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Studies of an electrical method for generating ultrasonic waves

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STUDIES OF AN ELECTRICAL METHOD FOR GENERATING ULTRASONIC WAVES

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

INTRODUCTION AND REVIEW OF LITERATURE

Ultrasonic waves have proved very valuable in the observation of both the velocity and the absorption coefficient of sound in various media. From the velocity measurements of ultrasonic waves in metals, it is possible to obtain elastic constants, such as the compressibility, and the specific heat [1]. Also, the observed attenuation depends greatly on the type and structure of the material. One of the widely used methods for investigation of the internal losses in solid state physics is the pulse-echo technique. This method is being used almost exclusively in internal friction investigation[2].

There are three methods usually used to generate ultrasonic waves:

a) Piezoelectric Method:

A piezoelectric transducer is placed in contact with the specimen and an oscillatory voltage is applied[3]. The transducer converts the electrical signal to a train of mechanical waves which then is propagated into the specimen. This method is used for the production of both continuous and pulsed waves.

b) Magnetostrictive Method:

In this method, a rod of magnetic material is subjected to an alternating magnetic field parallel to its length. The materials used are ferro-magnetic metals such as iron, nickel or cobalt. In this method, the application of an alternating magnetic field to the transducer generates mechanical vibrations which are transmitted into the specimen.

c) Electric Method:

In this method, an oscillating electric field is applied between the specimen and an electrode. This oscillating electric field gives rise to a force which acts on the surface of the specimen. This impressed electric force sets up vibrations in the specimen which in turn produces ultrasonic waves. This method has been used only to generate continuous waves.

The pulse-echo method has been one of the most important technical advances in ultrasonics in the past decade. In this method, a sinusoidal voltage pulse of between 1 and 5 μ sec duration is applied to a transducer which is attached to the sample. With Morse's[4] equipment the peak-to-peak voltage may be as much as several hundred volts. With this technique, the spatial decay of a pulse is measured, either as it passes between a sender and a receiver, or as it is reflected between two parallel faces of the sample. The chief advantages of this method lie in the simultaneous measurement of velocity and attenuation, the large frequency range possible, and that the directional properties in a single crystal may be studied.

The continuous wave method is useful for attenuation measurement in liquids but has also been used for such measurements in solids.

Bordoni[5] used the electric field method for generating continuous ultrasonic waves in solids. We investigate here the possibility of generating pulsed ultrasonic waves by applying an electric field to the surface of the sample.

The chief advantages of the electric method over other methods are:

1) The absence of a transducer, which eliminates elecromechanical resonance that occurs with the piezoelectric method. In that method, the transducer thickness must be chosen to be a multiple of the input wavelength, and design for a wide range of operation is difficult.
2) The design is simple, as there is no direct contact between the sample and the electrode. In other methods, oil or other bonding material is required between the generator and the sample.
3) Energy losses from absorption of the ultrasonic waves are small, because the sample is not in contact with the electrode, therefore attenuation measurements are more accurate for the electric method than for other methods.

Mathematical methods were used here for:

1) Calculation of the stress generated in a sample by piezoelectric crystals.
2) Analysis of the wave propagation in an isotropic circular cylinder.
3) Calculation of the stress distribution in a cylindrical sample using elastic theory. Stress as a function of radius and of the frequency is obtained.
4) Investigation of electrode shapes necessary for generating the stress distribution found in 3). An electric field applied between such a shaped electrode and the cylindrical rod will produce a stress distribution similar to the previous one.
5) Numerical calculation of these electrode shapes; The shape of an electrode is taken to be that of an equipotential surface in the electric field of 4).

FORCES GENERATED BY A PIEZOELECTRIC GENERATOR

According to the usual convention in specifying crystal cuts, the thickness is taken along the z axis of the crystal, the length along the x axis, while the width lies along the y axis as shown in figure 1 below, where ℓ represents the thickness of a piezo-electric generator along the z axis.

For mathematical simplicity without loss of generality, it can be assumed that the generator has infinite dimensions along the y and x axes. If we take z as the thickness direction and apply a conductive coating to the transducer surfaces normal to this axis, an electric field in the z direction can be established in the transducer.

From the definitions of elastic and piezoelectric constants, the following equations can be obtained [6]:

$$S_3 = S_{33}^E T_3 + d_{33} E_3 \qquad (2\text{-}1)$$

where S_{33}^E is the elastic constant along the z axiz,

d_{33} is the piezoelectric constant along the z axis,

S_3 and T_3 are respectively the strain and stress components along the z axis,

E_3 is the oscillatory electric intensity with angular frequency ω_1 along the z axis.

Substitute Eq. (2-1) into

$$\rho \frac{\partial^2 u_3}{\partial t^2} = \frac{\partial T_3}{\partial z} \qquad (2\text{-}2)$$

which is the equation governing the longitudinal vibrations of the bar [7].

There results:

$$\rho \frac{\partial^2 u_3}{\partial t^2} = \frac{\partial S_3}{S_{33}^E \partial z} - \frac{d_{33} \partial E_3}{S_{33}^E \partial z} \qquad (2\text{-}3)$$

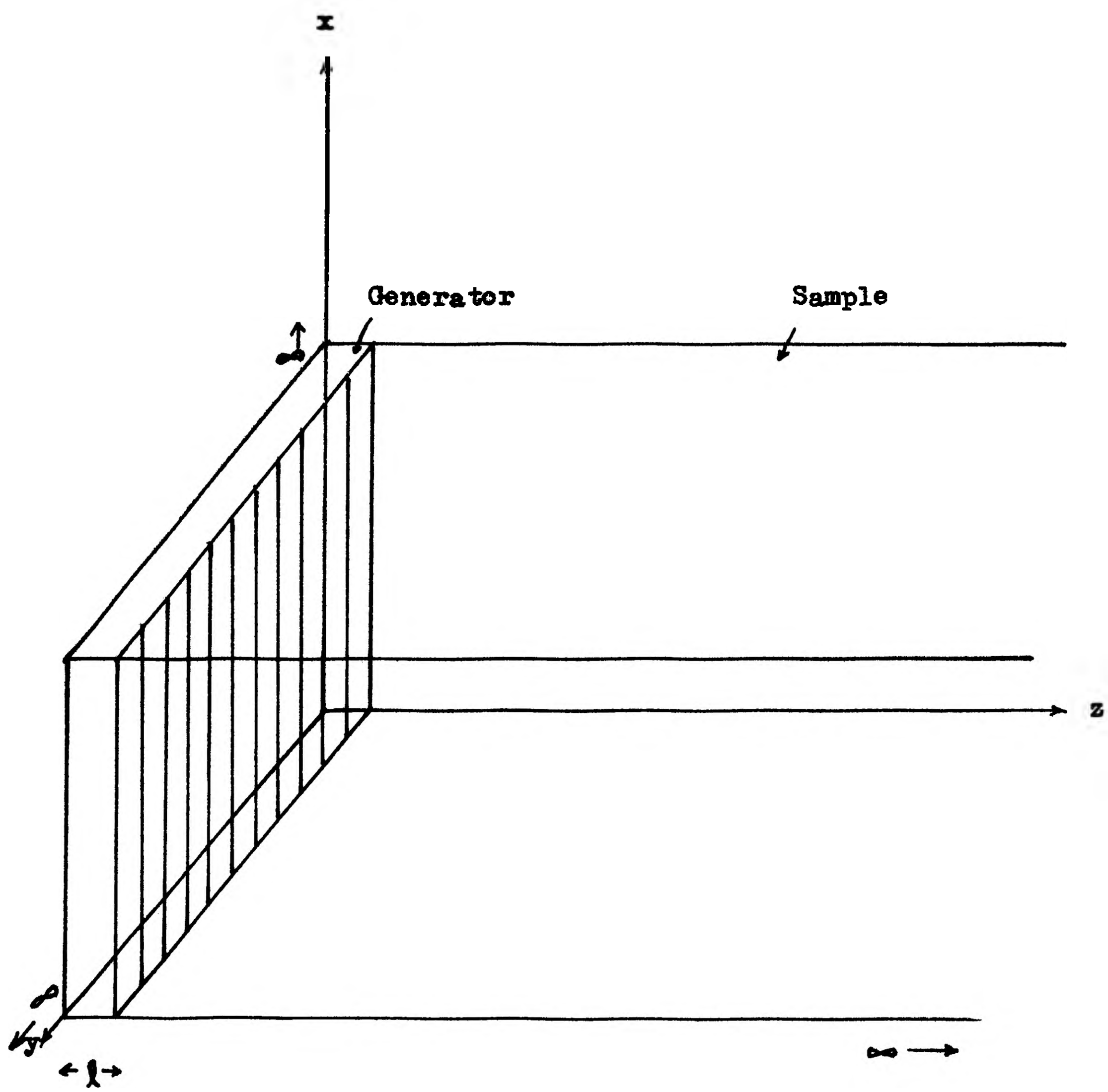


Fig. 1

Wave propagation from generator to sample along the z-axis

or

$$\rho \frac{\partial^2 u_3}{\partial t^2} = \frac{1}{S_{33}^E} \frac{\partial^2 u_3}{\partial z^2} - \frac{d_{33}}{S_{33}^E} \frac{\partial E_3}{\partial z} \qquad (2\text{-}4)$$

where u_3 is the displacement along the z axis,
ρ is the density of the generator.

Since the thickness is assumed small, the voltage gradient E_3 will be a constant throughout the thickness of the generator. The equation of motion for the generator is thus:

$$\rho \frac{\partial^2 u_3}{\partial t^2} = \frac{1}{S_{33}^E} \frac{\partial^2 u_3}{\partial z^2} \qquad (2\text{-}5)$$

In the same manner, if the generator is attached to the sample (see Fig. 1) the equation of motion of the sample is of the same form.

$$\rho' \frac{\partial^2 u_3'}{\partial t^2} = \frac{1}{S_{33}^{E'}} \frac{\partial^2 u_3'}{\partial z^2} \qquad (2\text{-}6)$$

Here u_3' is the displacement of the sample along the z axis,
ρ' and $S_{33}^{E'}$ are the density and elastic constant for the sample.

Before we can obtain the force generated by the piezoelectric generator, we must solve equations (2-5) and (2-6), which are the wave equations. We now set

$$V_1^2 = \frac{1}{\rho S_{33}^E}$$

the so-called wave velocity in the generator.

Eq. (2-5) now becomes

$$\frac{\partial^2 u_3}{\partial t^2} = V_1^2 \frac{\partial^2 u_3}{\partial z^2} \qquad (2\text{-}7)$$

The general solution of equation (2-7) is

$$u_3 = u_3 (z \pm v_1 t) \qquad (2\text{-}8)$$

where u_3 is any function which possesses finite second derivatives. We seek solutions in the form of a superposition of standing waves having frequency equal to the driving electric field angular frequency, ω_1. It is

$$u_3 = A \sin \frac{2\pi}{\lambda_1}(z - v_1 t) + B \sin \frac{2\pi}{\lambda_1}(z + v_1 t) + C \cos \frac{2\pi}{\lambda_1}(z - v_1 t) + D \cos \frac{2\pi}{\lambda_1}(z + v_1 t) \tag{2-9}$$

where $\lambda_1 = \frac{v_1}{\omega_1} 2\pi$ is the wave length in the generator and A, B, C and D are the arbitrary constants.

For simplicity we put

$$\frac{2\pi}{\lambda_1} = K_1 \tag{2-10}$$

$$\frac{2\pi}{\lambda_1} v_1 = \omega_1 \tag{2-11}$$

$$\begin{aligned} A + B &= L \\ B - A &= F \\ C + D &= G \\ C - D &= H \end{aligned} \tag{2-12}$$

Then (2-9) becomes

$$u_3 = (L \sin K_1 z + G \cos K_1 z) \cos \omega_1 t + (F \cos K_1 z + H \sin K_1 z) \sin \omega_1 t \tag{2-13}$$

Using the same method we consider the sample. The thickness of the sample is effectively infinite. Propagation will then occur only in the +z direction with velocity v_2. The solution in the form of a superposition of waves for the sample becomes

$$u_3' = M \sin \frac{2\pi}{\lambda_2}(z - v_2 t) + N \cos \frac{2\pi}{\lambda_2}(z - v_2 t) \tag{2-14}$$

Now if

$$\frac{2\pi}{\lambda_2} = K_2$$

$$\frac{2\pi}{\lambda_2} v_2 = \omega_2$$

then $$u_3' = (M \sin K_2 z + N \cos K_2 z) \cos \omega_2 t + (-M \cos K_2 z + N \sin K_2 z) \sin \omega_2 t \tag{2-15}$$

By using the following boundary conditions one can determine the arbitrary constants:

1) At z=0, for a free crystal, the stress is zero. From (2-1) it follows that

$$\left.\frac{\partial u_1}{\partial z}\right|_{z=0} = d_{33}E_3$$

2) When $z = \ell$, the stress is continuous across the interface,

$$\frac{S_3}{s_{33}^E} - \frac{d_{33}E_3}{s_{33}^E} = \frac{S_3'}{s_{33}^{E'}}$$

where $s_3^{E'}$ and d_{33}' are the elastic and piezoelectric constants along the z axis of the sample,

or

$$\frac{1}{s_{33}^E}\left(\frac{\partial u_3}{\partial z} - d_{33}E_3\right)_{z=\ell} = \frac{1}{s_{33}^{E'}} \cdot \frac{\partial u_3'(\ell, t)}{\partial z}$$

3) The displacement is continuous at $z = \ell$.

$$u_3(\ell, t) = u_3'(\ell, t)$$

From boundary condition 1)

$$\frac{d\ u_3}{d\ z} = L K_1 \cos K_1 z \cos \omega_1 t - F K_1 \sin K_1 z \sin \omega_1 t - G K_1 \sin K_1 z \cos \omega_1 t + H K_1 \cos K_1 z \sin \omega_1 t$$

$$\frac{d\ u_3(0,\ t)}{d\ z} = d_{33}E_3$$

$$(LK_1 \cos K_1 z - GK_1 \sin K_1 z)\cos \omega_1 t + (HK_1 \cos K_1 z - FK_1 \sin K_1 z) \sin \omega_1 t = d_{33}E_3$$

Put $E_3 = E_o \cos \omega_1 t$

where E_o is the maximum electric field.

Therefore, since the corresponding terms must be equal,

$$(LK_1 \cos K_1 z - GK_1 \sin K_1 z) = d_{33} E_o$$

and

$$HK_1 \cos K_1 z - FK_1 \sin K_1 z = 0$$

Putting $z = 0$, there results

$$L K_1 = d_{33} E_o$$

or

$$L = \frac{d_{33} E_o}{K_1} \qquad (2\text{-}16)$$

and

$$H K_1 = 0$$

or

$$H = 0 \qquad (2\text{-}17)$$

From boundary condition 2)

$$\frac{1}{s_{33}^E} (L K_1 \cos K_1 \ell - G K \sin K_1 \ell) \cos \omega_1 t + \frac{1}{s_{33}^E} (HK_1 \cos K_1 \ell - FK_1 \sin K_1 \ell) \sin \omega_1 t - \frac{1}{s_{33}^E} d_{33} E_o \cos \omega_1 t$$

$$= \frac{1}{s_{33}^{E'}} K_2 (M \cos K_2 - N \sin K_2 \ell) \cos \omega_2 t + \frac{1}{s_{33}^{E'}} (M \sin K_2 \ell + N \cos K_2 \ell) K_2 \sin \omega_2 t \qquad (2\text{-}18)$$

Since $\omega_1 = \omega_2$

the corresponding coefficients must be equal.

$$\frac{1}{s_{33}^E} (LK_1 \cos K_1 \ell - GK_1 \sin K_1 \ell) - \frac{K_2}{s_{33}^{E'}} M \cos K_2 \ell + \frac{K_2}{s_{33}^{E'}} N \sin K_2 \ell - \frac{d_{33} E_o}{s_{33}^E} = 0 \qquad (2\text{-}19a)$$

$$\frac{1}{s_{33}^{E}}\left(HK_1 \cos K_1\ell - FK_1 \sin K_1\ell\right) = \frac{K_2}{s_{33}^{E'}}\left(M \sin K_2\ell + N \cos K_2\ell\right) \tag{2-19b}$$

From equations (2-16) and (2-17), Eq.(2-19a) becomes

$$\frac{1}{s_{33}^{E}}\left(d_{33}E_o \cos K_1\ell - GK_1 \sin K_1\ell\right) - \frac{K_2}{s_{33}^{E'}} M \cos K_2\ell + \frac{K_2}{s_{33}^{E'}} N \sin K_2\ell - \frac{d_{33}E_o}{s_{33}^{E}} = 0 \tag{2-20}$$

and Eq. (2-19b) becomes

$$-\frac{FK_1}{s_{33}^{E}} \sin K_1\ell - \frac{K_2}{s_{33}^{E'}}\left(M \sin K_2\ell + N \cos K_2\ell \right) = 0 \tag{2-21}$$

From boundary condition 3), equations (2-13) and (2-15), it follows that

$$M \sin K_2\ell + N \cos K_2\ell = L \sin K_1\ell + G \cos K_1\ell \tag{2-22}$$

or

$$M \sin K_2\ell + N \cos K_2\ell = \frac{d_{33}E_o}{K_1} \sin K_1\ell + G \cos K_1\ell \tag{2-23}$$

and $$- M \cos K_2\ell + N \sin K_2\ell = F \cos K_1\ell + H \sin K_1\ell \tag{2-24}$$

or

$$- M \cos K_2\ell + N \sin K_2\ell = F \cos K_1\ell \tag{2-25}$$

The equations (2-20), (2-21), (2-23) and (2-25) give the values of M and N

$$M = \frac{\dfrac{d_{33}E_o}{s_{33}^{E}}\left(1 - \cos K_1\ell\right)\left(\dfrac{K_1}{s_{33}^{E}} \sin K_2\ell \sin K_1\ell + \dfrac{K_2}{s_{33}^{E'}} \cos K_2\ell \cos K_1\ell\right)}{\left(\dfrac{K_2}{s_{33}^{E'}}\right)^2 \cos^2 K_1\ell + \left(\dfrac{K_1}{s_{33}^{E}}\right)^2 \sin^2 K_1\ell} \tag{2-26}$$

$$N = \frac{-\frac{d_{33}E_o}{S_{33}^E}(1-\cos K_1\ell)\cdot\left(\frac{K_2}{S_{33}^{E'}}\sin K_2\ell\cos K_1\ell - \frac{K_1}{S_{33}^E}\cos K_2\ell\,\sin K_1\ell\right)}{\left(\frac{K_2}{S_{33}}\right)^2\cos^2 K_1\ell + \left(\frac{K_1}{S_{33}^E}\right)^2\sin^2 K_1\ell} \tag{2-27}$$

From the right side of Equation (2-18), the expression for the stress T_3 at $z=\ell$ becomes

$$T_3 = \frac{E_o K_2}{S_{33}^{E'}\,S_{33}^E}\;\frac{d_{33}(1-\cos K_1\ell)\left(\frac{K_2}{S_{33}^{E'}}\cos K_1\ell\cos\omega_1 t - \frac{K_1}{S_{33}^E}\sin K_1\ell\sin\omega_1 t\right)}{\left(\frac{K_2}{S_{33}^{E'}}\right)^2\cos^2 K_1\ell + \left(\frac{K_1}{S_{33}^E}\right)^2\sin^2 K_1\ell} \tag{2-28}$$

For simplicity, we express (2-28) in another form:

$$T_3 = \frac{K_2 E_o}{S_{33}^{E'}S_{33}^E}\;\frac{d_{33}\,(1-\cos K_1\ell)\,\sin(\phi-\omega_2 t)}{\left[\left(\frac{K_2}{S_{33}^{E'}}\right)^2\cos^2 K_1\ell + \left(\frac{1}{S_{33}^E}\right)^2 K_1^2\sin^2 K_1\ell\right]^{\frac{1}{2}}} \tag{2-29}$$

where

$$\phi = \tan^{-1}\frac{\left(\frac{K_2}{S_{33}^{E'}}\right)\cos K_1\ell}{\left(\frac{K_1}{S_{33}^E}\right)\sin K_1\ell} \tag{2-30}$$

From (2-29) the amplitude of the stress will be

$$\frac{K_2}{S_{33}^{E'}\;S_{33}^E}\;\frac{d_{33}E_o(1-\cos K_1\ell)}{\left[\left(\frac{K_2}{S_{33}^{E'}}\right)^2\cos^2 K_1\ell + \left(\frac{K_1}{S_{33}^E}\right)^2\sin^2 K_1\ell\right]^{\frac{1}{2}}}$$

Generally, the maximum value of the stress can be obtained when the thickness ℓ satisfies the condition

$$1-\cos K_1\ell = 2 \tag{2-31}$$

Therefore $\cos K_1 \ell = -1$

$$K_1 \ell = \pi, 3\pi, \cdots (2n - 1)\pi$$

$$\ell = \frac{(2n - 1)\pi}{K_1} = \frac{(2n - 1)}{2f} \cdot \left(\frac{1}{S_{33}^E}\right)^{\frac{1}{2}} \qquad (2\text{-}32)$$

where n = 1, 2, 3, ...

Therefore Eq.(2-32) is the resonance condition for the thickness of the piezoelectrical generator.

Now consider a quartz crystal as the generator and an aluminum crystal as the sample. In Eq.(2-32), the thickness of the quartz crystal for maximum stress can be found. Some typical values are given in Table I.

TABLE I

Thickness for quartz crystals for various resonant frequencies

Frequency	Thickness
1 megacycle	0.2690 cm
10 megacycles	0.0269 cm
100 megacycles	0.0026 cm
500 megacycles	0.0005 cm

From Eqs. (2-29) and (2-31), one obtains the equation

$$T_3 = \frac{K_2}{S_{33}^{E'} \; S_{33}^{E}} \cdot \frac{2\, d_{33} E_o S_{33}^{E'}}{K_2}$$

$$= \frac{2\, d_{33} E_o}{S_{33}^{E}}$$

which expresses the amplitude of the stress at resonanace, as a function of the electric intensity amplitude. In cgs units, the elastic and piezoelectric constants have the values

$$S_{33}^{E} = 127.9 \times 10^{-14} \text{ cm}^2/\text{dyne}$$
$$d_{33} = -6.76 \times 10^{-8} \text{ statcoulombs/dyne}$$

The ratio of the stress to electric intensity is given by T_3/E_o which numerically has the value 1.064×10^5 statcoulombs/cm^2.

WAVES IN CYLINDRICAL ROD

We now investigate wave propagation in an isotropic circular cylinder. Under this assumption we put the z-axis coincident with the axis of the cylinder.

Before we investigate further the wave forms, it is necessary to find the stress-strain equations and the vibration equations in cylindrical coordinates. For an isotropic body, only two elastic constants are independent, and the six stress-strain equations are[8]

$$
\begin{aligned}
T_1 &= (\lambda + 2\mu) S_1 + \lambda (S_2 + S_3) \\
T_2 &= (\lambda + 2\mu) S_1 + \lambda (S_1 + S_3) \\
T_3 &= (\lambda + 2\mu) S_1 + \lambda (S_1 + S_2) \\
T_4 &= \mu S_4 \\
T_5 &= \mu S_5 \\
T_6 &= \mu S_6
\end{aligned}
\qquad (3\text{-}1)
$$

where $T_1 = T_{xx}$, $T_2 = T_{yy}$, $T_3 = T_{zz}$

$T_4 = T_{yz} = T_{zy}$, $T_5 = T_{xz} = T_{zx}$, $T_6 = T_{xy} = T_{yx}$

are the stresses.

Here, in T_{ij}, the first letter designates the direction of the stress component and the second letter denotes the orientation of the face on which T_{ij} acts.

Also,

$$S_1 = \frac{\partial u_1}{\partial x}, \quad S_2 = \frac{\partial u_2}{\partial y}, \quad S_3 = \frac{\partial u_3}{\partial z}$$

$$S_4 = \frac{\partial u_3}{\partial y} + \frac{\partial u_2}{\partial z},$$

$$S_5 = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x},$$

$$S_6 = \frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y}$$

are the strains and u_1, u_2, u_3 are the displacements along the x, y, z directions respectively. The numbers λ and μ are the two Lame' elastic constants.

We use the equations of vibration in rectangular coordinates, x, y, z. These equations are:[9]

$$\rho \frac{\partial^2 u_1}{\partial t^2} = (\lambda+\mu)\frac{\partial \Delta}{\partial x} + \mu\nabla^2 u_1$$

$$\rho \frac{\partial^2 u_2}{\partial t^2} = (\lambda+\mu)\frac{\partial \Delta}{\partial y} + \mu\nabla^2 u_2 \qquad (3\text{-}2)$$

$$\rho \frac{\partial^2 u_3}{\partial t^2} = (\lambda+\mu)\frac{\partial \Delta}{\partial z} + \mu\nabla^2 u_3$$

where

$$\Delta = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} = S_1 + S_2 + S_3 \qquad (3\text{-}3)$$

and
$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad (3\text{-}4)$$

The equations of vibration in cylindrical coordinates, r, θ, z become[10]:

$$\rho \frac{\partial^2 u_z}{\partial t^2} = (\lambda+2\mu)\frac{\partial \Delta}{\partial z} - 2\mu\frac{\partial(r\varpi_\theta)}{r\partial r} + 2\mu\frac{\partial \varpi_r}{r\partial\theta}$$

$$\rho \frac{\partial^2 u_\theta}{\partial t^2} = (\lambda+2\mu)\frac{\partial \Delta}{r\partial\theta} - 2\mu\frac{\partial \varpi_r}{\partial z} + 2\mu\frac{\partial \varpi_z}{\partial r}$$

$$\rho \frac{\partial^2 u_r}{\partial t^2} = (\lambda+2\mu)\frac{\partial \Delta}{\partial r} - \frac{2\mu}{r}\frac{\partial \varpi_z}{\partial\theta} + 2\mu\frac{\partial \varpi_\theta}{\partial\theta}$$

(3-5)

in which
$$\Delta = \frac{\partial(r u_r)}{r\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z}$$

and
$$2\varpi_r = \frac{\partial u_z}{r\partial\theta} - \frac{\partial u_\theta}{\partial z}\ ; \quad 2\varpi_z = \frac{1}{r}\left(\frac{\partial r u_\theta}{\partial r} - \frac{\partial u_r}{\partial\theta}\right); \quad 2\varpi_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}$$

Love[11] has considered the problem of waves propagated along the axis of a cylinder and has shown that the displacements u_r, u_θ, u_z along the r, θ, z directions can be written in the forms:

$$u_r = R\, e^{j(Bz - \omega t)}$$
$$u_\theta = \Theta\, e^{j(Bz - \omega t)} \qquad (3\text{-}6)$$
$$u_z = Z\, e^{j(Bz - \omega t)}$$

where R, Θ and Z are, in general, functions of r and θ only, B is the propagation constant and j is the imaginary unit; $j = (-1)^{\frac{1}{2}}$.

STRESS DISTRIBUTION

It is assumed here that $u_\theta = 0$ and that the particle motions are independent of θ. Mason[12] points out that the solutions take the form

$$R = A\frac{\partial}{\partial r}J_0(hr) + CBJ_1(kr) = -AhJ_1(hr) + CBJ_1(kr) \qquad (4\text{-}1)$$

$$Z = AjBJ_0(hr) + \frac{jc}{r}\frac{\partial[rJ_1(kr)]}{\partial r} = j(ABJ_0(hr) + CKJ_0(kr)) \qquad (4\text{-}2)$$

where A and C are constants,

$$h^2 = \frac{\omega^2\rho}{(\lambda + 2\mu)} - B^2 \qquad (4\text{-}3)$$

and

$$k^2 = \frac{\omega^2\rho}{\mu} - B^2 \qquad (4\text{-}4)$$

The constants A and C are determined by using the following boundary conditions:

$T_{rz} = T_{rr} = T_{r\theta} = 0$ at $r = a$ (at the surface of the cylinder)

One obtains

$$A\left[\frac{2\mu\,\partial^2[J_0(ha)]}{\partial a^2} - \frac{\omega^2\rho\lambda}{\lambda+2\mu}J_0(ha)\right] - 2\mu CB\frac{\partial[J_1(Ka)]}{\partial a} = 0 \qquad (4\text{-}5)$$

and

$$\frac{2AB\,\partial[J_0(ha)]}{\partial a} + C\left(2B^2 - \frac{\omega^2\rho}{\mu}\right)J_1(Ka) = 0 \qquad (4\text{-}6)$$

For a given frequency, the phase velocity is

$$v = \omega/B \qquad (4\text{-}7)$$

It is found that the velocity in a rod for which the diameter is many times the wavelength, is close to the velocity

$$V = \left(\frac{\lambda + 2\mu}{\rho} \right)^{\frac{1}{2}} \qquad (4\text{-}8)$$

where ρ is the density of the rod

λ is Lame's elastic modulus

μ is Lame's shear modulus.

From Eq. (4-6) we obtain

$$C = \frac{2AB \frac{\partial J_0(ha)}{\partial a}}{\left(\frac{\omega^2\rho}{\mu} - 2B^2\right) J_0(Ka)} = \frac{2AB}{\left(\frac{\omega^2\rho}{\mu} - 2B\right) J_1(ka)} \frac{\partial J_0(ha)}{\partial a} \qquad (4\text{-}9)$$

where a is the radius of the cylindrical rod. The stress along the z axis is

$$T_{zz} = Y_0 \frac{\partial u_z}{\partial z} \qquad (4\text{-}10)$$

where Y_0 is Young's modulus.

From (3-6), (4-2) and (4-9) Eq. (4-10) becomes

$$T_{zz} = Y_0 jBu_z = jB\, Y_0 Z\, e^{j(Bz - \omega t)}$$

$$= -B^2 Y_0 A \left[J_0(hr) + \frac{2K J_0(Kr)}{\left(2B^2 - \frac{\omega^2\rho}{\mu}\right) J_1(Ka)} \cdot \frac{\partial J(ha)}{\partial a} \right] e^{j(Bz - \omega t)} \qquad (4\text{-}11)$$

Eliminating C, from (4-5) and (4-6), we have for the frequency equation

$$4\mu \frac{B^2 \partial J(ha)}{\partial a} \cdot \frac{\partial J_1(Ka)}{\partial a}$$

$$= \left(2B^2 - \frac{\omega^2\rho}{\mu}\right) J_1(Ka) \left(\frac{2\mu \partial^2 J(ha)}{\partial a^2} - \frac{\omega^2 \rho \lambda J(ha)}{\lambda + 2\mu} \right) \qquad (4\text{-}12)$$

Since

$$\frac{\partial J_0(ha)}{\partial a} = -h J_1(ha)$$

$$\frac{\partial J_1(Ka)}{\partial a} = K\left[J_0(Ka) - \frac{1}{Ka} J_1(Ka)\right]$$

$$\frac{\partial^2 J_0(ha)}{\partial a^2} = \frac{\partial}{\partial a}\frac{\partial J_0(ha)}{\partial a} = - h \frac{\partial J_1(ha)}{\partial a}$$

$$= -h^2\left[J_0(ha) - \frac{J_1(ha)}{ha}\right] \qquad (4\text{-}13)$$

Eq. (4-12) becomes

$$4\mu B^2 K \frac{\partial J_0(ha)}{\partial a}\left(J_0(Ka) - \frac{J_1(Ka)}{Ka}\right)$$

$$=\left[2B^2 - \frac{\omega^2\rho}{\mu}\right] J_1(Ka)\left[2\mu\left(J_0(ha) - \frac{J_1(ha)}{ha}\right)h^2 + \frac{\omega^2\rho\lambda}{\lambda+2\mu} J_0(ha)\right]$$

or

$$\frac{-2K \; \partial J_0(ha)}{\left(2B^2 - \frac{\omega^2\rho}{\mu}\right)J_1(Ka)\,\partial a} = \frac{2\mu h^2\left[J_0(ha) - \frac{J_1(ha)}{ha}\right] + \frac{\omega^2\rho\lambda}{\lambda+2\mu}J_0(ha)}{2\mu B\left(J_0(Ka) - \frac{J_1(Ka)}{Ka}\right)} \qquad (4\text{-}14)$$

From Eq. (4-3), (4-7) and (4-8) we find

$$h^2 = \frac{\omega^2\rho}{\lambda+2\mu} - \frac{\omega^2\rho}{\lambda+2\mu}$$

or $\qquad h = 0 \qquad\qquad (4\text{-}15)$

In the same way, we obtain

$$K^2 = \frac{\omega^2\rho}{\mu} - \frac{\omega^2\rho}{\lambda+2\mu} = \frac{\omega^2\rho(\lambda+\mu)}{\mu(\lambda+2\mu)} \qquad (4\text{-}16)$$

From Eqs. (4-11) , (4-14) and (4-15) the stress distribution at the z=0 plane becomes

$$T_{zz} = -B^2 Y_0 A\, e^{-j\omega t}\left[J_0(hr) - \frac{J_0(Kr)\left[2\mu h^2\left(J_0(ha) - \frac{J_1(ha)}{ha}\right) + \frac{\omega^2\rho\lambda}{\lambda+2\mu}J_0(ha)\right]}{2\mu B^2\left(J_0(Ka) - \frac{J_1(Ka)}{Ka}\right)}\right]$$

$$= -B Y_0 A\, e^{-j\omega t}\left\{1 - \frac{J_0(Kr)\frac{\omega^2\rho\lambda}{\lambda+2\mu}}{2\mu B^2\left(J_0(Ka) - \frac{J_1(Ka)}{Ka}\right)}\right\}$$

(4-17)

where A is an arbitrary constant. For simplicity, let

$$C_1 = -B^2 Y_o A = \frac{-\omega^2 \rho Y_o A}{\lambda + 2\mu} \qquad (4\text{-}18)$$

and

$$C_2 = \frac{B^2 Y_o A \dfrac{\omega^2 \rho \lambda}{\lambda + 2\mu}}{2\mu B^2 \left\{ J_o(ka) - \dfrac{J_1(ka)}{ka} \right\}} \qquad (4\text{-}19)$$

$$= \frac{\omega^2 \rho \lambda Y_o A}{2\mu(\lambda + 2\mu)\left\{ J_o(ka) - \dfrac{J_1(ka)}{ka} \right\}}$$

where C_1 and C_2 are both dependent on the cylinder radius and the frequency. A is an arbitrary constant. Therefore, Eq. (4-17) can be written in the following form:

$$T_{zz} = \left\{ C_1 + C_2 J_o(kr) \right\} \cos \omega t \qquad (4\text{-}20)$$

ELECTRODE SHAPES

When ultrasonic waves pass along the cylindrical rod, its stress distribution in a plane perpendicular to the axis may be represented by Eq.(4-20) for the z = 0 plane. Now let us assume an electrode placed near the end of the rod. By proper choice of the shape of this electrode, a voltage applied between this electrode and the cylindrical rod will produce a stress distribution at the z = 0 surface in the cylindrical rod in approximate conformity with (4-20). In other words, this method can be used to generate ultrasonic waves.

First we discuss the relation between stress and electrical intensity

$$T_{zz} = \frac{K\epsilon_o}{2} E_z^2 \qquad (5\text{-}1)$$

where $\epsilon_o = 8.85 \times 10^{-12}$ farad/meter

and K = dielectric constant,

We know that since the dielectric constant for free space is unity, Eq.(5-1) becomes

$$T_{zz} = (\epsilon_o/2)\, E_z^{\,2} \qquad (5\text{-}2)$$

When Eq.(5-2) is expressed in cgs units, it is

$$T_{zz} = (1/8\pi)\, E_z^2 \qquad (5\text{-}3)$$

From Gauss's theorem for a conductor, we have

$$E_z = 4\pi\sigma \qquad (5\text{-}4)$$

where σ is the surface charge density.

From (5-4) and (5-3) it follows that

$$\sigma = (\,T_{zz}/2\pi\,)^{\frac{1}{2}} \qquad (5\text{-}5)$$

From Eq.(5-5), the potential distribution in the region of space near the end of the rod can be obtained for those cases in which T_{zz} has the same sign for all values of $r \leqslant a$.

To find the potential distribution, the solution of Laplace's equation in the three dimensions is required. This problem was found too tedious for direct solution. Insted, it was solved by integrating Coulomb's Law over the charge distribution of Eq.(5-5). Let the rectangular coordinates of P be (b, 0, a) as shown in Fig. 2. Then

$$V = \int_0^{2\pi}\int_0^R \frac{\sigma\, r\, dr\, d\theta}{S} \qquad (5\text{-}6)$$

where σ is the surface charge density and R is the radius of the cylinder. Since

$$E_z = 4\pi\sigma$$

We have

$$V = \int_0^{2\pi}\int_0^R \frac{E_z\, r\, dr\, d\theta}{4\pi S} \qquad (5\text{-}7)$$

The potential at P , an arbitrary point on the xz plane, is

$$V = \frac{1}{4\pi}\int_0^{2\pi}\int_0 \frac{E_z\, r\, dr\, d\theta}{\sqrt{(b - r\cos\theta)^2 + r^2\sin^2\theta + a^2}} \qquad (5\text{-}8)$$

Putting

$$\theta = \pi - 2\phi$$

$$d\theta = -2\, d\phi$$

$$\cos\theta = \cos(\pi - 2\phi) = -(1 - 2\sin^2\phi) \qquad (5\text{-}9)$$

Since $a^2 + (b - r)^2 \geqslant 0$

and $a^2 + b^2 + r^2 - 2br \geqslant 0$

then $a^2 + b^2 + r^2 + 2br \geqslant 4br$

put

$$m^2 = \frac{a^2 + b^2 + r^2 + 2br}{4br}$$

then Eq. (5-8) becomes

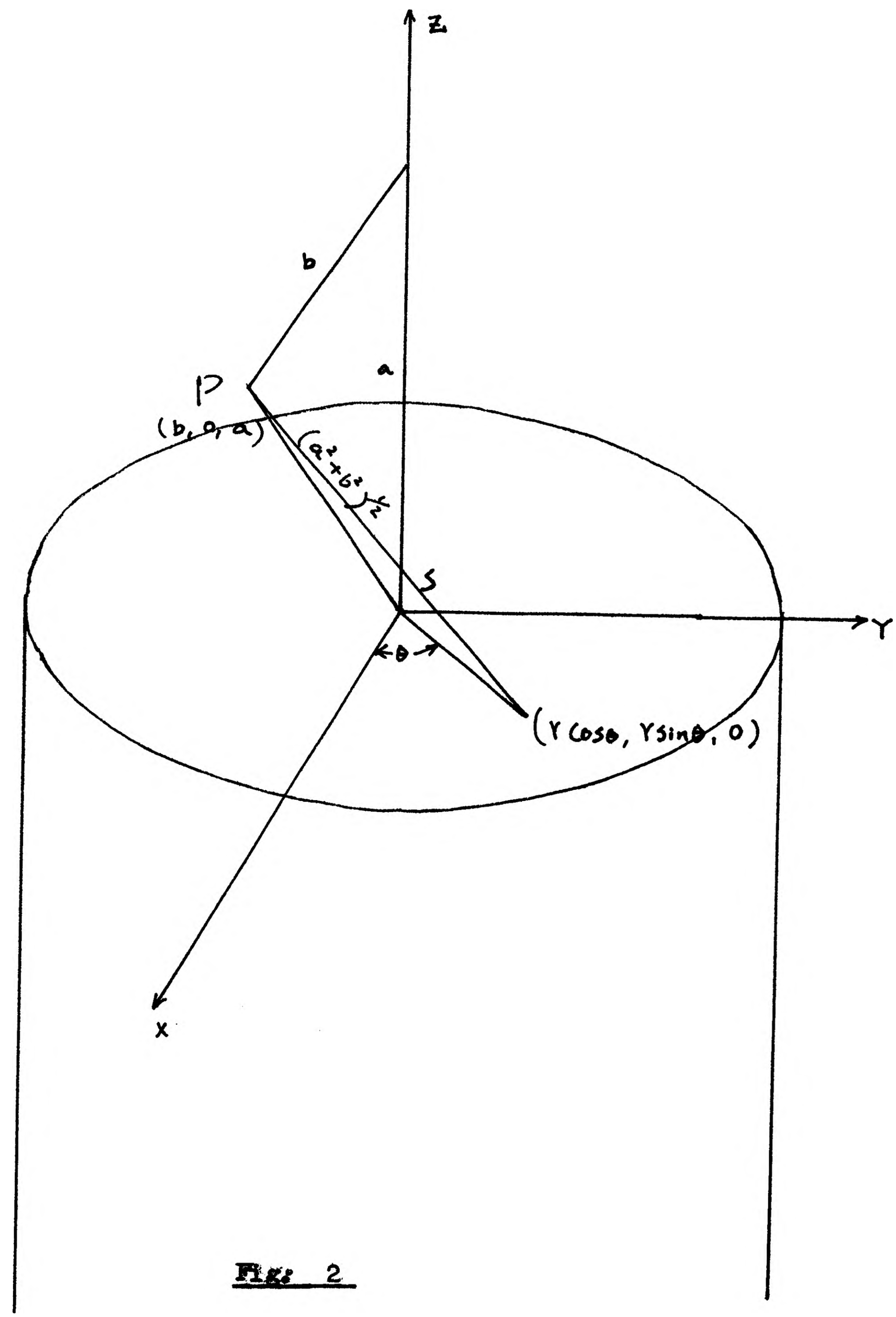


Fig. 2

Cylindrical coordinates used in calculating potential

$$V = \frac{1}{2\pi}\int_0^R \frac{E_z r\,dr}{(a^2+b^2+r^2+2br)^{1/2}}\left[F\left(\frac{1}{m},\frac{\pi}{2}\right)-F\left(\frac{1}{m},-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{2\pi}\int_0^R \frac{2E_z r K\left(\frac{1}{m}\right)dr}{(a^2+b^2+r^2+2br)^{1/2}} \qquad (5\text{-}10)$$

where $K(1/m) = F(1/m, \pi/2)$ is the Elliptic integral of the first kind.

The cylindrical bar is an equipotential region. It is assumed that its potential is zero. However, the contribution to the potential due to the neighboring electrode must be taken into account; this is represented by a term ϕ. Therefore, Eq.(5-8) becomes

$$V = \frac{1}{4\pi}\int_0^{2\pi}\int_0^R \frac{E_z r\,dr\,d\theta}{\sqrt{(b-r\cos\theta)^2+r^2\sin^2\theta+a^2}} + \phi$$

As $a \longrightarrow 0, \quad V = 0$

Hence,

$$-\phi = \frac{1}{4\pi}\int_0^{2\pi}\int_0^R \frac{E_z r\,dr\,d\theta}{\sqrt{(b^2+r^2-2br\cos\theta)}} \qquad (5\text{-}11)$$

$$V = \frac{1}{4\pi}\int_0^{2\pi}\int_0^R \frac{E_z r\,dr\,d\theta}{\sqrt{a^2+b^2+r^2-2br\cos\theta}} - \frac{1}{4\pi}\lim_{a\to 0}\int_0^{2\pi}\int_0^R \frac{E_z r\,dr\,d\theta}{\sqrt{a^2+b^2+r^2-2rb\cos\phi}} \qquad (5\text{-}12)$$

In order to test Eq.(5-12) one can show that it satisfies Laplace's Equation, $\nabla^2 V = 0$. For convenience Eq.(5-12) can be expressed in terms of rectangular coordinates(Fig. 3).

The distance S is

$$S = \lim_{y\to 0}\left[(x-x')^2+(y-y')^2+z^2\right] \qquad (5\text{-}13)$$

In rectangular coordinates Eq.(5-12) then becomes

$$V = \lim_{y\to 0}\frac{1}{4\pi}\iint \frac{E_z(x',y')\,dx'\,dy'}{\sqrt{(x-x')^2+(y-y')^2+z^2}}$$

$$-\frac{1}{4\pi}\lim_{\substack{y\to 0\\ z\to 0}}\iint \frac{E_z(x',y')\,dx'\,dy'}{\sqrt{(x-x')^2+(y-y')^2+z^2}} \qquad (5\text{-}14)$$

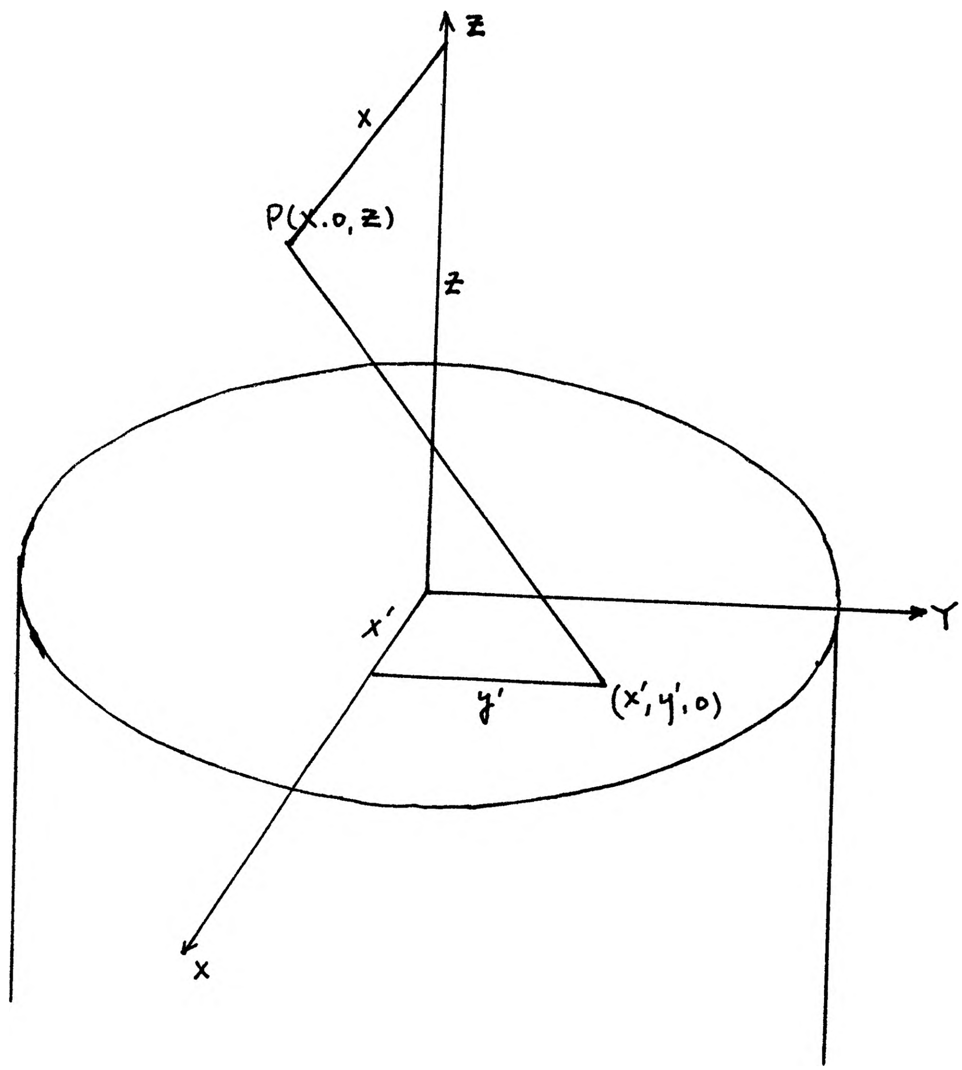


Fig. 3

Rectangular coordinates used for calculating potential

$$\frac{d^2V}{dx^2} = \frac{-1}{4\pi}\lim_{y\to 0}\iint \frac{E_z\left\{\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}-3\left[(x-x')^2+(y-y')^2+z^2\right]^{1/2}(x-x')^2\right\}dx'dy'}{\left[(x-x')^2+(y-y')^2+z^2\right]^3}$$

$$+\frac{1}{4\pi}\lim_{\substack{z\to 0\\ y\to 0}}\iint \frac{E_z\left\{\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}-3(x-x')\left[(x-x')^2+(y-y')^2+z^2\right]^{1/2}dx'dy'\right.}{\left[(x-x')^2+(y-y')^2+z^2\right]^3}$$

Here

$$\nabla^2V = \lim_{y\to 0}\frac{-1}{4\pi}\iint \frac{E_z\left\{3\left[(x-x')^2+(y-y')^2+z^2\right]^{\frac{1}{2}}\left[(x-x')^2+(y-y')^2+z^2-(x-x')^2-(y-y')^2-z^2\right]dx'dy'\right.}{\left[(x-x')^2+(y-y')^2+z^2\right]^3}$$

$$+\frac{1}{4\pi}\iint\lim_{\substack{z\to 0\\ y\to 0}}\frac{E_z\left\{3\left[(x-x')^2+(y-y')^2+z^2\right]^{\frac{1}{2}}\left[(x-x')^2+(y-y')^2+z^2-(x-x')^2-(y-y')^2+z^2\right]dx'dy'\right.}{\left[(x-x')^2+(y-y')^2+z^2\right]^3}$$

$\therefore\ \nabla^2 V = 0$

Therefore Eq. (5-12) is correct.

NUMERICAL INVESTIGATION OF ELECTRODE SHAPE

Once we find the potential distribution, the shape of our electrode is then determined by an equipotential surface. That is, if our electrode were not one of the equipotential surfaces, a current would flow on the surface of the electrode. However, we know that no current is flowing, thus our original statement is justified.

A numerical method was used to find an equipotential surface for the previous problem. Since this problem is symmetric with respect to z axis, for simplicity, we can condsider this to be a two-dimensional problem. First, we find the equipotential curve in the xz-plane. This plane curve can be rotated about the z axis. Then this surface of revolution is the equipotential surface.

Now consider the radius of the aluminum cylindrical rod as one centimeter and the medium around the cylindrical rod to be air. The frequency is taken as one megacycle. From (4-16), (4-18) and (4-19)
we have

$$c_1 = -0.694053 \times 10^{10} \text{ dynes/cm}^2$$
$$c_2 = 9.95305 \times 10^{10} \text{ dynes/cm}^2$$
$$k^2 = 5.5603 \text{ 1/cm}$$

From the roots of Bessel function and Eq.(4-20), if the value of r is larger than 0.4325 cm, the stress is opposite in sign to that for values of r less than 0.4325 cm.(See Fig. 4.)

When the stress is negative E_z of Eq.(5-3) becomes an imaginary number. Since we cannot deal with an imaginary field, the relative potential of every point was calculated only for that part of the charge distribution for which

$$r \leqslant 0.4325 \text{ cm}$$

Then from Eq.(5-10) and (5-12) we get

$$V = \frac{1}{\pi}\int_0^R \frac{E_z r K(\frac{1}{m})dr}{\sqrt{a^2+b^2+r^2+2br}} - \frac{1}{\pi}\int_0^R \frac{E_z r K(\frac{1}{m})dr}{\sqrt{b^2+r^2+2br}} \qquad (6\text{-}1)$$

where R is the radius of the rod. When the value of r is greater than 0.4325 cm, the value of E_z in Eq.(6-1) becomes zero. Therefore Eq.(6-1) can be written as

$$V = \frac{1}{\pi}\int_0^{0.4325} \frac{E_z r K(\frac{1}{m})dr}{\sqrt{a^2+b^2+r^2+2br}} - \frac{1}{\pi}\int_0^{0.4325} \frac{E_z r K(\frac{1}{m})dr}{\sqrt{b^2+r^2+2br}} \qquad (6\text{-}2)$$

The values of the above integrals were obtained for different values of a and b. These integrals could not be evaluated in terms of elementary functions but were evaluated using Simpson's Rule. For fixed values of a and b, corresponding values of potentials are listed in Table II.[13,14,15]

Now we choose one of the family of equipotential curves in the xz-plane. The value of potential at this surface is 116.4 statvolt. Fig. 5 indicates that for values of b near 0.4325 cm the slope is infinite. This is because the stress is zero for $r > 0.4325$ cm.

TABLE II

The value of the potential for corresponding values of a and b

b \ a	0.01	0.02	0.03	0.06	0.08	0.10	0.12 cm
0.00 cm	116.4	148.9	statvolts				
0.01	114.4	141.4					
0.02	112.3	140.2					
0.03	108.9	130.7					
0.04	105.1	122.4					
0.05	99.84	117.7					
0.06	99.12	117.3	128.4				
0.07	88.77	103.6	124.9				
0.08	88.64	101.8	117.4				
0.09	87.73	99.98	110.9				
0.10	87.74	99.59	106.4	138.4			
0.11			100.1	131.1			
0.12				128.1			
0.13				119.0	134.5		
0.14				118.9	131.1		
0.15				114.5	128.7		
0.16				107.1	120.3		
0.17					118.4		
0.18					113.2		
0.19					104.5		
0.20					98.7		
0.21						134.4	
0.22						127.7	
0.23						123.1	
0.24						119.4	
0.25							
0.26							
0.27							
0.28							129.3
0.29							127.1
0.30							110.5
0.31							110.2

b \ a	0.14	0.16	0.18 cm
0.32 cm	120.1	statvolts	
0.33	117.4		
0.34	114.0		
0.35	103.6		
0.36			
0.37		120.5	
0.38		117.4	
0.39		105.6	
0.40			120.0
0.41			116.2
0.42			110.1
0.43			105.7
0.44			
0.45			

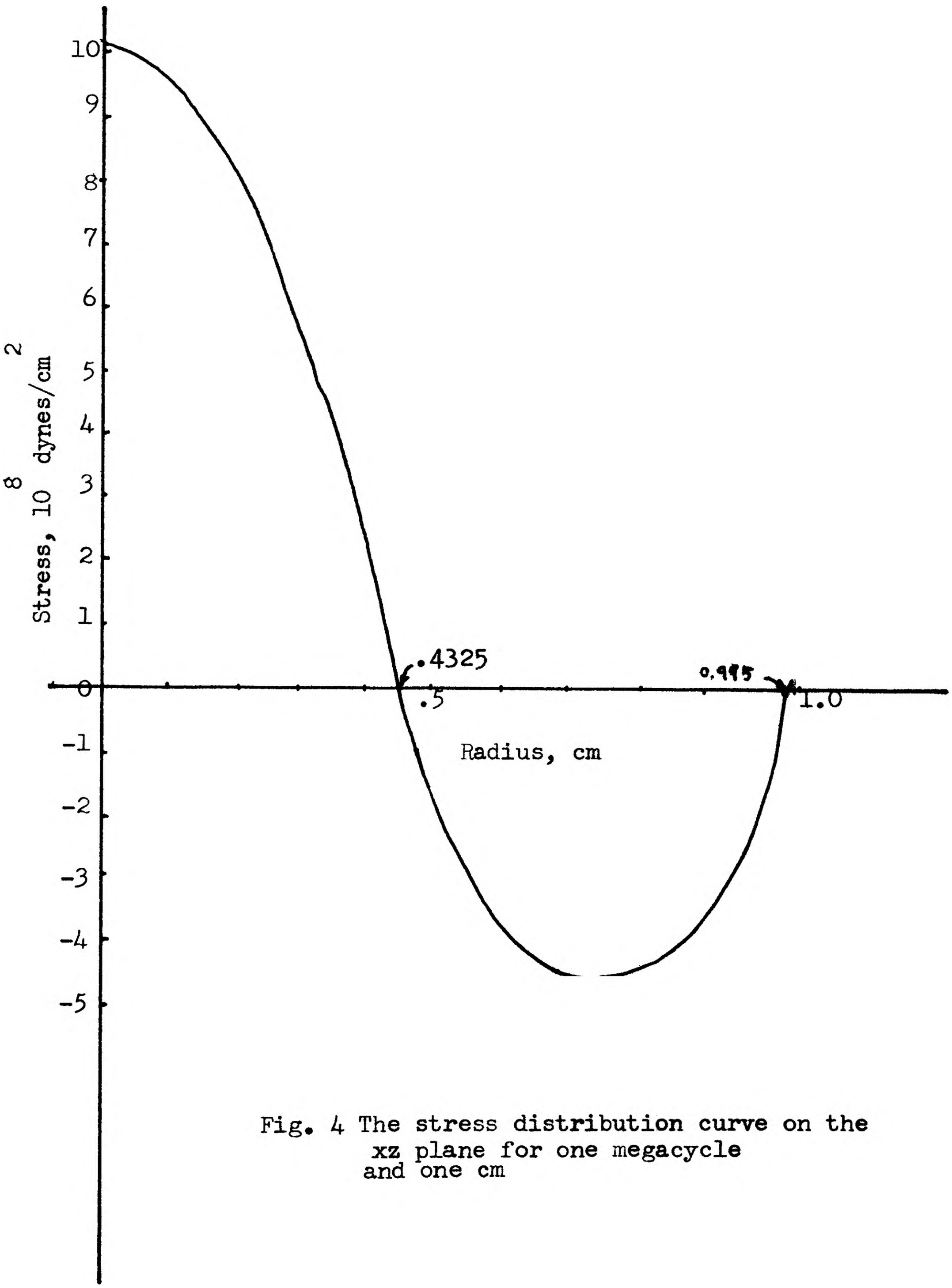
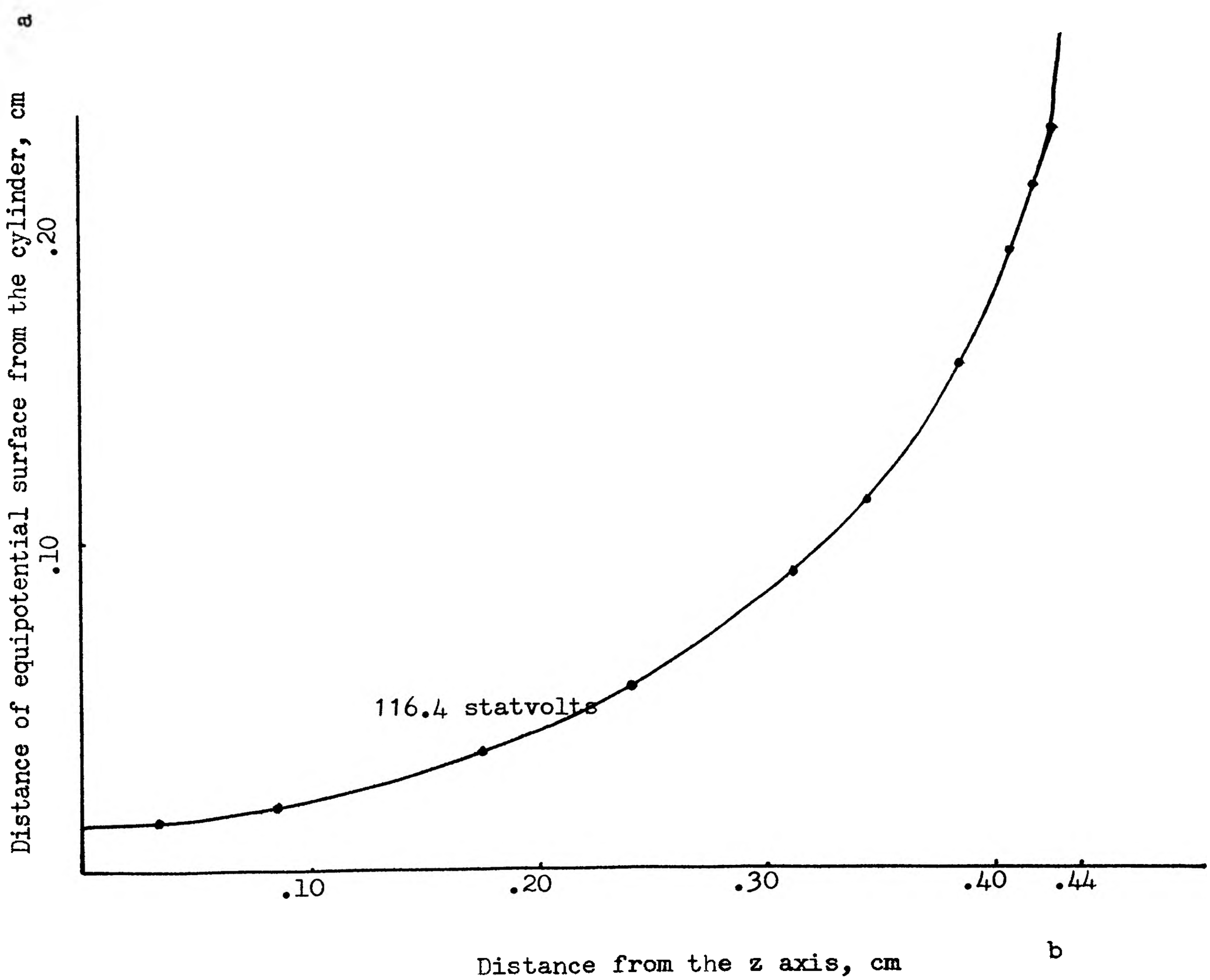


Fig. 4 The stress distribution curve on the xz plane for one megacycle and one cm

Fig. 5 An equipotential curve in the xz plane for one megacycle frequency and one centimeter radius of cylindrical rod



DISCUSSION

For the generation of ultrasonic waves by electric fields, the eletrode shape is designed as an equipotential surface. It has been shown that stress can be expressed in terms of Bessel functions. For a given frequency, the stress has one sign in some regions and the other sign in other regions. Hence , the shape of the equipotential surface will depend on the frequency and cylinder radius.

We will give the procedure for designing an electrode to produce the desired stress in a cylindrical rod of fixed radius:

a) From Eqs. (4-16), (4-18),(4-19) and (4-20) we know that the greater the frequency, the smaller are values of the roots of $J_o(kr)$, $r_1, r_2 \ldots$ The stress distribution for a higher frequency as is shown in Fig.6A. If an electrode is designed to give this stress distribution in a cylinder, it will have the shape shown in Fig.6B. Radii B and B" equal r_1, the first zero of the stress curve , C and C" equal r_2, the second zero,D and D" equal r_3, the third zero, E and E" equal r_4 , the fourth zero. Outer radius, F, equals R, the radius of the cylinder. Curve AB, CD and EF, when related to stress, can satisfy equations (4-20), (5-1) and(6-1). Then, the curvature is large along AB, CD and EF, becuase the stress gradient is large. We can consider distances BB", CC", DD" and EE" to be quite long, so that B"C" and D"E" will not materially affect the stress distribution. Note that, in Eq. (5-1), the stress T_{zz} is proportional E_z^2 , the square of electric field intensity. Therefore, in Fig. 6A. only positive values of stress have physical meaning, as negative values lead to imaginary field intensity. The resulting stress curve consists only of the parts of Fig. 6A above the zero axis, giving the effect of intermittent pulses of stress. The negative values of stress from r_1 to r_2, and

r_3 to r_4 are taken as zero stress, and the corresponding separation between the sample and the electrode is made large from B" to C" and D" to E", giving approximately zero stress.

b) When the frequency is small, the value of k is small but the difference between r_1 and r_2 becomes larger. Fig. 7A and 7B show the stress curve and corresponding electrode shape for low frequency. Here, the curvature is small in A'B' and C'D', because the stress gradient is lower.

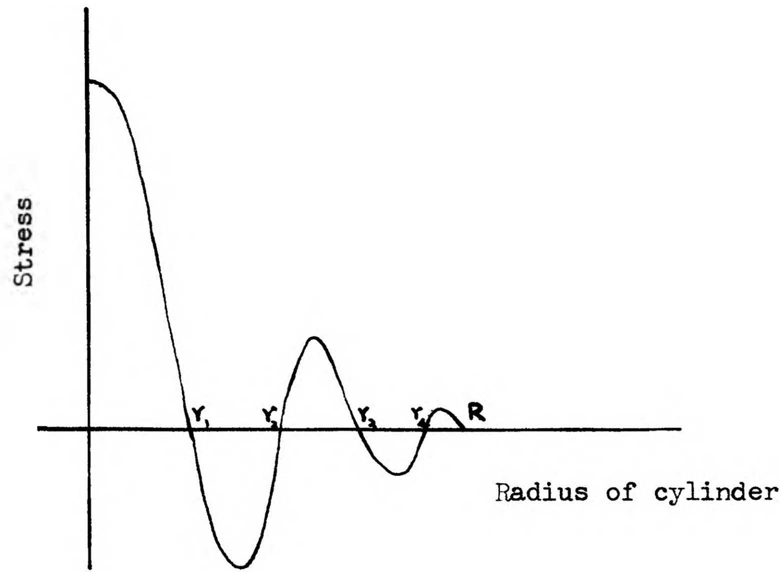


Fig. 6A

Stress versus radius on the top surface of cylindrical conductor for high frequency

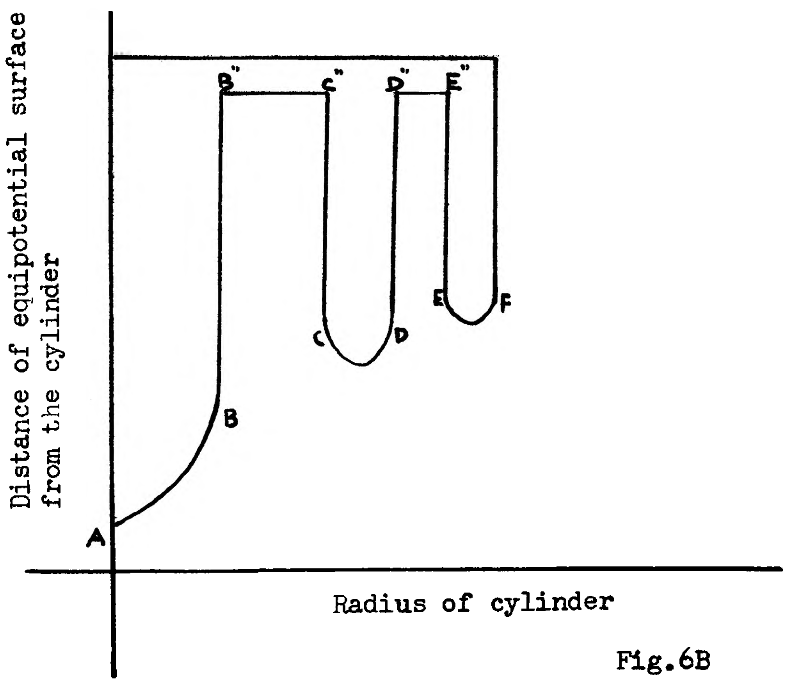


Fig.6B

Possible shape of electrode to give approximate stress distribution in cylinder for high frequency

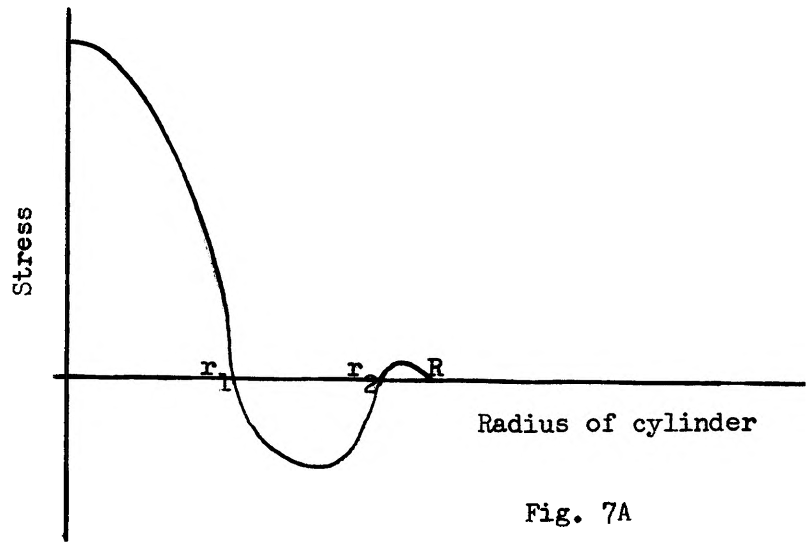


Fig. 7A

Stress versus radius on top surface of cylindrical conductor for low frequency

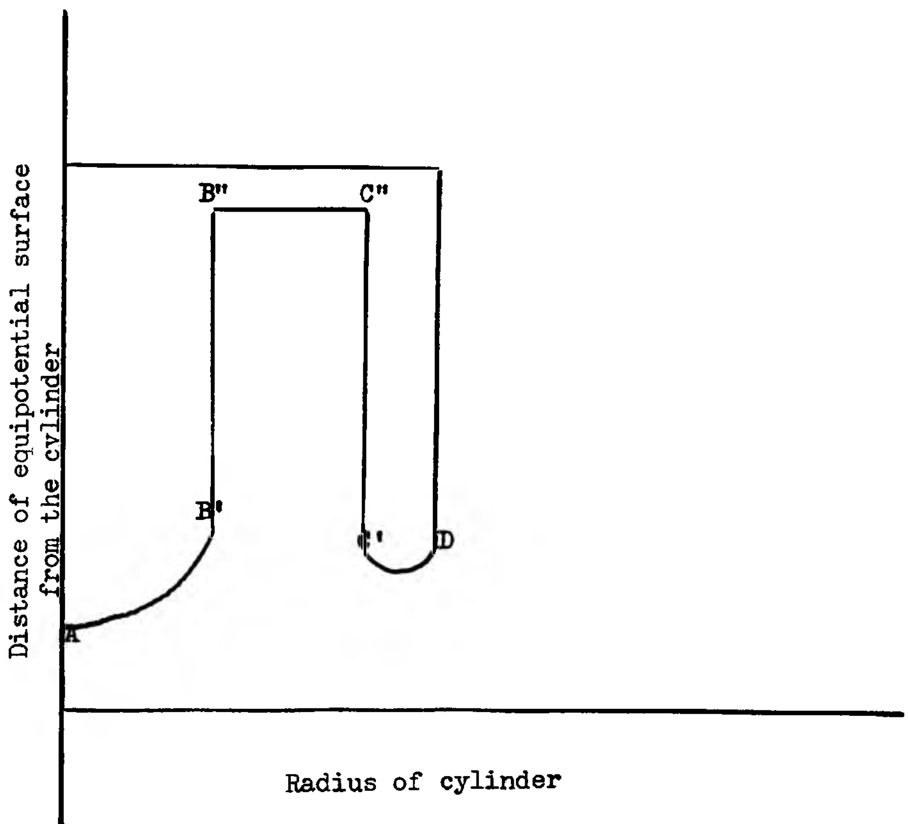


Fig.7B

Possible shape of electrode to give approximate stress distribution in cylinder for low frequency

BIBLIOGRAPHY

1. E.V. Condon, Handbook of Physics, pp.3-132(1951)
2. D.H. Niblett and J. Wilks, Adv. in Physics, pp.8-9(Jan. 1960)
3. E.G. Richardson, Ultrasonic Physics, Elsevier Publishing Company, pp.23-26(1952)
4. A. Granato and K. Lucke, J.Appl. Physics, 27, 583(1960)
5. P.G. Bordoni, J.Acoustics Society of America, 26, No. 4, 495 (July 1960)
6. W.P. Mason, Piezoelectric Crystals and Their Application to Ultrasonics, D. Van Nostrand Company, Inc., N.Y., p.84(1950)
7. K.S. Miller, Partial Differntial Equations in Engineering Problems, Prentice Hall, N.Y., p.14(1961)
8. W.P. Mason, Physical Acoustics and The Properties of Solids, D.Van Nostrand Co.Inc., N.Y., p. 13(1950)
9. W.P. Mason, Physical Acoustics and The Properties of Solids, D.Van Nostrand Co.Inc., N.Y., p. 18(1950)
10. A.E. Love, A Treatise on The Mathematical Theory of Elasticity, Dover, N.Y., p. 288(1944)
11. A.E. Love, A Treatise on The Mathematical Theory of Elasticity, Dover, N.Y. p. 290(1944)
12. W.P. Mason, Physical Acoustics and The Properties of Solids, D.Van Nostrand Co.Inc., N.Y., p.44(1950)
13. N.W. McLachlan, Bessel Functions for Engineers, Oxford Univ. Press(1934)
14. Table of The Bessel Functions for Complex Arguments, Columbia University Press, N.Y.(1943)
15. P.F. Byrd, Die Grundlehren Der Mathematischen Wissenschaften, Maxwell and Springer Ltd(1954)

VITA

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