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Axisymmetric Vibrations of Reinforced Orthotropic Shallow Spherical Caps

By Victor Birman[1] and George J. Simitses[2]

Abstract: Axisymmetric vibrations of reinforced shallow spherical caps manufactured from orthotropic materials are considered. The closed form solution is obtained for the natural frequency of the cap with a clamped and immovable circular edge by assuming that the motion component parallel to the cap boundary plane (in plane) is negligible. Parametric studies are performed to assess the effect of various geometric and structural parameters on the natural frequency of the cap and, most importantly, to identify the most influencing parameters of the problem. From the generated data, it is concluded that the national frequency increases with increasing extensional stiffness and eccentricity of reinforcements and to a lesser extent with increasing bending stiffness of reinforcements. Other important parameters include the base circle radius and the initial rise of the cap.

Introduction

Spherical shells and caps have numerous applications in aerospace, marine, and civil engineering structural systems. Satellites, protective structures of radars, pressure vessels, and curved bulkheads in submarine hulls are just a few examples. The number of reported studies considering statics and dynamics of isotropic spherical structures is considerable. However, the studies of nonisotropic spherical shells are relatively few.

Axisymmetric snap-through buckling of shallow spherical caps manufactured from a polar orthotropic material was studied by Varadan (1978). The caps were supposed to be clamped at the boundary. The analysis employed a two-term approximation for the transverse deflection mode shape. Nonlinear axisymmetric free and forced oscillations of such caps were studied by Varadan and Pandalai (1978). The effect of a circular hole on buckling was considered in Dumir et al. (1984). Static and dynamic analysis of geometrically nonlinear orthotropic shallow spherical caps on a linear Winkler-Pasternak foundation was presented by Nath and Jain (1986) by assuming that the deformation is axisymmetric.

Problems of dynamic buckling of orthotropic spherical caps subjected to step loading were investigated by Alwar and Reddy (1979), Ganapathi and Varadan (1982), Dumir et al. (1984), Nath and Jain (1986), and Jain and Nath (1987). These studies considered the effects of a central circular hole (Alwar and Reddy 1979; Dumir et al. 1984), viscous damping (Ganapathi and Varadan 1982), a Winkler-Pasternak elastic foundation and boundary conditions (Nath and Jain 1986; Jain and Nath 1987).

Axisymmetric free vibrations of a shear deformable orthotropic spherical

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shell were studied by Chao and Chern (1987), who employed a first-order shear deformation theory. Anisotropic spherical shells including orthotropic shells and shells assembled from several isotropic layers were also considered by Ambartsumyan (1974).

The bending stiffness of shells and caps can be greatly increased by appropriate stiffening. The studies of Simitses and Blackmon (1975) and Simitses and Cole (1970) represent an example of an approach to axisymmetric and asymmetric buckling of reinforced isotropic spherical shells, respectively. Reinforced cylindrical shells manufactured from composite materials were investigated by Block (1968), Bogdanovich and Koshkina (1983, 1984, 1986) and Birman 1988a, 1988b).

In this paper, small-amplitude axisymmetric vibrations of an orthotropic eccentrically stiffened thin spherical cap are considered. The material behavior of both the reinforcements and the cap is assumed to remain in the elastic range. The cap edge is clamped and immovable at the boundary plane and the motion is assumed to be strictly transverse without a component parallel to this plane.

Analysis

Consider a thin spherical cap reinforced in the circumferential and meridional directions (Fig. 1). The stiffeners in each direction are identical and uniformly spaced. The cap is shallow; its material is orthotropic. The principal material directions coincide with the corresponding coordinate directions of the triangular coordinate system used in Simitses and Blackmon (1975). The assumptions used in this paper coincide with those in Simitses and Blackmon (1975), i.e., those of small strains, moderate rotations, and the Donnell-Mushtari-Vlasov approximation. The spacing of stiffeners is such that their stiffnesses can be smeared out.

The analysis is limited to axisymmetric vibrations. Because the amplitude of motion is assumed to be small and the cap shallow, the component of motion in the planes parallel to the boundary plane can be neglected. Hence, the linear kinematic relations of point on the midsurface are:

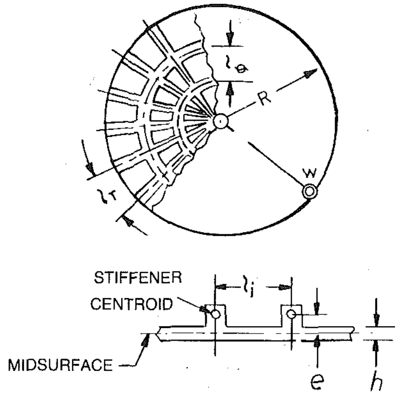

FIG. 1. Geometry of Cap

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$$\epsilon_{rr} = z_{,r}w_{,r} \qquad \epsilon_{\theta\theta} = \gamma_{r\theta} = 0$$

$$\kappa_{rr} = -w_{,rr} \qquad \kappa_{\theta\theta} = \frac{-w_{,r}}{r} \qquad \kappa_{r\theta} = 0 \quad (1)$$

where w = transverse displacement of the middle surface of the unreinforced cap; z = distance from the boundary plane to the undeformed middle surface; and r = radial polar coordinate. Moreover, ϵ_{rr}, $\epsilon_{\theta\theta}$, and $\gamma_{r\theta}$ are the reference surface strains, while κ_{rr}, $\kappa_{\theta\theta}$, and $\kappa_{r\theta}$ are reference surface changes in curvature and torsion.

The strains in a point at distance z from the middle surface are

$$\epsilon_r = \epsilon_{rr} + z\kappa_{rr} \quad (2a)$$

$$\epsilon_\theta = z\kappa_{\theta\theta} \quad (2b)$$

$$\gamma_{r\theta} = 0 \quad (2c)$$

The constitutive relations of a specially orthotropic material subject to axisymmetric deformations are

$$\sigma_r = Q_{rr}\epsilon_{rr} + z(Q_{rr}\kappa_{rr} + Q_{r\theta}\kappa_{\theta\theta}) \quad (3a)$$

$$\sigma_\theta = Q_{r\theta}\epsilon_{rr} + z(Q_{r\theta}\kappa_{rr} + Q_{\theta\theta}\kappa_{\theta\theta}) \quad (3b)$$

$$\sigma_{r\theta} = 0 \quad (3c)$$

where Q_{ij} are the reduced stiffnesses given by

$$Q_{rr} = \frac{E_r}{1 - \nu_{r\theta}\nu_{\theta r}} \quad (4a)$$

$$Q_{r\theta} = \frac{\nu_{\theta r}E_r}{1 - \nu_{r\theta}\nu_{\theta r}} \quad (4b)$$

$$Q_{\theta\theta} = \frac{E_\theta}{1 - \nu_{r\theta}\nu_{\theta r}} \quad (4c)$$

and σ_r, σ_θ, and $\sigma_{r\theta}$ are stress components at any material point. In Eq. 4, E_r and E_θ = moduli of elasticity of the cap skin material in the meridional and circumferential direction, respectively, and $\nu_{r\theta}$ and $\nu_{\theta r}$ = the Poisson's ratios. Note that Eq. 3 applies to the cap skin.

The stress-strain relations in the stiffeners are

$$\sigma_r^s = E_r\epsilon_r, \qquad \sigma_\theta^s = E_r\epsilon_\theta, \quad (5)$$

where σ_r^s and σ_θ^s = stresses in the meridional and circumferential stiffeners, respectively. In addition, all stiffeners are supposed to be manufactured from the same material with the modulus of elasticity in the direction of the stiffener axes E_r.

The stress resultants N_r, N_θ and stress couples M_r, M_θ can be represented as functions of the strains:

$$N_r = (A_{rr} + E_r^s)\epsilon_{rr} + E_r^s e_r \kappa_{rr} \quad (6a)$$

$$N_\theta = A_{r\theta}\epsilon_{rr} + E_\theta^s e_\theta \kappa_{\theta\theta} \quad (6b)$$

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$$M_r = E_r^s e_r \epsilon_{rr} + (D_{rr} + E_r^s e_r^2 + D_r^s)\kappa_{rr} + D_{r\theta}\kappa_{\theta\theta} \quad (6c)$$

$$M_\theta = D_{r\theta}\kappa_{rr} + (D_{\theta\theta} + E_\theta^s e_\theta^2 + D_\theta^s)\kappa_{\theta\theta} \quad (6d)$$

In Eq. 6, e_r and e_θ = the eccentricities of the corresponding stiffeners,

$$(A_{ij}, D_{ij}) = Q_{ij}\left(\frac{h,\ h^3}{12}\right) \quad (7)$$

h being the thickness of the cap skin, and

$$E_r^s = \frac{E_r A_r}{l_r} \qquad E_\theta^s = \frac{E_r A_\theta}{l_\theta} \quad (8a)$$

$$D_r^s = \frac{E_r I_r}{l_r} \qquad D_\theta^s = \frac{E_r I_\theta}{l_\theta} \quad (8b)$$

where I_r and I_θ = moments of inertia of the corresponding stiffeners about their centroidal axes; A_r and A_θ = stiffener cross-sectional areas; and l_r and l_θ = stiffener spacings. In the subsequent analysis, the height of the stiffeners (and their eccentricity) remains constant. However, the width of the radial stiffeners changes with r so that all parameters in Eq. 8 remain constant.

The potential and kinetic energies U and T of a cap experiencing axisymmetric deformations are given as follows:

$$U = \pi \int_0^R (N_r \epsilon_{rr} + M_r \kappa_{rr} + M_\theta \kappa_{\theta\theta}) r dr \quad (9a)$$

$$T = \pi \int_0^R \sigma h w_{,t}^2\ r dr \quad (9b)$$

where σ = effective mass density of the material of the cap including the "smeared out" reinforcements. The area of the stiffeners in the radial direction varies linearly with r, therefore the mass density can be represented as:

$$\sigma = \frac{\sigma_0 + \sigma_1 r}{R} \quad (10)$$

Substitution of Eqs. 1 and 6 into Eq. 9 yields the expressions for the potential and kinetic energies, which can be conveniently represented in a nondimensional form:

$$U_0 = \frac{UR^2}{Eh^5} = \pi \int_0^1 \left[(A_{rr} + E_r^s)\xi z_{,\xi}^2 w_{,\xi}^2 - 2E_r^s e_r R \xi z_{,\xi} w_{,\xi} w_{,\xi\xi} + (D_{rr} + E_r^s e_r^2 R^2 \right.$$

$$\left. + D_r^2)\xi w_{,\xi\xi}^2 + 2D_{r\theta} w_{,\xi} w_{,\xi\xi} + (D_{r\theta} + E_\theta^s e_\theta^2 R^2 + D_\theta^s)\frac{w_{,\xi}^2}{\xi} \right] d\xi \quad (11)$$

$$T_0 = \frac{TR^2}{Eh^5} = \pi R^4 \int_0^1 (\omega_0^2 + \omega_1^2 \xi)\xi w_{,\tau}^2 d\xi \quad (12)$$

where E = reference modulus.

The nondimensionalized parameters, identified by a bar, are defined as:

$$w = \frac{w}{h} \qquad (D_r^s, D_\theta^s) = \frac{D_r^s, D_\theta^s}{Eh^3}$$

$$\xi = \frac{r}{R} \qquad z = \frac{z}{h}$$

$$(e_r, e_\theta) = \frac{e_r, e_\theta}{R} \qquad R = \frac{R}{h} \quad \text{(13a)}$$

$$A_{rr} = \frac{A_{rr}}{Eh} \quad \text{(13b)}$$

$$(E_r^s, E_\theta^s) = \frac{E_r^s, E_\theta^s}{Eh} \quad \text{(13c)}$$

$$(D_{rr}, D_{r\theta}) = \frac{D_{rr}, D_{r\theta}}{Eh^3} \quad \text{(13d)}$$

The nondimensional time in Eq. 12 is:

$$\tau = \omega t \quad \text{(14)}$$

where ω = time-scale parameter that has the dimensions of frequency. Then

$$(\omega_0^2, \omega_1^2) = \frac{(\sigma_0, \sigma_1)h^2\omega^2}{E} \quad \text{(15)}$$

The approximation for the meridional curve used in this paper coincides with that of Huang (1964) and Simitses and Blackmon (1975):

$$z = e_0(1 - \xi^2) \quad \text{(16)}$$

The motion of the cap clamped at the reference surface can be represented by (Simitses and Blacmon 1975):

$$w = a_n(\tau)[J_0(k_n^1\xi) - J_0(k_n^1)] \quad \text{(17)}$$

where $J_1(k_n^1) = 0$. The boundary conditions

$$w = w_{,\xi} = 0 \qquad \text{at } \xi = 1 \quad \text{(18)}$$

are satisfied identically by the assumed displacement function, Eq. 17.

Substitution of Eqs. 16 and 17 into Eqs. 11 and 12 yields

$$U_0 = \pi a_n^2 U_0 \quad \text{(19a)}$$

$$T_0 = \pi a_{n,\tau}^2 T_0 \quad \text{(19b)}$$

U_0 and T_0 are given in Appendix I.

The total energy of a conservative system remains constant. Hence, the squared nondimensional frequency of motion is:

$$\Omega^2 = \frac{U_0}{T_0} \quad \text{(20)}$$

Note that the time scale parameter ω is arbitrary. If this parameter is known,

the corresponding value of Ω can be obtained from Eq. 20. Then the period of motion is available directly from

$$t_p = \frac{2\pi}{\omega\Omega} \quad (21)$$

Results and Review

In the chosen numerical examples, it was assumed that both the cap skin and the reinforcements were manufactured from the same orthotropic material. The geometry was assumed to be such that $E_r^s = E_\theta^s$; $D_r^s = D_\theta^s$; and $e_r = e_\theta$. The reference modulus E was taken as being equal to the modulus of elasticity of the material in the direction of the fibers. In this case

$$A_{rr} = \frac{1}{1 - \nu_{r\theta}\nu_{\theta r}} \approx 1$$

for almost all materials.

Then, if the time scale parameter is taken as $\omega = \sqrt{E_r/\sigma_0 h}$, the nondimensional analysis that yields the natural frequency Ω is material independent.

In the following examples, the geometric and stiffness parameters of the cap are: $E_r^s = E_r^0 = 0.5$; $D_r^s = D_r^0 = 12.5$; $e_r = e_\theta = 0.035$; $e_0 = 80$; $R = 250$; and $\omega_1^2 = 2.0$, if not indicated otherwise.

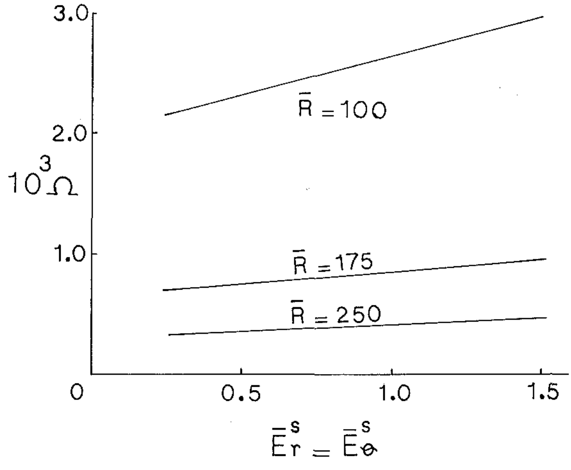

FIG. 2. Effect of Stiffener Extensional Stiffness on Natural Frequency

TABLE 1. Effect of Stiffener Bending Stiffness on Natural Frequency

$D_r^s = D_r^0$ (1)	5 (2)	15 (3)	25 (4)	35 (5)	45 (6)	55 (7)
$R = 100$	23.247	23.384	23.520	23.655	23.789	23.923
$R = 175$	7.518	7.563	7.608	7.652	7.697	7.741
$R = 250$	3.663	3.685	3.707	3.729	3.751	3.773

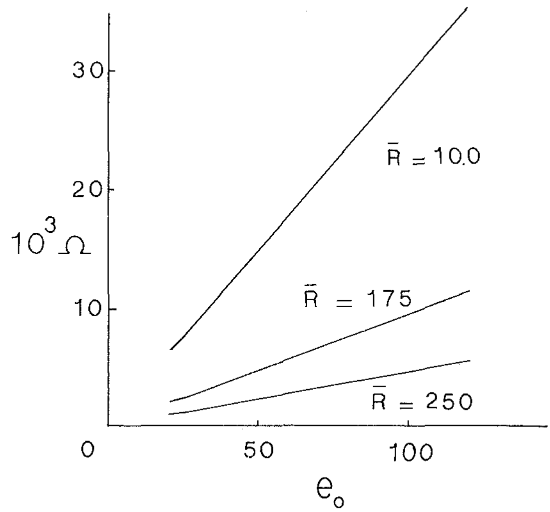

FIG. 3. Effect of Cap Rise Parameter on Natural Frequency

The effect of the extensional stiffness of the reinforcements on the nondimensional natural frequency is illustrated for three different values of R (Fig. 2). As the extensional stiffener stiffness increases, the total stiffness of the cap increases and, therefore, its natural frequency increases. This phenomenon is more pronounced for caps with smaller values for R. It can also be concluded that an increase in the stiffener spacing yields smaller frequencies. Note that the effect of varying the bending stiffnesses of reinforcements is less important, as seen from Table 1.

The cap rise parameter has a significant effect on the frequency, as shown in Fig. 3. This can be explained by the fact that an increase of e_0 makes the

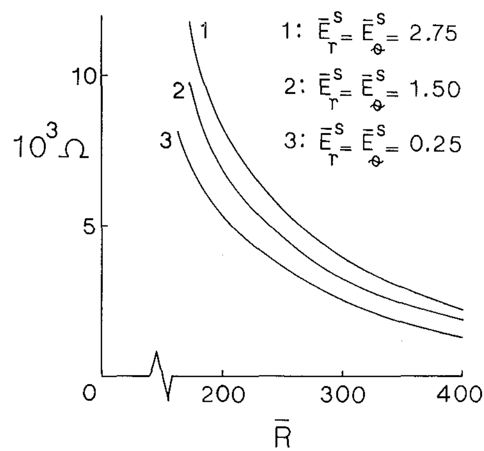

FIG. 4. Effect of Dimensionless Radius at Boundary Plane on Natural Frequency

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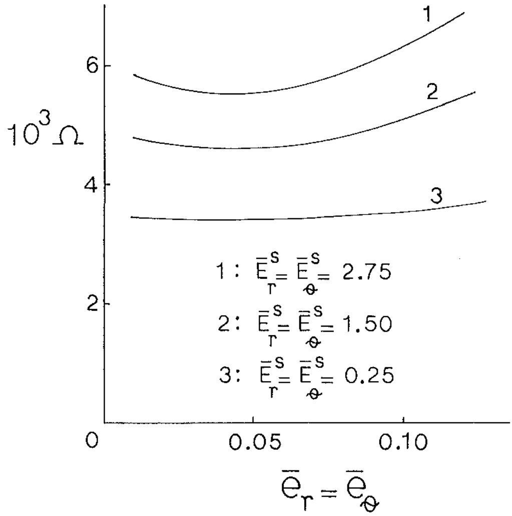

FIG. 5. Effect of Stiffener Eccentricities on Natural Frequency

cap less shallow. If other geometric parameters are kept constant, a less shallow (deeper) cap is usually stiffer.

Figs. 2 and 3 indicate that the dimensionless radius of the cap at the boundary plane R affects the natural frequency. This is shown in Fig. 4 for different values of nondimensional stiffener stiffnesses. All curves in Fig. 4 are similar and reflect the fact that smaller values of R result in much higher frequencies of otherwise identical caps.

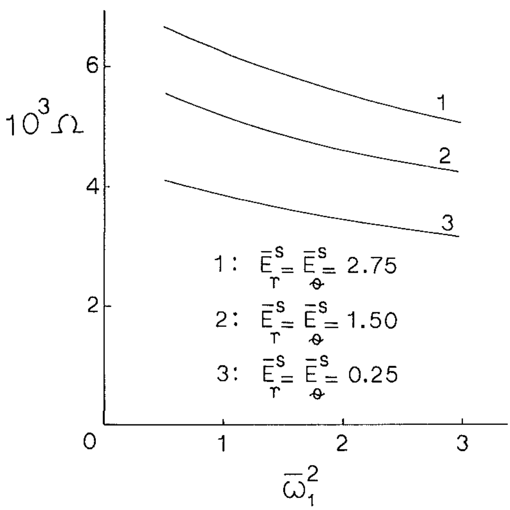

FIG. 6. Effect of Mass Distribution Parameter on Natural Frequency

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The eccentricity of the stiffeners is another important factor which influences the frequency, as follows from Fig. 5. The caps with larger extensional stiffnesses of reinforcements appear to be more sensitive to variations in stiffener eccentricities. This is expected because such cap reinforcements make a larger contribution to their total stiffness.

Finally, the effect of the mass distribution on frequencies is illustrated in Fig. 6. Redistribution of the mass to the top of the cap can increase its natural frequency. This can be explained by the larger stiffness of such caps.

Conclusions

The natural frequency of axisymmetric vibrations of reinforced shallow orthotropic caps was shown to increase with an increase of the extensional stiffness and eccentricity of reinforcements. An increase of the bending stiffness of reinforcements also results in larger natural frequencies but to a lesser degree. Natural frequencies can also be increased by an increase of the cap rise parameter and the concentration of the mass at the top of the cap. Conversely, an increase of the cap radius at the boundary plane as well as larger stiffener spacings resulted in smaller frequencies of vibration.

Appendix I.

$$U_0 = \frac{2}{3} e_0^2 k^2 (A_{rr} + E_r^s)[J_1^2(k) + J_2^2(k)] + 2Re_0 e_r E_r^s \left[\frac{k^3}{4}(J_0(k) + J_2(k))J_1(k) - i_1\right]$$

$$+ \frac{k^2}{4}(D_{rr} + E_r^s e_r^2 R^2 + D_r^s)\left[\frac{k^2}{2}(J_0^2(k) + J_1^2(k) - 2i_2 + i_3\right]$$

$$+ D_{r\theta} k^2 J_1^2(k) - \frac{k^2}{2}(D_{r\theta} + E_\theta^s e_\theta^2 R^2 + D_\theta^s)[J_0^2(k) + J_1^2(k) - 1]$$

where $J_\nu(k)$ are Bessel functions of the order ν; and $k = k_n^1$ are the zeroes of $J_1(x) = 0$. Moreover, i_j ($j = 1$, 2 and 3) are integrals, which are defined as:

$$i_1 = \int_0^k \psi^2 J_1(\psi) J_2(\psi) d\psi = 4 \sum_{p=0,1,2,\ldots}^{\infty} (2 + 2p) J_{2+2p}^2(k) - \frac{k^3}{4}[J_0(k) + J_2(k)]J_1(k)$$

$$i_2 = \int_0^k \psi J_0(\psi) J_2(\psi) d\psi = 2 \sum_{p=1,2,\ldots}^{\infty} J_p^2(k) - \frac{k^2}{2}[J_0^2(k) + J_1^2(k)]$$

$$i_3 = \int_0^k \psi J_2^2(\psi) d\psi = 2 \sum_{p=0,1,2,\ldots}^{\infty} (3 + 2p) J_{3+2p}^2(k)$$

$$T_0 = R^4 \omega_0^2 \frac{1}{2}[J_0^2(k) + J_1^2(k)] - \frac{2}{k} J_0(k) J_1(k)$$

$$+ \frac{1}{2} J_0^2(k) + R^4 \omega_1^2 \left[\frac{1}{k^3} i_4 - \frac{2}{k^3} J_0(k) i_5 + \frac{1}{3} J_0^2(k)\right]$$

Finally, i_4 and i_5 denote

$$i_4 = \int_0^k \psi^2 J_0^2(\psi) d\psi$$

$$i_5 = \int_0^k \psi^2 J_0(\psi) d\psi,$$

and can be calculated by using power series.

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Appendix III. Notation

The following symbols are used in this paper:

A_r, A_θ = stiffener cross-sectional areas;
$A_{rr}, A_{r\theta}, A_{\theta\theta}$ = orthotropic extensional stiffnesses (cap skin);
$\bar{A}_{rr}$ = parameters identified by bar denoting nondimensionalized parameters (see Eq. 13);
$a_n(t)$ = function of time (see Eq. 17);
$D_{rr}, D_{r\theta}, D_{\theta\theta}$ = orthotropic bending stiffnesses (cap skin);
D_r^s, D_θ^s = stiffener bending stiffnesses (smeared);
E_r, E_θ = stiffener Young's moduli;
E_r^s, E_θ^s = stiffener extensional stiffnesses (smeared);
e_r, e_θ = stiffener eccentricities;
e_0 = cap rise parameter (see Eq. 16);
h = cap skin thickness;
I_r, I_θ = stiffener second moments of area;
J_ν = Bessel functions of order ν;
k_n^1 = values of variable x that leads to $J_1(x) = 0$;
l_r, l_θ = stiffener spacings;
M_r, M_θ = cap stress couples (moment resultants);
N_r, N_θ = cap stress resultants;
$Q_{rr}, Q_{r\theta}, Q_{\theta\theta}$ = elements of orthotropic stiffness matrix;
R = radius of cap base circle;
r, θ = polar coordinates;
T = kinetic energy;
t = time variable;
t_p = period of oscillatory motion (see Eq. 21);
U = total potential;
w = transverse displacement of reference surface points;
Z = distance from reference surface;
z = initial shape of cap;
$\epsilon_{rr}, \epsilon_{\theta\theta}, \gamma_{r\theta}$ = extensional strains of reference surface points;
$\epsilon_r, \epsilon_\theta$ = extensional strains of any point;
$\kappa_{rr}, \kappa_{\theta\theta}, \kappa_{r\theta}$ = changes in curvature and torsion of reference surface points;
$\nu_{r\theta}, \nu_{\theta r}$ = orthotropic Poisson's ratios;
ξ = nondimensionalized radial coordinate = r/R;
σ = effective mass density of cap material;
τ = nondimensionalized time variable;
ψ = dummy variable of integration;
ω = time scale parameter; and
Ω = nondimensionalized frequency.

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