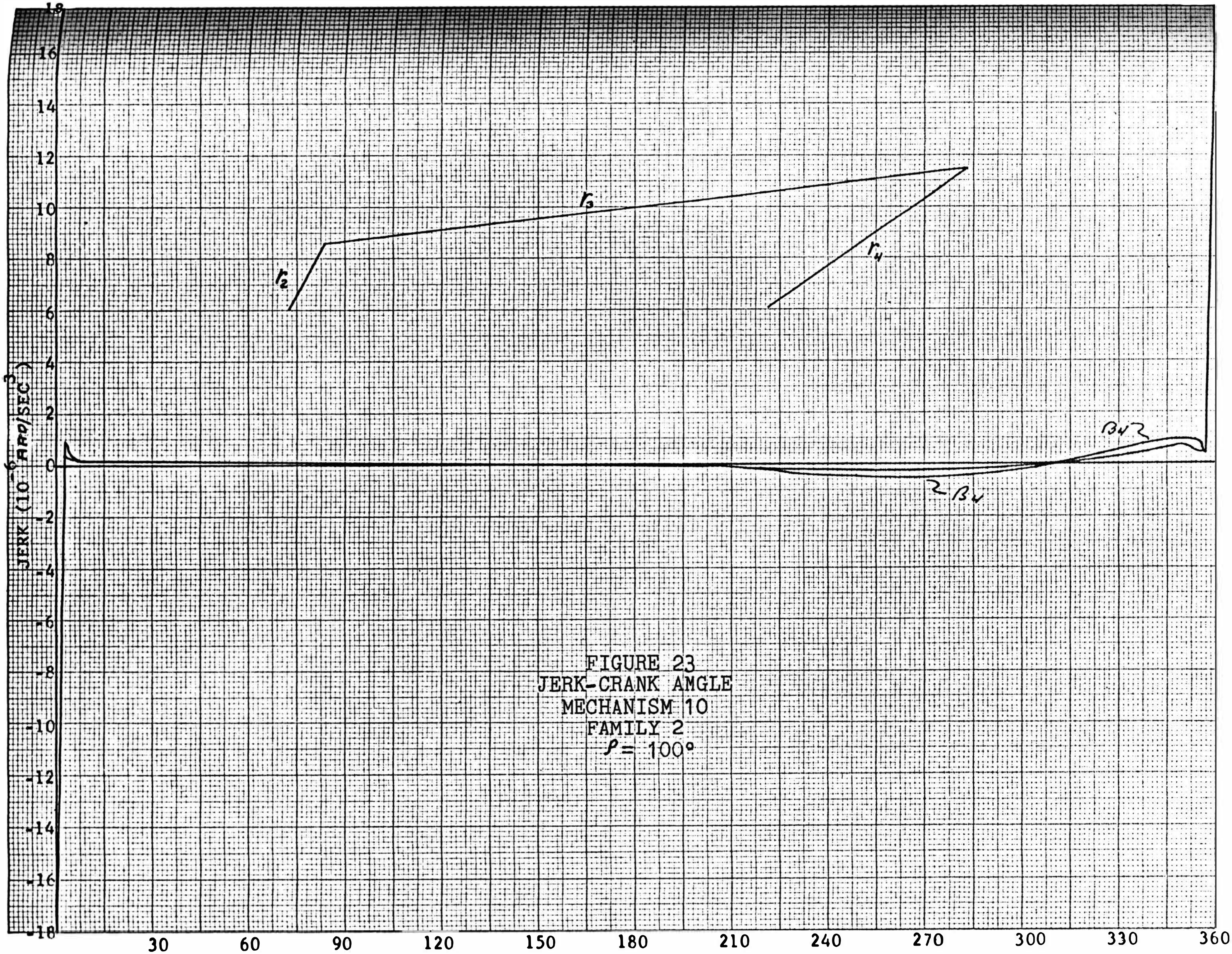


FIGURE 23
JERK-CRANK AMGLE
MECHANISM 10
FAMILY 2
$\rho = 100°$

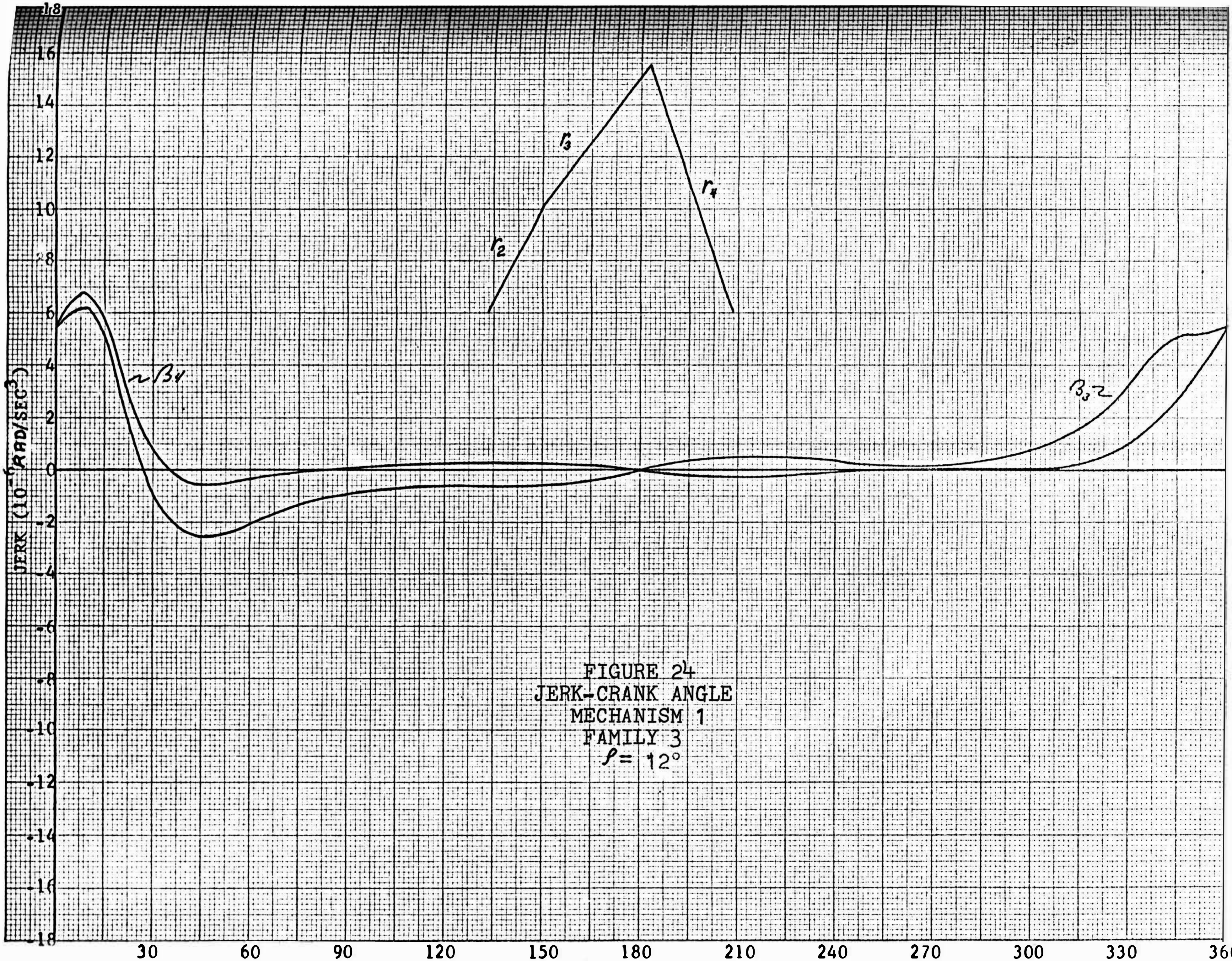


FIGURE 24
JERK-CRANK ANGLE
MECHANISM 1
FAMILY 3
$\rho = 12°$

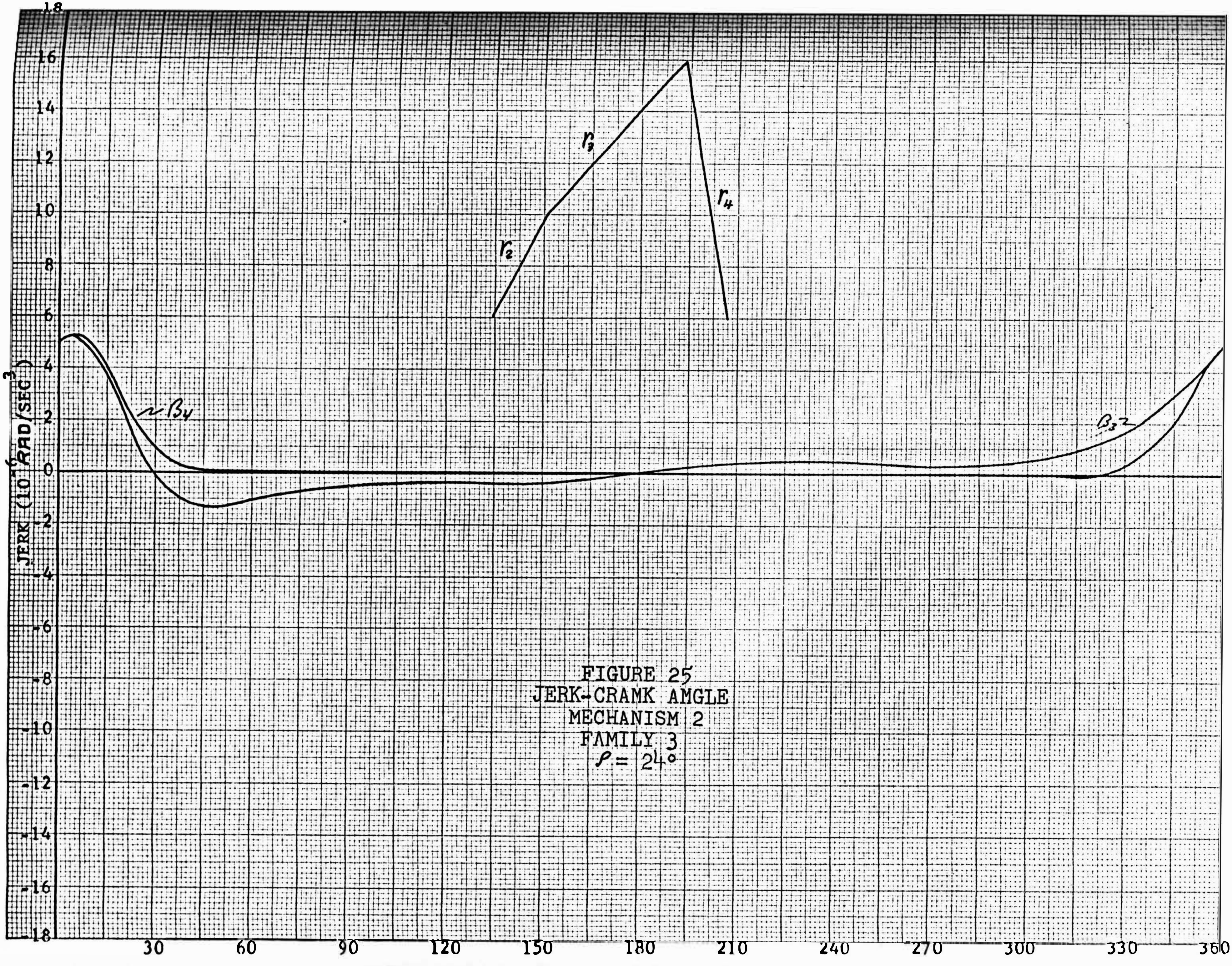


FIGURE 25
JERK-CRAMK AMGLE
MECHANISM 2
FAMILY 3
$\rho = 24°$

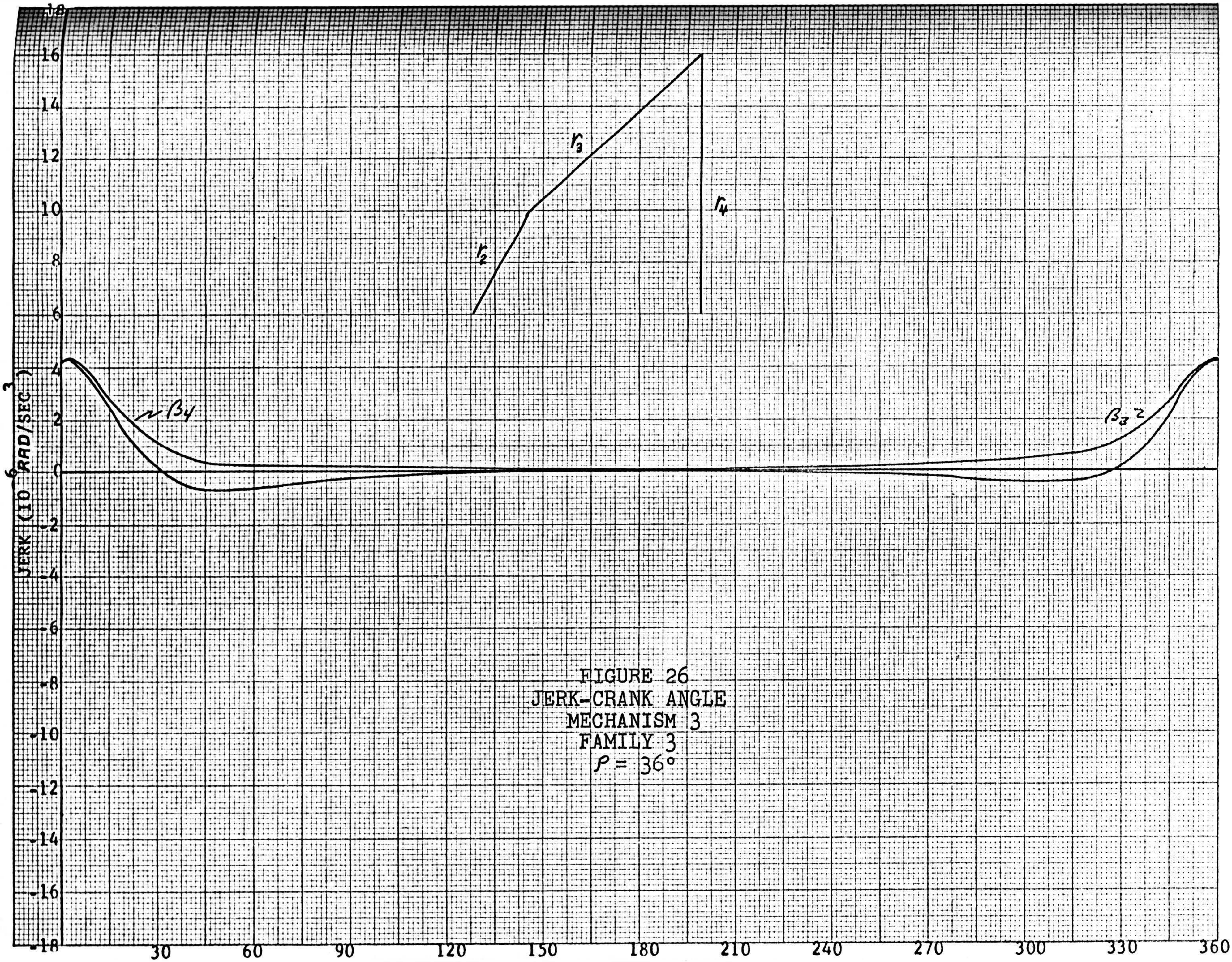


FIGURE 26
JERK-CRANK ANGLE
MECHANISM 3
FAMILY 3
$\rho = 36°$

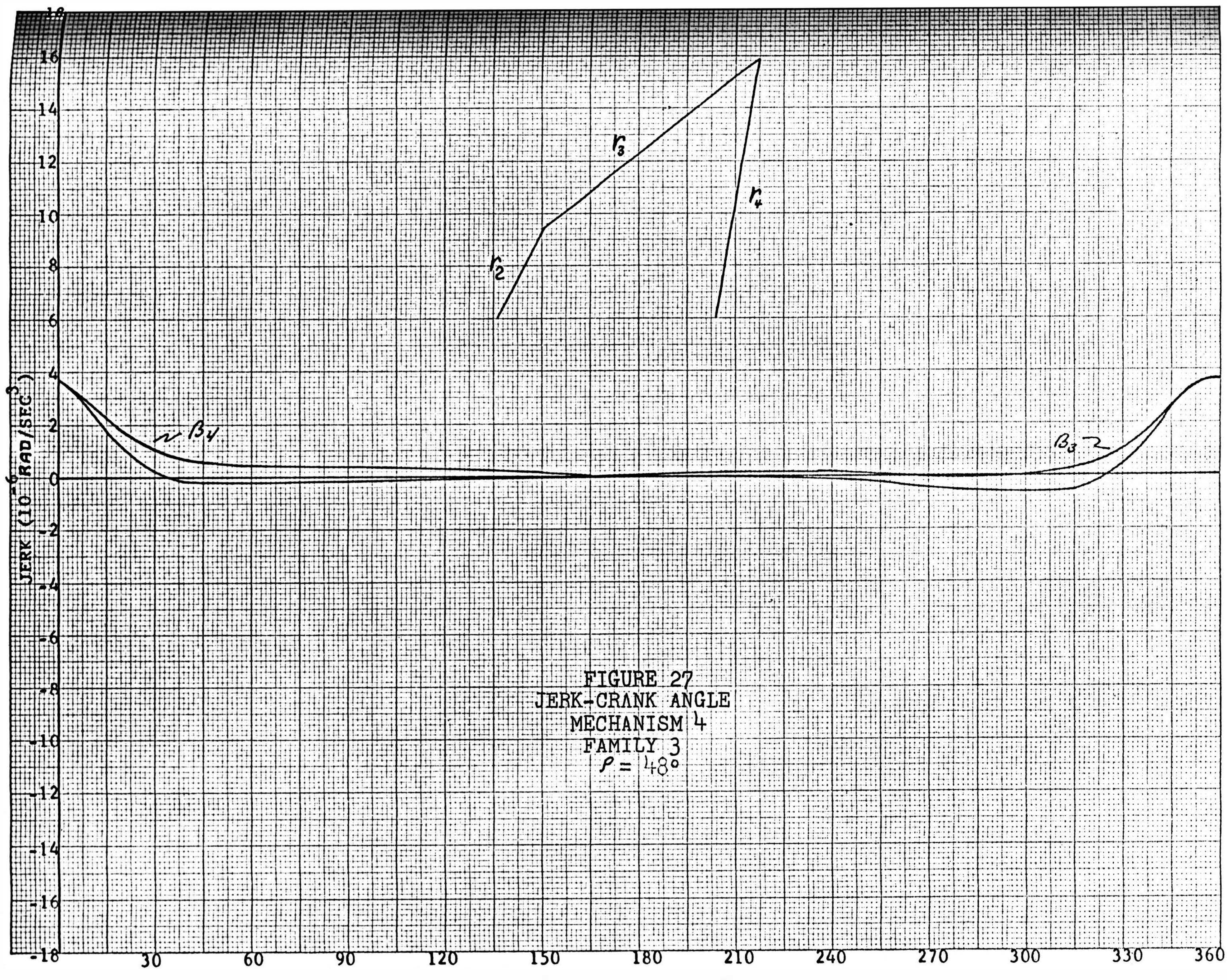


FIGURE 27
JERK-CRANK ANGLE
MECHANISM 4
FAMILY 3
$\rho = 48°$

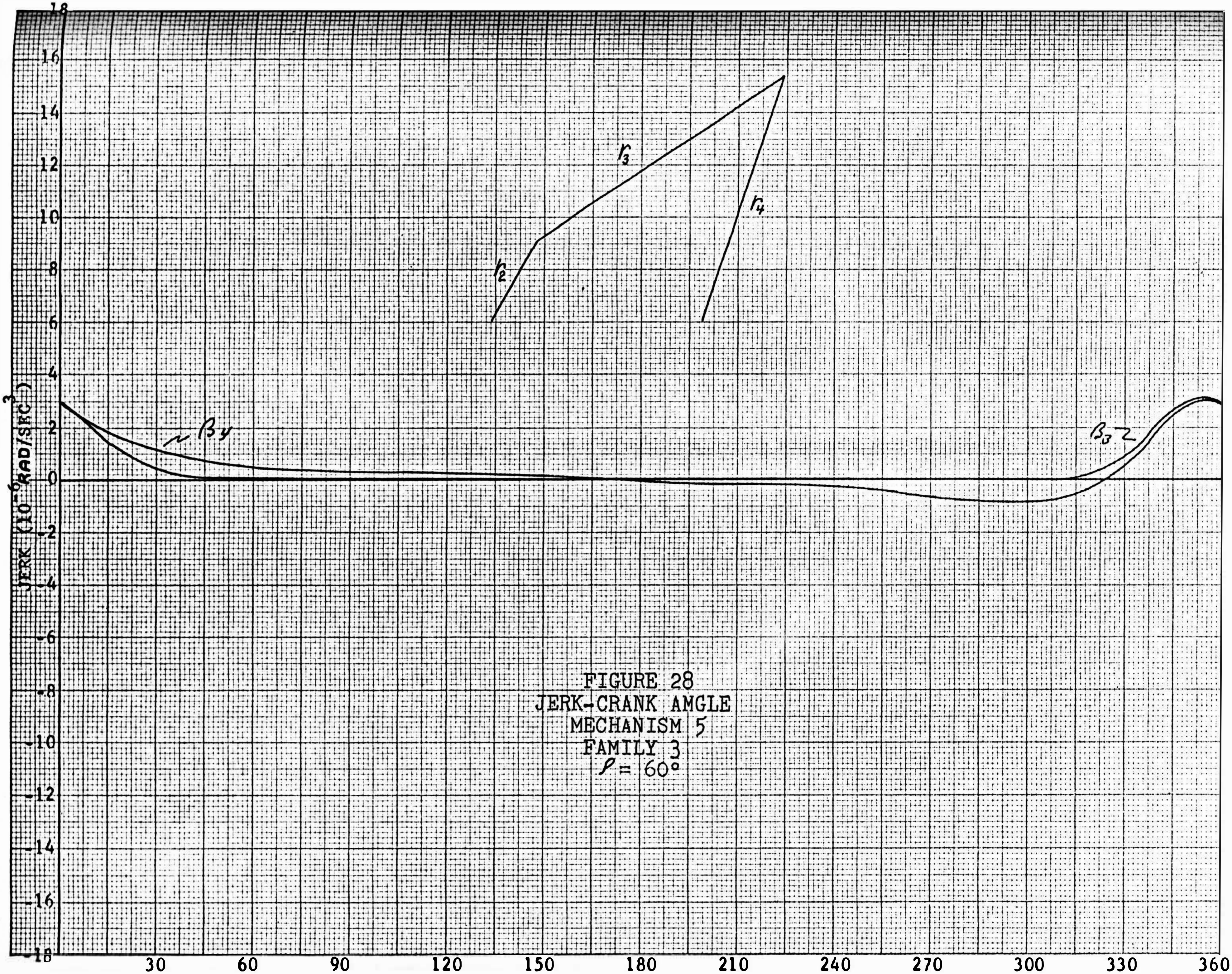


FIGURE 28
JERK-CRANK AMGLE
MECHANISM 5
FAMILY 3
$\rho = 60°$

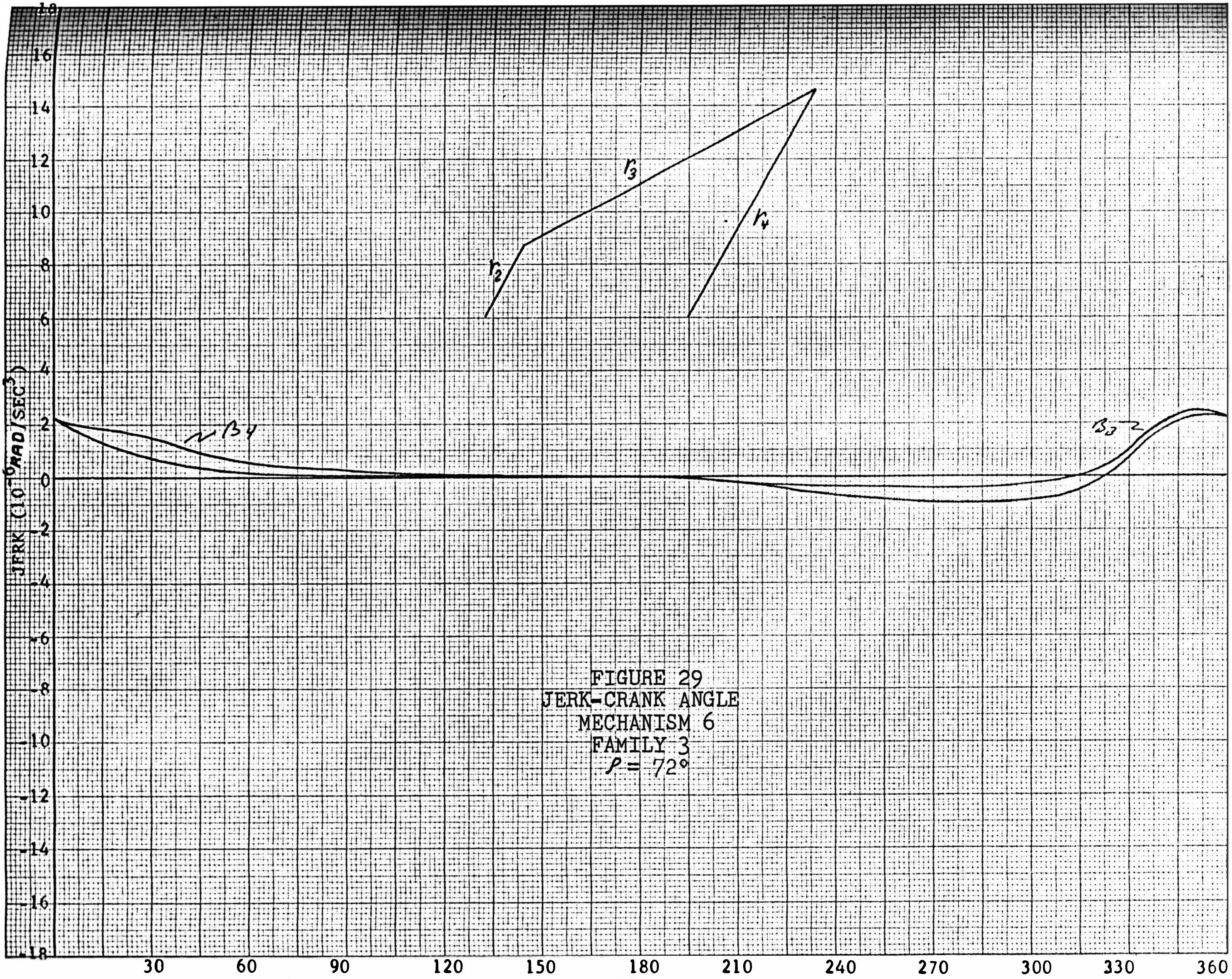


FIGURE 29
JERK-CRANK ANGLE
MECHANISM 6
FAMILY 3
$\rho = 72°$

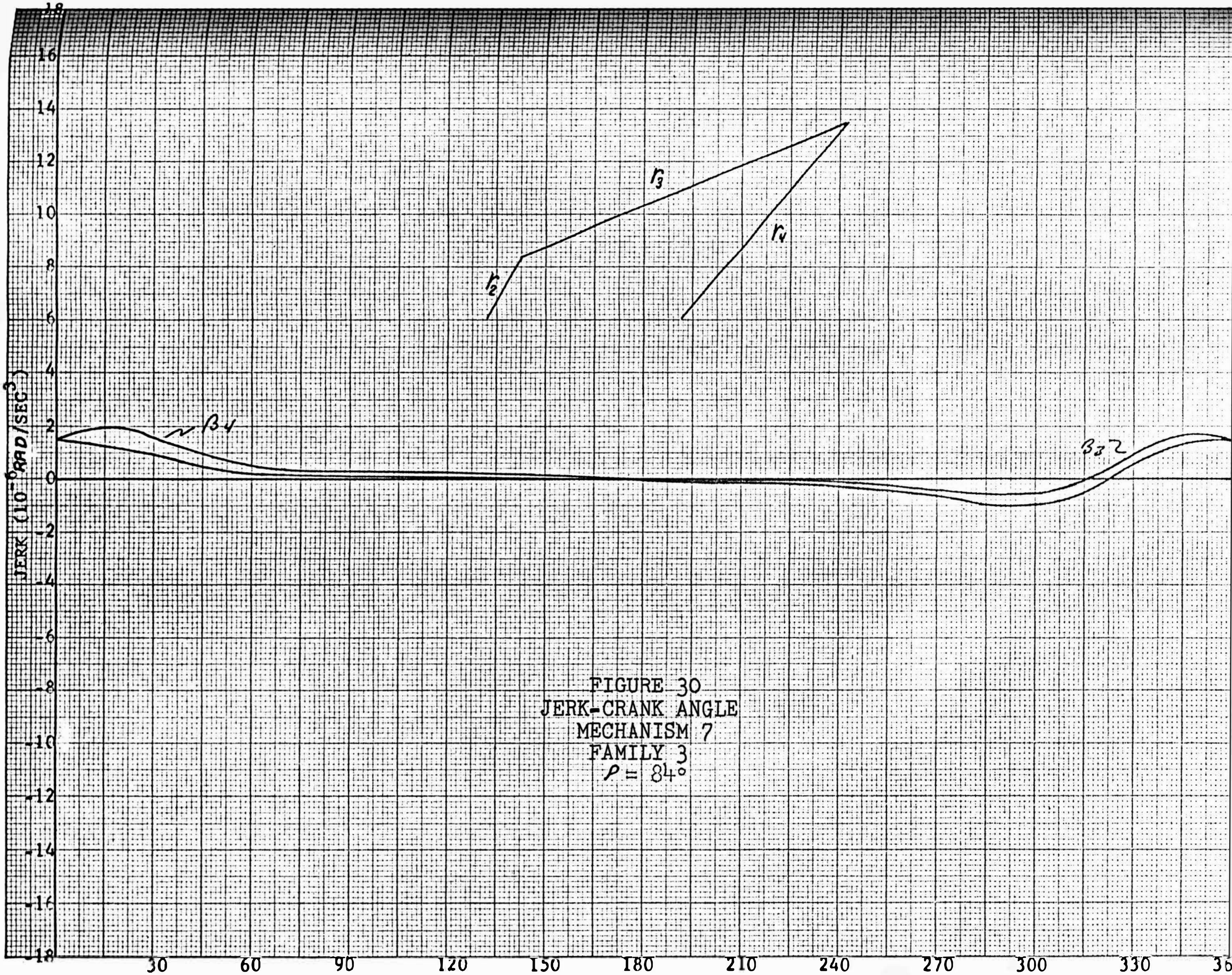


FIGURE 30
JERK-CRANK ANGLE
MECHANISM 7
FAMILY 3
$\rho = 84°$

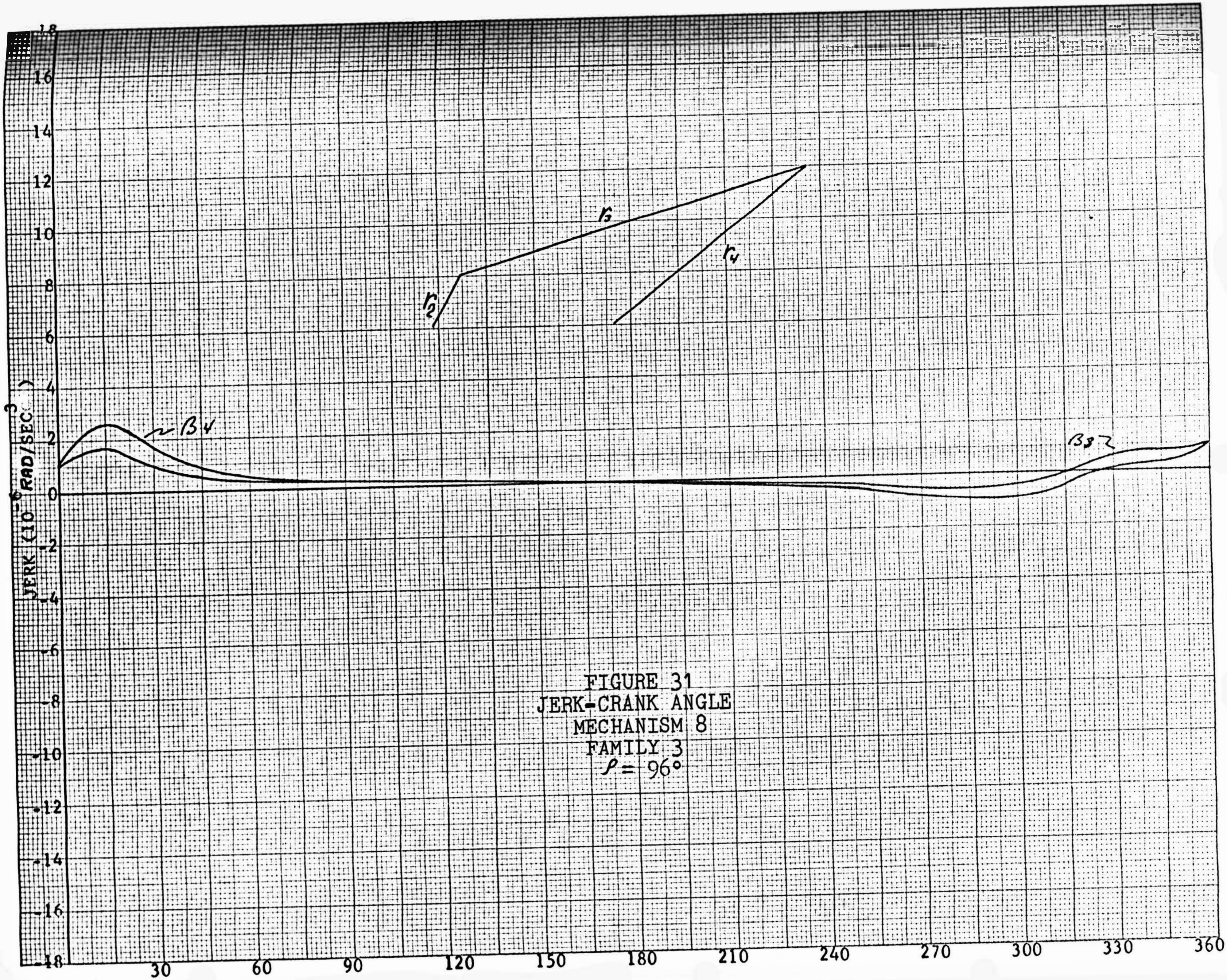


FIGURE 31
JERK-CRANK ANGLE
MECHANISM 8
FAMILY 3
$\rho = 96°$

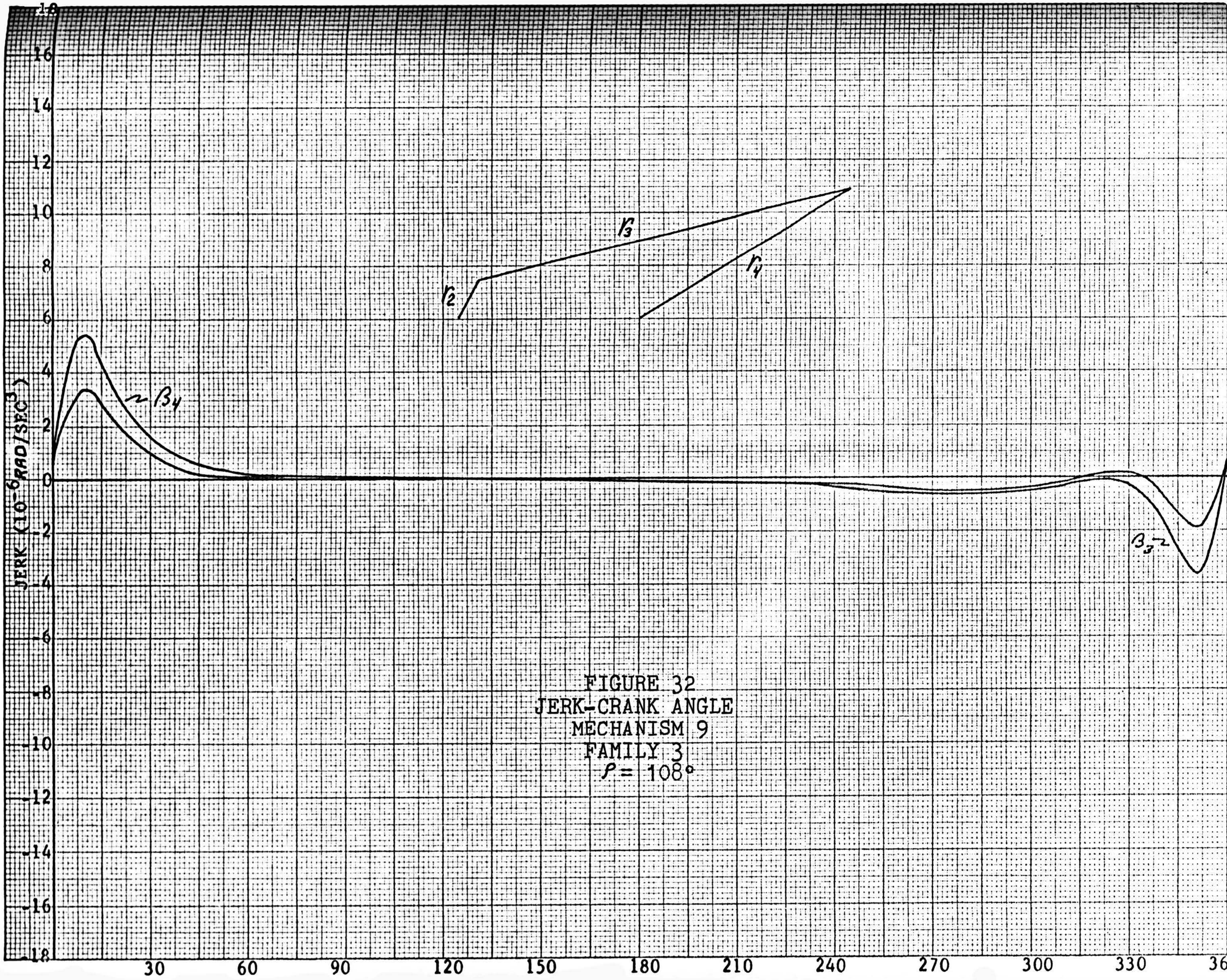


FIGURE 32
JERK-CRANK ANGLE
MECHANISM 9
FAMILY 3
$\rho = 108°$

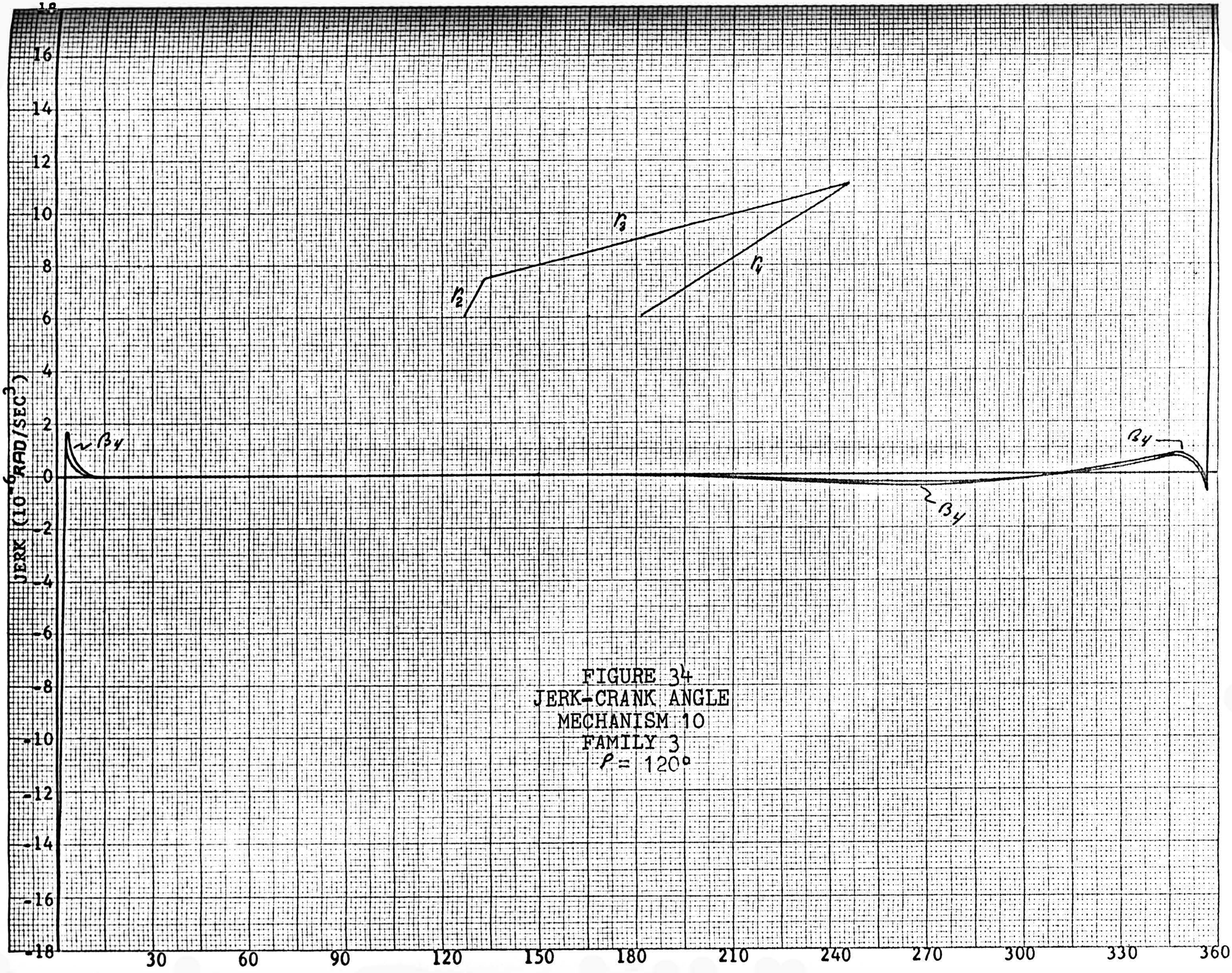


FIGURE 34
JERK-CRANK ANGLE
MECHANISM 10
FAMILY 3
$\rho = 120°$

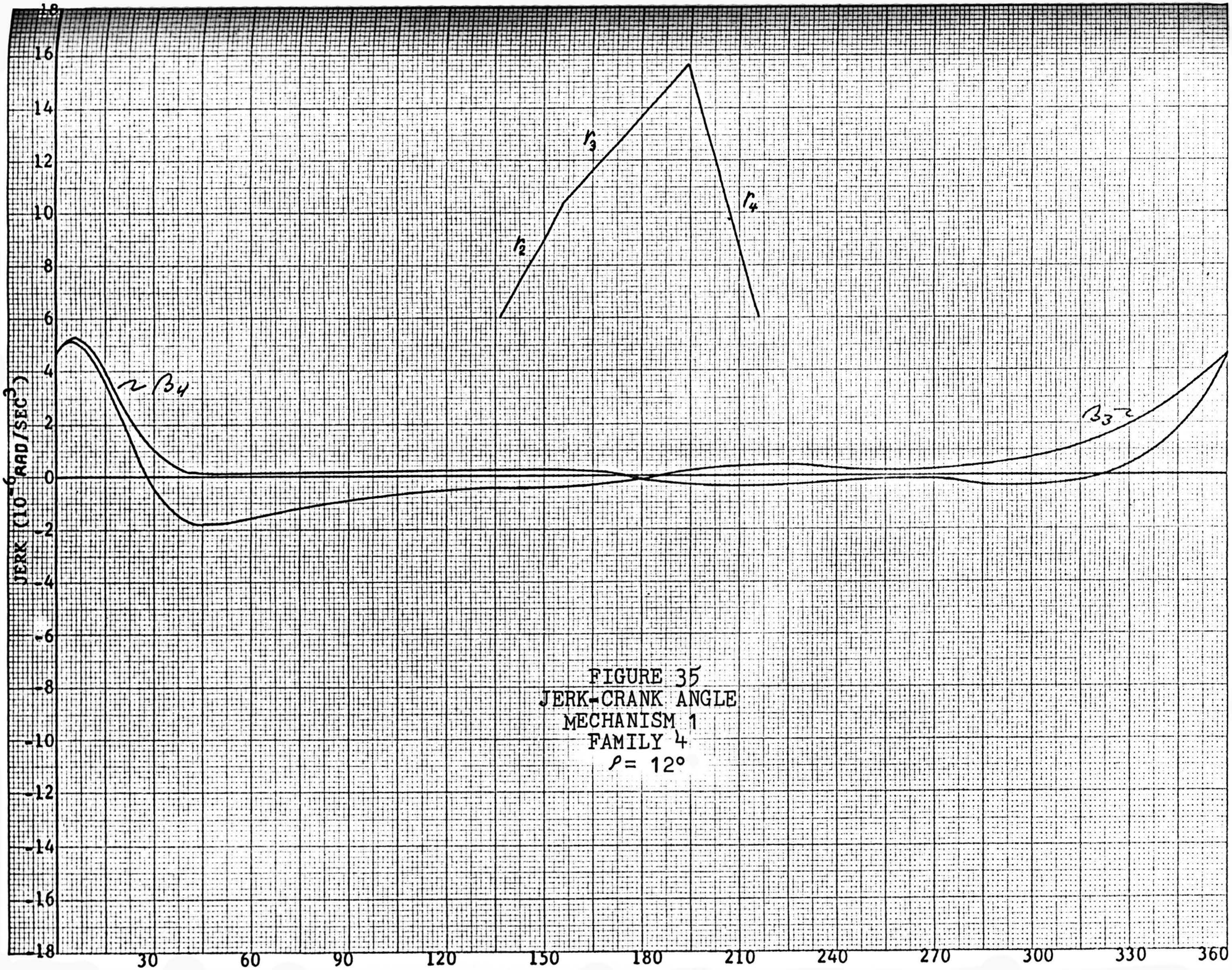


FIGURE 35
JERK-CRANK ANGLE
MECHANISM 1
FAMILY 4
$\rho = 12°$

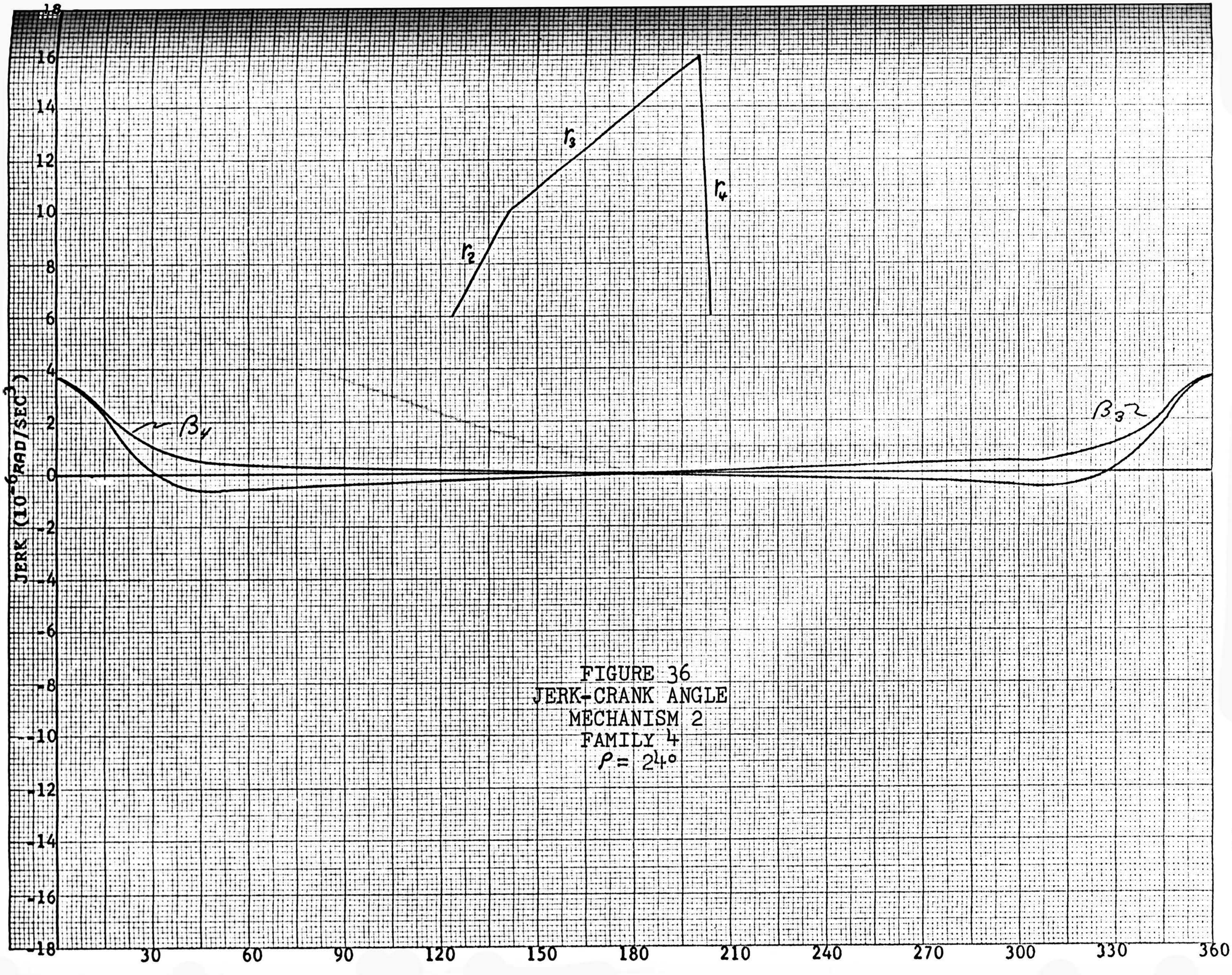


FIGURE 36
JERK-CRANK ANGLE
MECHANISM 2
FAMILY 4
$\rho = 24°$

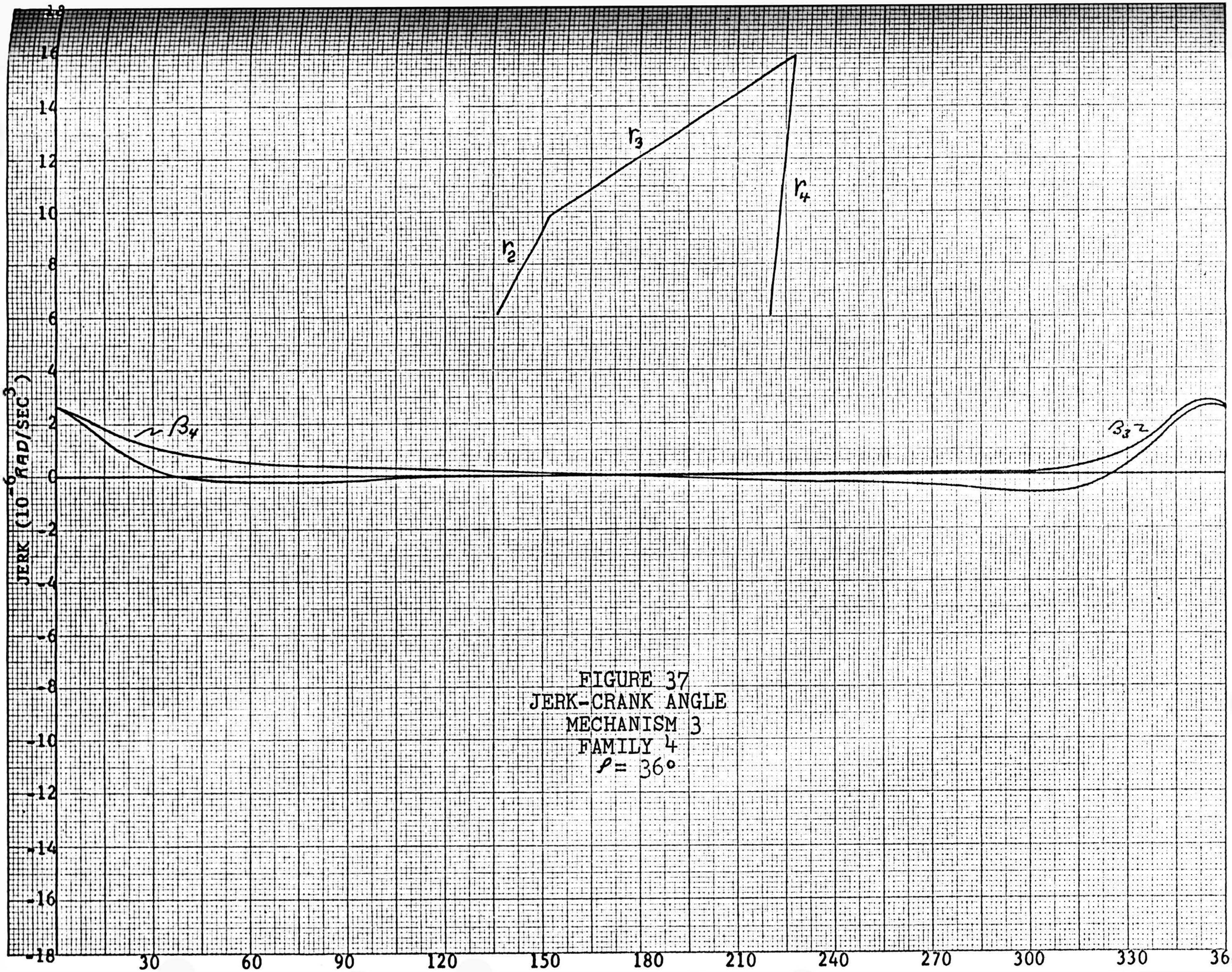


FIGURE 37
JERK-CRANK ANGLE
MECHANISM 3
FAMILY 4
$\rho = 36°$

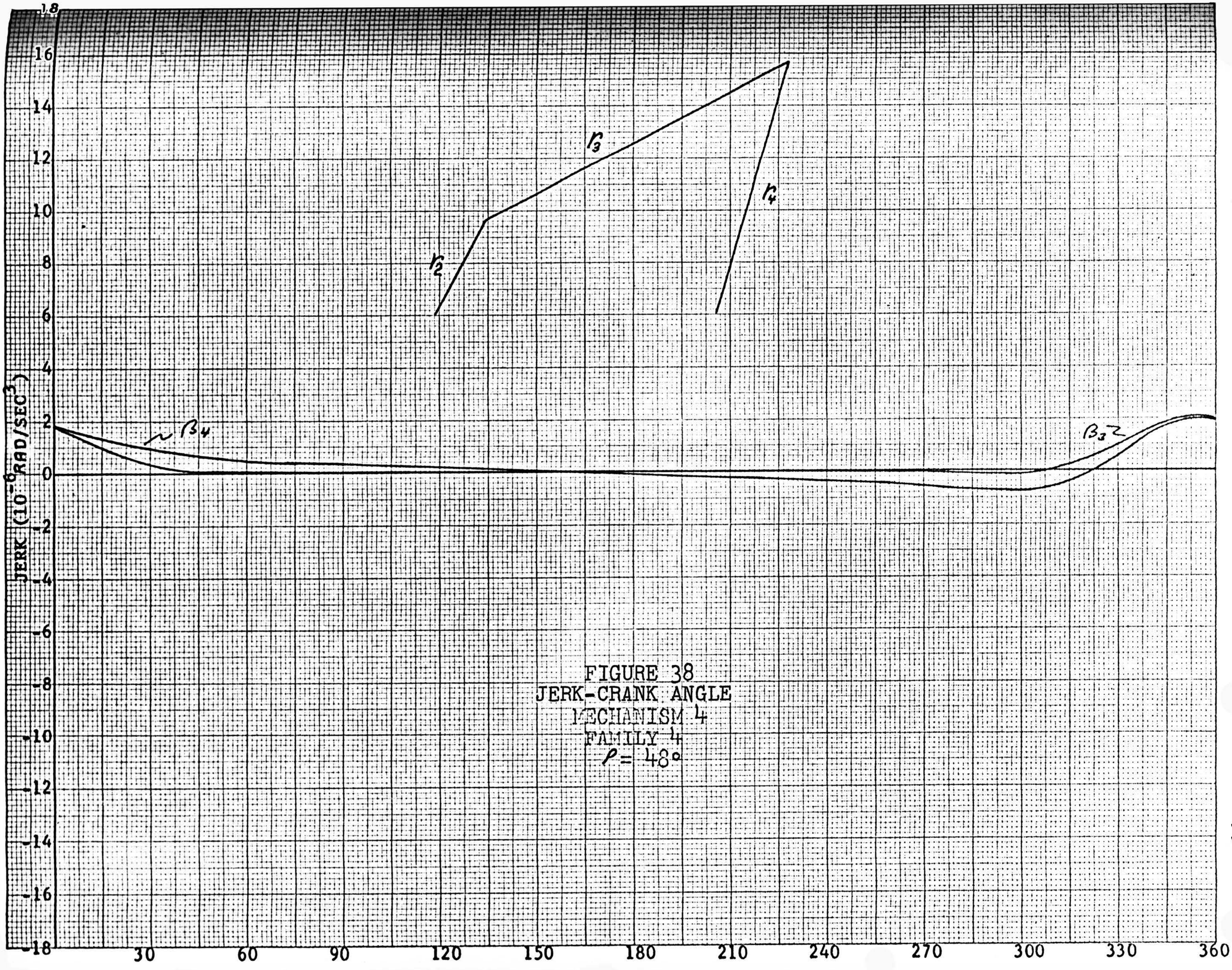
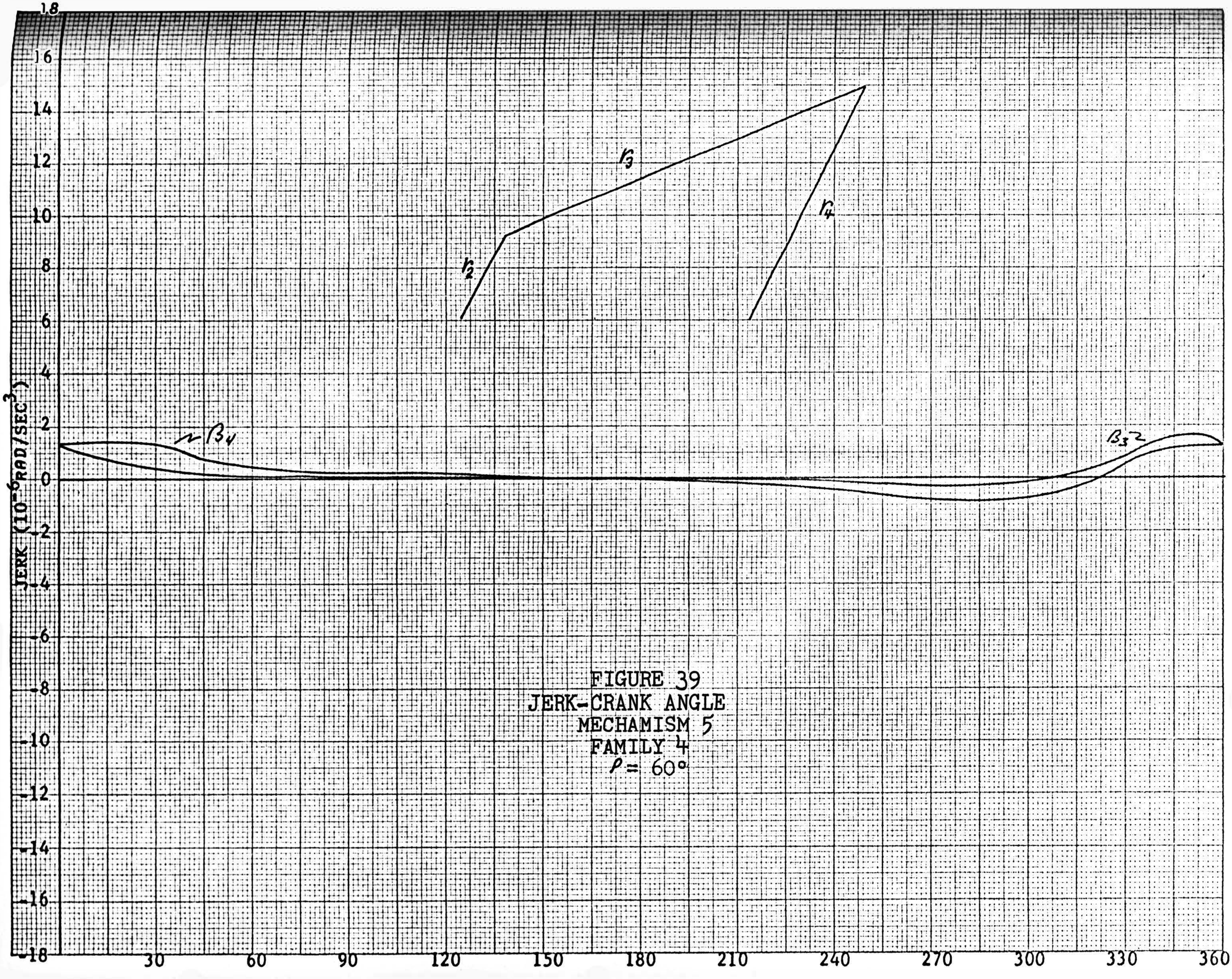


FIGURE 38
JERK-CRANK ANGLE
MECHANISM 4
FAMILY 4
$\rho = 48°$


JERK (10^{-6} RAD/SEC^3)
18
16
14
12
10
8
6
4
2
0
-2
-4
-6
-8
-10
-12
-14
-16
-18
30
60
90
120
150
180
210
240
270
300
330
360
γ_2
γ_3
γ_4
β_4
β_3
FIGURE 39
JERK-CRANK ANGLE
MECHAMISM 5
FAMILY 4
$\rho = 60°$

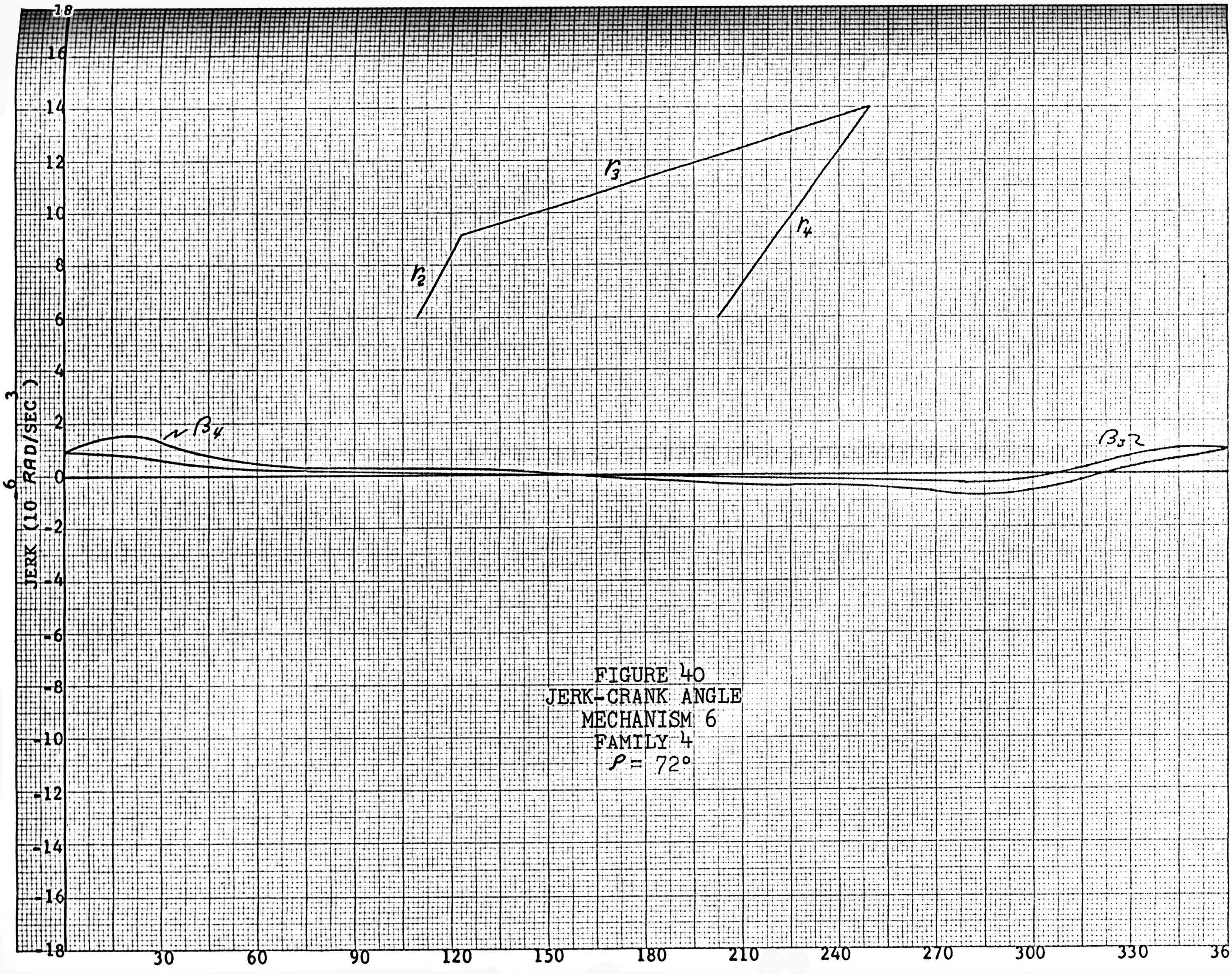


FIGURE 40
JERK-CRANK ANGLE
MECHANISM 6
FAMILY 4
$\rho = 72°$

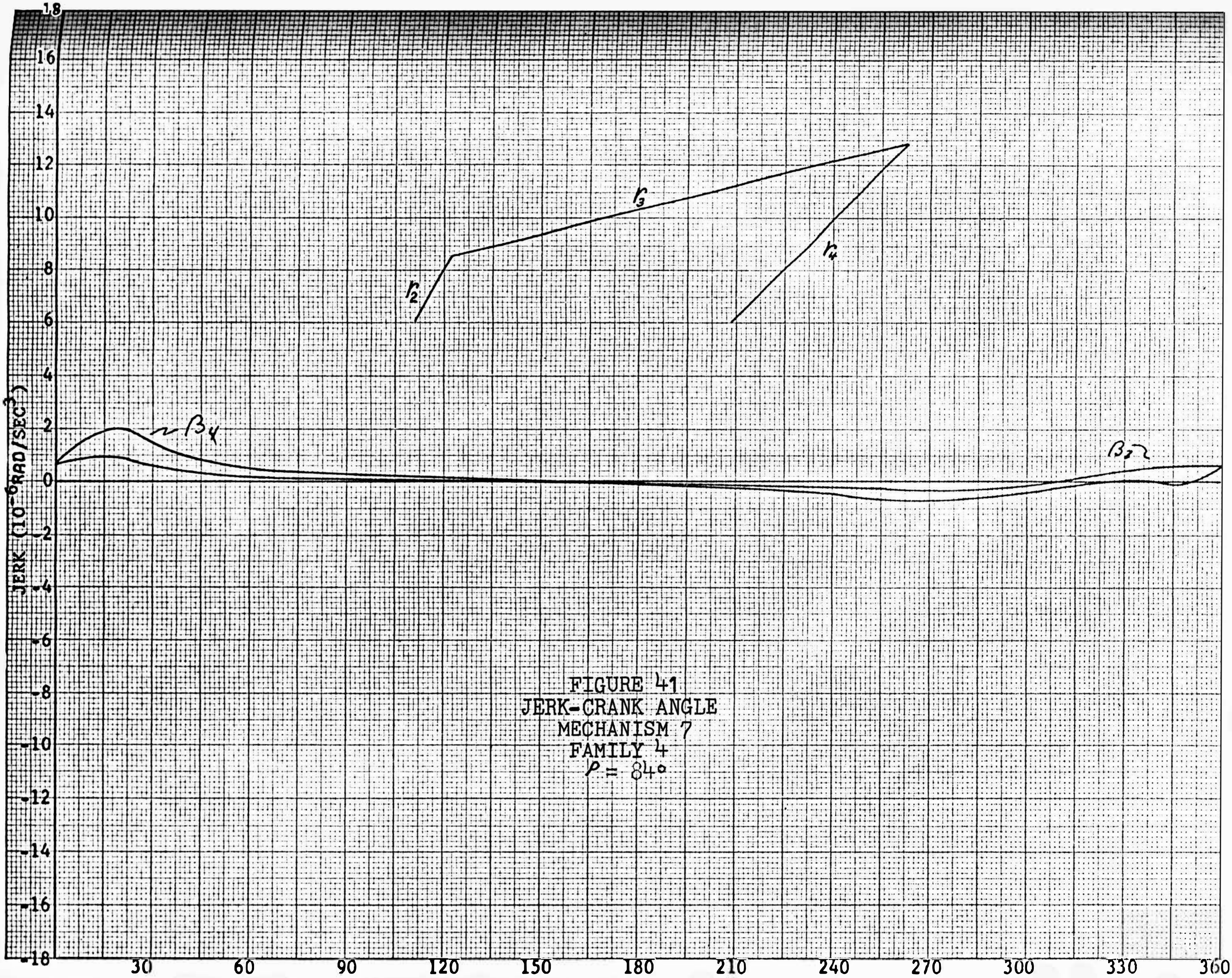


FIGURE 41
JERK-CRANK ANGLE
MECHANISM 7
FAMILY 4
$\rho = 84°$

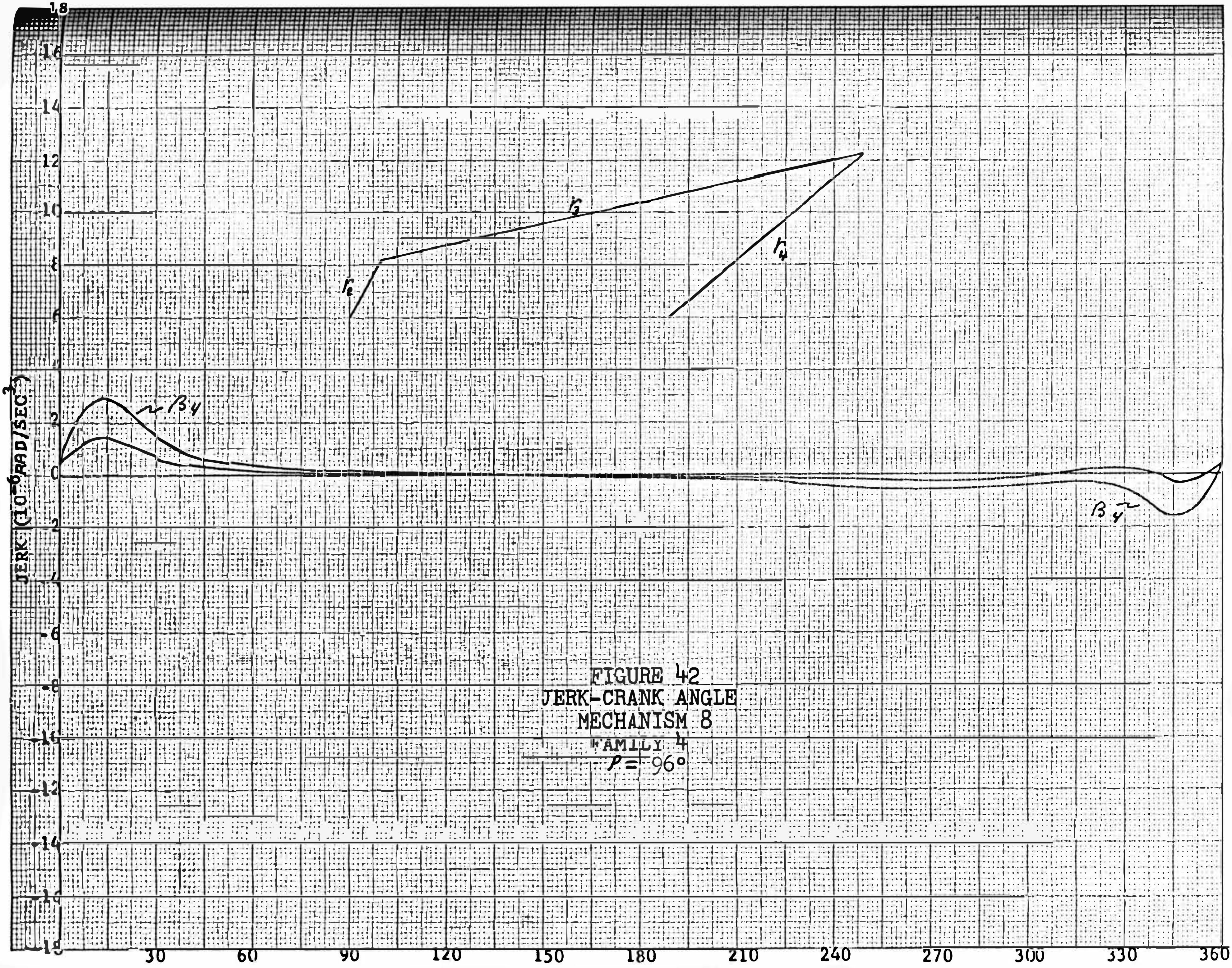


FIGURE 42
JERK-CRANK ANGLE
MECHANISM 8
FAMILY 4
$\rho = 96°$

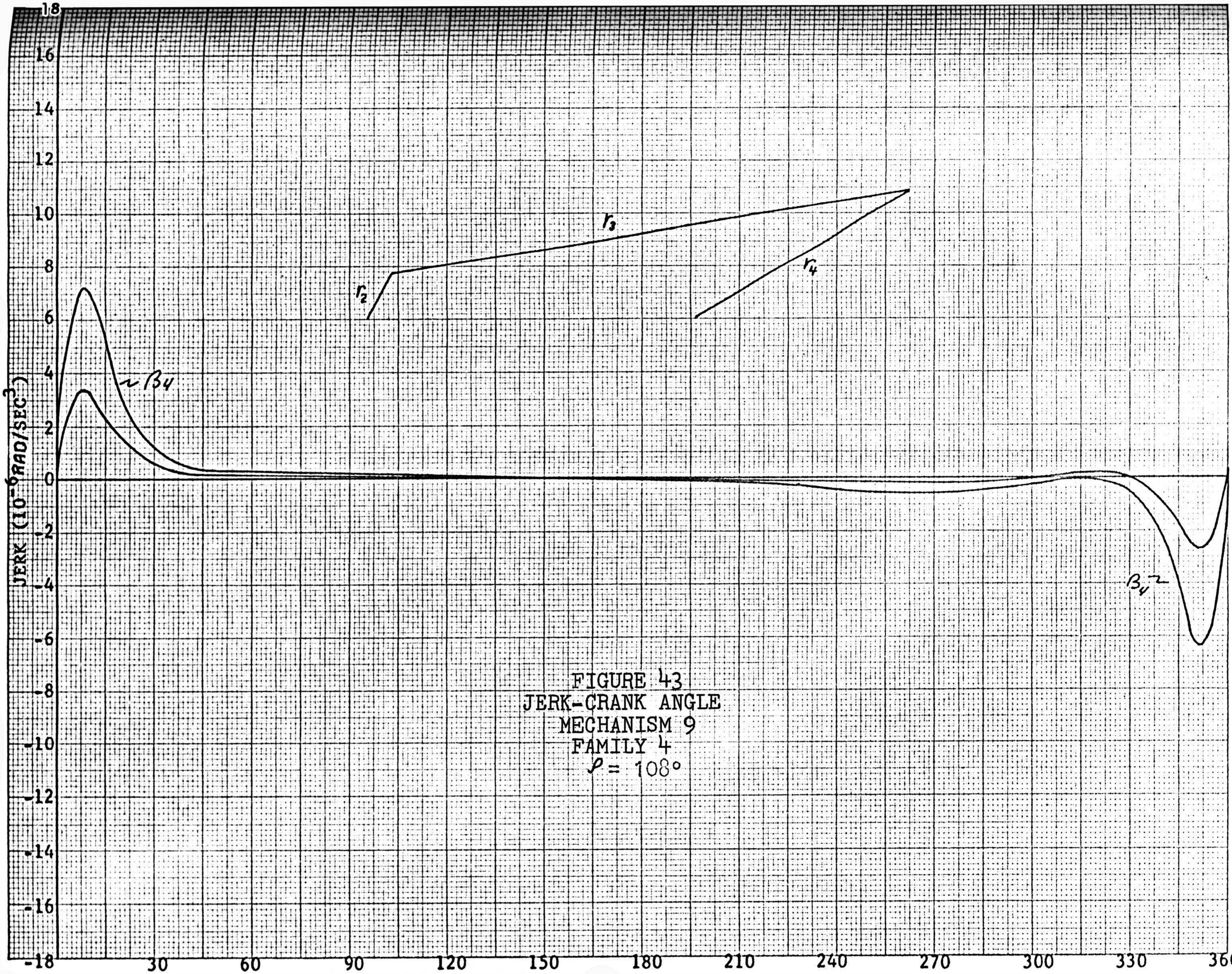


FIGURE 43
JERK-CRANK ANGLE
MECHANISM 9
FAMILY 4
ρ = 108°

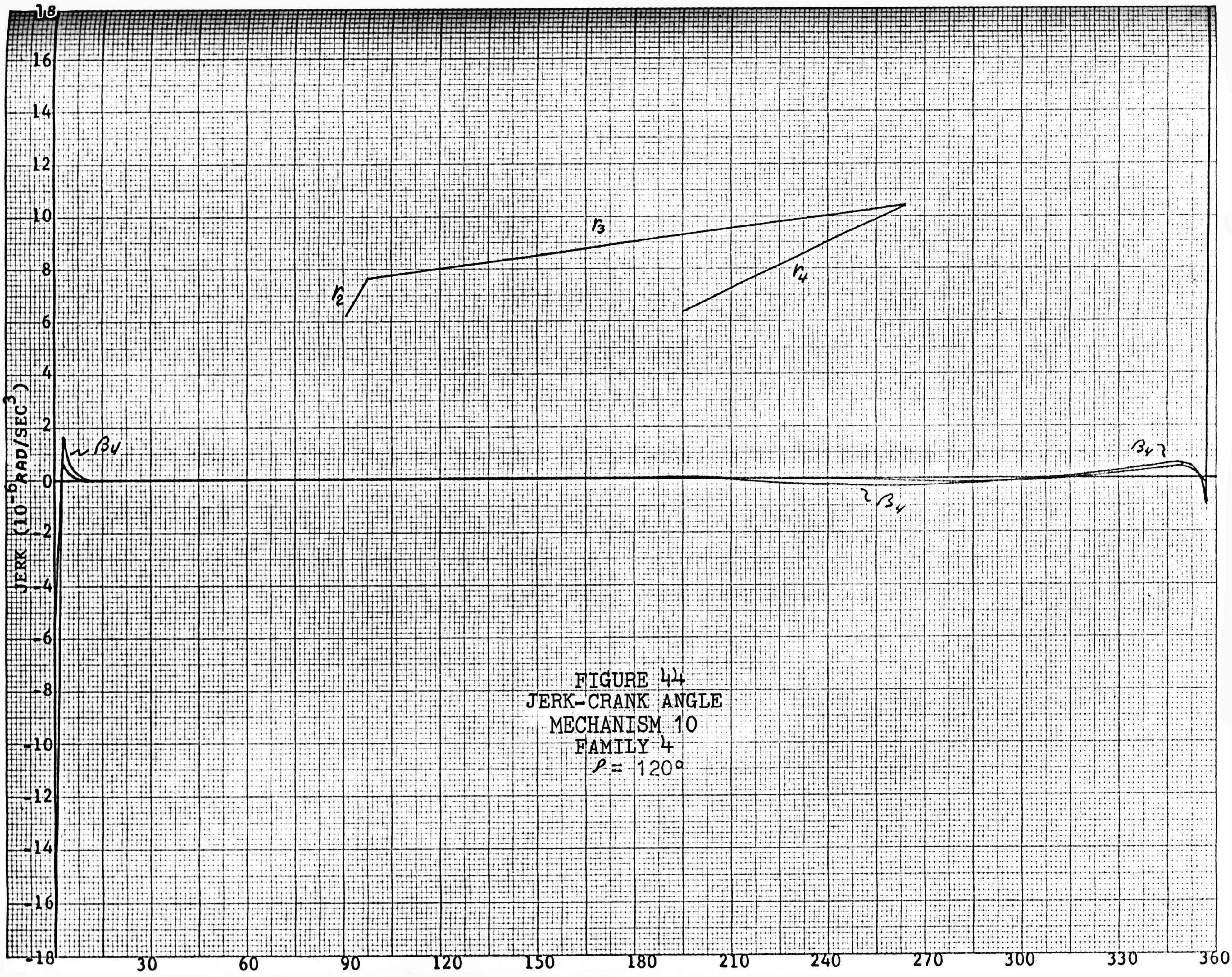


FIGURE 44
JERK-CRANK ANGLE
MECHANISM 10
FAMILY 4
$\rho = 120°$

that is noticed in each family is that as the angle increases the maximum value of jerk, in the lever and coupler of each mechanism, decreases until a minimum value is reached, then the jerk increases until a maximum that is essentially infinity is reached.

Considering just the first family one can see that for the first mechanism there is a very large value of maximum jerk occurring in both the lever and the coupler. In the second mechanism the value of maximum jerk in the lever and coupler is less than it was in the first mechanism. The value of maximum jerk in the lever continues to decrease until its minimum value is reached in the sixth mechanism. The value of jerk then increases for each mechanism until it becomes nearly infinite for the last mechanism. The maximum jerk in the coupler continues to decrease until a minimum value is reached in the seventh mechanism. Then it increases until it approaches infinity for the last mechanism. The maximum jerk in the lever and the coupler occurs at the same time in each of the first five mechanisms, then for the sixth mechanism the maximum jerk in the lever does not take place at the same time as the maximum jerk in the coupler. This is important because when the jerk in the lever and coupler have the same sign and occur at the same time the effect on the lever and crank pins is not as great as it would be if the maximum jerks occurred at different times or were of opposite sign. In the seventh mechanism a second point of

maximum jerk in each link is starting to appear and this point becomes readily discernible in the eighth mechanism. The eighth and ninth mechanisms both have very large values of maximum jerk that are of opposite sign. Then in the last mechanism the jerk is nearly infinite. To pick the best mechanism in this family one can start by eliminating mechanisms 1, 2, 3, 4, 8, 9, and 10 on the basis that the values of maximum jerk are too high. Of the three that are left, 5, 6, and 7, seven can be eliminated because on this mechanism the second point of maximum jerk is beginning to appear. As a margin of safety one wouldn't want to design near this mechanism. Number six can be ruled out because the maximum jerk in the lever doesn't occur at the same time that it does in the coupler. This leaves mechanism number five as the best choice. Actually the maximum jerks occurring in the links of this mechanism are not much greater than the maximum jerks occurring in mechanism six.

Family two shows the same trend as the first family except that the lowest values of maximum jerk in the lever and coupler occur in mechanisms four and six respectively. The fourth family has the lowest values of maximum jerk in the lever and coupler occurring in the fifth and sixth mechanisms. In families two and four the fifth mechanism is the best one. Mechanism seven and eight in the third family have the lowest values of maximum jerk in the lever and the coupler. The four families are compared in the curves of maximum jerk

versus angle ρ in Figures 45-52. The curves for families one and two are quite similar as are the curves for families three and four. These similarities are based largely on the angle of oscillation θ. The angle θ for families one and two is 80° and θ is 60° for families three and four. This means that for one revolution of the crank the lever moves through a total angle of 160° for the first two families while in the last two families the lever moves through a total angle of 120° in the same amount of time. Obviously the accelerations in the lever for the first two families will be greater than the accelerations for the second two families. Then it follows that the jerk will be greatest in families one and two. The curves of maximum jerk for the coupler show the same trend.

The original problem is to find a method of synthesis for a four-bar mechanism so that the mechanism that is synthesized has a low value of jerk. A definite trend has already been shown for families one, two, and four. This trend indicates that the best mechanism for each family is the fifth or middle one. For family three the best mechanism is number seven. This choice is based on the same considerations used to establish mechanism five as the best in family one. Family three differs from the other families in another aspect namely the value of $\theta - 2\alpha$ is negative whereas it is positive for the other families. The significance, of the sign of the quantity $\theta - 2\alpha$, is best understood with the

TABLE III

VALUES OF MAXIMUM JERK FOR EACH MECHANISM

FAMILY ONE

ρ	β_3 (10^{-6} RAD/SEC^3)	β_4 (10^{-6} RAD/SEC^3)
10	18,815,544	18,934,864
20	12,171,059	12,171,059
30	8,781,712	8,743,192
40	6,577,420	6,399,724
50	4,848,140	4,412,380
60	3,275,117	4,247,452
70	3,070,284	5,498,340
80	4,829,566	8,788,330
90	12,765,751	23,879,861
100	35,232,653 x 10^{20}	67,951,070 x 10^{20}

TABLE III (continued)

VALUES OF MAXIMUM JERK FOR EACH MECHANISM

FAMILY TWO

ρ	β_3 (10^{-6} RAD/SEC^3)	β_4 (10^{-6} RAD/SEC^3)
10	12,944,119	12,942,052
20	7,300,857	7,264,774
30	4,771,037	4,583,066
40	3,221,400	2,764,326
50	2,077,343	3,107,027
60	1,152,772	3,776,279
70	2,242,116	5,236,496
80	3,742,440	9,061,439
90	10,324,151	26,070,560
100	722,422,220	1,902,587,000

TABLE III (continued)

VALUES OF MAXIMUM JERK FOR EACH MECHANISM

FAMILY THREE

ρ	β_3 (10^{-6} RAD/SEC^3)	β_4 (10^{-6} RAD/SEC^3)
12	6,307,828	6,837,034
24	5,191,102	5,272,564
36	4,325,725	4,325,725
48	3,662,822	3,627,976
60	3,070,036	2,989,400
72	2,461,432	2,308,427
84	1,729,504	1,921,305
96	1,661,263	2,577,396
108	3,521,717	5,464,507
120	213,734,220	338,109,200

TABLE III (continued)

VALUES OF MAXIMUM JERK FOR EACH MECHANISM

FAMILY FOUR

ρ	β_3 (10^{-6} RAD/SEC^3)	β_4 (10^{-6} RAD/SEC^3)
12	5,161,442	5,323,240
24	3,606,038	3,606,038
36	2,705,972	2,666,667
48	2,061,092	1,950,194
60	1,521,422	1,377,514
72	1,012,232	1,528,156
84	957,776	1,925,191
96	1,413,945	2,917,902
108	3,468,073	7,402,887
120	155,161,470	341,635,270

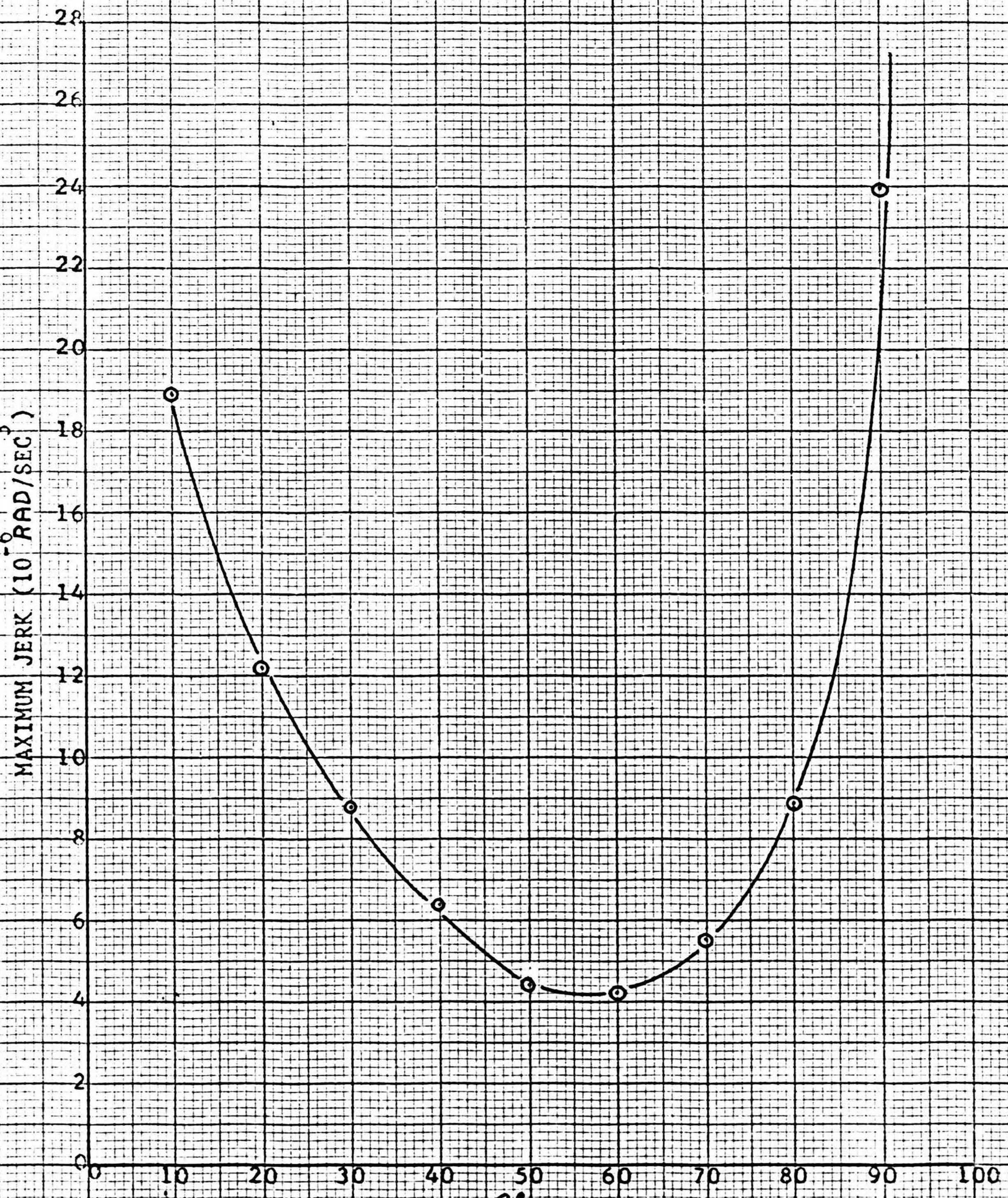


FIGURE 45
MAXIMUM JERK – ρ
LEVER
FAMILY ONE

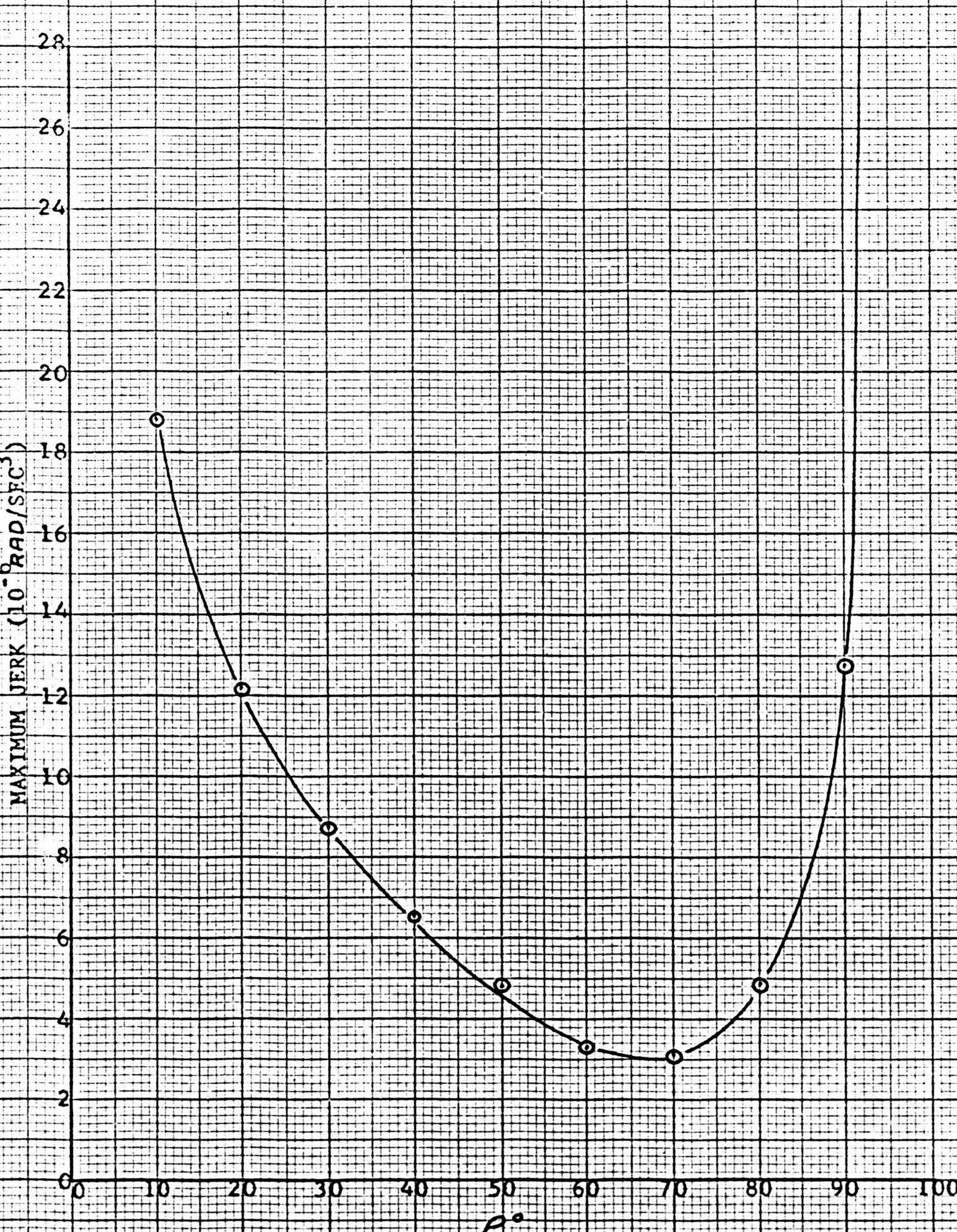


FIGURE 46
MAXIMUM JERK - ρ
COUPLER
FAMILY ONE

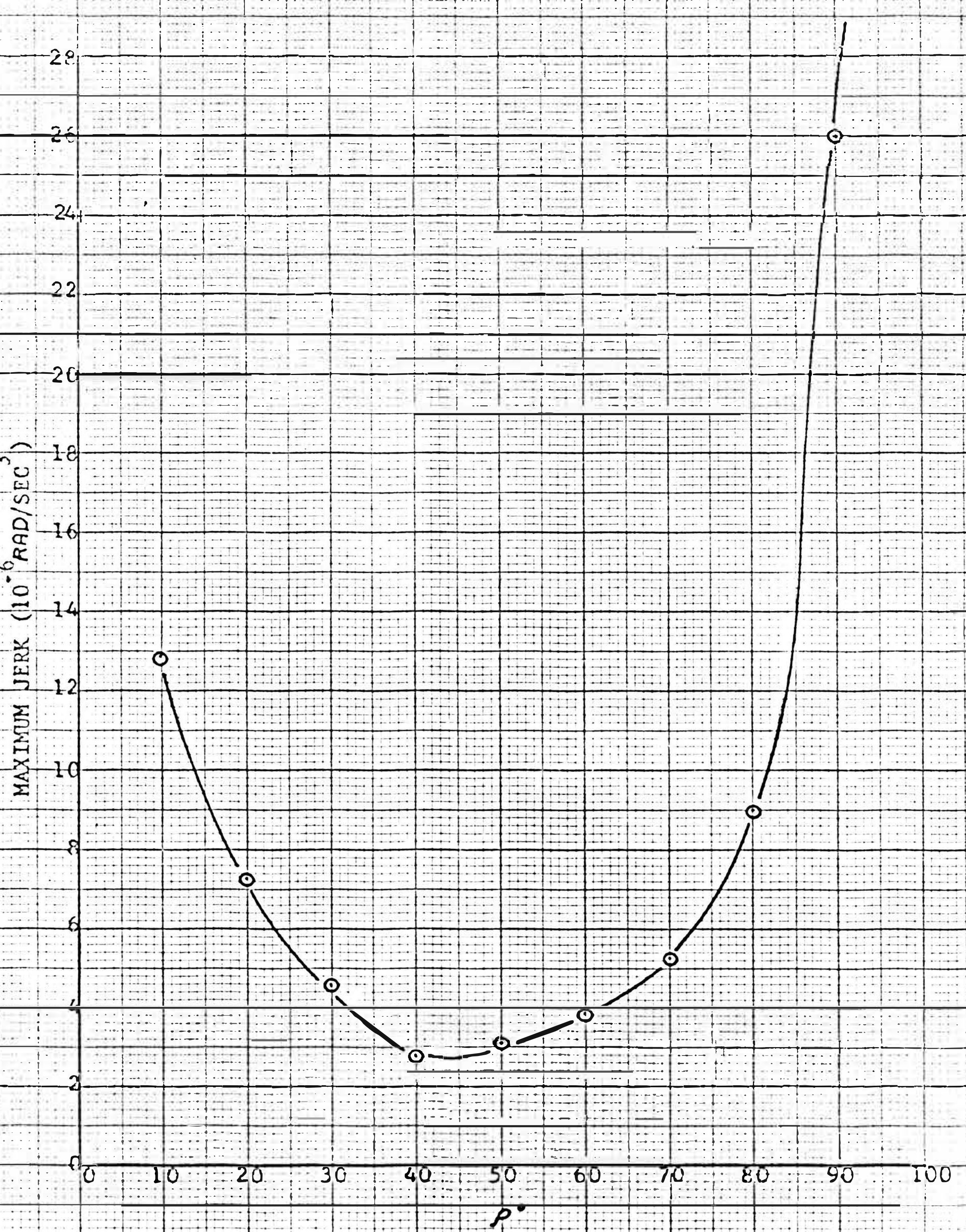


FIGURE 47
MAXIMUM JERK - ρ
LEVER
FAMILY TWO

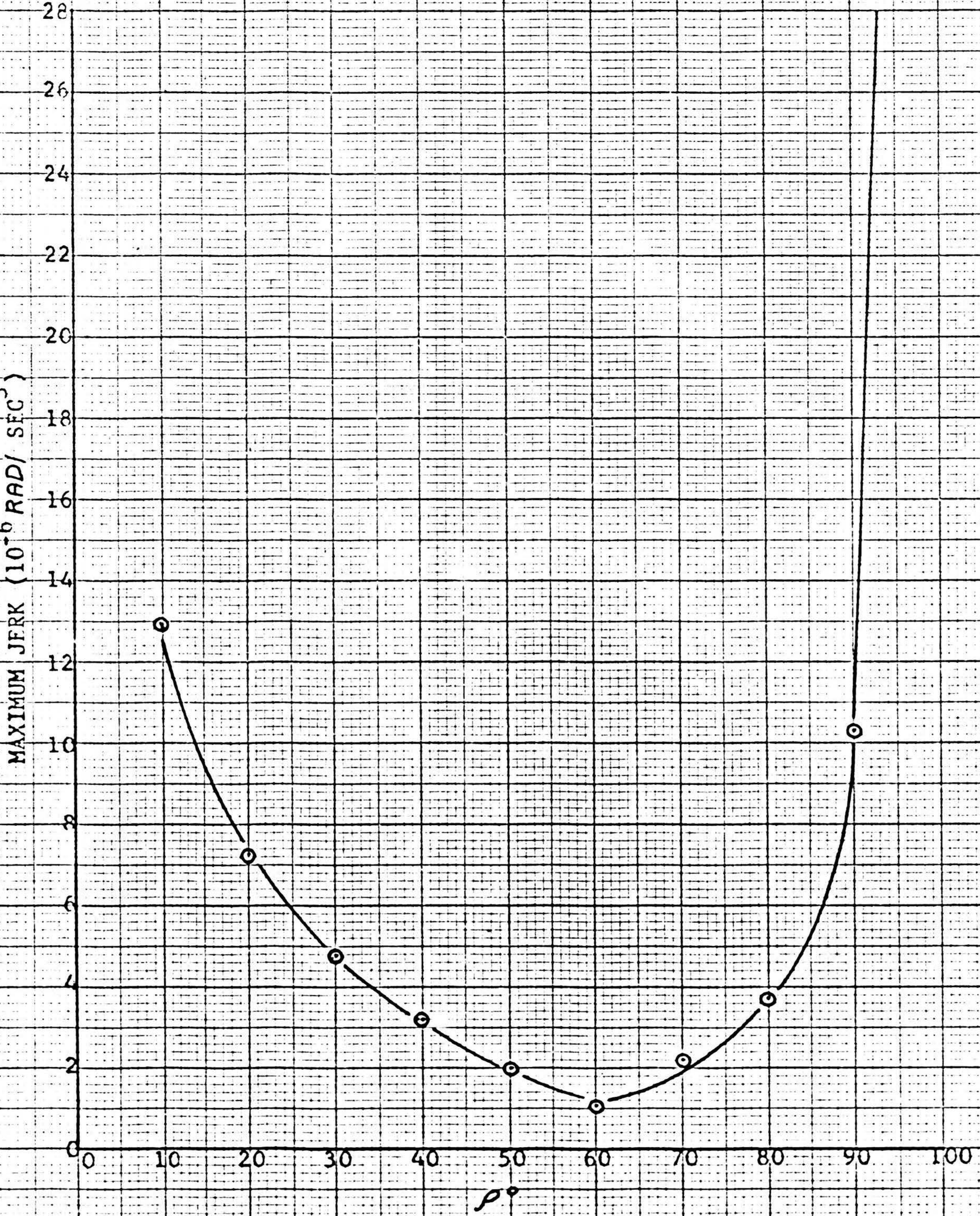


FIGURE 48
MAXIMUM JERK - ρ
COUPLER
FAMILY TWO

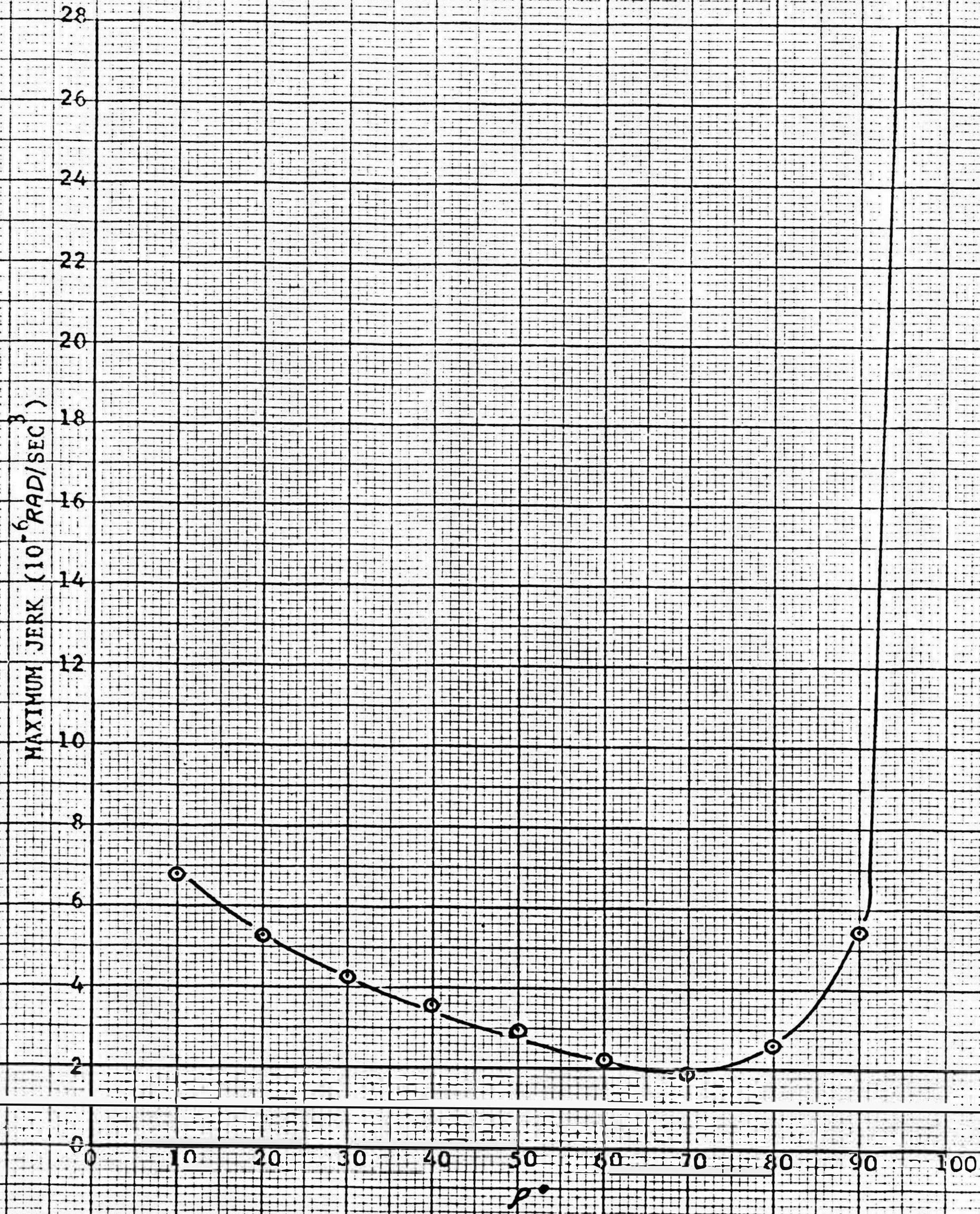
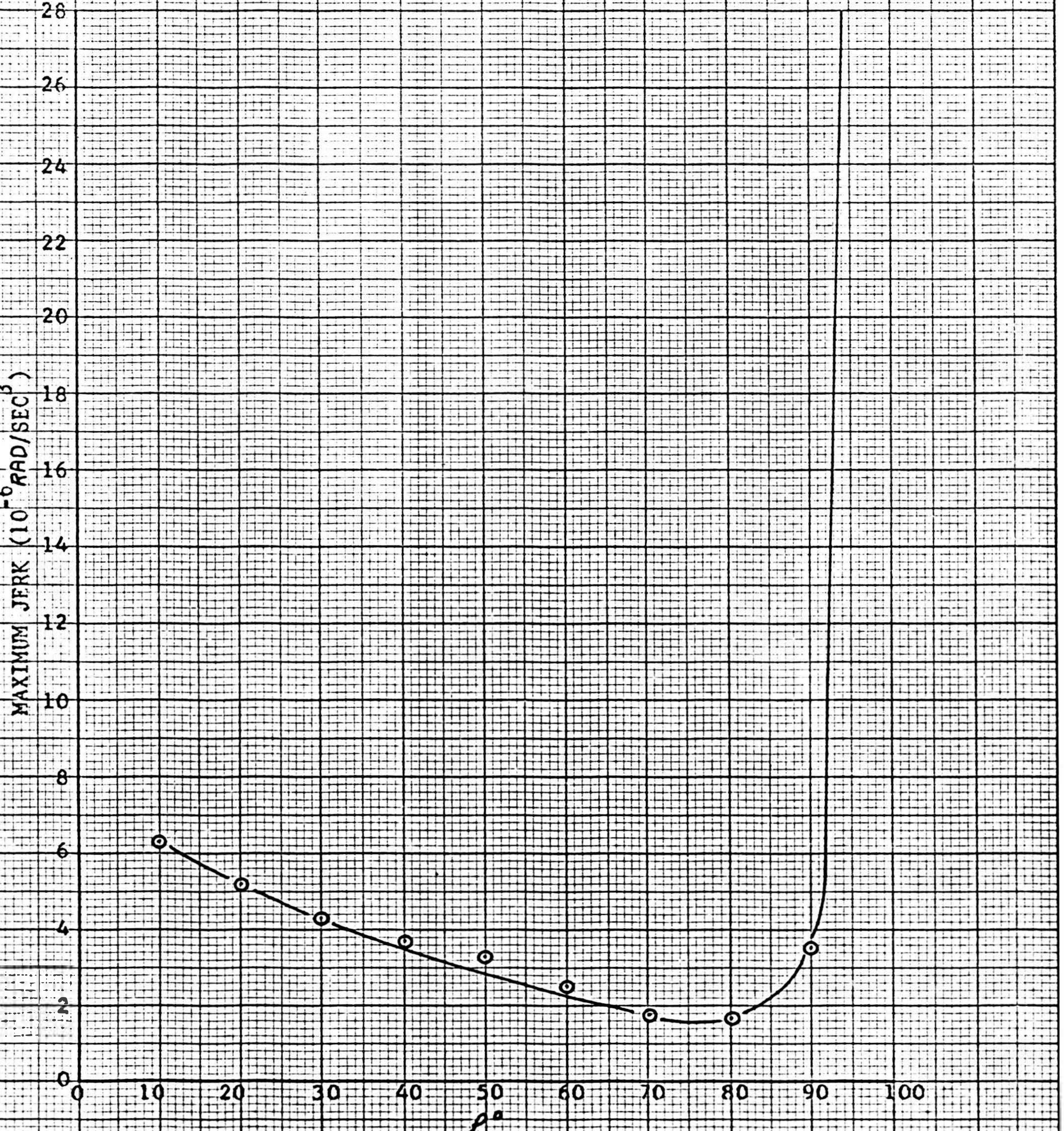


FIGURE 49
MAXIMUM JERK — ρ
LEVER
FAMILY THREE



FAMILY 50
MAXIMUM JERK - ρ
COUPLER
FAMILY THREE

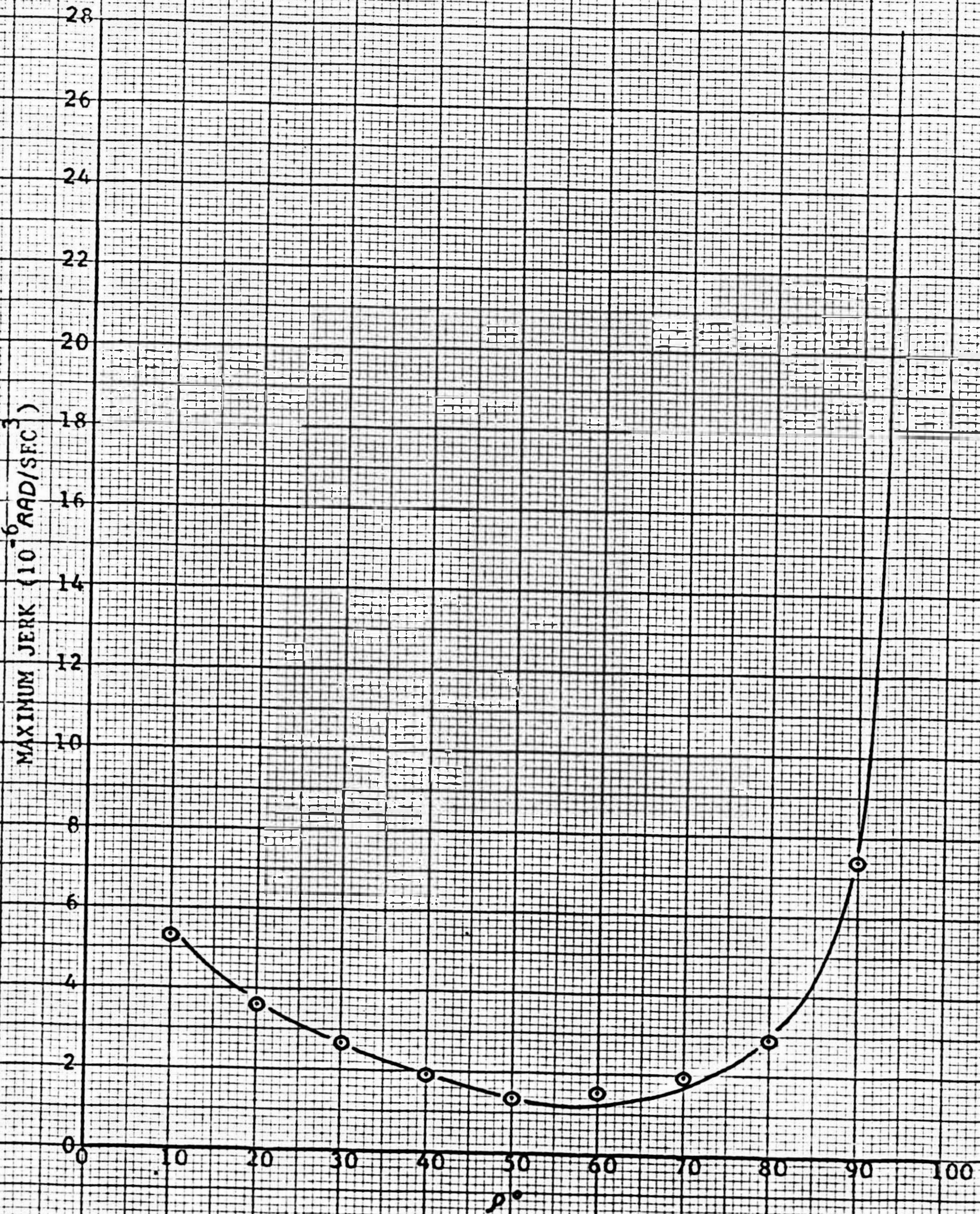


FIGURE 51
MAXIMUM JERK – ρ
LEVER
FAMILY FOUR

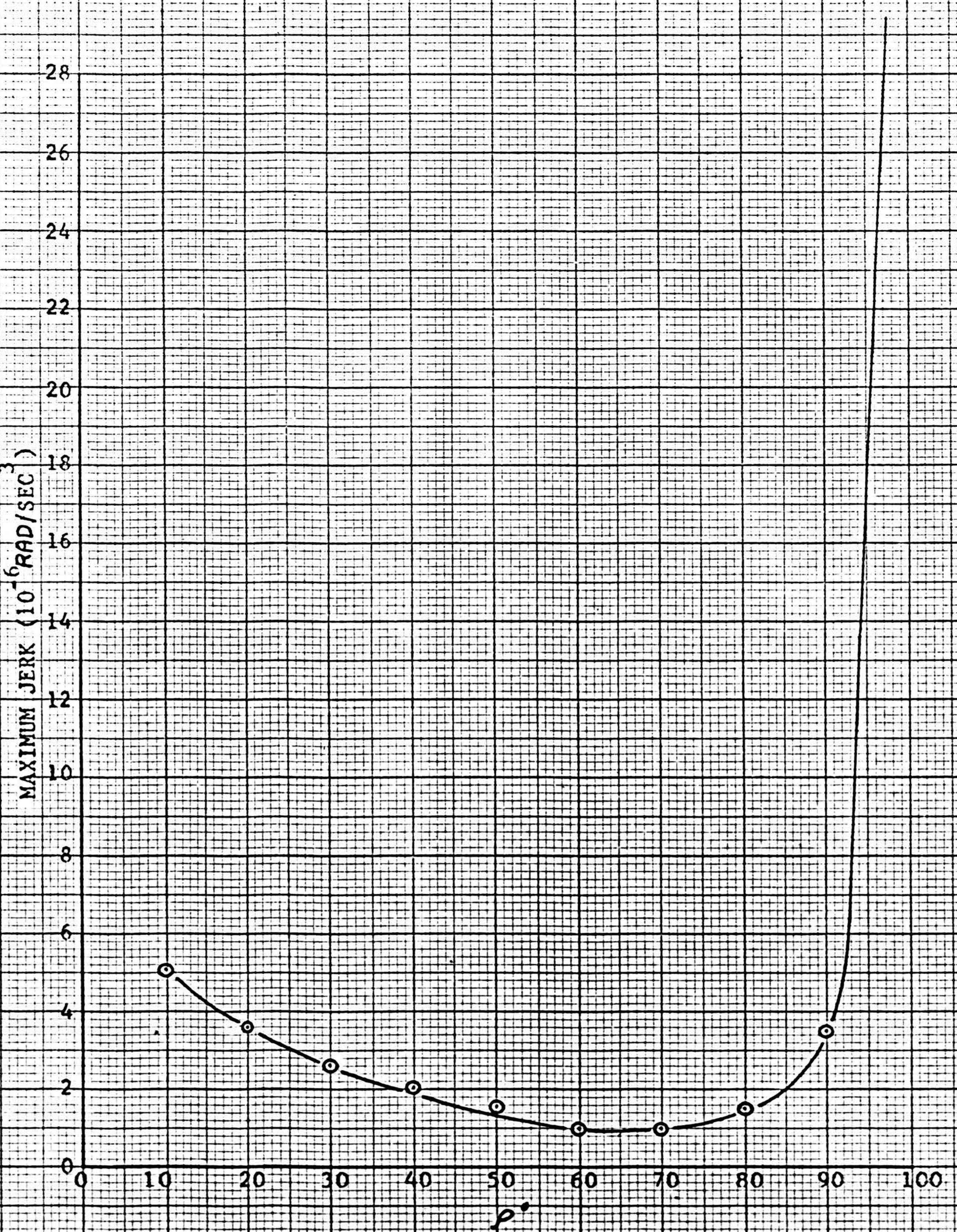


FIGURE 52
MAXIMUM JERK - ρ
COUPLER
FAMILY FOUR

help of either of the geometric constructions (Figure 64 or 65) which were the basis for the equations used to determine the dimensions of the mechanisms. However the geometric construction for method one, Figure 64, will be used because the illustration is clearer. When $\theta - 2\alpha$ is negative the center of the arc, which is the locus of all points M, lies inside the triangle QB_1B_2 and on line QD. With the center of the arc in this position the distance MQ, which is r_1, decreases as ρ increases. MQ (r_1) decreases, with an increasing ρ, only when $\theta - 2\alpha$ is negative. It would be difficult to prove, but not unreasonable to assume, that the different proportions of the links for family three account for the fact that the lowest values of maximum jerk in the lever and coupler occurs in mechanisms seven and eight rather than in mechanisms five and six. On the basis of the assumed validity of this assumption it can be further assumed that in all cases where $\theta - 2\alpha$ is negative the mechanism determined when $\rho = \frac{7}{10}(180 - \theta)$ will be the best mechanism or close enough to the best mechanism to be considered the best.

The designer is now faced with two situations the first being when $\theta - 2\alpha$ is positive and the second occurring when $\theta - 2\alpha$ is negative. On the basis of the trends already exhibited and the assumptions made, a "rule of thumb" can be formulated that will aid the designer in synthesizing a mechanism with a low value of jerk. It is to be understood that the rule of thumb is not to be considered an exact method

because the assumptions are not backed by rigorous proofs. When $\theta - 2\alpha$ is positive we shall design for the mechanism determined by $\rho = \frac{5}{10}(180 - \theta)$ and when $\theta - 2\alpha$ is negative we shall design for the mechanism determined by $\rho = \frac{7}{10}(180 - \theta)$.

No formal proof will be given for the following methods of synthesis as the methods are basically the same as shown in Appendix A.

I Geometric Synthesis

A. Method one: θ, α, and r_4 specified

(1) Start by picking any point, Q, for the lever pivot and then construct the isosceles triangle QB_1B_2 with the included angle θ.

(2) Extend a line from the midpoint of the base B_1B_2 through the apex. On this line locate point C so that angle B_1CB_2 is equal to 2α.

(3) Scribe an arc, with center at C and with a radius of CB_1, from point B_1 to the arcs intersection with the extension of line B_2Q (point A). See Figures 53 and 54.

(4) If $\theta - 2\alpha$ is positive locate the point where the bisector of angle ACB_1 intersects the arc. This is point M the crank pivot. If $\theta - 2\alpha$ is negative a line drawn from C, at an angle of $\rho = \frac{7}{10}(180 - \theta)$ degrees counter-clockwise from the line CB_1, intersects the arc at the point M which is the crank pivot.

(5) Now by direct measurement the neccesary measurements can be made to determine the links from the following relationships.

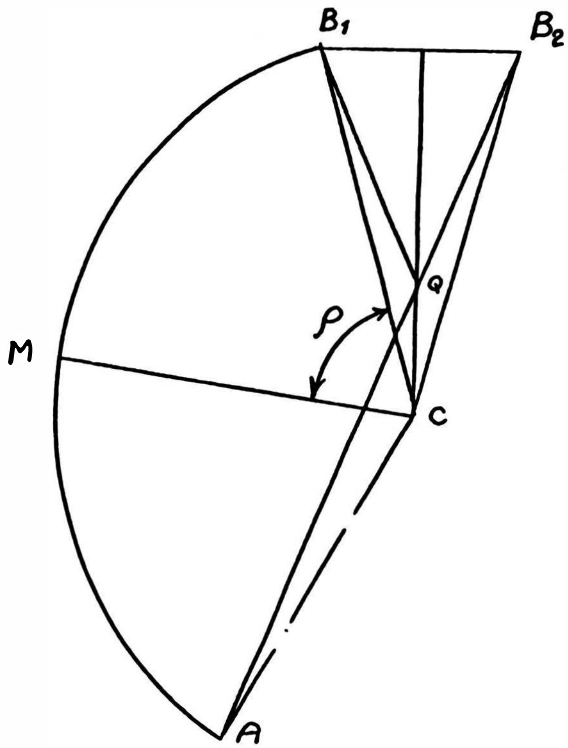


FIGURE 53

LAYOUT FOR METHOD ONE

$\theta - 2\alpha$ POSITIVE

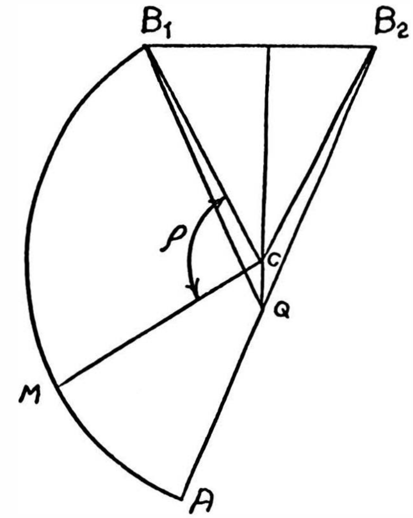


FIGURE 54

LAYOUT FOR METHOD ONE

$\theta - 2\alpha$ NEGATIVE

$$r_1 = MQ$$

$$r_2 = \frac{MB_2 - MB_1}{2} \qquad r_3 = \frac{MB_2 + MB_1}{2}$$

B Method two: θ, α , and r_1 specified.

(1) Start the layout by choosing the distance MQ which is r_1.

(2) Draw line MG at angle $\theta/2 - \alpha$ (clockwise if $\theta/2 - \alpha$ is positive) with line MQ.

(3) Draw line QG at an angle of $180 + \theta/2$ degrees counter-clockwise if $\theta/2 - \alpha$ is positive (Figure 55). If $\theta/2 - \alpha$ is negative draw line QG at an angle of $\theta/2$ degrees counter-clockwise. (Figure 56).

(4) The intersection of these two lines locate point G. Now locate point G' so that G and G' are symetrically located with respect to line MQ.

(5) With G as center and MG as the radius, scribe an arc from G to where the arc intersects the extension of line MQ. This arc will always be above the line MQ. Now if $\theta - 2\alpha$ is positive, draw a line GD at an angle of $\rho = \frac{5}{10}(180-\theta)$ degrees clockwise from line MG (Figure 55). If $\theta - 2\alpha$ is negative, draw a line GD at an angle of $\rho = \frac{7}{10}(180-\theta)$ degrees clockwise from line MG (Figure 56). In both cases D is a point on the arc and it represents one of the extreme positions of the lever pin.

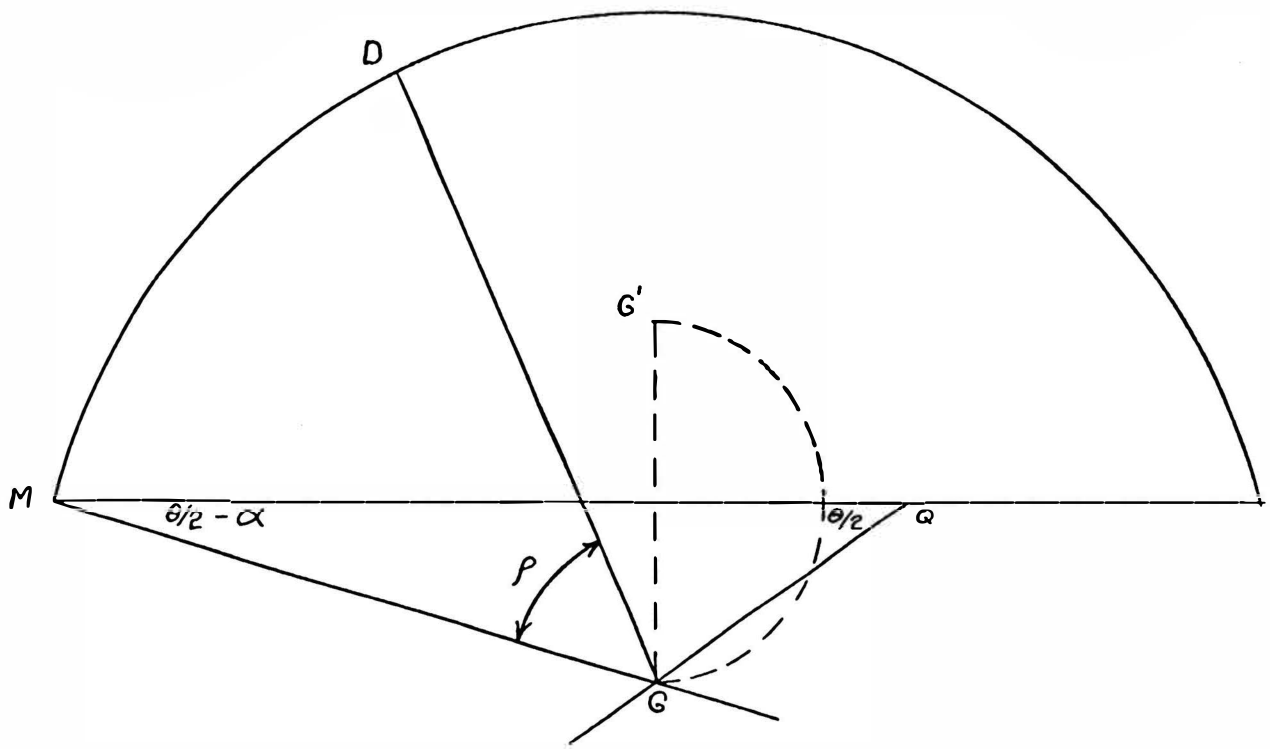


FIGURE 55

LAYOUT FOR METHOD TWO

$\theta - 2\alpha$ POSITIVE

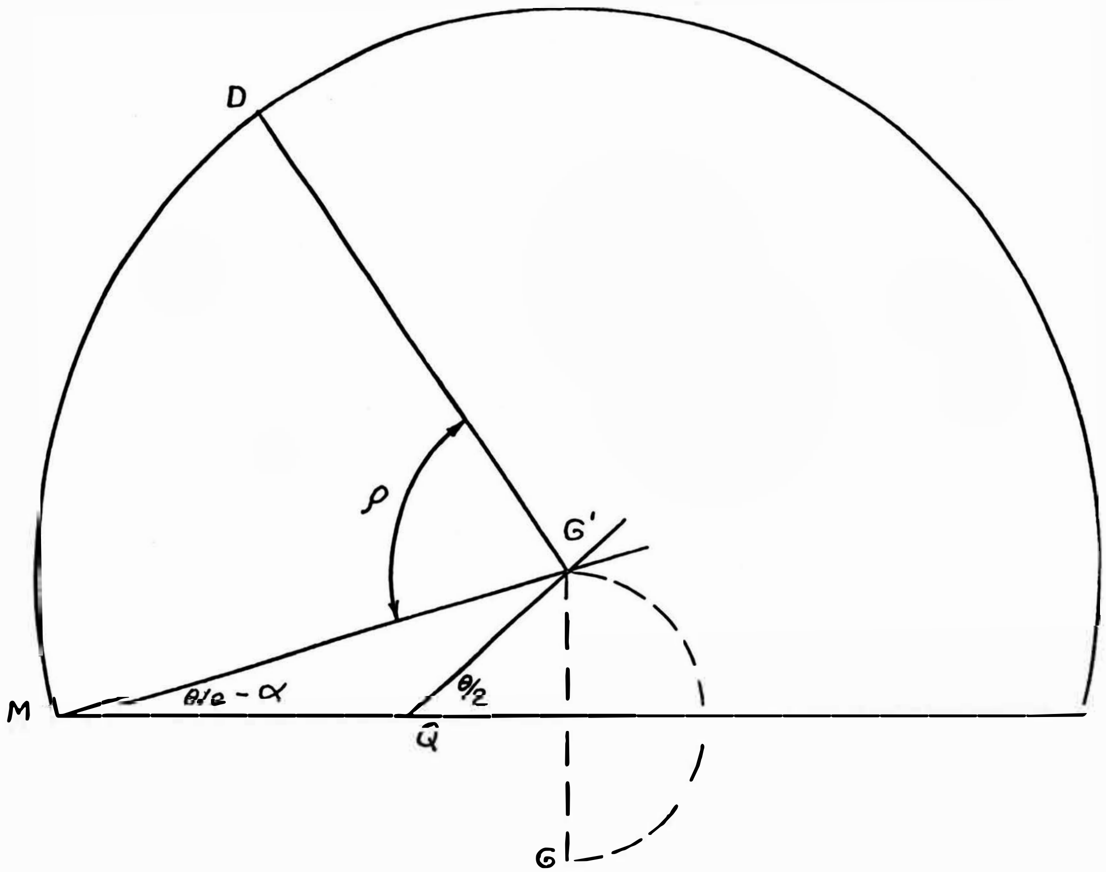


FIGURE 56

LAYOUT FOR METHOD TWO

$\theta - 2\alpha$ NEGATIVE

(6) With G' as center and MG as the radius scribe a second arc from point M around to the point where the first arc intersects the extension of line MQ. This second arc is always above the line MQ.

(7) Scribe an arc, with center at Q, and with radius QD, between the two arcs whose centers are at F and G'. See Figures 57 and 58. This third arc strikes the first arc at point D which is already established and it strikes the second arc at the point that will be called E. Point E is the other extreme position of the lever pin.

(8) Now we can measure the distances MD and ME and determine the links from the following relationships.

$$r_2 = \frac{ME - MD}{2} \qquad\qquad r_3 = \frac{ME + MD}{2}$$

$$r_4 = QD$$

II. Analytical Synthesis

The analytical equations already derived in Appendix A can still be used. The only consideration is that ρ, for both methods one and two, depends on the value of $\theta - 2\alpha$. If $\theta - 2\alpha$ is positive $\rho = \frac{5}{10}(180-\theta)$. If $\theta - 2\alpha$ is negative $\rho = \frac{7}{10}(180-\theta)$.

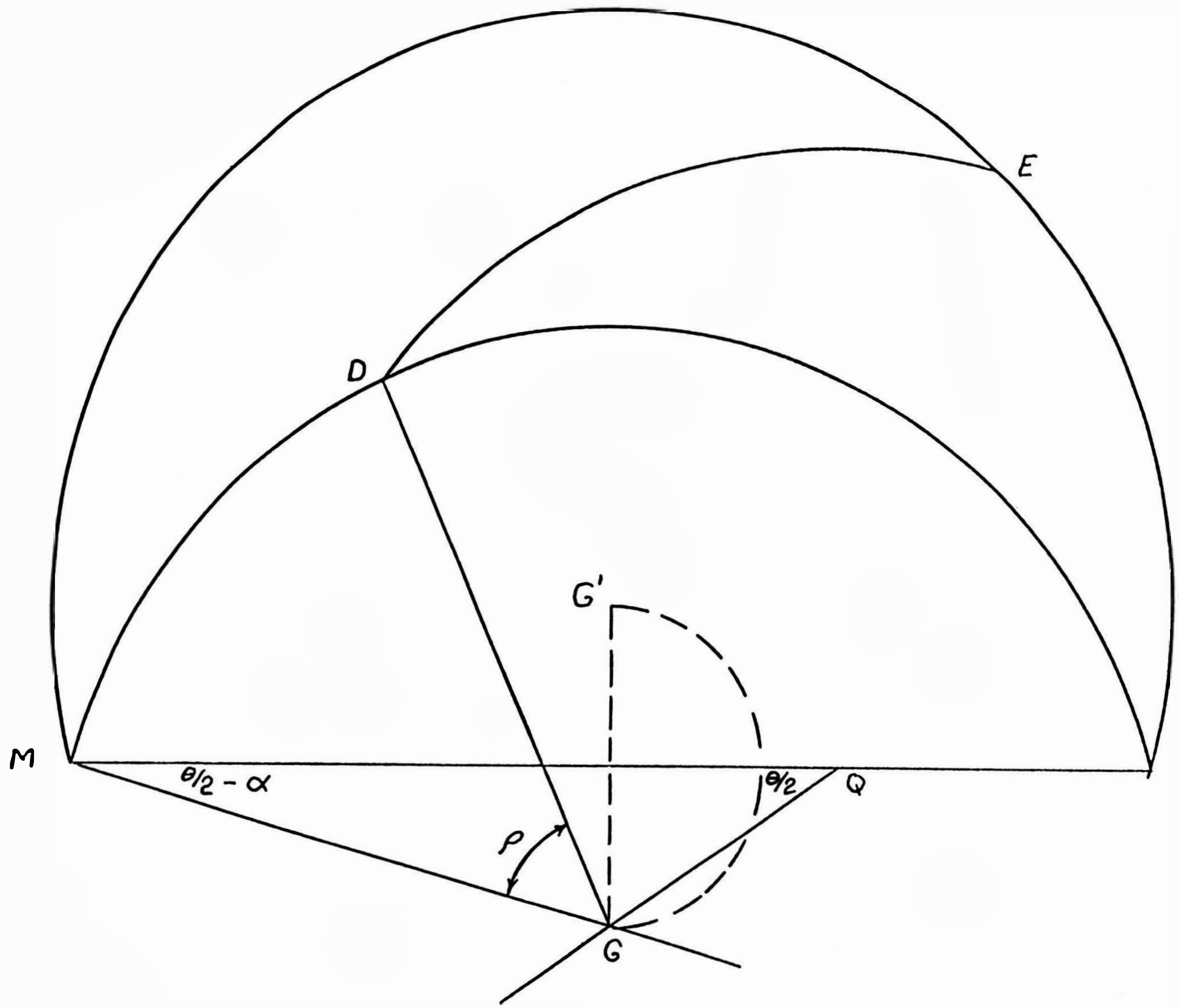


FIGURE 57

DETERMINATION OF POINT E

$\theta - 2\alpha$ POSITIVE

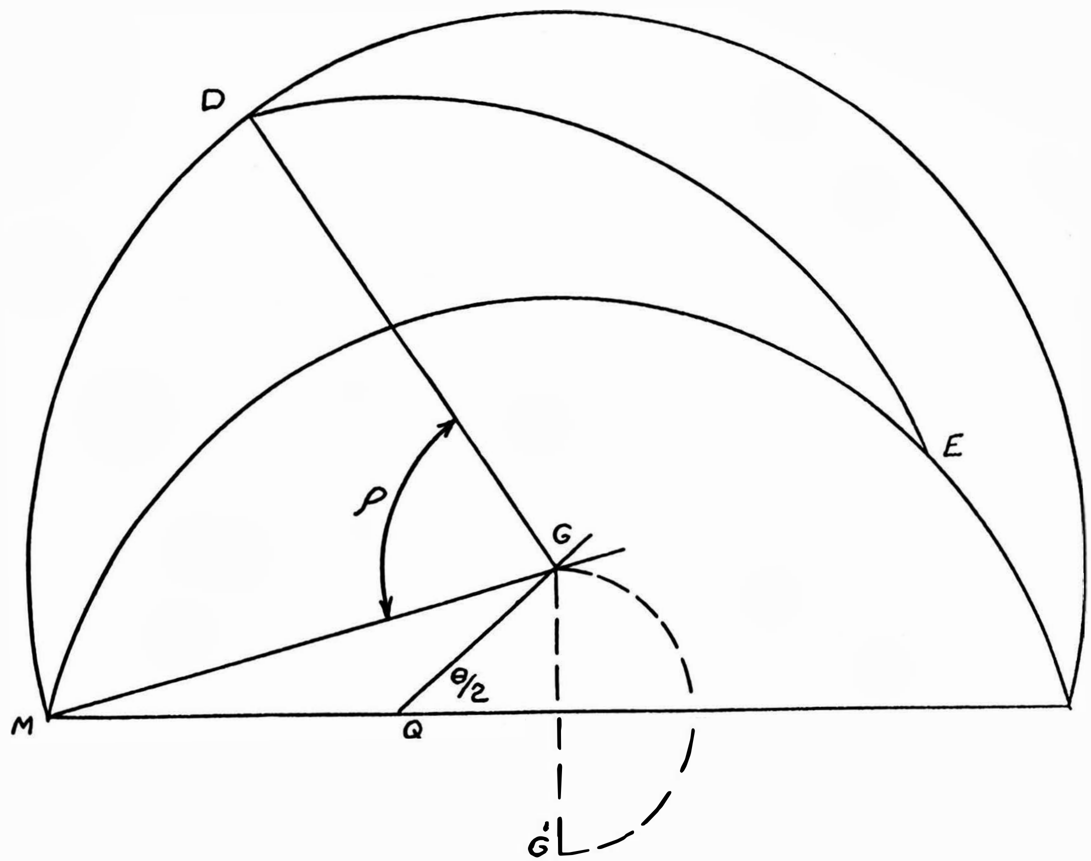


FIGURE 58

DETERMINATION OF POINT E

$\theta - 2\alpha$ NEGATIVE

III CONCLUSION

The methods of synthesis that were developed will make it possible for a designer to synthesize a crank-lever mechanism so that it will have a low value of jerk. These methods of synthesis are actually rules of thumb because it has not been proven that for any angle θ and any angle α the methods will determine the mechanism that is absolutely the best from the stand point of jerk. Instead it is intended that these methods allow the designer to synthesize a mechanism that is approximately the one with the lowest value of jerk.

The whole procedure that led to the final determination of the methods of synthesis could be applied to any of the other four-bar mechanisms such as the drag link, double rocker, and crank-shaper mechanism. The procedure would be as follows.

(1) Determine equations for jerk from the position vector equations.

(2) Write a computer program to calculate the jerk in a mechanism for any position of the mechanism.

(3) From the calculated values of jerk determine some criterion that will predict or determine the mechanism with the lowest value of jerk.

Further investigations might answer many interesting questions that came about during the investigations for this thesis.

Some of them are:

(1) What factors determine the position of a mechanism at the time maximum jerk occurs?

(2) What effect does the proportions of the mechanism have on the jerk?

(3) Is there a definite relationship between r_1, r_2, r_3, and r_4 that is the same for the mechanism with the lowest value of jerk in all possible families?

These questions and many more would make absorbing subjects for investigations.

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3. Hall, op. cit. p. 33-39.

4. Martin, G.H. (1958) Calculating Velocities and Accelerations in Four-Bar Linkages, Machine Design, vol. 30, No. 8 p. 146.

APPENDIX A

THE CRANK-LEVER MECHANISM (3)

In Figure 59 a crank-lever mechanism is shown with the notation that will be used in the discussion. As the crank (link 2) rotates, the lever (link 4) oscillates through the angle θ. B_1 and B_2 are the two extreme positions of the pin at the end of the lever. A_1 and A_2 are the corresponding crank positions.

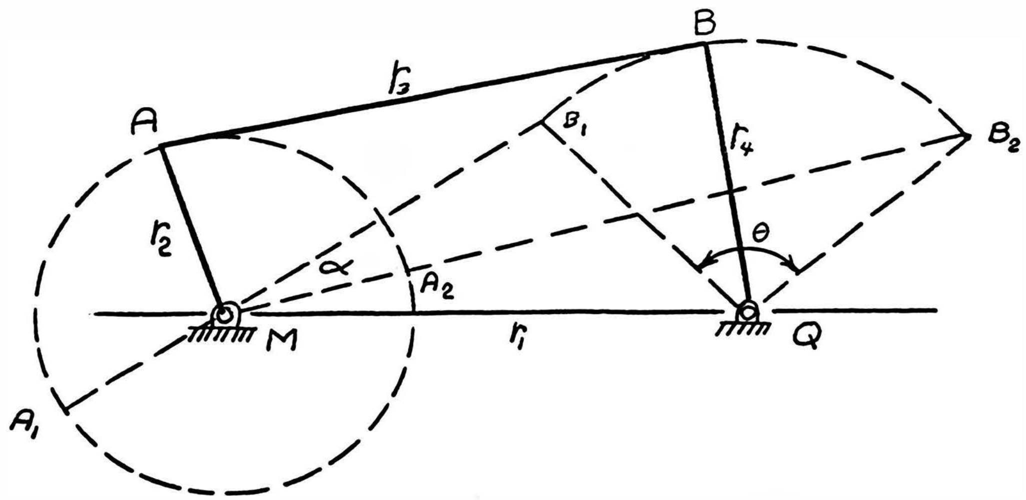

FIGURE 59

CRANK-LEVER MECHANISM

Notice that the two swings of the lever do not take place during equal rotation angles. This means that the four-bar functions as a "quick-return" mechanism. If the crank turns at constant speed, the time ratio of the two swings of the lever is

$$T.R. = \frac{180 + \alpha}{180 - \alpha}$$

The most common design problem will be one in which the angle of oscillation, θ, and the angle α (or the time ratio, which determines α) are specified.....

DESIGNING CRANK-LEVER MECHANISMS FOR SPECIFIED θ AND α : FIRST METHOD

To design a crank-lever for a specified θ and α we can proceed as illustrated in Figure 60.

(1) Start a layout by picking any point Q for the lever pivot and drawing an isosceles triangle QB_1B_2 with angle B_1QB_2 equal to θ.

(2) Through B_2 draw a line B_2X. Through B_1 draw a line B_1Y at an angle α with B_2X. The two lines intersect at M, a possible location for the crank pivot.

(3) The lengths, r_2 of the crank and r_3 of the connecting rod, can be determined from the following relations:

$$r_3 + r_2 = MB_2$$
$$r_3 - r_2 = MB_1$$

(4) The mechanism designed is shown in Figure 61.

The above process is simple and leads quickly to a design. Since line B_2X was arbitraraly drawn, a great many different designs could be worked out. [Italics not in the original] For positive α the locus of all possible locations for the pivot M is the dark portion of the...circle shown in Figure 62....The light portions of the circle are not possible locations for M because the lever would be required to swing past the line of pivots MQ an impossibility in the crank-

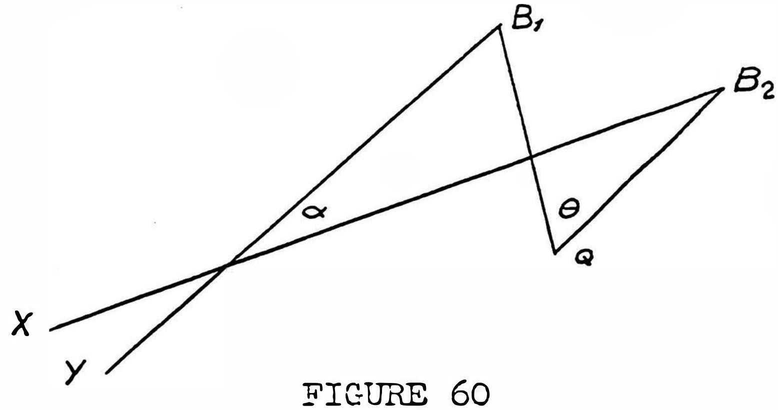

FIGURE 60

DESIGN LAYOUT

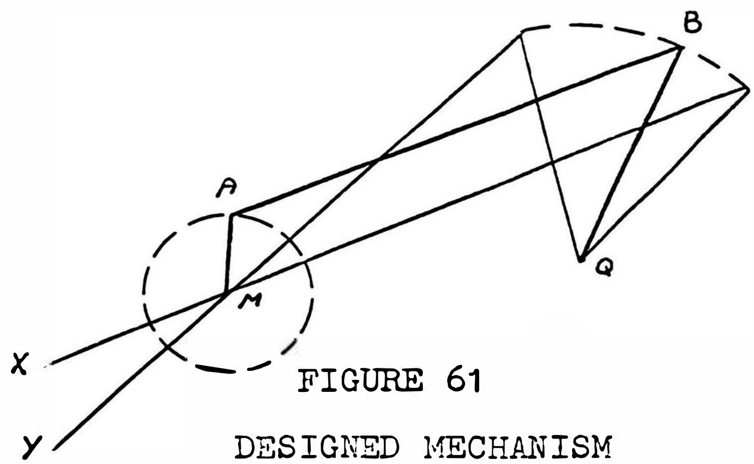

FIGURE 61

DESIGNED MECHANISM

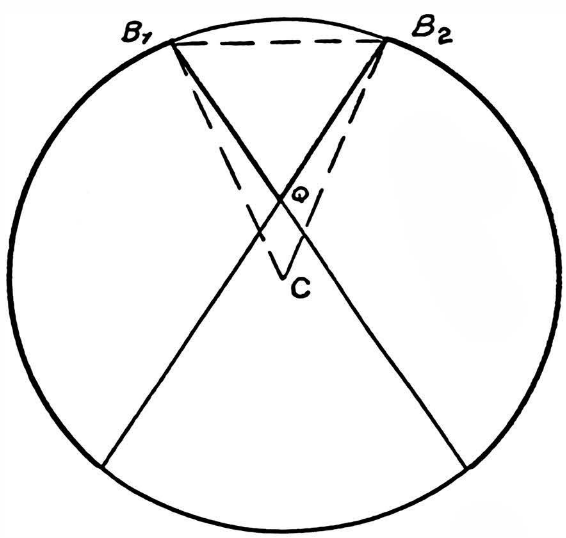

FIGURE 62

LOCUS OF ALL POINTS M

lever mechanism.

DESIGNING CRANK-LEVER MECHANISM FOR SPECIFIED θ AND α : SECOND METHOD

[Before explaining the design procedures Mr. Hall discusses similarly varying triangles and uses them to prove the validity of his design procedures which follow in part.]

(1) Start a layout by choosing a pivot distance MQ. See Figure 63.

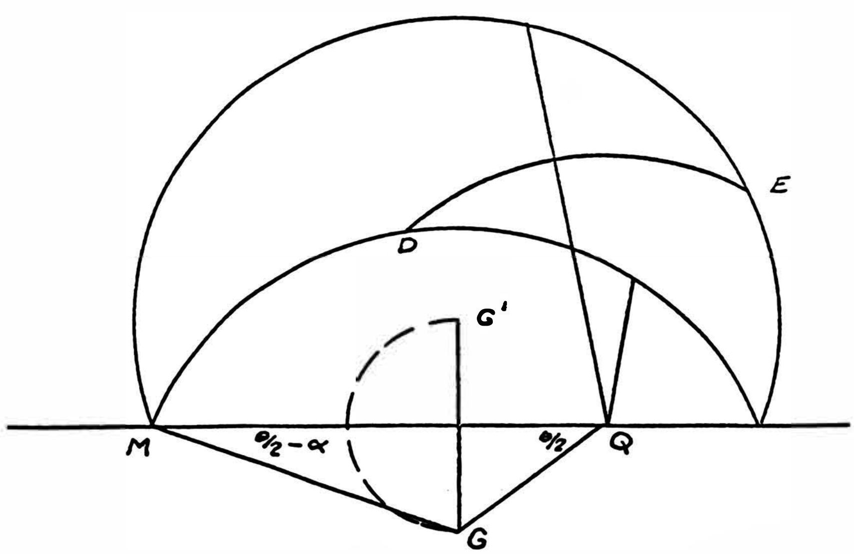

FIGURE 63

DESIGN LAYOUT METHOD TWO

(2) Draw line MG at angle $\theta/2 - \alpha$ (clockwise is positive) with line MQ....

(3) Draw line QG at angle $\theta/2 = 40$ degrees (counter-clockwise with line QM). Point G, the center of the circle locus for B_1 has now been located. The locus for B_2 is the circle with center G' and radius G' M, where G' and G are

symetrically located with respect to line MQ.... The usable portions of the locus are shown by heavy lines. Locating B_1 or B_2 outside these limits would require the impossible action of swinging the lever past the MQ line.

....

In the two methods outlined by Hall the end result in each is a layout from which different mechanisms can be picked off. From the geometry of the layouts the mechanisms can also be determined analytically.

Consider Figure 62. From the geometry of the Figure equations can be written for the links r_1, r_2, and r_3. There are two dark portions of the circle which are the loci for M. One of these dark arcs can be disregarded because the mechanism designed from one arc is the mirror image of the mechanism designed from the other arc.

Let us now consider Figure 64. The angle ϕ can be determined from triangle ACQ. It can be seen that angle ACQ = $\theta/2$ and that angle $\psi = 180 - \theta/2 - (\phi + \alpha)$. Thus

$$\phi = 180 - \theta/2 - \alpha - \psi$$

From triangle CQB_2, angle $QB_2C = \psi$

$$\psi = 180 - (180 - \theta/2) - \alpha = \theta/2 - \alpha$$

Therefore

$$\phi = 180 - \theta$$

The chords MB_1 and MB_2 can be determined because the angles that their arcs subtend are known. Thus

$$MB_1 = 2R \sin \rho/2$$

$$MB_2 = 2R \sin (\rho/2 + \alpha)$$

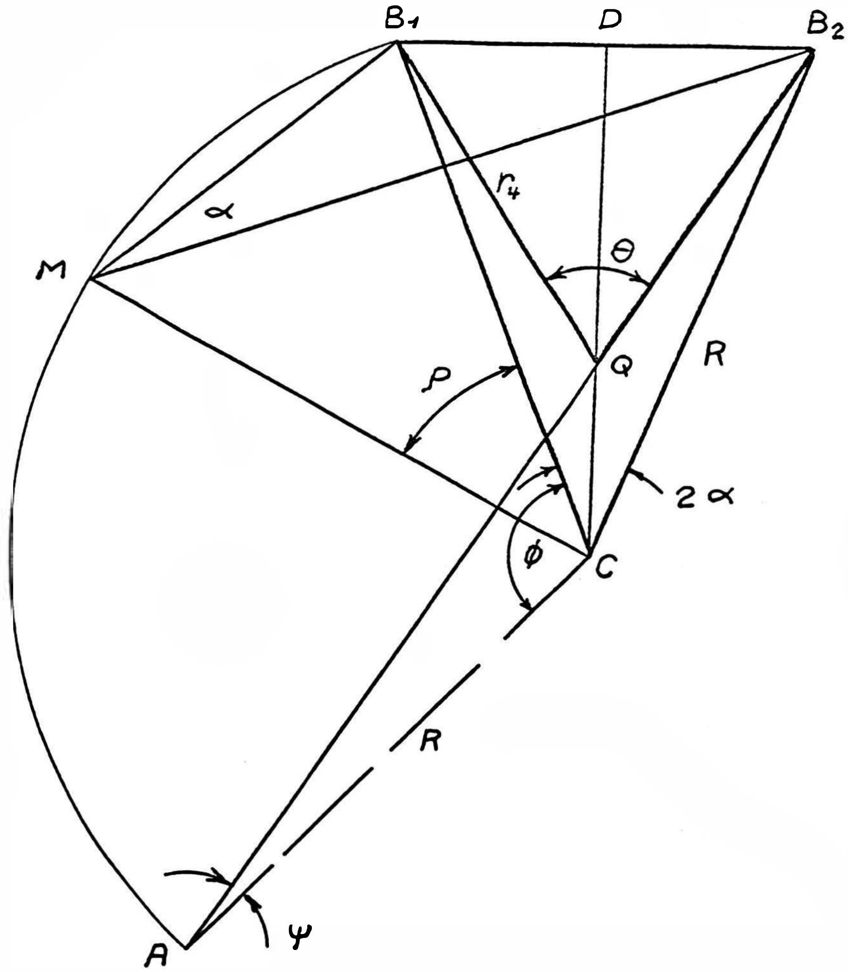

FIGURE 64

GEOMETRIC CONSTRUCTION FOR METHOD ONE

C = Center of arc which is the locus of M
Q = Pivot of the lever r_4
R = Radius of the arc
θ = Angle of oscillation of the lever r_4
B_1, B_2 = Extreme positions of the lever pin
M = Pivot for the crank r_2
ϕ = Angles subtended by the arc of the locus
L = Line QC
CD = Perpendicular Bisector of B_1B_2
ρ = Angle that is a fractional part of angle ϕ and is subtented by the arc MB_1
α = Angle that is determined by the time ratio

We have the relationship that

$$r_3 + r_2 = MB_2$$

$$r_3 - r_2 = MB_1$$

From this it can be found that

$$r_2 = \frac{MB_2 - MB_1}{2} \qquad\qquad r_3 = \frac{MB_2 + MB_1}{2}$$

All that remains is to find the distance MQ which is r_1. This can be done by considering the triangle MB_2Q. The angle δ must be determined then MQ can be found by using the law of cosines.

$$\measuredangle MB_1C = 90 - \rho/2$$

$$\measuredangle CB_1Q = 180 - (180 - \theta/2) - \alpha = \theta/2 - \alpha$$

$$\measuredangle QB_1B_2 = 90 - \theta/2$$

$$\measuredangle MB_1B_2 = \measuredangle MB_1C + \measuredangle CB_1Q + \measuredangle QB_1B_2 = 180 - \rho/2 - \alpha$$

$$\measuredangle B_1B_2M = 180 - \alpha - (180 - \rho/2 - \alpha) = \rho/2$$

$$\delta + \rho/2 = 90 - \theta/2$$

Thus

$$\delta = 90 - \theta/2 - \rho/2$$

By the law of cosines

$$r_1 = MQ = (r_4^2 + (MB_2)^2 - 2(r_4)(MB_2)\cos\delta)^{1/2}$$

In summary, to find a family of mechanisms containing N mechanisms with θ and α specified and r_4 specified we use the following formulas.

$$\rho = \frac{\phi}{N} = \frac{180 - \theta}{N}$$

$$R = \frac{r_4 \sin(\theta/2)}{\sin \alpha}$$

$$MB_1 = 2R \sin(\rho/2)$$

$$MB_2 = 2R \sin(\rho/2 + \alpha)$$

$$\delta = 90 - (\theta/2 + \rho/2)$$

$$r_1 = (r_4{}^2 + (MB_2)^2 - 2r_4 (MB_2) \cos \delta)^{\frac{1}{2}}$$

$$r_2 = \frac{MB_2 - MB_1}{2}$$

$$r_3 = \frac{MB_2 + MB_1}{2}$$

For the first calculation use ρ, for the second calculation use 2ρ, for the third calculation use 3ρ, and so forth until for the Nth calculation the value $N\rho$ is used which is equal to $180 - \theta$. This method is readily adaptable to the digital computer.

From Hall's second method analytical expressions for determining the lengths of the links can be derived also. The procedure for determining the configuration of a mechanism from Figure 63 consists of scribing an arc, whose center is at Q, between the locus of B_1 and B_2. The points D and E where this arc strikes the locus of B_1 and B_2 are the extreme positions of the lever pins. The distances MD and ME can be determined and from this r_1 and r_2 can be found. In Figure 63 the angles subtended by the arcs of the locus B_1 and B_2 must be determined. Figure 63 can be redrawn and labled as shown in Figure 65. From Figure 65 the following relationships can be established.

$$\measuredangle MGQ = 180 - (\theta/2 - \alpha) - \theta/2 = 180 + \alpha - \theta$$

$$\measuredangle GCQ = \theta/2 - \alpha \qquad \text{from symmetry}$$

Thus $\measuredangle CGQ = 180 - \theta/2 - \theta/2 - \theta/2 - (180 - 2\theta) - (\theta/2 - \alpha)$

$$\measuredangle CGQ = \alpha$$

Thus $\measuredangle MGC = \measuredangle MGQ - \measuredangle CGQ$

$$\measuredangle MGC = 180 + \alpha - \theta - \alpha = 180 - \theta$$

This is the angle subtended by the arc of the locus B_1. The angle subtended by the arc that is the locus of B_2 is angle AG'B and can be determined as follows.

$$\measuredangle AG'Q = 180 - \theta/2 - (\theta/2 - \alpha) = 180 - \theta + \alpha$$

$$\measuredangle QG'B = 180 - (\theta/2 - \alpha) - \theta/2 - \theta - (180 - 2\theta) = \alpha$$

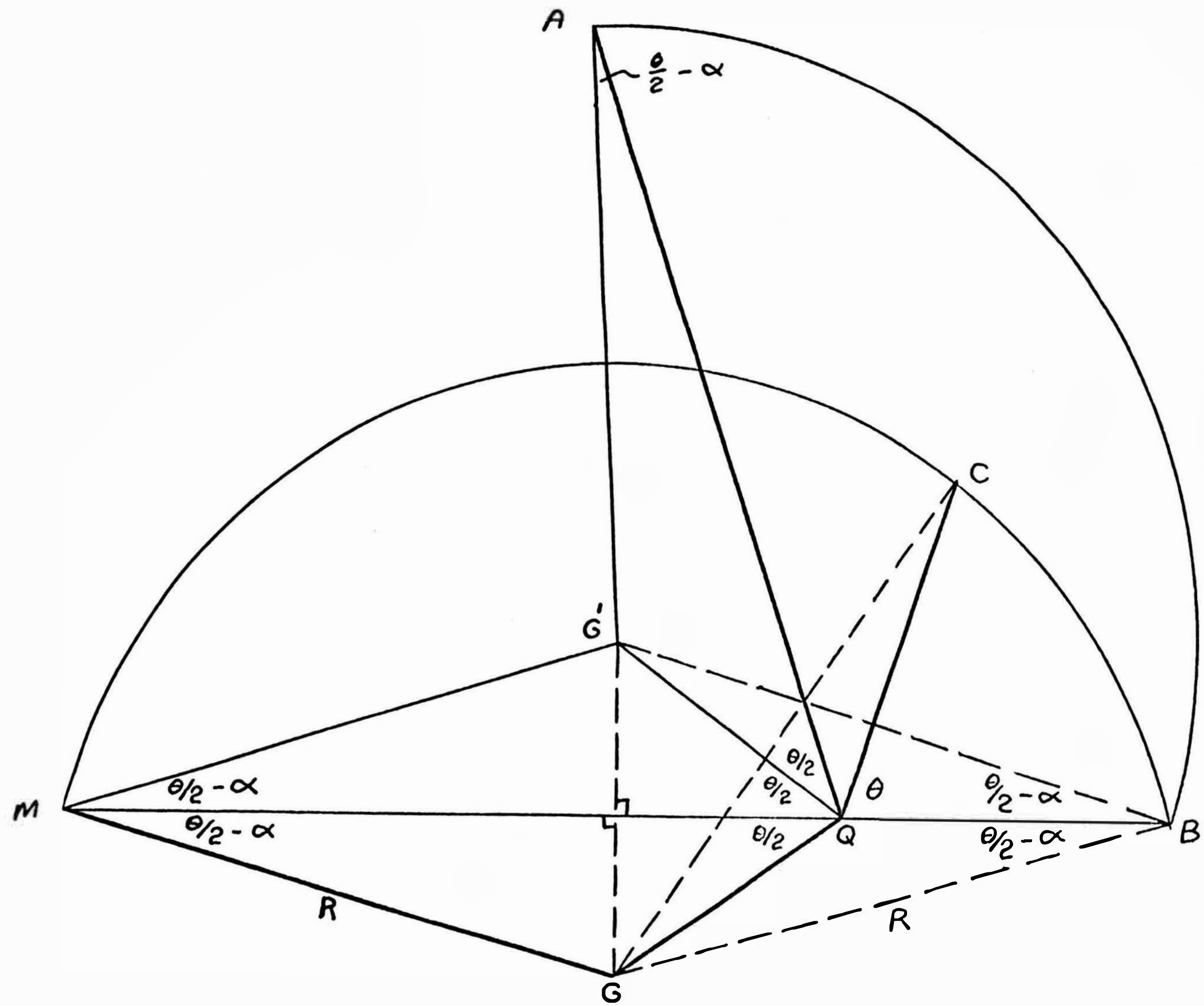

FIGURE 65

GEOMETRIC CONSTRUCTION FOR METHOD TWO

Thus

$\angle AG'B = \angle AG'Q - \angle QG'B$

$AG'B = 180 - \theta + \alpha - \alpha = 180 - \theta$

Now it is clear that the two arcs have the same radius and subtend the same angle. Now the links r_2, r_3, and r_4 can be determined. In Figure 66 the angle $\rho = \frac{180-\theta}{N}$ where N is the desired number of mechanisms. MD is the chord of the arc subtending angle ρ. Thus

$$MD = 2R \sin(\rho/2)$$

From triangle MG'E

$ME = 2R \cos(\psi - \theta/2 + \alpha)$

Where $\psi = 90 - \left(\frac{\rho + \theta}{2}\right)$ from triangle MGD. By taking triangle MGQ and using the law of sines it can be found that

$$\frac{R}{\sin(\theta/2)} = \frac{r_1}{\sin(180 + \alpha - \theta)}$$

Thus

$$R = \frac{r_1 \sin(\theta/2)}{\sin(180 + \alpha - \theta)}$$

$$r_2 + r_3 = ME$$

$$r_2 - r_3 = ME$$

Thus

$$r_2 = \frac{ME - MD}{2} \qquad r_3 = \frac{ME + MD}{2}$$

And from triangle MQE using the law of cosines

$$r_4 = \left((r_1)^2 + (ME)^2 - 2(r_1)(ME)\cos\psi\right)^{1/2}$$

Thus to find a family of N mechanisms when θ and α are specified and r_1 is specified the following equations can be used.

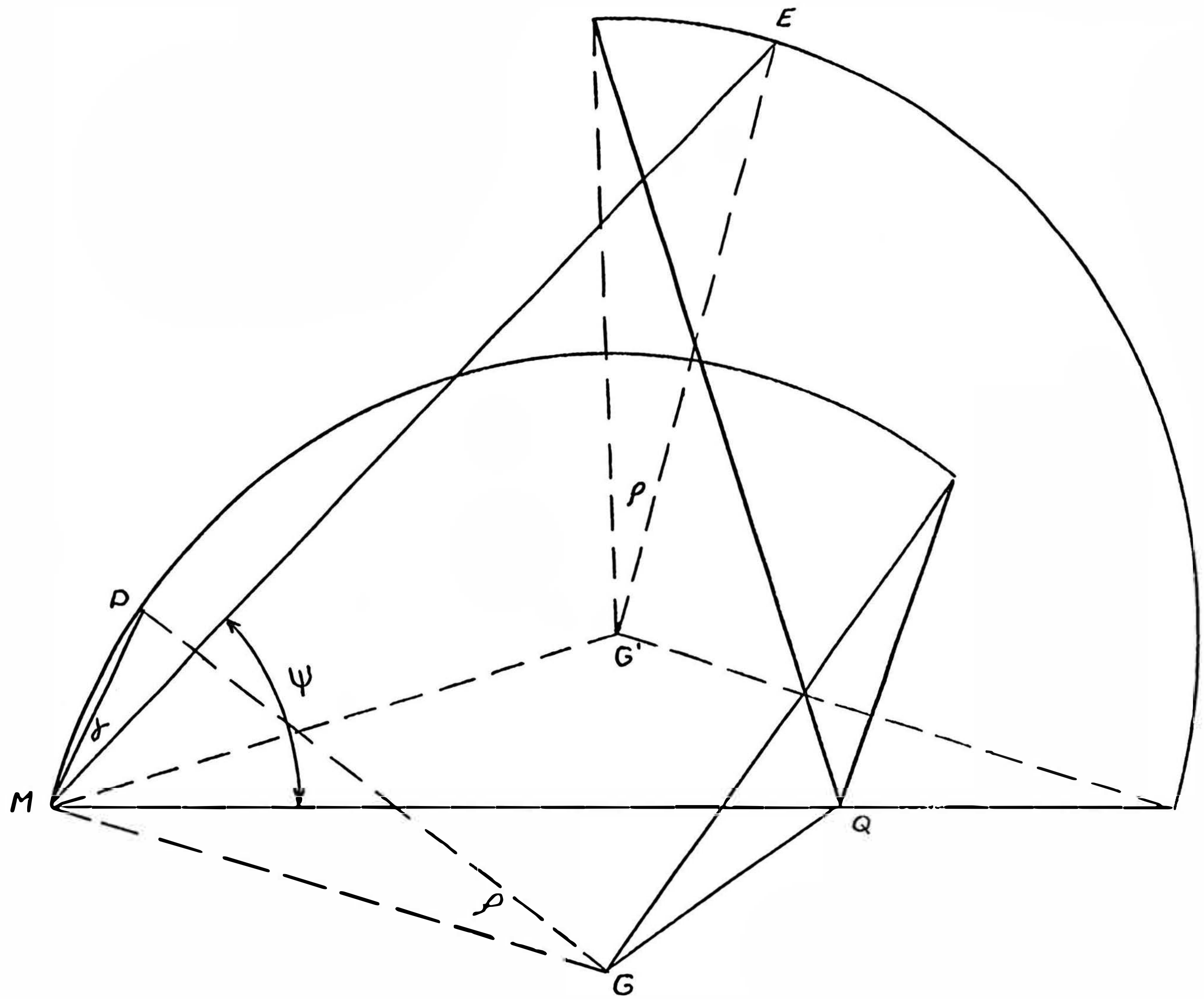

FIGURE 66

GEOMETRY FOR ANGLE RELATIONS

$$\rho = \frac{180 - \theta}{N}$$

$$R = \frac{r_1 \sin \theta/2}{\sin (180 + \alpha - \theta)}$$

$$\psi = 90 - \frac{\rho + \theta}{2}$$

$$MD = 2R \sin (\rho/2)$$

$$ME = 2R \left| \cos (\psi - \theta/2 + \alpha) \right|$$

$$r_2 = \frac{ME - MD}{2}$$

$$r_3 = \frac{ME + MD}{2}$$

$$r_4 = \left((r_1)^2 + (ME)^2 - 2(r_1)(ME) \cos \psi \right)^{1/2}$$

The value of $\rho = \frac{180 - \theta}{N}$ is used for the first calculation, twice that value for the second calculation, three times that value for the third calculation, and so forth until the Nth calculation when the value of ρ becomes $180 - \theta$. This method also can be used on the digital computer with good results.

APPENDIX B

DERIVATION OF EQUATIONS FOR DETERMINING θ_3 AND θ_4

In the derivation of the equations for jerk it was assumed that the angles θ_3 and θ_4 were known. These angles can be determined analytically from the geometry of the linkage. In Figure 67 all angles are considered positive when measured counter-clockwise. Link 2 is the driver and its angular position θ_2 is assumed to be known.

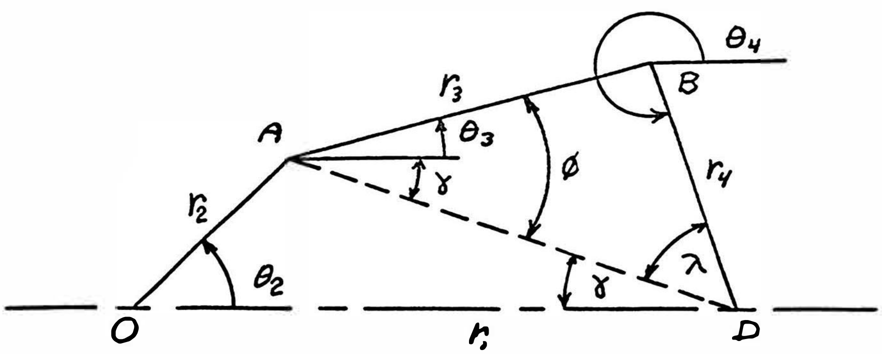

FIGURE 67

FOUR-BAR LINKAGE

Consider triangle OAD. From the law of cosines

$$a = ((r_1)^2 + (r_2)^2 - 2(r_1)(r_2)\cos\theta 2)^{\frac{1}{2}}$$

and from the law of sines

$$\frac{\sin\gamma}{r_2} = \frac{\sin\theta 2}{a}$$

Thus

$$\gamma = \sin^{-1}\left(\frac{r_2}{a}\sin\theta 2\right)$$

Next, for triangle ABD,

$$(r_4)^2 = ((r_3)^2 + a^2 - 2(r_3)(a)\cos\phi)^{1/2}$$

$$\phi = \cos^{-1}\left(\frac{(r_3)^2 + a^2 - (r_4)^2}{2r_3 a}\right)$$

Now from the configuration of the linkages it is seen that

$$\theta_3 = \phi - \gamma$$

Next, from triangle ABD,

$$\frac{\sin\lambda}{r_3} = \frac{\sin\phi}{r_4}$$

Thus

$$\lambda = \sin^{-1}\left(\frac{r_3}{r_4}\sin\phi\right)$$

From the configuration of the linkage

$$\theta_4 = 2\pi - (\lambda + \gamma)$$

Thus from the known lengths of the links, the values of θ_3 and θ_4 can be determined for any choosen value of θ_2.

APPENDIX C
COMPUTER FLOW-CHARTS

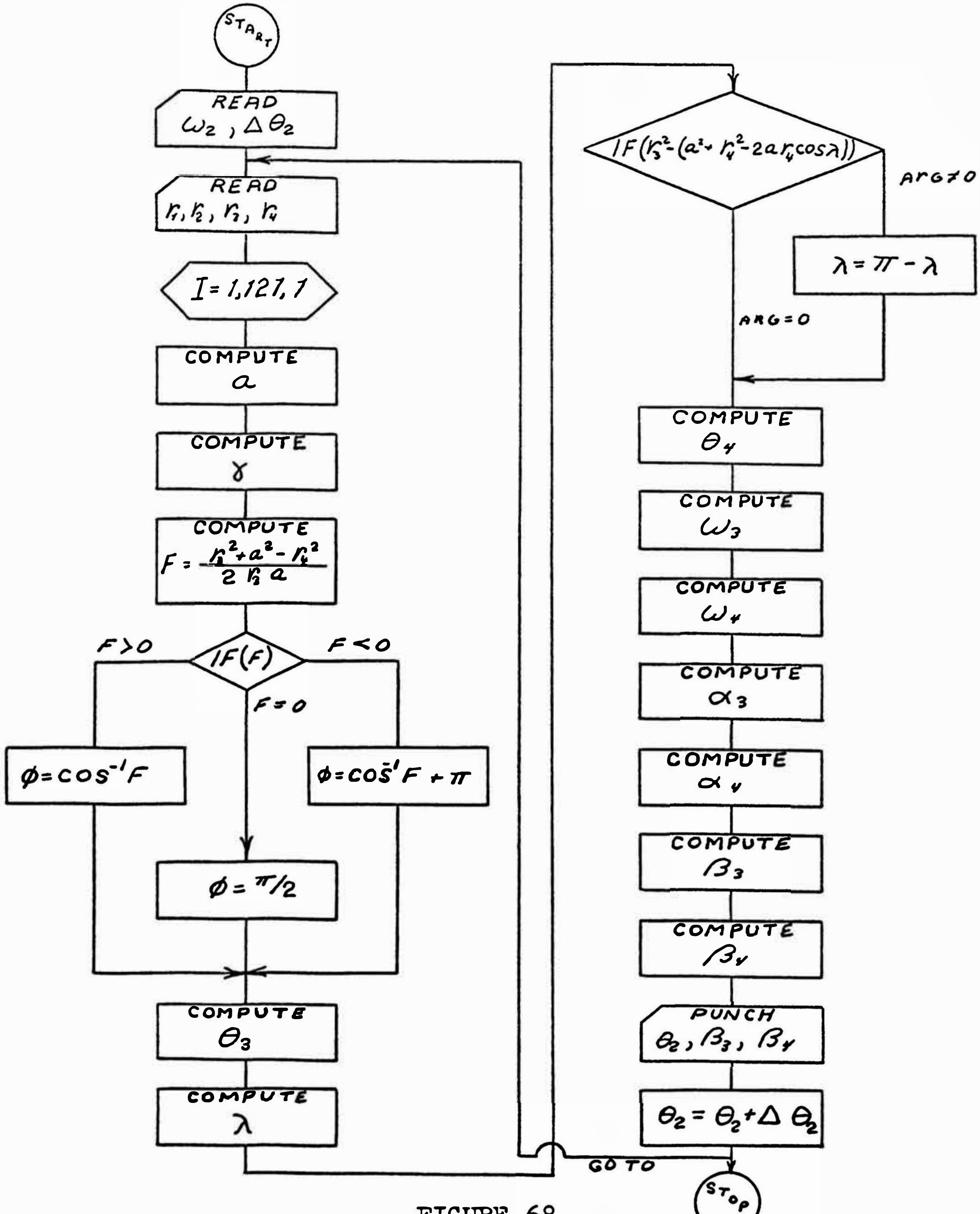

FIGURE 68

FLOW-CHART FOR COMPUTING JERK ON THE IBM 1620

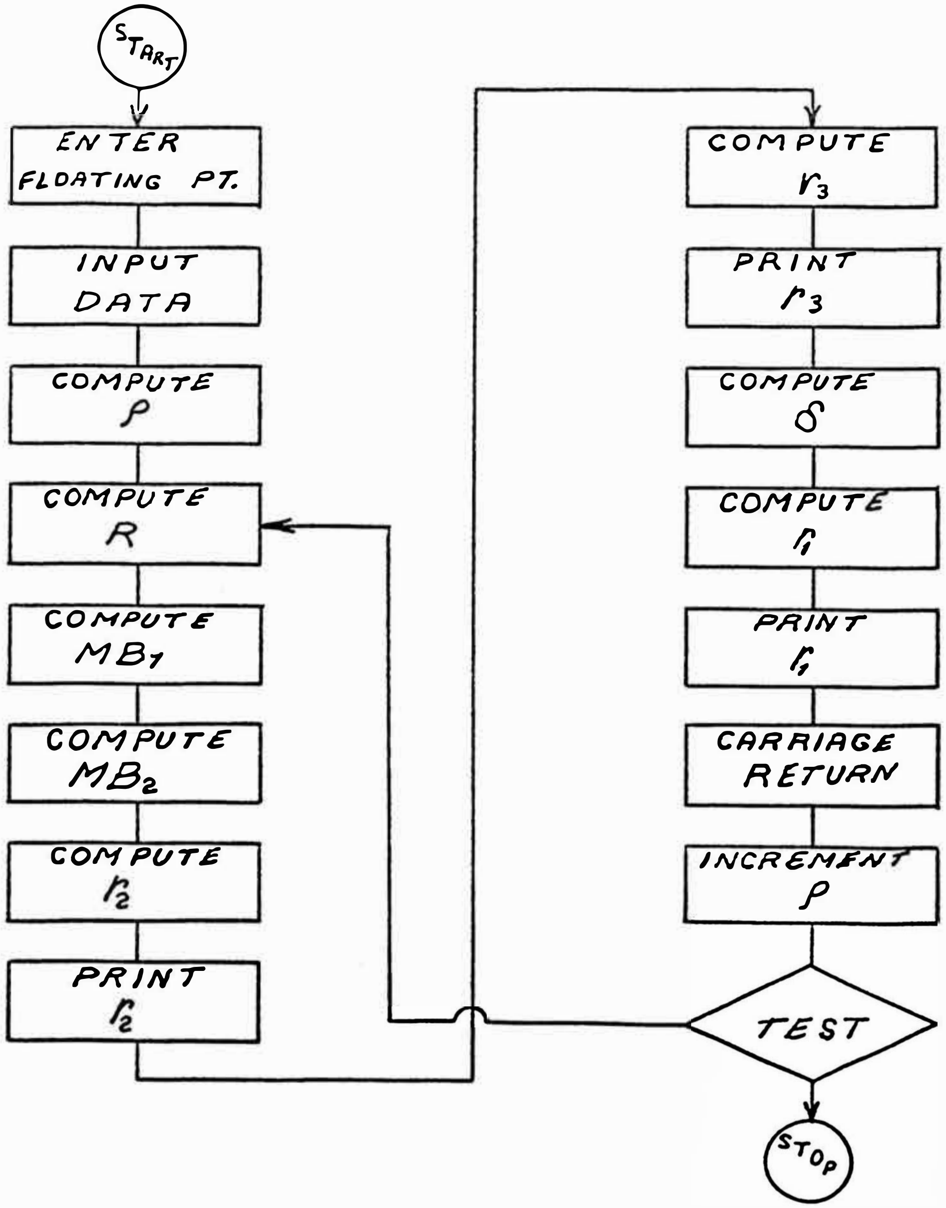

FIGURE 69

FLOW-CHART FOR DETERMINING THE DIMENSIONS OF THE LINKS BY METHOD ONE, ON THE ROYAL MCBEE LGP-30

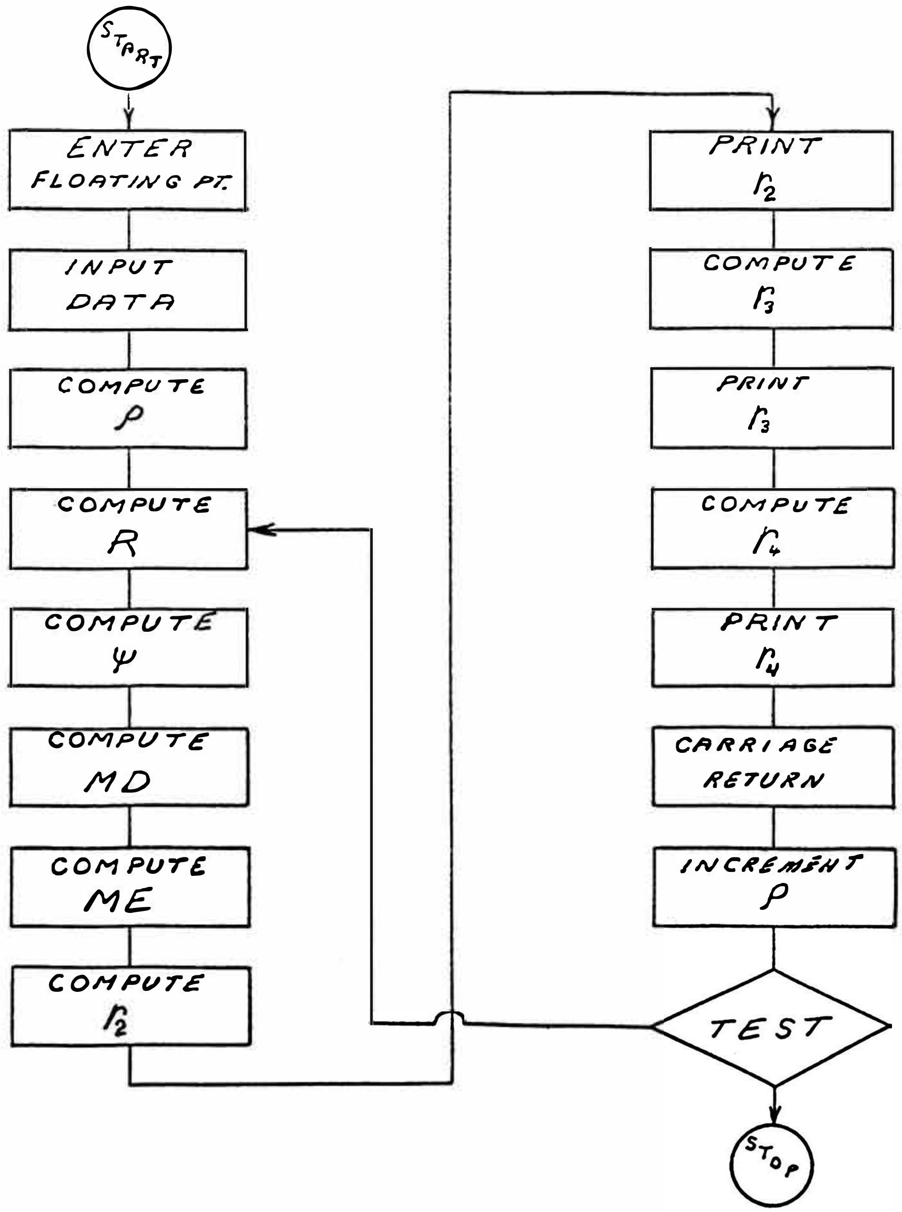

FIGURE 70

FLOW-CHART FOR DETERMINING THE DIMENSIONS OF THE LINKS BY METHOD TWO, ON THE ROYAL MCBEE LGP-30

VITA

The author was born on November 3, 1940, in Chicago, Illinois. He attended primary schools in Illinois and Missouri. He attended Ruskin High School in Hickman Mills, Missouri and graduated in 1958. In September 1958 he enrolled in Missouri School of Mines and received a Bachelor of Science Degree in Mechanical Engineering in August, 1962. He entered the Graduate School of the University of Missouri School of Mines and Metallurgy in September 1962 to work towards a Master of Science Degree in Mechanical Engineering.