1964

The electrical analogy of critical speed for the transverse vibrations of a shaft with one, two and three masses

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THE ELECTRICAL ANALOGY OF CRITICAL SPEED FOR THE TRANSVERSE VIBRATIONS OF A SHAFT WITH ONE , TWO AND THREE MASSES .

BY

ASHOK Z. PATEL, 1940

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THESES

submitted to the faculty of the

UNIVERSITY OF MISSOURI AT ROLLA

in partial fulfillment of the requirements for the

Degree of

MASTER OF SCIENCE, MECHANICAL ENGINEERING

ROLLA , MISSOURI

1964

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ABSTRACT

The object of this thesis is to determine the critical speed of a shaft by an electrical analogy. The shaft considered is of constant diameter and is simply supported at the ends by fixing it in self aligning bearings. The weight of the shaft is assumed to be proportionally distributed in the attached masses.

The original system is first of all converted to an equivalent simple spring mass system by calculating the influence coefficients and hence from these determining the spring constants. The spring mass system obtained is then converted to an equivalent electrical circuit.

A sine-square wave frequency generator was used to produce the sine wave voltage which simulates the cyclic external force on the mechanical system.

In the case of the three mass system two of the capacitances were negative. The solution for negative capacitance was obtained by making two separate amplifier circuits employing inverse feedback in the amplifier circuit.

The method used in this thesis makes the problem easy for up to two masses, but for more than two masses it becomes involved by the introduction of negative capacitances. The author has given the solution for the three mass system to show the complications introduced by increasing the number of masses.

ACKNOWLEDGEMENT

The completion of a study and experiment depends upon assistance and cooperation of many persons. The guidance given by Prof. Charles L. Edwards and Prof. Baumgartner is gratefully acknowledged. The author is indebted to Prof. Murray for checking the work done in this thesis and making the corrections. The author is deeply appreciative of the sincere encouragement he received from both faculty and fellow students during the course of study.

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LIST OF SYMBOLS

ω_n	Natural circular frequency (rad/sec.)
ω_e	Electrical frequency (rad./sec.)
f_n	Natural frequency (cps.)
f_e	Electrical frequency (cps.)
E	Modulus of elasticity (p.s.i.)
I	Moment of inertia ($in.^4$)
d	Diameter of shaft (in.)
L'	Length of shaft (in.)
k, k_{ij}	Spring constants (lb./in.)
m	Mass (lb.- sec^2/in.)
x	Displacement (in.)
L	Inductance (hy.)
C, C_{ij}	Capacitance (farads)
q	Charge (Coulombs)
i	Current (amps.)
T,t	Time (sec.)
F	Force (lbs.)
E'	Voltage (volts)
a_{ij}	Influence coefficient(in.)
w	Weight (lbs.)
g	Gravitational acceleration (in./sec^2)

INTRODUCTION

An analogy is a recognized relationship of consistent mutual similarity between the equations and elements appearing in two or more fields of knowledge, and an identification of quantities which play mutually similar roles in these equations for the purpose of transfer of knowledge of the behavior of the elements between these fields. Analog devices were employed in surveying and other cartographic projects by the Babylonians as early as 3800 B.C.

The mechanical system which is dealt with is composed of a constant diameter shaft with one, two and three masses attached respectively. In the electrical analogy, an equivalent circuit consists of inductance, capacitance and resistance. The chosen mechanical system is first converted to a simple spring mass lumped system by finding the influence coefficients, then the equivalent electrical circuit is formed. It will be observed that the differential equations for all of the cases are fully analogous.

If the mechanical system is allowed to operate at or near one of the natural frequencies the deflection becomes very large. In the equivalent electrical circuit the critical speed occurs when the voltage across the corresponding capacitance becomes a maximum. The system is considered in transverse vibrations only.

The study of mechanical vibrations and the electrical analogy of the mechanical system made the subject interesting to the author and that lead him to select this topic.

REVIEW OF LITERATURE

The method of solution for critical speed problems in a mechanical system was developed by Stodola who considered influence coefficients.

Church (1)* discusses the fundamentals of the electrical analogy for mechanical systems. He gives a great deal of discussion about the electrical equivalents to mechanical systems by dimensionless analysis.

Kemler and Freberg (3) discuss the electrical analogy and gives the solutions of various problems.

Karplus and Soroka (2) discuss the electrical analogy of a simply supported beam with three concentrated masses by the influence coefficient method. The solution of the problem using negative capacitance is introduced employing inverse feedback in the amplifier circuit.

Olson (4) discusses the dynamical analogies and gives general ideas about the chosen problem.

* Refer to Bibliography for all references.

DISCUSSION

In the first case a simply supported shaft with a concentrated mass of one pound is considered. The diameter of the shaft is 0.25" and the over all length is 20". The mass is placed at equal distances from both the ends. The properties of the shaft are as follows:
The modulus of elasticity of the shaft is 30×10^6 p.s.i. The moment of inertia of the shaft is;

$$I = \frac{\pi}{64}(d)^4 = \frac{\pi}{64}\left(\frac{1}{4}\right)^4 = 1.92 \times 10^{-4} \text{ in.}^4$$

The spring constant of the shaft at the disc is;

$$k = \frac{48EI}{L^3} = \frac{48 \times 30 \times 10^6 \times 1.92 \times 10^{-4}}{(20)^3} = 34.56 \text{ lb/in.}$$

The natural frequency of the shaft is:

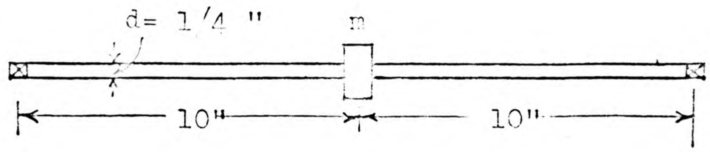
$$\omega_n = \sqrt{k/m} = \sqrt{34.56 \times 386} = 115.5 \text{ rad/sec.}$$

The above system can be reduced to a spring mass system which can be represented by the equivalent electrical circuit, as shown in fig.1.

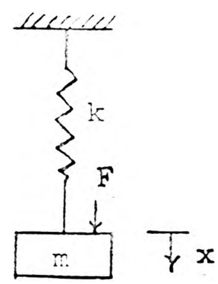
The equation of motion for a spring mass system displaced by a distance x , due to the force F is;

$$m\frac{d^2x}{dt^2} + kx - F = 0$$

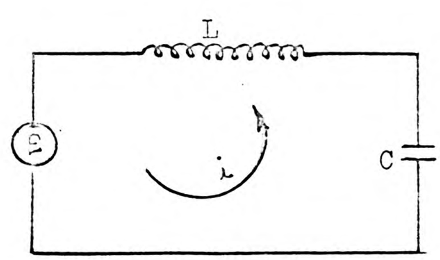
According to Kirchhoff's voltage law, the sum of the voltage drops around a closed loop must be zero. Writing the sum of the voltage drops around the loop,



(a) One Mass System.



(b) Equivalent Spring Mass System.



(c) Equivalent Electrical Circuit.

Figure 1

$$L\frac{di}{dt} + \frac{1}{C}q - E = 0 \text{ , but } q = \int i\,dt \text{ , } \frac{dq}{dt} = i \text{ , } \frac{d^2q}{dt^2} = \frac{di}{dt} \text{ ,}$$

therefore,

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q - E = 0.$$

If q in the electrical system is analogous to x in the mechanical system, by then comparing the terms of the voltage equation in the electrical circuit to the equation of motion in the mechanical system, the following terms become equivalent.

$$\frac{d^2x}{dt^2} = \frac{d^2q}{dt^2} \text{ , } m = L \text{ , } k = \frac{1}{C} \text{ , } F = E.$$

Thus the electrical circuit satisfies the motion equation, hence it is an equivalent system. Table 1 describes the the properties of the elements in a mechanical system and the corresponding electrical components for a force voltage analogy.

To find the equivalent quantities in the electrical circuit, Buckingham's pi theorem was used. This theorem states that, if there are N quantities to be considered and M fundamental dimensions, the number of independent dimensionless groups, designated as π_1, π_2 etc., that will be formed is N - M.

The amplitude of the displacement at the mass is:

$$x = f(m, k, F, \omega_n)$$

There are three fundamental dimensions in this system, they are: length in inches (L), force in pounds (F), and time in seconds (T).

By Buckingham's pi theorem, there are 5 quantities to be considered and 3 fundamental dimensions, thus the number of independent dimensionless groups will be five minus three i.e. 2, pi groups. To find these the quantities m, k and x are selected, giving each quantity the exponents of a, b and c respectively. Considering these quantities with each of the remaining quantities in turn:

$$\pi_1 = m^a\, k^b\, x^c \omega_n = (\, F\, L^{-1}\, T^2\,)^a\, (F\, L^{-1}\,)^b\, (L)^c\, (T)^{-1}$$

$$\pi_2 = m^a\, k^b\, x^c F \quad = (\, F\, L^{-1}\, T^2\,)^a\, (F\, L^{-1}\,)^b\, (L)^c\, (F)$$

The algebraic sum of the exponents of each fundamental dimension must equal zero. Considering the π_1 group;

F exponents : $a + b = 0$

L exponents : $- a - b + c = 0$

T exponents : $2a - 1 = 0$

By solving the above three equations, values of a , b and c are found to be $a = 1/2$, $b = - 1/2$, $c = 0$. Substituting these exponents in the first form of π_1 ;

$$\pi_1 = \omega_n \sqrt{m/k}$$

Similarly the value of π_2 can be found as $\pi_2 = F/kx$. The ratios π_1, π_2 are converted to an equivalent electrical circuit terms by the substitution of analogous terms given in Table I.

$$\pi_1 = \omega_n \sqrt{m/k} = \omega_e \sqrt{L\, C}$$

$$\pi_2 = \frac{F}{k\, x} = \frac{EC}{q}$$

When the electrical circuit components are selected to have the same dimensionless ratios as their mechanical counterparts, the two systems are completely equivalent.

Letting the inductance L = 3.73 mhy., and the capacitance C = 0.1 μF

The experimental reading of the electrical frequency is taken on the frequency generator, when the amplitude of the sine wave curve in the oscilloscope, is a maximum, giving a reading of 8230 cps.

$$\omega_e = 8230\ (2\pi) = 51684.4 \text{ rad/sec.}$$

Converting this frequency to mechanical frequency, which is the critical speed, by dimensionless group:

$$\pi_1 = \omega_n \sqrt{m/k} = \omega_e \sqrt{L\ C}$$

$$= \omega_e \sqrt{3.73 \times 10^{-3} \times 0.1 \times 10^{-6}}$$

$$= \omega_e\ 10^{-5} \sqrt{3.73} = 51684.4\ (10^{-5}) 1.932$$

$$= 0.999$$

$$\omega_n = \sqrt{k/m} \times 0.999 = 115.5 \times 0.999$$

$$\omega_n = 115.384 \text{ rad/sec., } f_n = 18.373 \text{ cps.}$$

Thus the critical speed of the one mass system is $\omega_n = 115.384$ rad/sec., by the electrical analogy.

The theoretical result is:

$$\omega_n = 115.5 \text{ rad/sec.,} \quad f_n = 18.4 \text{ cps.}$$

The electrical analogy result agrees with the theoretical results with the deviation of 0.15%.

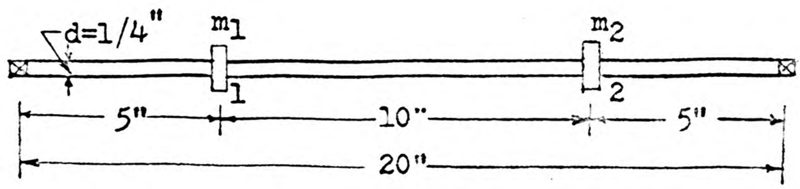
The Two Mass System

In this case a system with two masses m_1 and m_2 of 1.26 lb., and 1 lb., respectively, are fixed on a 0.25" diameter steel shaft supported by self aligning bearings at the ends. It is assumed that the weight of the shaft is included in the attached masses m_1 and m_2 .

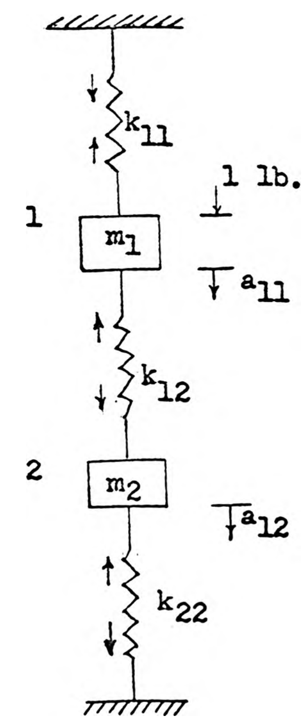
Before converting this system to the equivalent electrical system it is better to convert it into a simple spring mass system by caculating the influence coefficients and hence find the spring constants.

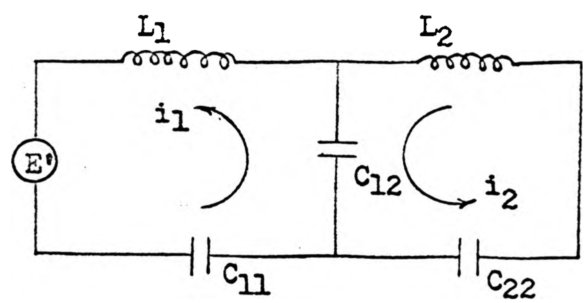
There are two masses in the system which give four influence coefficients $a_{11}, a_{22}, a_{12}, a_{21}$, the subscripts on the influence coefficients are so arranged that the second subscript indicates the point at which the unit load is applied, while the first indicates the point at which the corresponding deflection is being considered. For small deflections Maxwell's reciprocity rule states that $a_{12} = a_{21}$. So actually there are only three independent influence coefficients. The two masses and three influence coefficients may be represented by the two masses and three spring constants, which can be converted to an electrical circuit as shown in the figure 2 (c).

A unit load is applied to each mass in turn in figure 2(b) and the equations of static equilibrium are written in each case for the entire system and for each mass individually, thereby obtaining four equations. Applying a 1 lb. load at 1, the deflection of 1 is a_{11} and that of 2 is a_{21}. These two deflections will be resisted by two springs k_{11} and k_{22}. For static equilibruim of the entire system, the sum of



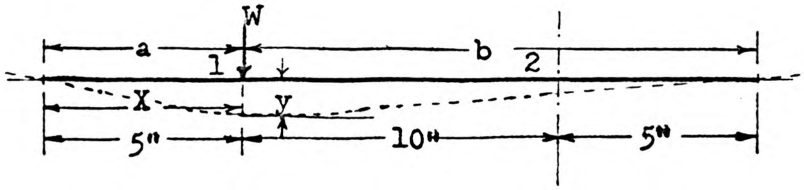
(a) Two Mass System.





(c) Equivalent Electrical Circuit.

(b) Equivalent Spring Mass System.



(d) Simply Supported Beam.

Figure 2.

the restoring forces in the system equals the applied force 1 lb. Let a_{11} be greater than a_{12}.

$$a_{11}\,k_{11} + k_{12}\,(a_{11} - a_{12}) - k_{12}\,(a_{11} - a_{12}) + k_{22}\,a_{12} = 1$$

$$a_{11}\,k_{11} + a_{12}\,k_{22} = 1 \quad \text{-------------(1)}$$

Considering static equilibrium at mass m_1,

$$a_{11}\,k_{11} + (a_{11} - a_{12})\,k_{12} = 1 \quad \text{-----------(2)}$$

Similarly if 1 lb. load is applied at 2, and the static equilibrium equation is written for the entire system and for the equilibrium at 2, two similar equations are obtained:

$$a_{22}\,k_{22} + a_{21}\,k_{11} = 1 \quad \text{-------------(3)}$$

$$(a_{22} - a_{21})\,k_{21} + a_{22}\,k_{22} = 1 \quad \text{-----------(4)}$$

To calculate the influence coefficients, consider the original system which is a simply supported beam with two masses on it. A load W is applied at point 1 at a distance a from the left end and b from the right end of a simply supported beam, as shown in fig. 2(d). The deflection y at any distance X from the left end support is given by,

$$y = \frac{W\,b\,X\,(a^2 + 2\,a\,b - X^2)}{6\,E\,I\,L'}$$

The load applied to find the influence coefficients is a unit load therefore,the deflection y is the influence coefficient. Let X = a =5" and b=15",then y is the influence coefficient a_{11}:

$$a_{11} = \frac{b\,X\,(a^2 + 2\,a\,b - X^2)}{6\,E\,I\,L'} = \frac{15(5)(150)}{6\,E\,I\,(20)} = \frac{375}{4EI}$$

$$a_{11} = \frac{375}{4 \times 30\times10^6 \times 1.92\times10^{-4}} = 0.01625''$$

To calculate a_{22} and a_{12} a unit load is applied at 2. When b = 5",
a = 15" , X = 15" ,

$$a_{22} = \frac{b\ X\ (2\ a\ b\ )}{6\ E\ I\ L} = \frac{5\ (15)(150)}{6\ EI(20)}$$

$$= \frac{375}{4\ E\ I} = 0.01625''$$

$a_{11} = a_{22}$ by symmetry.

To calculate a_{12} , let a = 15", b = 5" , and X = 5" ,

$$a_{12} = \frac{5(5)(15^2 + 150 - 25)}{6\ E\ I\ (20)} = \frac{875}{12\ EI}$$

$$= \frac{875}{12 \times 30\times10^6 \times 1.92\times10^{-4}}$$

$$a_{12} = 0.01265''$$

By Maxwell's reciprocity rule $a_{12} = a_{21}$.

The equations (1),(2),(3),(4) are solved for k_{11} , k_{22} , k_{12} by substituting the values of a_{11} , a_{22} , a_{12} . Subtracting equation (4) from (2),

$$a_{11}\ k_{11} - a_{11}\ k_{22} = 0 \quad \text{- - - - - - - - - - (5)}$$

$$k_{11} = k_{22}$$

Subtracting (5) from (1),

$$k_{22} = \frac{1}{(a_{11} + a_{12})} = \frac{3\ E\ I}{500} = \frac{3 \times 30\times10^6 \times 1.92\times10^{-4}}{500}$$

$$k_{22} = 34.6 \text{ lb/in.} = k_{11}$$

From equation (4),

$$a_{22}\,k_{22} + (a_{22} - a_{21})\,k_{21} = 1$$

$$k_{21} = \frac{21\,E\,I}{1000} = \frac{21 \times 30\times10^{6} \times 1.92\times10^{-4}}{1000} = 121 \text{ lb/in.}$$

Equivalent system:

At any instant displacements of masses m_1 and m_2 are x_1 and x_2 respectively, due to the force F. The equations of motion of the system shown in figure 2 (b) are:

$$m_1 \frac{d^2 x_1}{dt^2} + k_{11}\,x_1 + k_{12}(x_1 - x_2) - F = 0$$

$$m_2 \frac{d^2 x_2}{dt^2} + k_{22}\,x_2 - k_{12}(x_1 - x_2) = 0$$

According to Kirchhoff's voltage law, the sum of the voltage drops around a closed loop must be zero. Considering the first loop of figure 2(c),

$$L_1 \frac{d^2 q_1}{dt^2} + \frac{1}{C_{12}}(q_1 - q_2) + \frac{1}{C_{11}}\,q_1 - E = 0$$

Considering the second loop,

$$L_2 \frac{d^2 q_2}{dt^2} + \frac{1}{C_{22}}\,q_2 - \frac{1}{C_{12}}(q_1 - q_2) = 0$$

If $q_1 = x_1$ and $q_2 = x_2$, comparing the corresponding terms in the equations of motion in the mechanical system to the equations in the electrical system,

$$m_1 = L_1, \quad m_2 = L_2, \quad k_{11} = \frac{1}{C_{11}},$$

$$k_{22} = \frac{1}{C_{22}}, \quad k_{12} = \frac{1}{C_{12}}, \quad F = E$$

Therefore, the electrical circuit satisfies the motion equations, hence it is an equivalent system.

Applying Buckingham's pi theorem to calculate the equivalents in the electrical circuit for the two mass system.

$$x_1 \text{ or } x_2 = f\,(m_1, m_2, k_{11}, k_{22}, k_{12}, F, \omega_n)$$

There are eight variables and three fundamental dimensions, so the number of independent dimensionless groups, designated as $\pi_1, \pi_2 \cdots$ etc., that will be formed are (8-3)= 5.

$$\pi_1 = m_1^a k_{11}^b x^c m_2 = (F L^{-1} T^2)^a (F L^{-1})^b (L)^c (F L^{-1} T^2)$$

$$\pi_2 = m_1^a k_{11}^b x^c k_{12} = (F L^{-1} T^2)^a (F L^{-1})^b (L)^c (F L^{-1})$$

$$\pi_3 = m_1^a k_{11}^b x^c k_{22} = (F L^{-1} T^2)^a (F L^{-1})^b (L)^c (F L^{-1})$$

$$\pi_4 = m_1^a k_{11}^b x^c F = (F L^{-1} T^2)^a (F L^{-1})^b (L)^c (F)$$

$$\pi_5 = m_1^a k_{11}^b x^c \omega_n = (F L^{-1} T^2)^a (F L^{-1})^b (L)^c (T^{-1})$$

To illustrate the procedure of obtaining the dimensionless ratio the procedure was followed using one of these quantities, say π_5. The algebraic sum of the exponents of each fundamental dimension must equal zero.

F exponents: $a + b = 0$

L exponents: $-a - b + c = 0$

T exponents: $2a - 1 = 0$

Solving the above three equations, $a = 1/2$, $b = -1/2$, $c = 0$, therefore,

$$\pi_5 = \omega_n \sqrt{m_1 / k_{11}} = \omega_e \sqrt{L_1 C_{11}}$$

Treating the other π values in a similar manner the following values are obtained;

$$\pi_1 = \frac{m_2}{m_1} = \frac{L_2}{L_1} \; ; \; \pi_2 = \frac{k_{12}}{k_{11}} = \frac{C_{11}}{C_{12}} \; ; \; \pi_3 = \frac{k_{22}}{k_{11}} = \frac{C_{11}}{C_{22}} \; ; \; \pi_4 = \frac{F}{k_{11} x} = \frac{E\,C_{11}}{q}$$

To calculate the equivalent electrical quantities corresponding to the

mechanical properties, the above dimensionless pi groups were calculated. Assume $C_{11} = 1\mu F$ and $L_1 = 4.44$ hy.

$$\pi_2 = \frac{k_{12}}{k_{11}} = \frac{C_{11}}{C_{12}} \; ; \quad C_{12} = \frac{34.6}{121}(10)^{-6} = 0.286\,\mu F.$$

$$\pi_3 = \frac{k_{22}}{k_{11}} = \frac{C_{11}}{C_{22}} = 1 \; ; \quad C_{11} = C_{22} = 1 \;\mu F.$$

$$\pi_1 = \frac{m_2}{m_1} = \frac{L_2}{L_1} \; ; \quad L_2 = \frac{4.44}{1.26} = 3.52 \text{ hy.}$$

The electrical system components are : $C_{11} = 1\mu F.$, $C_{22} = 1\mu F.$, $C_{12} = 0.286\mu F.$, $L_1 = 4.44$ hy., $L_2 = 3.52$ hy.

The calculation of the critical speed by the influence coefficient method is done in order to make a check with the electrical analogy.

Suppose now that masses m_1 and m_2 are located at points 1 and 2 and that the system is vibrating freely at its natural frequency ω_n, then $F_1 = m_1 y_1 \omega_n^2$, $F_2 = m_2 y_2 \omega_n^2$, where F_1 and F_2 are the disturbing forces due to masses m_1 and m_2 caused by rotation, y_1, and y_2 are the deflections at 1 and 2, and are given by;

$$y_1 = a_{11} F_1 + a_{12} F_2$$

$$y_2 = a_{21} F_1 + a_{22} F_2$$

inserting the values of F_1 and F_2 in the above equations,

$$y_1 = m_1 y_1 \omega_n^2 a_{11} + m_2 y_2 \omega_n^2 a_{12}$$

$$y_2 = m_1 y_1 \omega_n^2 a_{21} + m_2 y_2 \omega_n^2 a_{22}$$

These equations can be written in the form of determinants as:

$$\begin{vmatrix} a_{11} m_1 \omega_n^2 - 1 & a_{12} m_2 \omega_n^2 \\ a_{12} m_1 \omega_n^2 & a_{22} m_2 \omega_n^2 - 1 \end{vmatrix} = 0$$

Solving the determinant,

$$(a_{11} m_1 \omega_n^2 - 1)(a_{22} m_2 \omega_n^2 - 1) - a_{12}^2 m_1 m_2 \omega_n^4 = 0$$

$$\omega_n^4 m_1 m_2 a_{12}^2 - (m_1 m_2 a_{11} a_{22} \omega_n^4 - m_1 \omega_n^2 a_{11} - m_2 \omega_n^2 a_{22} + 1) = 0$$

$$\omega_n^4 (m_1 m_2 a_{12}^2 - m_1 m_2 a_{11}^2) + \omega_n^2 (m_1 a_{11} + m_2 a_{22}) - 1 = 0$$

$$\omega_n^2 = \frac{-(m_1 a_{11} + m_2 a_{22}) \pm \sqrt{(m_1 a_{11} + m_2 a_{22})^2 + 4(m_1 m_2 a_{12}^2 - m_1 m_2 a_{11}^2)}}{2(m_1 m_2 a_{12}^2 - m_1 m_2 a_{11}^2)}$$

$$= \frac{-(m_1 a_{11} + m_2 a_{22})}{2(m_1 m_2 a_{12}^2 - m_1 m_2 a_{11}^2)}$$

$$\pm \sqrt{\left[\frac{m_1 a_{11} + m_2 a_{22}}{2(m_1 m_2 a_{12}^2 - m_1 m_2 a_{11}^2)}\right]^2 + \frac{1}{(m_1 m_2 a_{12}^2 - m_1 m_2 a_{11}^2)}}$$

$$= \frac{-g(w_1 a_{11} + w_2 a_{22})}{2(w_1 w_2 a_{12}^2 - w_1 w_2 a_{11}^2)}$$

$$\pm \sqrt{\left[\frac{g(w_1 a_{11} + w_2 a_{22})}{2(w_1 w_2 a_{12}^2 - w_1 w_2 a_{11}^2)}\right]^2 + \frac{g^2}{(w_1 w_2 a_{12}^2 - w_1 w_2 a_{11}^2)}}$$

$$= -\frac{386(0.01625)(1.26 + 1)}{2(1.26)(0.01265^2 - 0.01625^2)} \pm \sqrt{\left[\frac{14.2(10)^4}{2.62}\right]^2 - 113.8(10)^7}$$

$$= 5.322 \times 10^4 \pm \sqrt{16.843 \times 10^8}$$

$$\omega_n^2 = (5.322 \pm 4.105)10^4$$

$$\omega_{n_1}^2 = 12170 \quad , \quad \omega_{n_2}^2 = 94270$$

$$\omega_{n_1} = 110.8 \text{ rad/sec.,} \qquad \omega_{n_2} = 307 \text{ rad/sec.}$$

$$f_{n_1} = \frac{110.8}{2\pi} = 17.643 \text{ cps.}$$

$$f_{n_2} = \frac{307}{2\pi} = 48.885 \text{ cps.}$$

The electrical analogy results: $f_{e_1} = 74$ cps., $f_{e_2} = 230$ cps.

These results were obtained when the voltage across the capacitance C_{22} was maximum which in the mechanical system means the maximum amplitude of mass m_2 or m_1 , as it is a symmetrical system. Converting these electrical frequencies to mechanical frequencies by using the dimensionless form:

$$\omega_n \sqrt{m_1/k_{11}} = \omega_e \sqrt{L_1 C_{11}}$$

$$\omega_n = \omega_e \sqrt{\frac{4.44 \times 10^{-6} \times 34.6 \times 386}{1.26}}$$

$$\omega_n = 0.217\, \omega_e$$

$$f_{n_1} = 0.217\ (74) = 16.209 \text{ cps.}$$

$$f_{n_2} = 0.217\ (230) = 49.91 \text{ cps.}$$

The theoretical results:

$$f_{n_1} = 17.643 \text{ cps.}$$

$$f_{n_2} = 48.885 \text{ cps.}$$

The electrical analogy results agree with the theoretical results with the deviation of 8.4% and 2.3%.

The Three Mass System

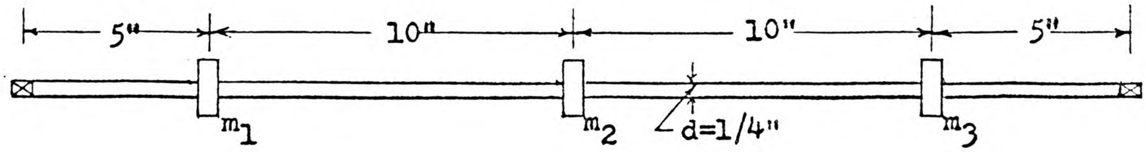
In this case a mechanical system with three masses m_1, m_2 and m_3 of 1.26 lb., 1.0 lb., and 0.92 lb., respectively are fixed on a shaft, which is supported by self aligning bearings, as shown in fig. 3(a)

Just as it was done for the two mass system, it is better to convert this system into a simple spring mass system, and then to an equivalent electrical system. The same technique for finding the influence coefficients and spring constants is applied.

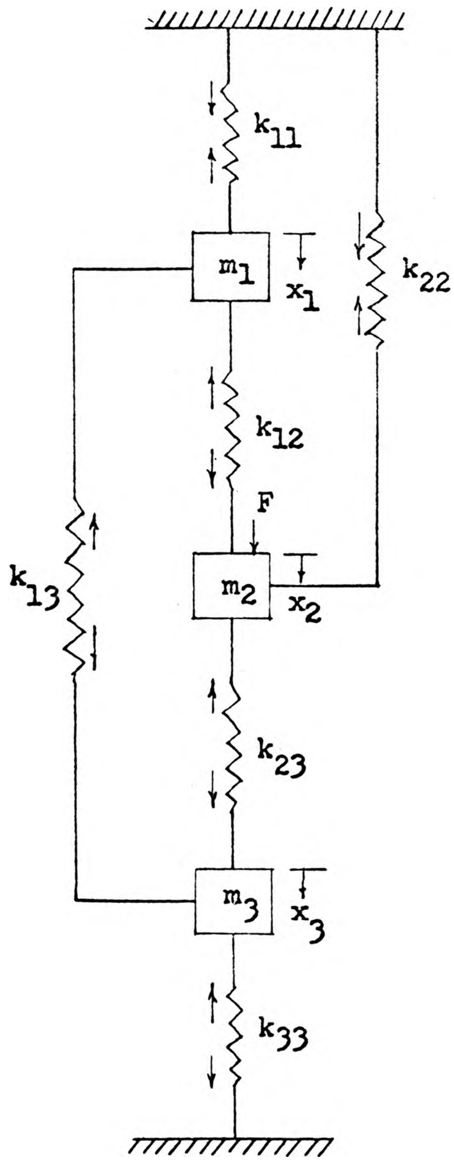
There will be nine influence coefficients as there are three masses. They are a_{11}, a_{22}, a_{33}, a_{12}, a_{21}, a_{13}, a_{31}, a_{23}, a_{32}, but for small deflections Maxwell's reciprocity rule states that $a_{12} = a_{21}$, $a_{13} = a_{31}$ etc., so that actually there are only six independent influence coefficients. The three masses and six influence coefficients may be represented by the three masses and six springs of the equivalent lumped system shown in the figure 3.

If a unit load is applied to each mass in turn in figure 3 and the equation of static equilibrium written in each case for the entire system and for each mass individually, six equations are obtained. By the same argument as in the two mass system the following six equations can be written:

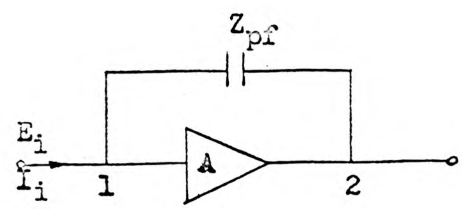
$$a_{11} k_{11} + a_{21} k_{22} + a_{31} k_{33} = 1$$
$$a_{12} k_{11} + a_{22} k_{22} + a_{32} k_{33} = 1$$
$$a_{13} k_{11} + a_{23} k_{22} + a_{33} k_{33} = 1$$
$$a_{11} k_{11} + (a_{11} - a_{21}) k_{12} + (a_{11} - a_{31}) k_{13} = 1$$



(a) Three Mass System.



(b) Equivalent Spring Mass System.



(c) Method of Obtaining Negative Capacitance.

Figure 3.

$$(a_{22} - a_{12}) k_{21} + a_{22} k_{22} + (a_{22} - a_{32}) k_{23} = 1$$

$$(a_{33} - a_{13}) k_{31} + (a_{33} - a_{23}) k_{32} + a_{33} k_{33} = 1$$

All the influence coefficients can be calculated by the same formula used in the two mass system. The influence coefficients were calculated by the use of the IBM - 1620 computer. The computed values are;

$$a_{11} = 0.0302'' , \quad a_{22} = 0.0977'' , \quad a_{33} = 0.0302'' ,$$

$$a_{13} = 0.0205'' , \quad a_{12} = 0.0471'' , \quad a_{23} = 0.0471''$$

After determining the above values of influence coefficients, the first three equations of static equilibrium can be solved for k_{11}, k_{22}, k_{33} by the use of the computer. The values determined were $k_{11}=96.64$ lb/in., $k_{22} = - 82.83$ lb/in., $k_{33} = 96.64$ lb/in. Considering the remaining three equations of static equilibrium and solving for k_{13} k_{12}, k_{23} ,

$$(0.0302)96.64 - 0.0169 k_{12} + 0.0097 k_{13} = 1$$

$$0.0506 k_{21} - (0.0977)82.83 - 0.0506 k_{23} = 1$$

$$0.0097 k_{31} - 0.0169 k_{32} + (0.0302) 96.64 = 1$$

$$0.0169 k_{12} - 0.0097 k_{13} = 1.92 \quad \text{- - - - - - - - - -(6)}$$

$$0.0506 k_{12} + 0.0506 k_{23} = 9 \quad \text{- - - - - - - - - -(7)}$$

$$0.0169 k_{32} - 0.0097 k_{31} = 1.92 \quad \text{- - - - - - - - - -(8)}$$

Subtracting (8) from (6),

$$0.0169 k_{12} - 0.0169 k_{32} = 0$$

$$k_{12} - k_{32} = 0 , \quad k_{12} = k_{32}$$

From equation (7) ,

$$0.0506 k_{12} + 0.0506 k_{12} = 9$$

$$k_{12} = \frac{9}{0.1012} = 89 \text{ lb/in.} = k_{23}$$

From equation (6),

$$(0.0169)89 - 0.0097\, k_{13} = 1.97$$

$$k_{13} = -48 \text{ lb/in.}$$

Finally the spring constants are:

$k_{11} = 96.64$ lb/in. $= k_{33}$, $k_{12} = 89$ lb/in. $= k_{23}$, $k_{13} = -48$ lb/in.,
$k_{22} = -82.83$ lb/in.

Equivalent system:

At any instant the displacements of the masses m_1 , m_2 and m_3 are x_1 , x_2 and x_3 respectively, due to the force F acting on mass m_2. The equations of motion of the system shown in figure 3(b) are:

$$m_1 \frac{d^2x_1}{dt^2} + k_{11}\, x_1 + k_{12}\,(x_1 - x_2) + k_{13}\,(x_1 - x_3) = 0$$

$$m_2 \frac{d^2x_2}{dt^2} + k_{22}\, x_2 - k_{12}\,(x_1 - x_2) + k_{23}\,(x_2 - x_3) - F = 0$$

$$m_3 \frac{d^2x_3}{dt^2} + k_{33}\, x_3 - k_{13}\,(x_1 - x_3) - k_{23}\,(x_2 - x_3) = 0$$

According to Kirchhoff's voltage law, the sum of the voltage drops around a closed loop must be zero. Considering the first loop of figure 4 (a) with inductance L_1,

$$L_1 \frac{d^2q_1}{dt^2} + \frac{1}{C_{11}}\, q_1 + \frac{1}{C_{13}}(q_1 - q_3) + \frac{1}{C_{12}}(q_1 - q_2) = 0$$

Considering the second loop with inductance L_2,

$$L_2 \frac{d^2q_2}{dt^2} - \frac{1}{C_{12}}(q_1 - q_2) + \frac{1}{C_{23}}(q_2 - q_3) + \frac{1}{C_{22}} q_2 - E = 0$$

The third loop with inductance L_3 ,

$$L_3 \frac{d^2q_3}{dt^2} + \frac{1}{C_{33}} q_3 - \frac{1}{C_{23}}(q_2 - q_3) - \frac{1}{C_{13}}(q_1 - q_3) = 0$$

If $q_1 = x_1$, $q_2 = x_2$, $q_3 = x_3$;

$m_1 = L_1$, $m_2 = L_2$, $m_3 = L_3$, $k_{11} = \frac{1}{C_{11}}$, $k_{22} = \frac{1}{C_{22}}$, $k_{33} = \frac{1}{C_{33}}$,$k_{12} = \frac{1}{C_{12}}$,

$k_{23} = \frac{1}{C_{23}}$, $k_{13} = \frac{1}{C_{13}}$.

Therefore the electrical circuit is analogous to the mechanical system, therefore it is an equivalent system.

Applying Buckingham's pi theorem to calculate the equivalents in the electrical circuit.

$$x_1 , x_2 \text{ or } x_3 = f(m_1 , m_2 , m_3 , k_{11} , k_{22} , k_{12} , k_{13} , F_1 , \omega_n)$$

Because $k_{33} = k_{11}$ and $k_{23} = k_{12}$, k_{33} and k_{23} need not be included.

There are ten variables and the number of fundamental dimensions is three, so the number of independent dimensionless groups π_1 , π_2.... etc., will be $10 - 3 = 7$.

$$\pi_1 = m_1^a k_{11}^b x^c \omega_n = (F L^{-1} T^2)^a (F L^{-1})^b (L)^c T^{-1}$$

$$\pi_2 = \ldots\ldots = \ldots\ldots\ldots$$

$$= \ldots\ldots\ldots$$

$$\ldots\ldots\ldots$$

$$\pi_7 = \ldots\ldots$$

By carrying out the same procedure to find the dimensionless groups

for the two mass system, the dimensionless groups for the three mass system can be written as:

$$\frac{m_2}{m_1} = \frac{L_2}{L_1}, \quad \frac{m_3}{m_1} = \frac{L_3}{L_1}, \quad \frac{k_{12}}{k_{11}} = \frac{C_{11}}{C_{12}}, \quad \frac{k_{22}}{k_{11}} = \frac{C_{11}}{C_{22}},$$

$$\frac{k_{13}}{k_{11}} = \frac{C_{11}}{C_{13}}, \quad \omega_n \sqrt{m_1 / k_{11}} = \omega_c \sqrt{L_1 C_{11}}, \quad \frac{F}{k_{11} x} = \frac{E\, C_{11}}{q}$$

The mechanical properties of the system are as follows :

$m_1 = 1{,}26$ lb., $m_2 = 1$ lb., $m_3 = 0.92$ lb., $k_{11} = 96{,}64$ lb/in. $= k_{33}$,

$k_{22} = -\ 82.83$ lb/in., $k_{12} = 89$ lb/in. $= k_{23}$, $k_{13} = -\ 48$ lb/in.

Assuming $C_{11} = 1\mu F$, $L_1 = 4.44$ hy.

$$\frac{m_1}{m_2} = \frac{L_1}{L_2}; \quad L_2 = \frac{4.44}{1.26} = 3.52 \text{ hy.}$$

$$\frac{m_3}{m_1} = \frac{L_3}{L_1}; \quad L_3 = \frac{0.92}{1.26}\, 4.44 = 3.24 \text{ hy.}$$

$$\frac{k_{13}}{k_{11}} = \frac{C_{11}}{C_{13}}; \quad C_{13} = -\frac{96.64}{48}(10)^{-6} = -\ 2.01\ \mu F.$$

$$\frac{k_{22}}{k_{11}} = \frac{C_{11}}{C_{22}}; \quad C_{22} = -\frac{96.64}{82.83}(10)^{-6} = -\ 1.167\ \mu F.$$

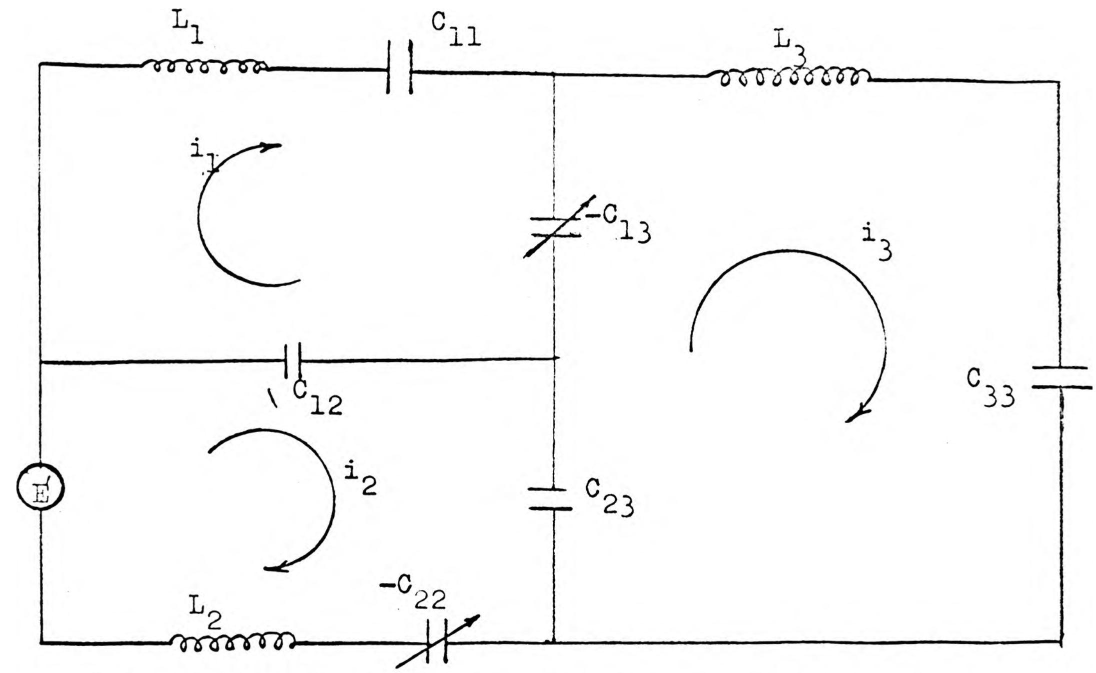
$$\frac{k_{11}}{k_{12}} = \frac{C_{12}}{C_{11}}; \quad C_{12} = \frac{96.64}{89}(10)^{-6} = 1.085\ \mu F.$$

As $k_{11} = k_{33}$, $C_{11} = C_{33} = 1\ \mu F.$

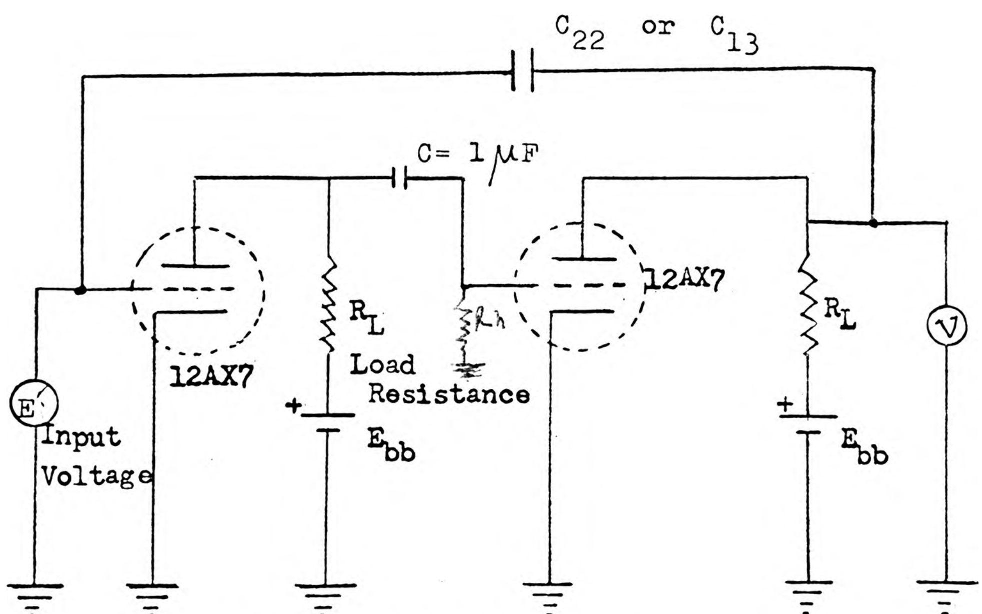
$k_{12} = k_{23}$,, $C_{12} = C_{23} = 1.085\ \mu F.$

The electrical circuit components are as follows:

$L_1 = 4.44$ hy., $L_2 = 3.52$hy., $L_3 = 3.24$ hy., $C_{11} = 1\mu F.$,



(a) Equivalent Electrical Circuit of Three Mass System



(b) Negative Capacitance Circuit

Figure 4

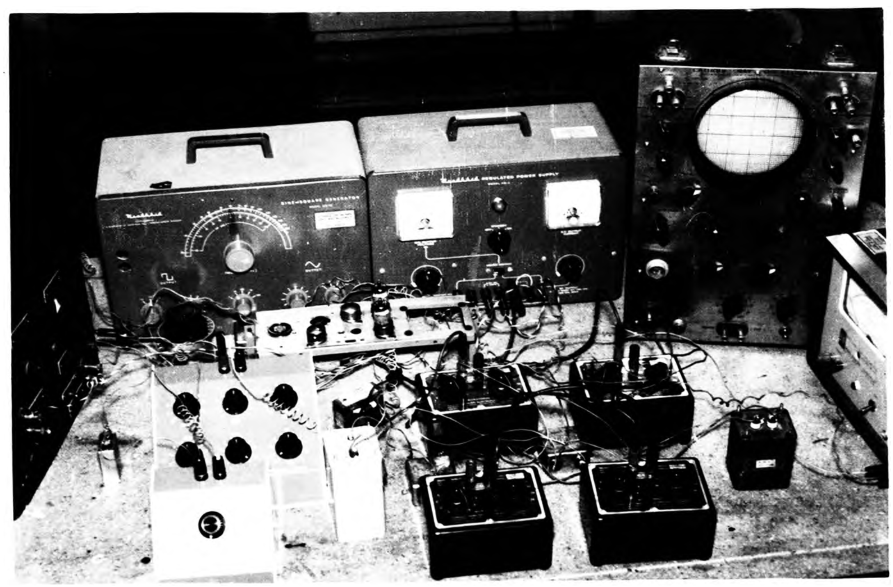


Figure 5

Experimental Setup for Three Mass System. Top View.

$C_{22} = -1.167\,\mu F.$, $C_{33} = 1\,\mu F.$, $C_{12} = 1.085\,\mu F.$, $C_{23} = 1.085\,\mu F.$, $C_{13} = -2.01\,\mu F.$

Before going further it is necessary to explain the strange quantity that was encountered while calculating electrical equivalent quantities, which is negative capacitance. A negative capacitance is a reactive circuit element whose reactance varies inversely as the frequency, but is positive in sign. At any given frequency a negative capacitance will present the same type of reactance as an inductance. However, if the frequency is increased, the reactance of the inductance will increase in value while that of the negative capacitance will decrease. This must not occur in the desired circuit as frequency is to be changed. So the variable units for the elements representing the negative stiffneses should be used. Then each time the exciting frequency is changed, the magnitude of the negative unit would also be changed to bring its impedance at that frequency to the appropriate value.

A negative capacitance can be made to be substantially independent of the amplifier tube characteristics and supply voltage variations and of frequency by employing inverse feedback in the amplifier.

In figure 3(c) is shown an amplifier with an impedance Z_{pf} connected between input and output terminals, 1 and 2 respectively, of the two stage amplifier. The input current is found to be

$$I_i = (E_i - A\,E_i) / Z_{pf} \qquad \text{--------- (9)}$$

Where A is the amplification and E_i is the input voltage, the input impedance Z_n is,

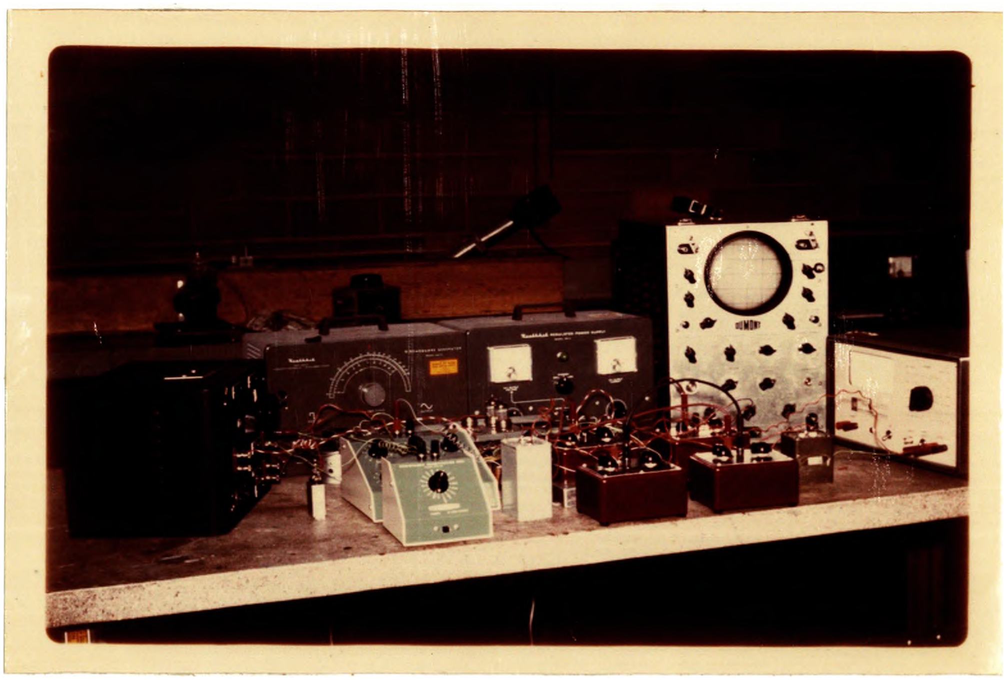


Figure 6

Experimental Setup for Three Mass System. Front View.

$$Z_n = \frac{E_i}{I_i} = \frac{Z_{pf}}{(1 - A)} \quad \text{------------} (10)$$

Equation (10) shows that if the gain A is real i.e. has no phase shift, and greater than unity, Z_n will be a negative multiple of Z_{pf}. If Z_{pf} is a capacitor of value C, equation (10) yields

$$Z_n = \frac{1}{j\omega C_{pf}(1 - A)}$$

The input impedance is that of a capacitance of value C_{pf} (1 -A) which for A greater than unity and real is negative.

Positive gain i.e. no phase shift , can be obtained by using a two stage amplifier. If one stage is used the phase shift between input and output is 180 degrees, that means when the input voltage increases, the output voltage decreases so the gain will be negative(-A). By using the same type of circuit in connection with the first stage, positive gain is obtained. It was observed that the final gain A was 2. The complete circuit is the two stage amplifier.

The calculation of the critical speed of the three mass system by the influence coefficient method is done by the use of the computer.

Suppose that masses m_1 ,m_2, and m_3 are located at points 1 , 2 and 3 respectively, and that the system is vibrating freely at its natural frequency ω_n . Then $F_1 = m_1 y_1 \omega_n^2$, $F_2 = m_2 y_2 \omega_n^2$, $F_3 = m_3 y_3 \omega_n^2$ where F_1 , F_2 , F_3 are the disturbing forces at masses m_1 , m_2 and m_3 respectively, caused by rotation. The deflections at 1, 2 and 3 respectively are y_1 , y_2 , and y_3 and are given by:

$$y_1 = a_{11} F_1 + a_{12} F_2 + a_{13} F_3$$
$$y_2 = a_{21} F_1 + a_{22} F_2 + a_{23} F_3$$
$$y_3 = a_{31} F_1 + a_{32} F_2 + a_{33} F_3$$

inserting the values of F_1 , F_2 , and F_3 ,

$$y_1 = a_{11} m_1 y_1 \omega_n^2 + a_{12} m_2 y_2 \omega_n^2 + a_{13} m_3 y_3 \omega_n^2 = 0$$
$$y_2 = a_{21} m_1 y_1 \omega_n^2 + a_{22} m_2 y_2 \omega_n^2 + a_{23} m_3 y_3 \omega_n^2 = 0$$
$$y_3 = a_{31} m_1 y_1 \omega_n^2 + a_{32} m_2 y_2 \omega_n^2 + a_{33} m_3 y_3 \omega_n^2 = 0$$

These three equations can be written in the form of determinants as follows:

$$\begin{vmatrix} (a_{11} m_1 \omega_n^2 - 1) & a_{12} m_2 \omega_n^2 & a_{13} m_3 \omega_n^2 \\ a_{21} m_1 \omega_n^2 & (a_{22} m_2 \omega_n^2 - 1) & a_{23} m_3 \omega_n^2 \\ a_{31} m_1 \omega_n^2 & a_{32} m_2 \omega_n^2 & (a_{33} m_3 \omega_n^2 - 1) \end{vmatrix} = 0$$

If this determinant is zero then the above three eouations can be solved for the displacement. For different values of ω_n, the determinant was solved by means of the computer and the values of ω_n were noted for which the determinant became zero. These values of ω_n are the three critical speeds, they are $\omega_{n_1} = 50.835$ rad/sec., $\omega_{n_2} = 193.01$ rad/sec., $\omega_{n_3} = 321.32$ rad/sec., i.e. $f_{n1} = 8.094$ cps., $f_{n2} = 30.734$ cps., $f_{n_3} = 51.165$ cps.

The electrical analogy results are:

$f_{e1} = 25$ cps., $f_{e2} = 90$ cps., $f_{e_3} = 150$ cps.

These results were obtained when the voltage across the capacitance was maximum, which in the mechanical system means the maximum amplitude of mass m_1, m_2 or m_3. Converting the obtained electrical freouency to the mechanical frequency, which is the critical speed, in dimensionless form:

$$\omega_n \sqrt{m_1 / k_{11}} = \omega_e \sqrt{L_1 C_{11}}$$

$$\omega_n = \omega_e \sqrt{\frac{L_1 \; C_{11} \; k_{11}}{m_1}}$$

$$\omega_n = \omega_e \sqrt{\frac{4.44 \; (96.64)386}{1000000(1.26)}} \qquad = \; 0.3625 \; \omega_e$$

$$f_{n_1} = (0.3625) \; f_{e_1} = (0.3625)25 = 9.0625 \text{ cps.},$$

$$f_{n_2} = (0.3625) \; f_{e_2} = (0.3625)90 = 32.625 \text{ cps.},$$

$$f_{n_3} = (0.3625) \; f_{e_3} = (0.3625)150 = 54.375 \text{ cps.}$$

The theoretical results are :

$$f_{n_1} = 8.094 \text{ cps.}, \quad f_{n_2} = 30.734 \text{ cps.}, \quad f_{n_3} = 51.165 \text{ cps.}$$

The electrical analogy results agree with the theoretical results with the deviations of 10.6% , 6.14% , and 6.28%.

CONCLUSION

It can be concluded that the solution of a mechanical vibration problem can be easily obtained by electrical analogy. The solution of the problem of critical speed of transverse vibration of the shaft up to two masses can be easily obtained by the electrical analogy. For more than two masses the solution by electrical analogy becomes cumbersome while the solution of a mechanical system by computer is much easier.

By the influence coefficients method the actual system is converted to a spring mass system which makes it easier to convert it into an electrical system.

From the values of influence coefficients and spring constants it can be seen that for the two mass system the influence coefficients a_{11} and a_{22} were equal because of symmetry, and so the spring constants k_{11} and k_{22} were equal. Similarly for the three mass system influence coefficient a_{11} equals a_{33} and a_{12} equals a_{23} due to symmetry and so k_{11} equals k_{33} and k_{12} equals k_{23}.

Two of the spring constants k_{13} and k_{22} were negative so while finding electrical equivalents, two corresponding quantities came out negative which is a negative capacitance.

From the circuit diagram for each case it can be seen that the number of loops gives that number of degrees of freedom.

From the two stage it was observed that the final gain was two which is real and greater than unity.

The tests made on the one, two and three mass system show that

calculated frequencies for the mechanical system and the equivalent frequencies from the electrical system check very closely. The reasons for not getting the exact values of frequencies are inherent resistance of the inductors, inaccuracy of component values and the calibration error of the instruments. The theoretical critical speed calculated mathematically neglects many variables such as gyroscopic effect, bearing elasticity etc.

A simply supported beam problem up to three concentrated masses can be solved by the same method followed for the critical speed of the shaft up to three masses.

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APPENDIX

TABLE 1

Mechanical			Electrical Equivalents (Force-Voltage Analogy)		
Force	-	F	Voltage	-	E
Mass	-	m	Inductance	-	L
Compliance (1/ spring constant)		1/k	Capacitance	-	C
Velocity	-	v	Current	-	i
Damping	-	c	Resistance	-	R
Displacement	-	x	Charge	-	q

```
C  C****6293FORMO     MEX015 A.Z.PATEL                04/09/64  0020  0005    - 4/ 9/64      PG
C      ELECTRICAL ANALOGY OF CRITICAL SPEEDS OF SHAFT
       DIMENSION AA(3,3),A(3,4)
       READ 100,E,EL,C,B,D
       PUNCH 150
       FI=(3.1415927/64.)*(D**4)
       P=1./(6.*E*FI*EL)
       L=1
       DO 15J=1,3
       X=5.
       DO 25I=1,L
       AA(I,J)=B*X*(C**2+2.*C*B-X**2)*P
       AA(J,I)=AA(I,J)
    25 X=X+10.
       L=L+1
       C=C+10.
    15 B=B-10.
       DO 24I=1,3
       DO 26J=1,3
    26 A(I,J)=AA(I,J)
    24 A(I,4)=1.
       PUNCH 200,A(1,1),A(1,2),A(1,3),A(2,2)
       PUNCH 160
       PUNCH 200,A(2,3),A(3,3)
   100 FORMAT(5E14.8)
   150 FORMAT(7X,6HA(1,1),13X,6HA(1,2),13X,6HA(1,3),13X,6HA(2,2))
   160 FORMAT(7X,6HA(2,3),13X,6HA(3,3))
   200 FORMAT(4E18.8)
       STOP
       END
        30.0E6          30.            5.             25.            .25

   C****6293FORMO    MEX015 A Z PATEL              04/09/64 FORMO 04/09/64-
        A(1,1)            A(1,2)            A(1,3)            A(2,2)
    0.30180486E-01    0.47081574E-01    0.20522737E-01    0.97784808E-01
        A(2,3)            A(3,3)
    0.47081574E-01    0.30180486E-01

       STOP  END OF PROGRAM AT  STATEMENT 0200 + 01 LINES
```

```
C     ELECTRICAL ANALOGY OF CRITICAL SPEEDS OF SHAFT
      DIMENSION AA(3,3),A(3,4)
      READ 100,E,FL,C,B,D
      PUNCH 150
      FI=(3.1415827/64.)*(D**4)
      P=1./(6.*E*FI*FL)
      L=1
      DO 15J=1,3
      X=5.
      DO 25I=1,L
      AA(I,J)=B*X*(C**2+2.*C*B-X**2)*P
      AA(J,I)=AA(I,J)
   25 X=X+10.
      L=L+1
      C=C+10.
   15 B=B-10.
      DO 24I=1,3
      DO 26J=1,3
   26 A(I,J)=AA(I,J)
   24 A(I,4)=1.
      CALL GAUJOR (A,3,4,3,4)
      PUNCH 200,(A(I,4),I=1,3)
      CALL EXIT
  100 FORMAT(5E14.8)
  150 FORMAT(7X,6HK(1,1),12X,6HK(2,2),12X,6HK(3,3))
  200 FORMAT(3E18.8)
      END
        30.0E6          30.           5.            25.           .25
       K(1,1)            K(2,2)            K(3,3)
     .96640796E+02   -.82834971E+02    .96640797E+02
```

```
C   C****7923FORTRAN2 MEXO15 A Z PATEL                    04/17/64  0050  0105
C        ELECTRICAL ANALOGY OF CRITICAL SPEEDS OF SHAFT
         DIMENSION AA(3,3),A(25,25)
         READ 100,E,EL,C,B,R
         PUNCH 150
         D=0.0
         W=5.0
         DW=3.0
         FI=(3.1415927/4.)*(R**4)
         P=1./(6.*E*FI*EL)
         L=1
         DO 15J=1,3
         X=5.
         DO 25I=1,L
         AA(I,J)=B*X*(C**2+2.*C*B-X**2)*P
         AA(J,I)=AA(I,J)
      25 X=X+10.
         L=L+1
         C=C+10.
      15 B=B-10.
         P1=1.26/386.
         P2=1./386.
         P3=.92/386.
         DO 35K=1,120
         DO 26I=1,3
         DO 26J=1,3
      26 A(I,J)=AA(I,J)
         A(2,2)=A(2,2)*P2-1./(W**2)
         A(2,1)=A(2,1)*P1
         A(2,3)=A(2,3)*P3
         A(1,1)=A(1,1)*P1-1./(W**2)
         A(1,2)=A(1,2)*P2
         A(1,3)=A(1,3)*P3
         A(3,1)=A(3,1)*P1
         A(3,2)=A(3,2)*P2
         A(3,3)=A(3,3)*P3-1./(W**2)
         CALL DETERM(A,3,D)
```

```
      VALUE=(W**6)*D
      PUNCH 200,W,VALUE
   35 W=W+DW
      STOP
  100 FORMAT (5E14.8)
  150 FORMAT (6X,1HW,13X,5HVALUE)
  200 FORMAT(F9.1,3X,E18.8)
      END
        30.0E6            30.            5.            25.          .125
      W               VALUE
      5.0          -.98941467E+00
      8.0          -.97293791E+00
     11.0          -.94893583E+00
     14.0          -.91749798E+00
     17.0          -.87874152E+00
     20.0          -.83281139E+00
     23.0          -.77987989E+00
     26.0          -.72014663E+00
     29.0          -.65383770E+00
     32.0          -.58120551E+00
     35.0          -.50252841E+00
     38.0          -.41810981E+00
     41.0          -.32827788E+00
     44.0          -.23338472E+00
     47.0          -.13380567E+00
     50.0          -.29938753E-01
     53.0           .77796532E-01
     56.0           .18895957E+00
     59.0           .30309010E+00
     62.0           .41970871E+00
     65.0           .53831808E+00
     68.0           .65840418E+00
     71.0           .77943717E+00
     74.0           .90087268E+00
     77.0           .10221531E+01
     80.0           .11427090E+01
     83.0           .12619597E+01
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86.0	.13793165E+01
89.0	.14941813E+01
92.0	.16059512E+01
95.0	.17140181E+01
98.0	.18177687E+01
101.0	.19165947E+01
104.0	.20098811E+01
107.0	.20970156E+01
110.0	.21773949E+01
113.0	.22504169E+01
116.0	.23154897E+01
119.0	.23720295E+01
122.0	.24194656E+01
125.0	.24572403E+01
128.0	.24848123E+01
131.0	.25016579E+01
134.0	.25072753E+01
137.0	.25011844E+01
140.0	.24829303E+01
143.0	.24520864E+01
146.0	.24082553E+01
149.0	.23510733E+01
152.0	.22802108E+01
155.0	.21953772E+01
158.0	.20963205E+01
161.0	.19828339E+01
164.0	.18547558E+01
167.0	.17119736E+01
170.0	.15544270E+01
173.0	.13821090E+01
176.0	.11950720E+01
179.0	.99342961E+00
182.0	.77735670E+00
185.0	.54709753E+00
188.0	.30296661E+00
191.0	.45351272E-01
194.0	-.22528375E+00

197.0	-.50839337E+00
200.0	-.80334899E+00
203.0	-.11094343E+01
206.0	-.14258417E+01
209.0	-.17516711E+01
212.0	-.20859226E+01
215.0	-.24274934E+01
218.0	-.27751800E+01
221.0	-.31276631E+01
224.0	-.34835152E+01
227.0	-.38411881E+01
230.0	-.41990144E+01
233.0	-.45552012E+01
236.0	-.49078310E+01
239.0	-.52548389E+01
242.0	-.55940394E+01
245.0	-.59230939E+01
248.0	-.62395258E+01
251.0	-.65406976E+01
254.0	-.68238259E+01
257.0	-.70859624E+01
260.0	-.73239974E+01
263.0	-.75346712E+01
266.0	-.77145022E+01
269.0	-.78598817E+01
272.0	-.79669774E+01
275.0	-.80318322E+01
278.0	-.80502118E+01
281.0	-.80177543E+01
284.0	-.79298725E+01
287.0	-.77817530E+01
290.0	-.75683885E+01
293.0	-.72845813E+01
296.0	-.69248626E+01
299.0	-.64835438E+01
302.0	-.59547486E+01
305.0	-.53322917E+01
308.0	-.46098396E+01

311.0	-.57807094E+01
314.0	-.28380245E+01
317.0	-.17746268E+01
320.0	-.53317145E+00
323.0	.74414825E+00
326.0	.22152565E+01
329.0	.38383298E+01
332.0	.56220033E+01
335.0	.75750283E+01
338.0	.97065950E+01
341.0	.12026072E+02
344.0	.14543276E+02
347.0	.17268243E+02
350.0	.20211340E+02
353.0	.23383251E+02
356.0	.26795128E+02
359.0	.30458387E+02
362.0	.34384617E+02

VITA

The author, Ashok Z. Patel, was born on February 15, 1940, in Palitana, Gujarat, India. He received his primary education in Palitana and secondary education in Junagadh, Gujarat, India. From June 1957 to April 1958 he attended the Bahauddin College (Science College), Junagadh. In June 1958 he enrolled at the Birla Vishvakarma Mahavidyalaya(Engineering College), V.V.Nagar, Gujarat, India, and graduated with Bachelor of Engineering degree in Mechanical Engineering in June 1962. After taking three months training he came to U.S.A. to do graduate work. In January 1963 he enrolled at the Missouri School of Mines and Metallurgy to do graduate work leading to a Master of Science degree in Mechanical Engineering.