A circularly polarized corner reflector antenna

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A CIRCULARLY POLARIZED CORNER

REFLECTOR ANTENNA

BY

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A

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ABSTRACT

Image theory is used to develop the far field and polarization equations for a ninety degree corner reflector antenna. These equations are in terms of spherical coordinates, the length of the dipole element, the distance of the dipole from the corner of the reflector, and the angle of tilt of the dipole element with the apex of the corner.

The equations are simplified to give the far fields in the vertical and horizontal planes for the special case of a half-wavelength dipole. An expression to calculate the distance of the dipole from the corner necessary to give circular polarization broadside to the antenna in terms of the tilt angle of the dipole is derived. This distance is plotted against the tilt angle. Plots are made of the far field in the vertical and horizontal planes for various values of distance and tilt angle, maintaining circular polarization broadside.

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LIST OF SYMBOLS

a - mathematical function used to simplify equations; see page 14

$\bar{A}$ - instantaneous (function of time) electromagnetic vector potential

A - mathematical function used to simplify equations; see page 13

$\bar{A}_1$ - a vector in the direction of the current of the real dipole

$\bar{a}_n$ - unit vector normal to a plane

$\bar{a}_r$ - unit vector in the r-direction of spherical coordinates

$\bar{a}_{tan}$ - unit vector tangent to a plane

$\bar{a}_x$ - unit vector in the x-direction of Cartesian coordinates

$\bar{a}_y$ - unit vector in the y-direction of Cartesian coordinates

$\bar{a}_z$ - unit vector in the z-direction of Cartesian coordinates

$\bar{a}_\theta$ - unit vector in the θ-direction of spherical coordinates

$\bar{a}_{\theta_n}$ - unit vector in the θ-direction of a special spherical coordinate system in which the $\theta=0$ axis coincides with the nth dipole element of the antenna array

$\bar{a}_\phi$ - unit vector in the ϕ-direction of spherical coordinates

AR - axial ratio

b - mathematical function used to simplify equations; see page 14

B - mathematical function used to simplify equations; see page 13

c - mathematical function used to simplify equations; see page 13

d - mathematical function used to simplify equations; see page 14

D - mathematical function used to simplify equations; see page 13

dr - distance of dipole from corner of reflection; dr is in radians unless specified as wavelengths in specific uses

LIST OF SYMBOLS

e - Naperian base, 2.71828

ϕ - mathematical function used to simplify equations; see page 14

E or $\mathcal{E}$ - peak value of a sinusoidal electrical field, but is an equation which may take on negative values

$\hat{E}$ - peak value of a sinusoid, may be positive only

E' - prime indicates that the coordinate system has been rotated

E or $\mathcal{E}$ - instantaneous (function of time) electric field; IBM script letter denotes function of time

$\dot{E}$ - dot above symbol denotes complex value

$\bar{E}$ - bar above symbol denotes space vector

$\underline{E}$ - bar below symbol denotes matrix vector

j - $\sqrt{-1}$

$\bar{J}$ - instantaneous electric current density vector

L - electrical length of dipole in radians

M_1, M_2, M_3, M_4, - mathematical functions used to simplify equations; see page 14

n - an integer; 1, 2, 3, ...

$\bar{P}$ - a vector perpendicular to the plane of vector $\bar{A}_1$ and $\bar{R}$

Q - a mathematic function used to simplify equations; see page 15

q_r - real electric charge

q_i - image electric charge

r - radius in spherical coordinates

r' - distance between a point charge and the origin

R - a mathematical function used to simplify equations; see page 15

$\bar{R}$ - position vector to some point in space

S - a mathematical function used to simplify equations; see page 15

LIST OF SYMBOLS

S_1, S_2 - semimajor and semiminor axes of an ellipse

t - time

T - a mathematical function used to simplify equations; see page 15

$\bar{u}$ - velocity of an electric charge

V - instantaneous electrical potential of a moving point charge

V_r - electrical potential of a static real charge

V_i - electrical potential of a static image charge

$\underline{W}$ - coefficient matrix of an elliptical polarization equation

x - first coordinate of the Cartesian coordinate system

x' - x-component of the distance between a point charge and the origin

y - second coordinate of the Cartesian coordinate system

y' - y-component of the distance between a point charge and the origin

z - third coordinate of the Cartesian coordinate system

z' - z-component of the distance between a point charge and the origin

α - the angle between two conducting planes

β - the angle of tilt between the dipole and the apex of the corner reflector; angle between the dipole and the positive z-axis

β'' - the angle between the second dipole and the positive z-axis

β''' the angle between the third dipole and the positive z-axis

δ - any phase, angle of the electric field

ε - electric permittivity of a medium

LIST OF SYMBOLS

θ - the second coordinate of the standard spherical coordinate system; the angle between the z-axis and the radius of a point

θ'' - the second coordinate of a special spherical coordinate system rotated π radians about the z-axis of a superimposed Cartesian coordinate system.

θ''' the second coordinate of a special spherical coordinate system rotated $\pi/2$ radians clockwise about the z-axis of a superimposed Cartesian coordinate system

θ_n - the second coordinate of a special spherical coordinate system in which the $\theta=0$ axis coincides with the nth dipole element of the antenna array

λ - (1) wavelength of electromagnetic radiation; (2) characteristic value or eigenvalue of a matrix. Unless specified otherwise, λ stands for wavelengths.

π - 3.14159 radians

ρ - electric charge density

τ - angle of rotation of a plane coordinate axes

ϕ - the third coordinate of the standard spherical coordinate system; the angle between the x-axis and the projection of the radius to a point on the x-y plane

ϕ'' - the third coordinate of a special spherical coordinate system rotated π radians about the z-axis of a superimposed Cartesian coordinate system

ϕ''' - the third coordinate of a special spherical coordinate system rotated $\pi/2$ radians clockwise about the z-axis of a superimposed Cartesian coordinate system

Ψ_n - the phase angle of the electric field caused by the displacement of the nth dipole element from the origin

ω - radian frequency

∇ - del operator

I. INTRODUCTION

In certain applications an antenna is required to send or receive circularly polarized radiation. This may be done by several types of antennas, the turnstile being one, the helical antenna another. Recently an engineer at McDonnell Aircraft Corporation discovered that rotation of the dipole element of a corner reflector antenna resulted in elliptically polarized radiation. By adjusting the distance of the dipole element from the corner, circular polarization was obtained.

The corner reflector antenna provides a simple means of obtaining circular polarization. It is easier to construct than the helix and does not require the complicated feed of the turnstile antenna. Adjustments are simple, as any degree of polarization may be attained by merely rotating the dipole element or changing the distance between the dipole and the corner.

The purpose of this thesis is to develop the far field and polarization equations for a dipole element of any length in front of a ninety degree corner reflector. These equations are then used to investigate the field patterns of the antenna when the dipole element is a half wave in length and is adjusted to give circularly polarized radiation in the broadside direction.

II. REVIEW OF LITERATURE

The first work on corner reflector antennas, published by Kraus,[1,2] applies image theory to a dipole parallel to the apex of a corner reflector. Field patterns and gain are discussed. Other authors have investigated this type of corner reflector antenna quite thoroughly. For example Wilson and Cottony[3] have done extended work in measuring the field patterns of a finite size corner reflector antenna.

Most theoretical investigations on reflector antennas are based on the concept of images. Javid and Brown[4] have given a discussion of this concept in connection with Green's function. Kraus[5] discusses it strictly from the view of satisfying boundary conditions.

Kraus[5] has also given a thorough discussion on methods of determining elliptical polarization.

The earliest work on a corner reflector antenna in which the dipole element was tilted with respect to the apex of the corner was done by Woodward[6] and Klopfenstein[7]. Woodward demonstrated that circular polarization could be obtained from a corner reflector antenna and compared plots of power gain to other types of antennas. Klopfenstein analized the antenna for any position of the dipole element relative to the corner and any angle to the apex of the reflector. However, both assumed a very short dipole.

III. THEORY

A. Image Theory

The analysis of the dipole backed by a corner reflector may be simplified by assuming infinite, perfectly conducting, grounded planes and using image theory.

According to image theory the reflector will have the same effect on the field as three image elements spaced so that they, together with the real element form the corners of a square. This is illustrated in figure 1. An analysis of the resulting four element array will give the required

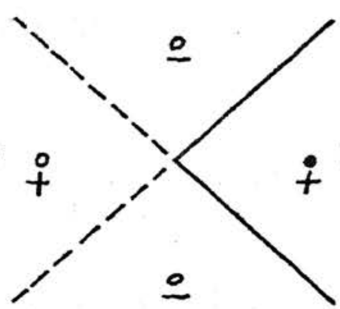

Figure 1. Corner reflector antenna showing the position of the three image elements.

field patterns. The justification of image theory is given in the following paragraphs.

If an electric charge is situated at a point $\bar{r}_r{}'$ above a perfectly conducting grounded plane at z = 0 as in figure 2, the potential from the charge is

$$V_r = \frac{q_r}{4\pi\varepsilon|\bar{r}-\bar{r}_r{}'|} = \frac{q_r}{4\pi\varepsilon[(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}} \tag{1}$$

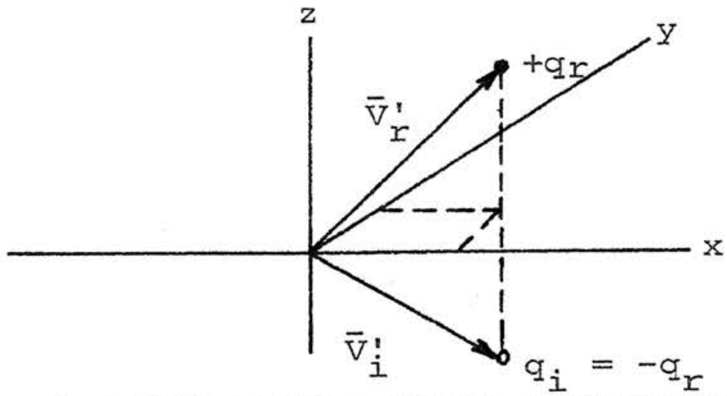

Figure 2. A point charge above a perfectly conducting grounded plane and its image charge.

However, the potential at the grounded plane is zero. A second point charge q_i is to be placed so that its potential together with V_r gives zero everywhere on the xy plane with the conducting sheet removed. Let V_i be the potential from the second charge. Then at z = 0

$$V_r + V_i = 0$$

$$V_i = -V_r = \frac{-q_r}{4\pi\varepsilon[(x-x')^2 + (y-y')^2 + (z')^2]^{1/2}} \qquad (2)$$

Thus $q_i = -q_r$ and may be located at either x', y', and +z', or x', y', and -z' but must agree with physical conditions. If +z' is chosen, the position of q_i coinsides with that of q_r, giving zero field everywhere. As this does not agree with the physical condition the other solution must be chosen, i.e. q_i is located at x', y', and -z'. The second charge is then equal to the negative of the original charge and located at a point directly opposite of the conducting plane as if it were a mirror image.

This idea may be extended to two conducting planes meeting at an angle as in figure 3. The charge must be projected to the opposite side of each plane. If it is pro-

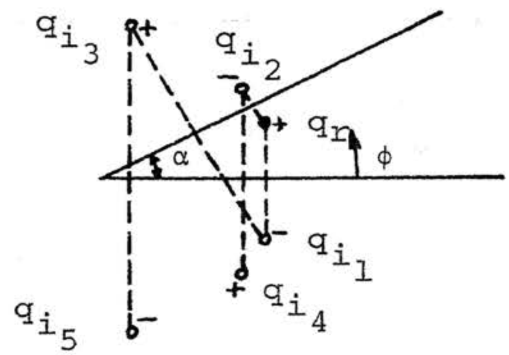

Figure 3. Images caused by an electric charge near two conducting planes.

jected first through the $\phi=0$ plane, both the charge and its first image must be projected through the plane $\phi=\alpha$. But now there is no longer charge equilibrium about the $\phi=0$ plane, so the last two images must be projected through that plane, and so on. When α is a submultiple of 180°, a finite number of images may be used. If $n = \frac{180}{\alpha}$ the number of images equals 2n-1.

Now consider a moving point charge such as would be found in an antenna current. The complex electric field $\dot{E} = -\nabla\dot{V} - j\omega\dot{\bar{A}}$ caused by the moving charge must have no components tangent to the boundary of any conductor within the field. This is the same as saying that the components of $\nabla\dot{V}$ and $\dot{\bar{A}}$ tangent to conducting boundaries must disappear. $\dot{V}$ is a function of charge density ρ and $\dot{\bar{A}}$ a function of current density $\dot{\bar{J}}$. According to image theory a charge moving near a conducting plane has an image charge as in figure 4.

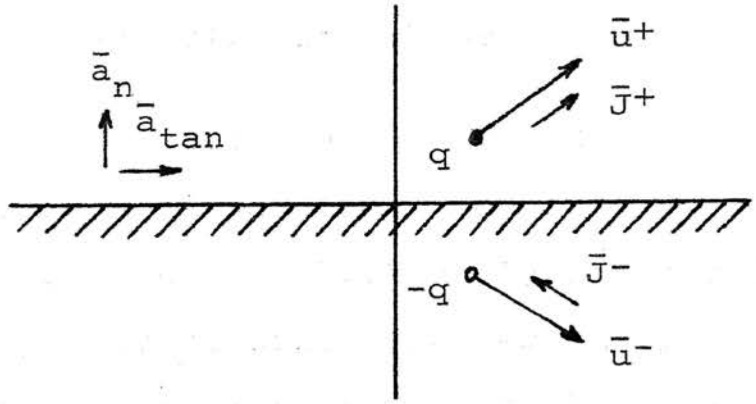

Figure 4. A moving point charge and its image.

The potential resulting from the real and image charges is zero along the plane, making the tangential component of the gradient of $\dot{V}$ zero. The velocity $\bar{u}_r$ of the real charge is

$$\bar{u}_r = u_n \bar{a}_n + u_{tan} \bar{a}_{tan} \tag{3}$$

where u_n and u_{tan} are normal and tangential components of velocity respectively and $\bar{a}_n$ and $\bar{a}_{tan}$ are unit vectors normal and tangential to the conducting plane. The velocity $\bar{u}_i$ of the image charge is

$$\bar{u}_i = -u_n \bar{a}_n + u_{tan} \bar{a}_{tan} \tag{4}$$

The total current density is

$$\bar{J} = \bar{J}^+ + \bar{J}^- = q\, \bar{u}_r - q\, \bar{u}_i$$

$$= q\, u_n \bar{a}_n + q\, u_{tan} \bar{a}_{tan} + u_n \bar{a}_n - u_{tan} \bar{a}_{tan}$$

$$= 2q\, u_n \bar{a}_n \tag{5}$$

It can now be seen that the tangential component of J and thus of $\dot{\bar{A}}$ is zero. Hence the component of $\dot{\bar{E}}$ tangent to the boundary of the conducting plane is zero and image theory may be applied to antenna currents.

B. Elliptical Polarization Theory.

When the electric field vector $\bar{E}$ of an electromagnetic wave rotates in a plane normal to the direction of travel, the wave is said to be elliptically polarized. A wave of this type may also be considered as the sum of two out of phase linearly polarized waves normal to each other. Thus if $E_\theta = \hat{E} \sin (\omega t + \delta_1)$ and $E = E_\phi \sin (\omega t + \delta_2)$, E_θ and E_ϕ are normal. Further, if $\delta_1 \neq \delta_2$ the sum of the two waves is an elliptically polarized wave. The general form of the equation of an ellipse with arbitrary orientation, but centered at the origin is

$$ax^2 - bxy + cy^2 = d \tag{6}$$

If the major and minor axes of the ellipse coincide with the axes of some coordinate system, the equation may be written

$$\frac{x'^2}{S_1^{\,2}} + \frac{y'^2}{S_2^{\,2}} = 1 \tag{7}$$

where S_1 and S_2 are the semimajor and semiminor axis (not necessarily respectively). Figure 5 shows an elipse with its major and minor axes at an angle τ with the x and y axes. For any phase angle, the two electric fields E_θ and E_ϕ may be written in the form

$$\begin{aligned} E_\theta &= E_{\theta_1} \cos \omega t + E_{\theta_2} \sin \omega t \\ E_\phi &= E_{\phi_1} \cos \omega t + E_{\phi_2} \sin \omega t \end{aligned} \tag{8}$$

These two equations may be solved simultaneously and the

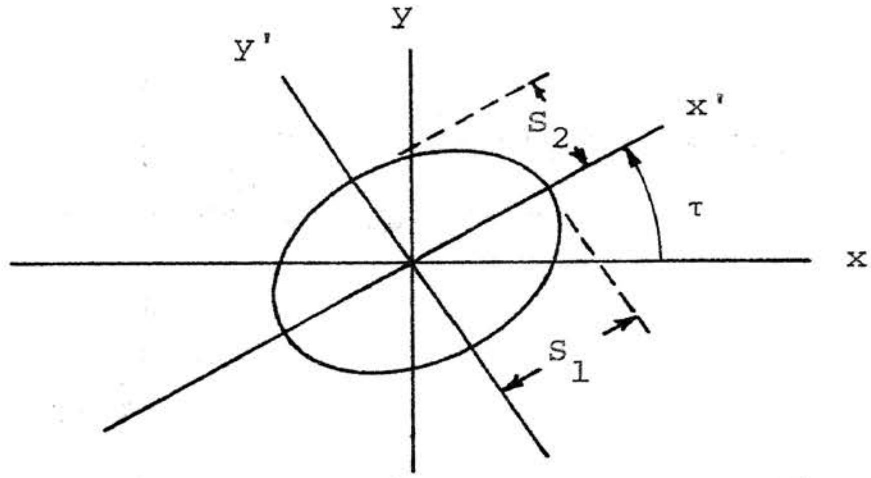

Figure 5. An ellipse with arbitrary orientation. The semimajor and semiminor axes and the tilt angle are labeled S_1, S_2 and τ respectively.

result put in the form of equation (6). If the axes are then rotated, the form of equation (7) may be obtained.

The ratio of the semimajor and semiminor axis is called the axial ratio. If the form of equation (7) is used, the axial ratio may be found by the equation

$$AR = \left(\frac{S_1}{S_2}\right)^{\pm 1} \tag{9}$$

Some authors use the larger divided by the smaller so AR goes from one to infinity. Others divide the smaller by the larger, thereby limiting the values of AR to the range between zero and one. The latter will be chosen here to simplify plotting later on.

To completely specify elliptical polarization, the direction of rotation of the $\bar{E}$ vector must be included, that is, whether the rotation is clockwise or counterclockwise as the wave approaches. Figure 6 shows that if E_θ leads E_ϕ by

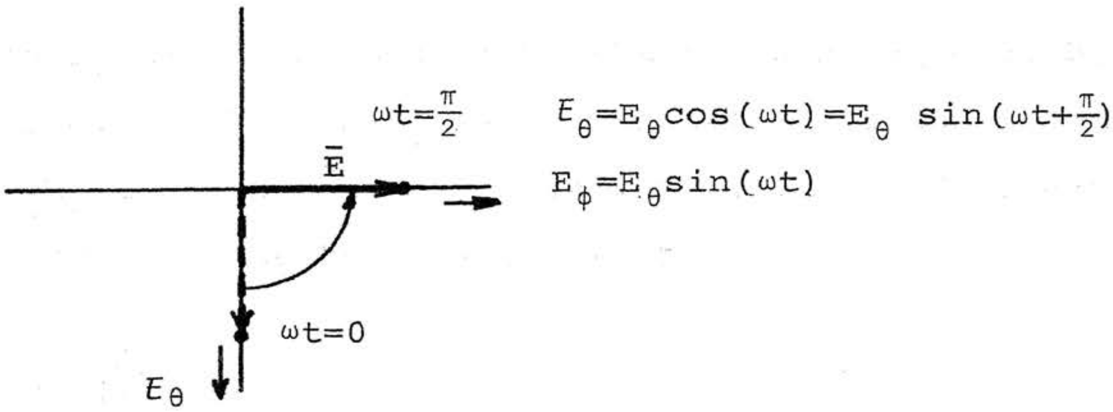

Figure 6. Direction of polarization. The phase of E_θ leads E_ϕ by $\frac{\pi}{2}$

$\frac{\pi}{2}$ in phase, the rotation will be in the direction from $\bar{a}_\theta$ to $\bar{a}_\phi$ where $\bar{a}_\theta$ and $\bar{a}_\phi$ are unit vectors in the θ and ϕ directions respectively. This is the counterclockwise direction. The fields may be written as the complex quantities

$$\dot{E}_\theta = \hat{E}_\theta\ e^{j(\omega t\ +\ \frac{\pi}{2})}$$
$$\dot{E}_\phi = \hat{E}_\phi\ e^{j\omega t} \tag{10}$$

Now form the ratio of $\dot{E}_\theta$ to $\dot{E}_\phi$

$$\frac{\dot{E}_\theta}{\dot{E}_\phi} = \frac{\hat{E}_\theta\ e^{j(\omega t\ +\ \frac{\pi}{2})}}{\hat{E}_\phi\ e^{j\omega t}} = \frac{\hat{E}_\theta}{\hat{E}_\phi}\ e^{+j\frac{\pi}{2}} = j(+\ \frac{\hat{E}_\theta}{\hat{E}_\phi}) \tag{11}$$

If E_θ lags E_ϕ by $\frac{\pi}{2}$ the polarization is in the opposite direction and the ratio of $\dot{E}_\theta$ to $\dot{E}_\phi$ is

$$\frac{\dot{E}_\theta}{\dot{E}_\phi} = \frac{\hat{E}_\theta\ e^{j(\omega t-\frac{\pi}{2})}}{\hat{E}\ \ e^{j\ t}} = \frac{\hat{E}_\theta}{\hat{E}_\phi}\ e^{-j\frac{\pi}{2}} = j(-\ \frac{\hat{E}_\theta}{\hat{E}_\phi}) \tag{12}$$

Now E_θ may be written in the form E_θ cos ωt where $E_\theta = \pm\dot{\bar{E}}_\theta$ $= \hat{E}_\theta\ e^{jn\pi}$ and n = 0 or 1. The same holds true for $\dot{E}_\phi$.

If E_θ and E_ϕ have the same sign, the ratio $E_\theta/E_\phi = \hat{E}_\theta/\hat{E}_\phi$ is positive. If E_θ and E_ϕ have opposite signs the ratio $E_\theta/E_\phi = -\hat{E}_\theta/\hat{E}_\phi$ is negative. Equations (11) and (12) may then be rewritten as the one equation

$$\frac{\dot{E}_\theta}{\dot{E}_\phi} = j\left(\frac{E_\theta}{E_\phi}\right) \tag{13}$$

Thus it can be seen that if the ratio E_θ/E_ϕ is positive, polarization is counterclockwise approaching (or $\bar{a}_\theta$ rotated toward $\bar{a}_\phi$). If E_θ/E_ϕ is negative, rotation is clockwise. If the axial ratio AR is defined as $(E_\theta/E_\phi)^{\pm 1}$ instead of $(\hat{E}_\theta/\hat{E}_\phi)^{\pm 1}$, i.e. AR is permitted to take on negative values, then the direction of polarization may be determined from the sign of AR.

IV. FAR FIELD EQUATIONS FOR THE GENERAL CASE

A. General Field Equations.

The antenna under consideration consists of a single dipole in front of a 90° corner reflector. The dipole is rotated to an angle β with the apex of the corner reflector and in a plane parallel to the apex. The analysis is best carried out using spherical coordinates with the $\theta=0$ axis coinciding with the apex of the corner. The axes $\theta=\frac{\pi}{2}$, $\phi=0$ (x axis) coincides with a line perpendicular to the apex and bisecting the antenna element. The antenna may be represented by an array of the real element and three image elements as shown in figure 7.

The far field of the array is found by summing the far fields of each individual dipole. When a single dipole in free space is lined up with the $\theta=0$ axis, it has the far field pattern

$$\dot{E} = \bar{a}_\theta \left[\frac{\cos\left(\frac{L}{2}\cos\theta\right) - \cos\left(\frac{L}{2}\right)}{\sin\theta}\right] e^{j\omega t} \tag{14}$$

where $\dot{E}$ = the complex field

L = length of dipole in radians or $\frac{2\pi \times \text{length}}{\text{wavelength}}$

The problem then is essentially trigonometry, i.e. writing the field of each element in terms of its own private coordinate system, changing to the coordinate system of the array, and finally adding the four fields. The total

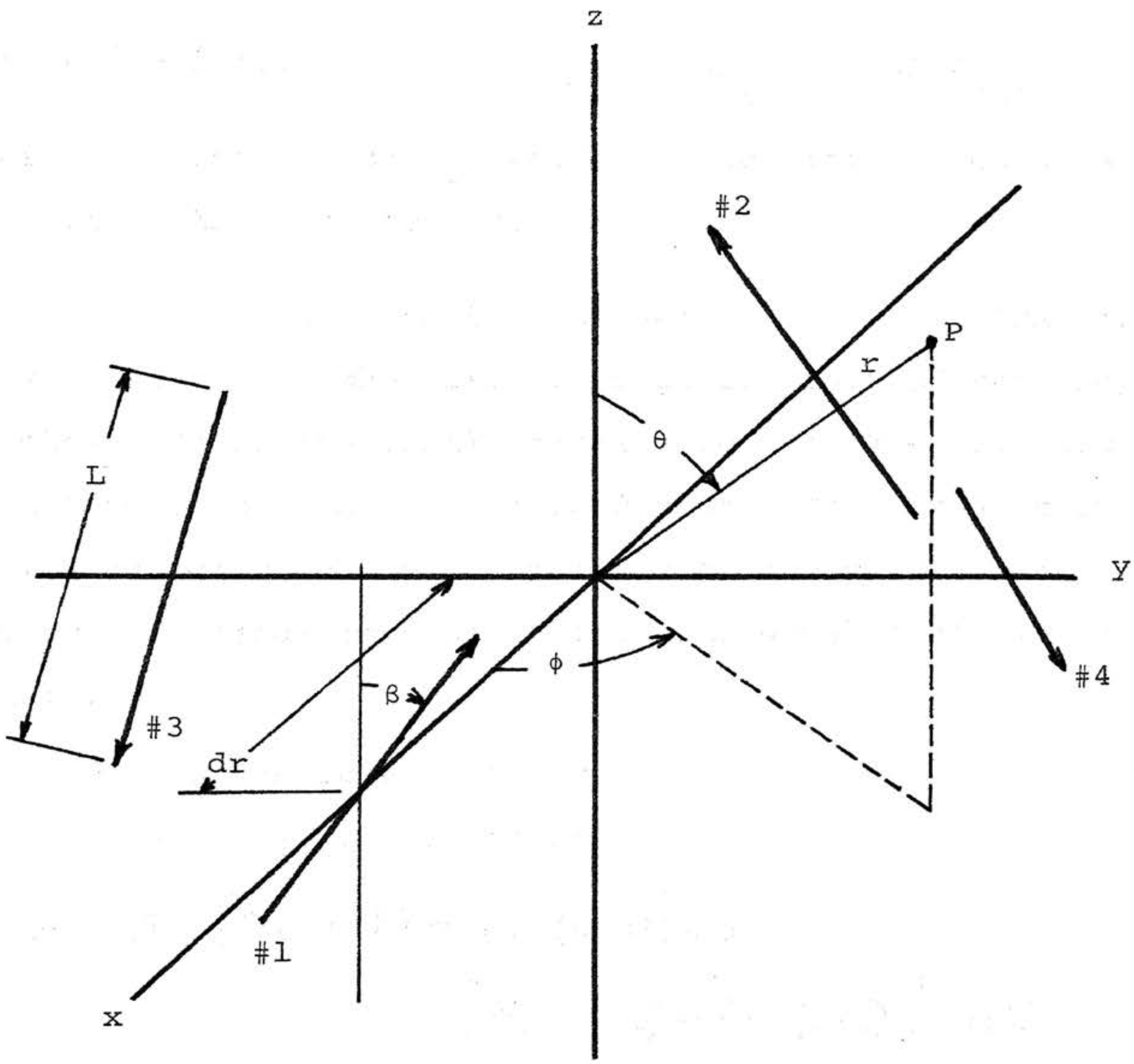

Figure 7. Corner reflector array showing spherical coordinates. The arrows on the dipoles point in the direction of current.

complex field is

$$\dot{\bar{E}} = \sum_{n=1}^{4} \dot{\bar{E}}_n \tag{15}$$

where $\dot{\bar{E}}_n = \dot{\bar{E}}_n(\theta,\phi,\beta,\, \bar{a}_\theta,\, \bar{a}_\phi,\, \omega t) = \dot{E}_{\theta_n}(\theta_n,\, \omega t)\, \bar{a}_{\theta_n}$.

The variables θ_n and $\bar{a}_{\theta_n}$ refer to the private coordinate system of the nth element.

To find the instantaneous values of the fields, one must take either the real or imaginary part of the complex fields. It is the usual practice to take the real part and this is the convention used here. The mathematical details are carried out in Appendix A, the results are stated below. E stands for the instantaneous value of electric field.

$$\begin{aligned} E_\theta &= A \cos \omega t + B \sin \omega t \\ E_\phi &= C \cos \omega t + D \sin \omega t \end{aligned} \tag{16}$$

where
$$A = [M_1 (a-b) + M_2 (a+b)] \cos \Psi_1 - [M_3 (a+c) + M_4 (a-c)] \cos \Psi_4$$

$$B = [-M_1 (a-b) + M_2 (a+b)] \sin \Psi_1 - [M_3 (a+c) - M_4 (a-c)] \sin \Psi_4$$

$$C = (M_2 - M_1)\, d \cos \Psi_1 + (M_3 - M_4)\, \phi \cos \Psi_4$$

$$D = (M_2 + M_1)\, d \sin \Psi_1 + (M_3 + M_4)\, \phi \sin \Psi_4$$

$$M_1 = \frac{\cos\left[\frac{L}{2}\left(\sin\theta \sin\phi \sin\beta + \cos\theta\cos\beta\right)\right] - \cos\left(\frac{L}{2}\right)}{1 - \left(\sin\theta \sin\phi \sin\beta + \cos\theta \cos\beta\right)^2}$$

$$M_2 = \frac{\cos\left[\frac{L}{2}\left(\cos\theta \cos\beta - \sin\theta \sin\phi \sin\beta\right)\right] - \cos\left(\frac{L}{2}\right)}{1 - \left(\cos\theta \cos\beta - \sin\theta \sin\phi \sin\beta\right)^2}$$

$$M_3 = \frac{\cos\left[\frac{L}{2}\left(\sin\theta \cos\phi \sin\beta - \cos\theta \cos\beta\right)\right] - \cos\left(\frac{L}{2}\right)}{1 - \left(\sin\theta \cos\phi \sin\beta - \cos\theta \cos\beta\right)^2}$$

$$M_4 = \frac{\cos\left[\frac{L}{2}\left(\sin\theta \cos\phi \sin\beta + \cos\theta \cos\beta\right)\right] - \cos\left(\frac{L}{2}\right)}{1 - \left(\sin\theta \cos\phi \sin\beta + \cos\theta \cos\beta\right)^2}$$

where

$a = \sin\theta \cos\beta$

$b = \cos\theta \sin\theta \sin\beta$

$c = \cos\theta \cos\phi \sin\beta$

$d = \cos\phi \sin\beta$

$\not{e} = \sin\phi \sin\beta$

$\Psi_1 = dr \sin\theta \cos\phi$

$\Psi_4 = dr \sin\theta \sin\phi$

L = lenght of dipole in radians or $\frac{2\pi \text{length}}{\text{wavelength}}$

dr = distance of dipole from origin in radians or

$\frac{2\pi \text{distance}}{\text{wavelength}}$

The equations apply only to the area in front of the corner reflector, that is, the quarter space $0 \leqq \theta \leqq \pi$, $-\frac{\pi}{4} \leqq \theta \leqq \frac{\pi}{4}$. Due to the reflector the fields are zero everywhere else.

B. Polarization Equations for the General Case

The polarization equations for the corner reflector antenna may be developed from the instantaneous far field equations. The equations from page 13 are

$$E_\theta = A \cos \omega t + B \sin \omega t$$
$$E_\phi = C \cos \omega t + D \sin \omega t \tag{17}$$

If these equations are solved simultaneously to eliminate ωt, the result is in the form of equation (6) on page 7, that is

$$Q\, E_\theta{}^2 - 2RE_\theta E_\phi + SE_\phi{}^2 = T \tag{18}$$

where $Q = C^2 + D^2$

$R = AC + BD$

$S = A^2 + B^2$

$T = (AD-BC)^2$

and A, B, C, and D are as defined starting on page 13. This is the general equation of an ellipse. The details in going from equation (17) to (18) are in Appendix B.

The cross product term of equation (18) may be eliminated by rotating the axes to coincide with the major and minor axes of the ellipse. This may be done by several methods, Lagrange's for one and a method using matrix vectors for another. The latter method is given in Appendix B, the result is

$$\frac{E_\theta'^2}{2T \big/ Q+S+\sqrt{(Q-S)^2+4R^2}} + \frac{E_\phi'^2}{2T \big/ Q+S-\sqrt{(Q-S)^2+4R^2}} = 1 \tag{19}$$

Since this is in the same form as equation (7) on page 7, the peak values of the fields in the new coordinate system are

$$\hat{E}_\theta' = \left[\frac{2T}{Q+S+\sqrt{(Q-S)^2+4R^2}}\right]^{\frac{1}{2}}$$

$$\hat{E}_\phi' = \left[\frac{2T}{Q+S-\sqrt{(Q-S)^2+4R^2}}\right]^{\frac{1}{2}} \tag{20}$$

The equation for the axial ratio is then

$$AR = \left(\frac{\hat{E}_\theta'}{\hat{E}_\phi'}\right)^{\pm 1} \tag{21}$$

where the plus or minus exponent is chosen so that $AR \leq 1$. It should be noted from equation (18) that if R is zero then the major and minor axes of the ellipse are already parallel to the coordinate axes and so $\hat{E}_\theta' = \hat{E}_\theta$ and $\hat{E}_\phi' = \hat{E}_\phi$

V. FAR FIELD EQUATIONS FOR A HALF-WAVE ELEMENT

A. Antenna Parameters for Circular Polarization

Equations for the far field of an arbitrary length corner reflector antenna have been developed. Now they will be applied to investigate the fields of a half-wavelength element. The investigation will be restricted to the planes vertical and horizontal through the center of the dipole

Since circular polarization is the objective of this antenna, the equations producing circular polarization broadside from the antenna will be developed first. Substitution of $L=\pi$, $\theta=\frac{\pi}{2}$ and $\phi=0$ into equation (19) gives the polarization equation

$$\frac{E_\theta^{\,2}}{4\left[\cos\beta\cos dr - \frac{\cos\left(\frac{\pi}{2}\sin\ \right)}{\cos\beta}\right]^2} + \frac{E_\phi^{\,2}}{4(\sin\beta\sin dr)^2} = 1 \tag{22}$$

$$\text{with } E_\theta = 2\cos\beta\cos dr - \frac{2\cos\left(\frac{\pi}{2}\sin\beta\right)}{\cos\beta} \tag{23}$$

$$E_\phi = 2\sin\beta\sin dr$$

The details are carried out in Appendix C. Circular polarization occurs when the denominators of (22) are equal.

$$4\left[\cos\beta\cos dr - \frac{\cos\left(\frac{\pi}{2}\sin\ \right)}{\cos\beta}\right]^2 = 4(\sin\beta\sin dr)^2 \tag{24}$$

Solving for dr (See Appendix C)

$$dr = \cos^{-1}\left[\cos\left(\frac{\pi}{2}\cos\beta\right) \pm \sin\beta\sqrt{1 - \frac{\cos^2\left(\frac{\pi}{2}\sin\beta\right)}{\cos^2\beta}}\right] \tag{25}$$

This equation shows that for any β there are four possible values of dr between zero and one wavelength that will give circular polarization.

Figure 8 shows the distance dr of the dipole from the apex of the corner plotted against the angle β of the dipole from the vertical. The range of β is $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$ radians. Except for the direction of rotation, the graph is symmetrical with respect to the $\beta=0$ axis and the dr = .5 wavelength axis. Equation (25) shows that if the graph were extended for distances greater than one wavelength, the four branches would be repeated at one wavelength intervals. The direction was determined from the axial ratio equation, i.e.

$$Ar = \left(\frac{E_\theta}{E_\phi}\right)^{\pm 1} = \pm 1 \tag{26}$$

As was shown on page 10 a positive AR indicates counterclockwise rotation (wave approaching) while a negative AR indicates the opposite.

Figure 9 shows the magnitude (or peak value) of the unnormalized electric field and dr plotted simultaneously against β. The range of β is 0 to $\frac{\pi}{2}$ radians and that of dr is 0 to .5 wavelengths from the corner. The maximum of $\hat{E}$ for the lower branch is .164 and occurs at β = 54.9 degrees and dr = .0160 wavelengths. The maximum of $\hat{E}$ for the second branch is 1.482 and occurs when β = 52.7 degrees and dr = .309 wavelengths. Both of these curves are periodic with distance, occurring along with the branches at distances of $2\pi n \pm dr$ from the corner (where n is an

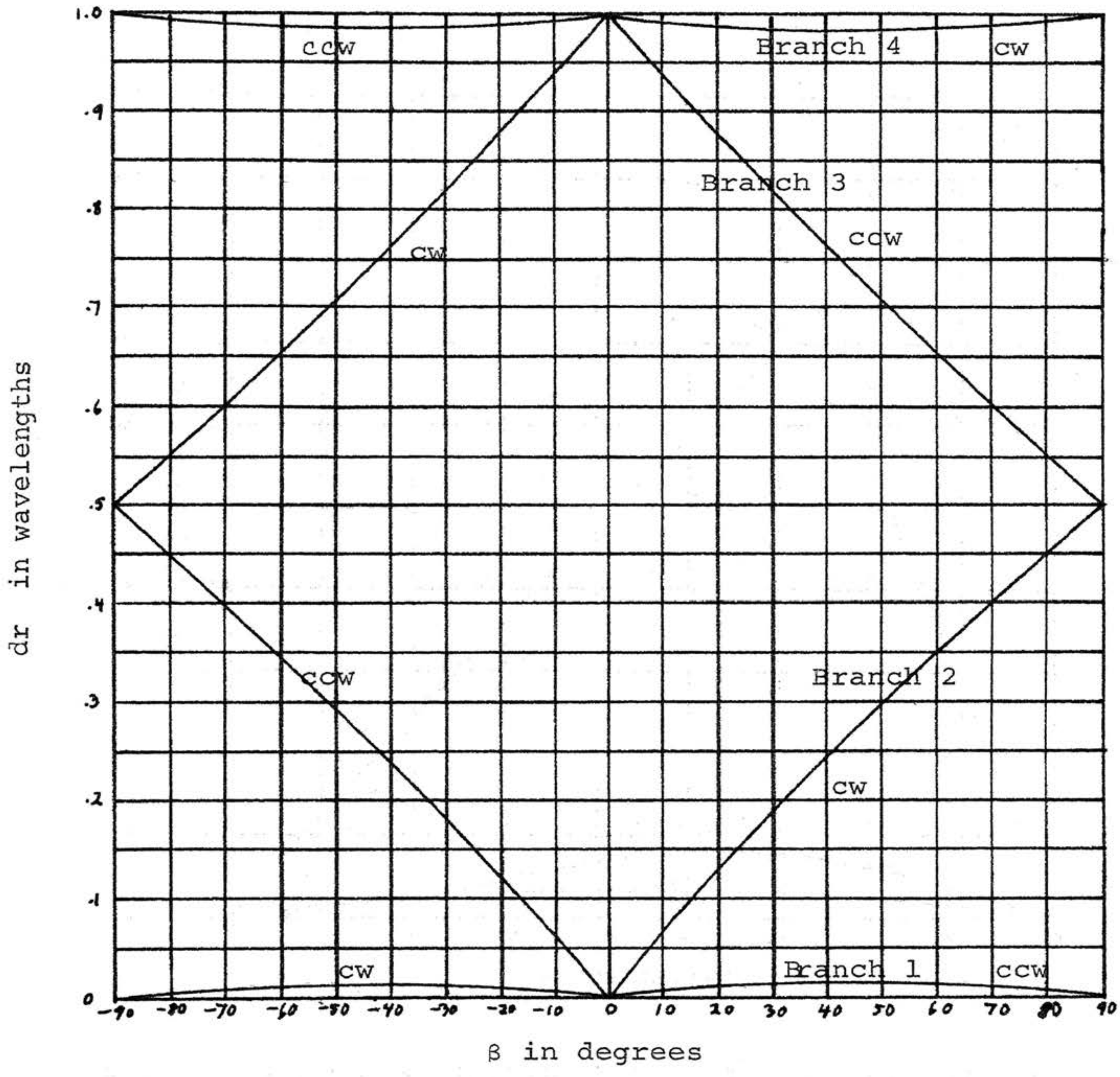

Figure 8. Distance of dipole from corner reflector for circular polarization as a function of dipole tilt

cw = clockwise ccw = counterclockwise

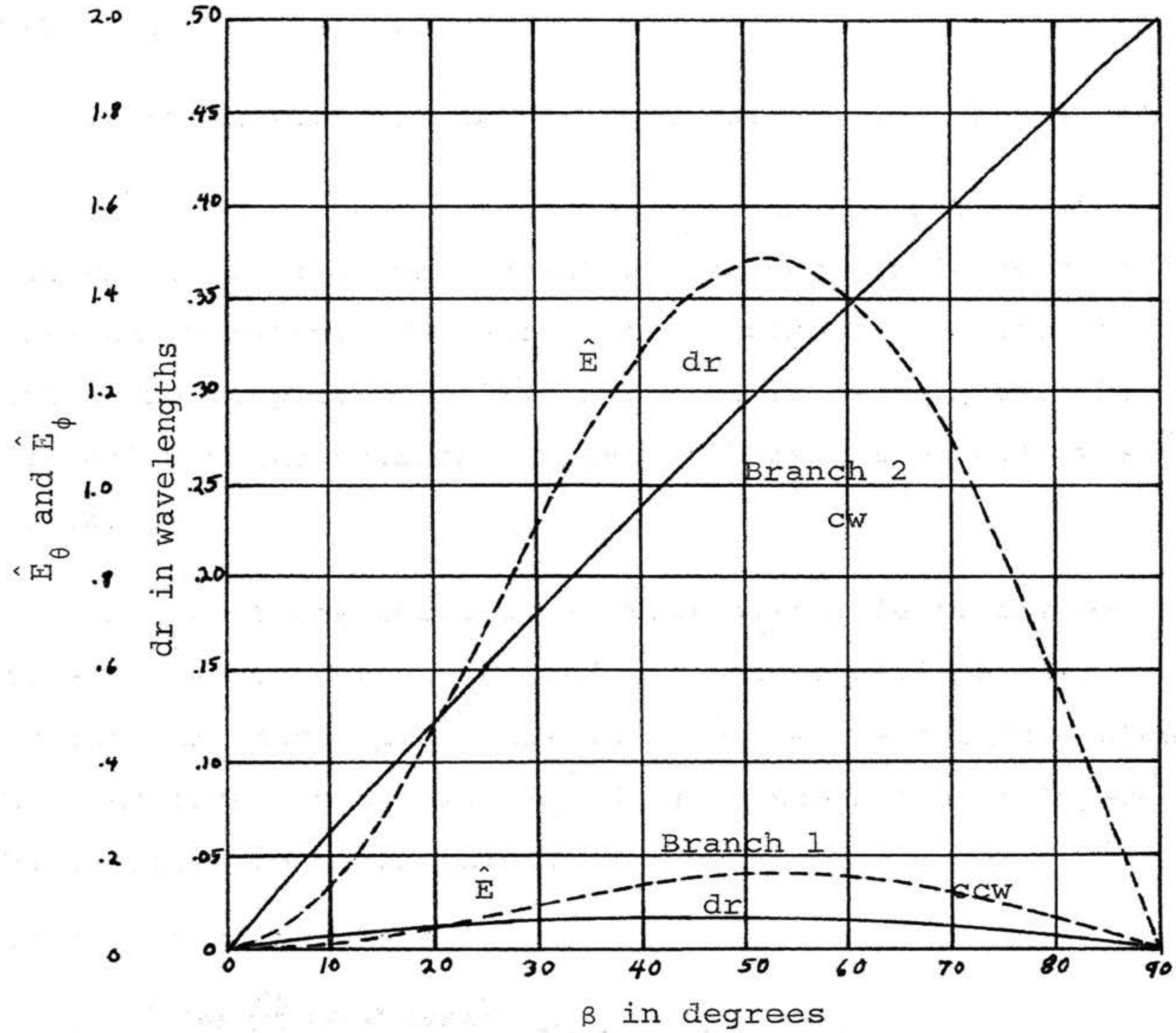

Figure 9. $\hat{E}_\theta$ and $\hat{E}_\phi$ versus β for circular polarization

cw = clockwise ccw = counterclockwise

integer). The first branch is purely theoretical as it is too close to the corner to permit placing a physical element between the walls of the reflector. However, the results of this branch apply to corresponding branches at greater distances from the corner.

B. Far Field Patterns in the Vertical and Horizontal Planes

To calculate and display on a graph the far field patterns of an antenna everywhere in space would be an immense undertaking. Therefore, it is common practice in the literature to plot the field patterns in two planes, the vertical and the horizontal planes through the bore axis, $\phi=0$ and $\theta=\frac{\pi}{2}$.

The equations for the vertical plane fields are obtained by substituting $\phi=0$ and $L=\pi$ into equations (16). For the horizontal plane fields, $\theta=\frac{\pi}{2}$, $L=\pi$ were substituted into the same equations. The details are shown in Appendix C, the results are given below.

For the vertical plane

$$E_\theta = \frac{2\cos\left(\frac{\pi}{2}\cos\theta\cos\beta\right)}{1-\cos^2\theta\cos^2\beta}\sin\theta\cos\beta\cos(dr\sin\theta) - \frac{\cos\left[\frac{\pi}{2}\cos(\theta+\beta)\right]}{\sin(\theta+\beta)} - \frac{\cos\left[\frac{\pi}{2}\cos(\theta-\beta)\right]}{\sin(\theta-\beta)} \tag{27}$$

$$E_\phi = \frac{2\cos\left(\frac{\pi}{2}\cos\theta\cos\beta\right)}{1-\cos^2\theta\cos^2\beta}\sin\beta\sin(dr\sin\theta)$$

For the horizontal plane

$$E_\theta = 2\left[\frac{\cos\left(\frac{\pi}{2}\sin\phi\,\sin\beta\right)}{1-\sin^2\phi\,\sin^2\beta}\cos\left(d_r\cos\phi\right) - \frac{\cos\left(\frac{\pi}{2}\cos\phi\,\sin\beta\right)}{1-\cos^2\phi\,\sin^2\beta}\cos\left(d_r\sin\phi\right)\right]\cos\beta$$

$$E_\phi = 2\left[\frac{\cos\left(\frac{\pi}{2}\sin\phi\,\sin\beta\right)}{1-\sin^2\phi\,\sin^2\beta}\cos\phi\,\sin\left(d_r\cos\phi\right) + \frac{\cos\left(\frac{\pi}{2}\cos\phi\,\sin\beta\right)}{1-\cos^2\phi\,\sin^2\beta}\sin\phi\,\sin\left(d_r\sin\phi\right)\right]\sin\beta \tag{28}$$

Figures 10 through 29 show plots of the vertical and horizontal field components and axial ratios in the vertical and horizontal planes for branches 2 through 5 of figure 8. The plots are for every 15° of rotation of β between 0 and 90° and use the values of dr from figure 8 to obtain circular polarization along the bore. In addition, the fields at the maximum values of $\hat{E}$ on the bore from figure 9 were included in these plots. The plots of like fields ($\hat{E}_\theta$ or $\hat{E}_\phi$, vertical or horizontal) are displayed on the same graph to show the change in fields as the angle β increases. Each set of fields is normalized so that the maximum value of $\hat{E}_\theta$ and $\hat{E}_\phi$ in both planes together is approximately equal to one for each set of parameters.

Inspection of these plots shows that as the dipole is moved farther from the corner, side lobes begin to appear.

Along the third branch the side lobes begin to dominate, and in the fourth and fifth branches the main lobe virtually disappears. The side lobes do not all have the same direction of polarization. The third, fourth, and fifth branches are useless if circular polarization is desired in a strong, main lobe broadside. The ideal pattern would have a single lobe with circular polarization throughout this lobe. The lower angles of β in the first branch (and consequently the smallest values of dr) give the best patterns and the most nearly uniform axial ratios. Thus the smaller distances of the dipole from the corner give the best all around field patterns.

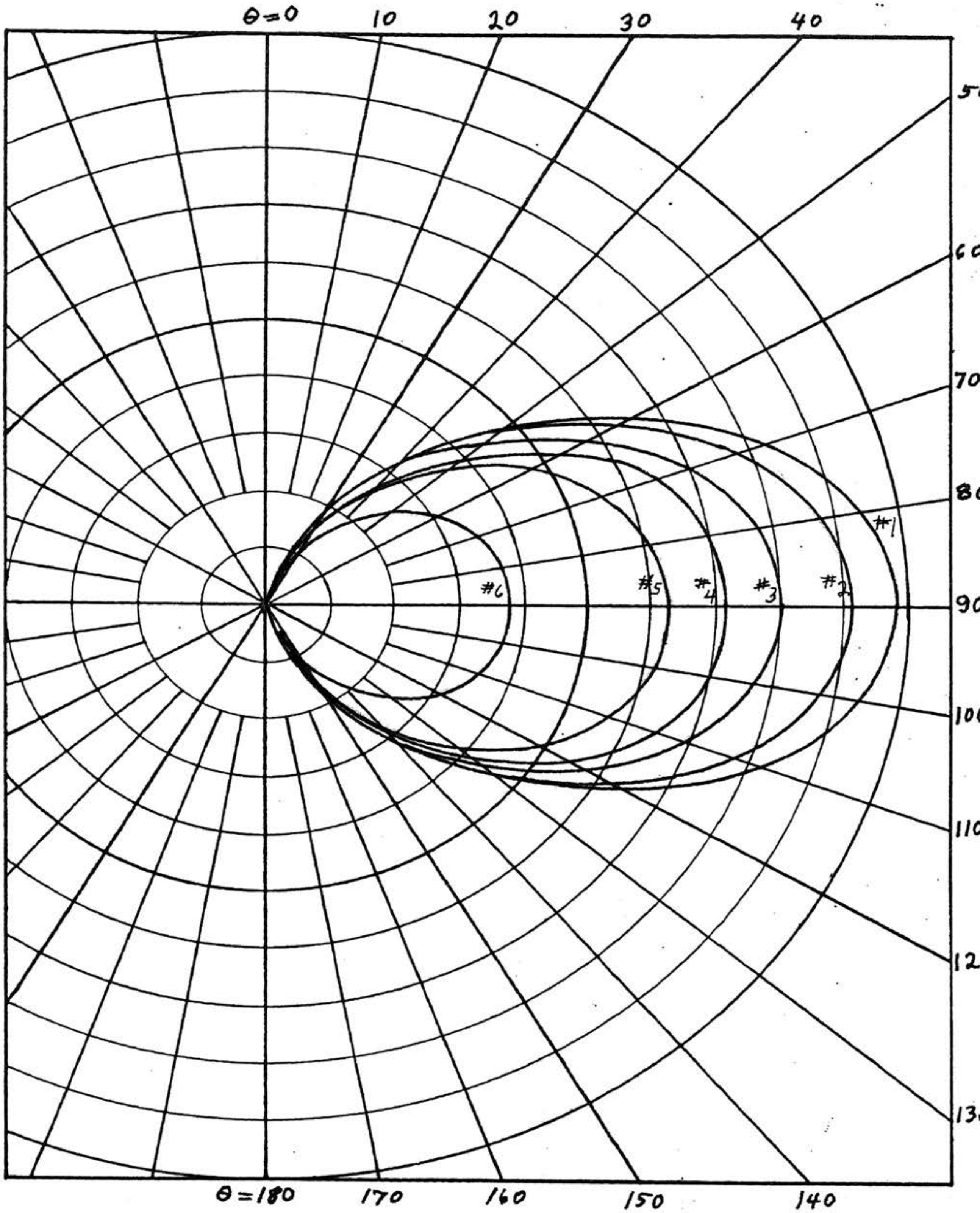

Figure 10. Field Patterns: Branch 2, Vertical Plane, E_θ

#1	β = 15°	dr = .092λ	#4	β = 52.7°	dr = .309λ
#2	β = 30°	dr = .181λ	#5	β = 60°	dr = .348λ
#3	β = 45°	dr = .267λ	#6	β = 75°	dr = .425λ

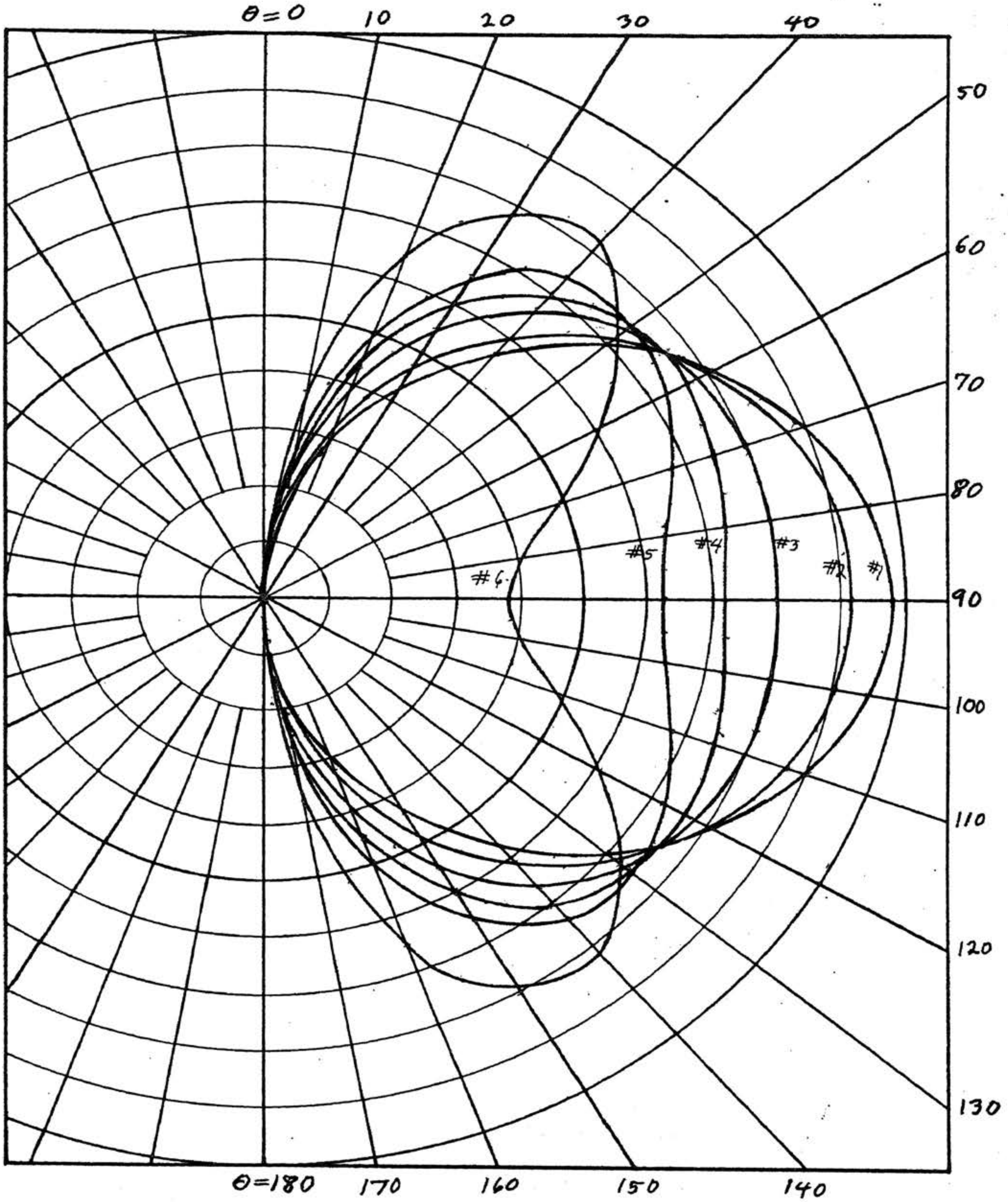

Figure 11. Field Patterns: Branch 2, Vertical Plane, E_ϕ

#1	$\beta = 15°$	dr = .092	#4	$\beta = 52.7°$	dr = .309λ
#2	$\beta = 30°$	dr = .181	#5	$\beta = 60°$	dr = .348λ
#3	$\beta = 45°$	dr = .267	#6	$\beta = 75°$	dr = .425λ

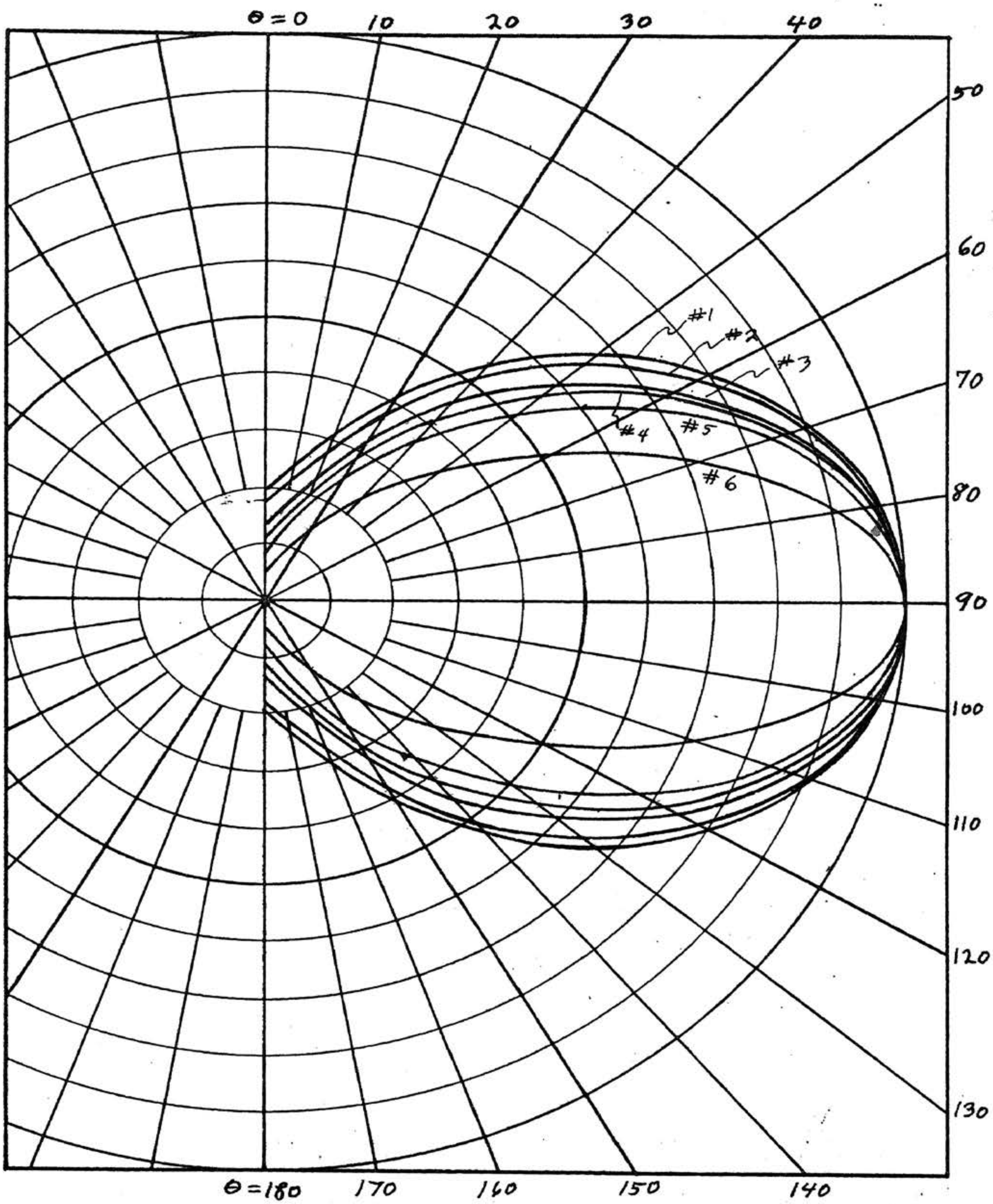

Figure 12. Field Patterns: Branch 2, Vertical Plane, AR

#1	β = 15°	dr = .092λ	#4	β = 52.7°	dr = .309λ
#2	β = 30°	dr = .181λ	#5	β = 60°	dr = .348λ
#3	β = 45°	dr = .267λ	#6	β = 75°	dr = .425λ

Polarization is Clockwise

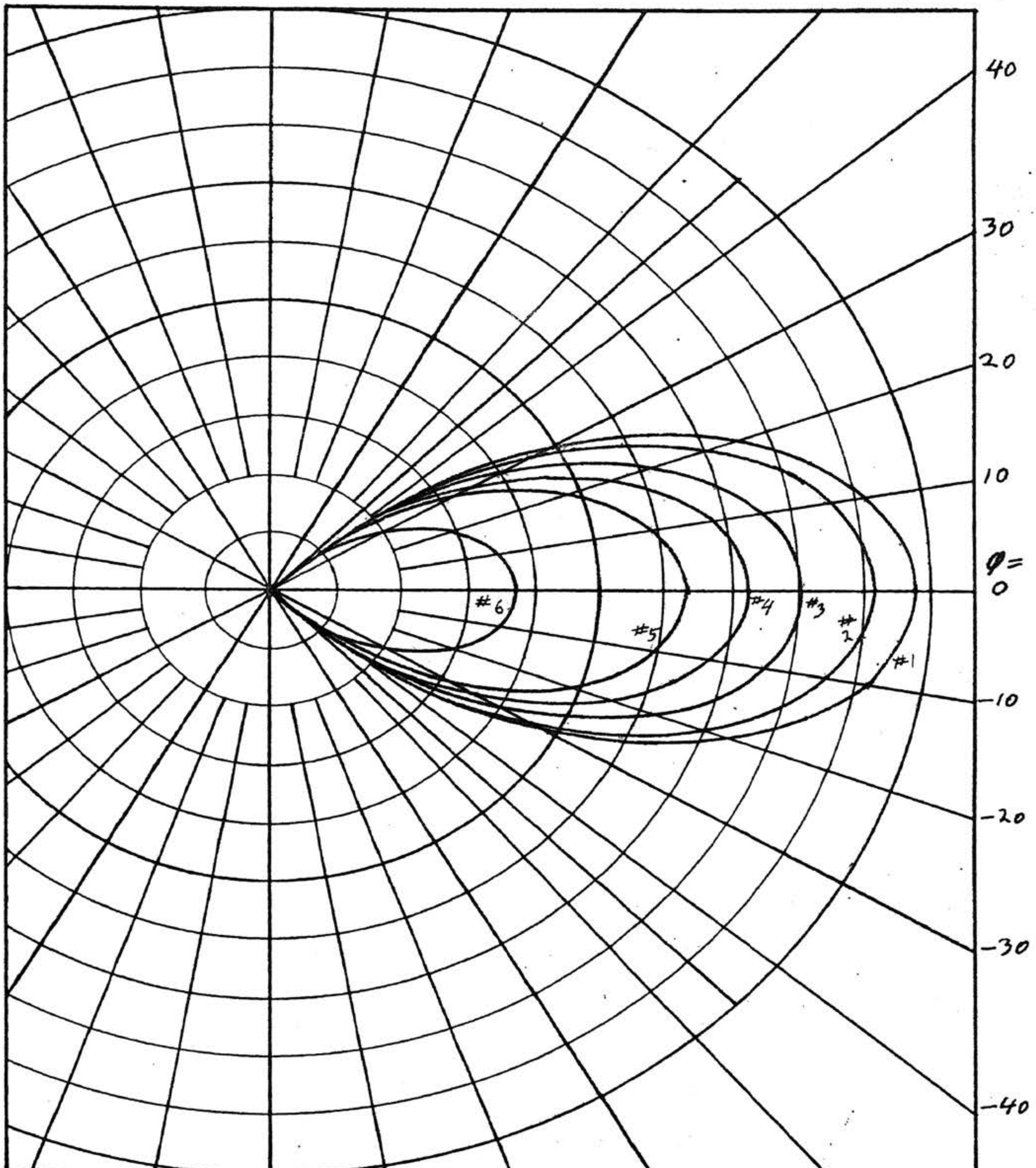

Figure 13. Field Patterns: Branch 2, Horizontal Plane, E_θ

#1	$\beta = 15°$	$dr = .092\lambda$	#4	$\beta = 52.7°$	$dr = .309\lambda$
#2	$\beta = 30°$	$dr = .181\lambda$	#5	$\beta = 60°$	$dr = .348\lambda$
#3	$\beta = 45°$	$dr = .267\lambda$	#6	$\beta = 75°$	$dr = .425\lambda$

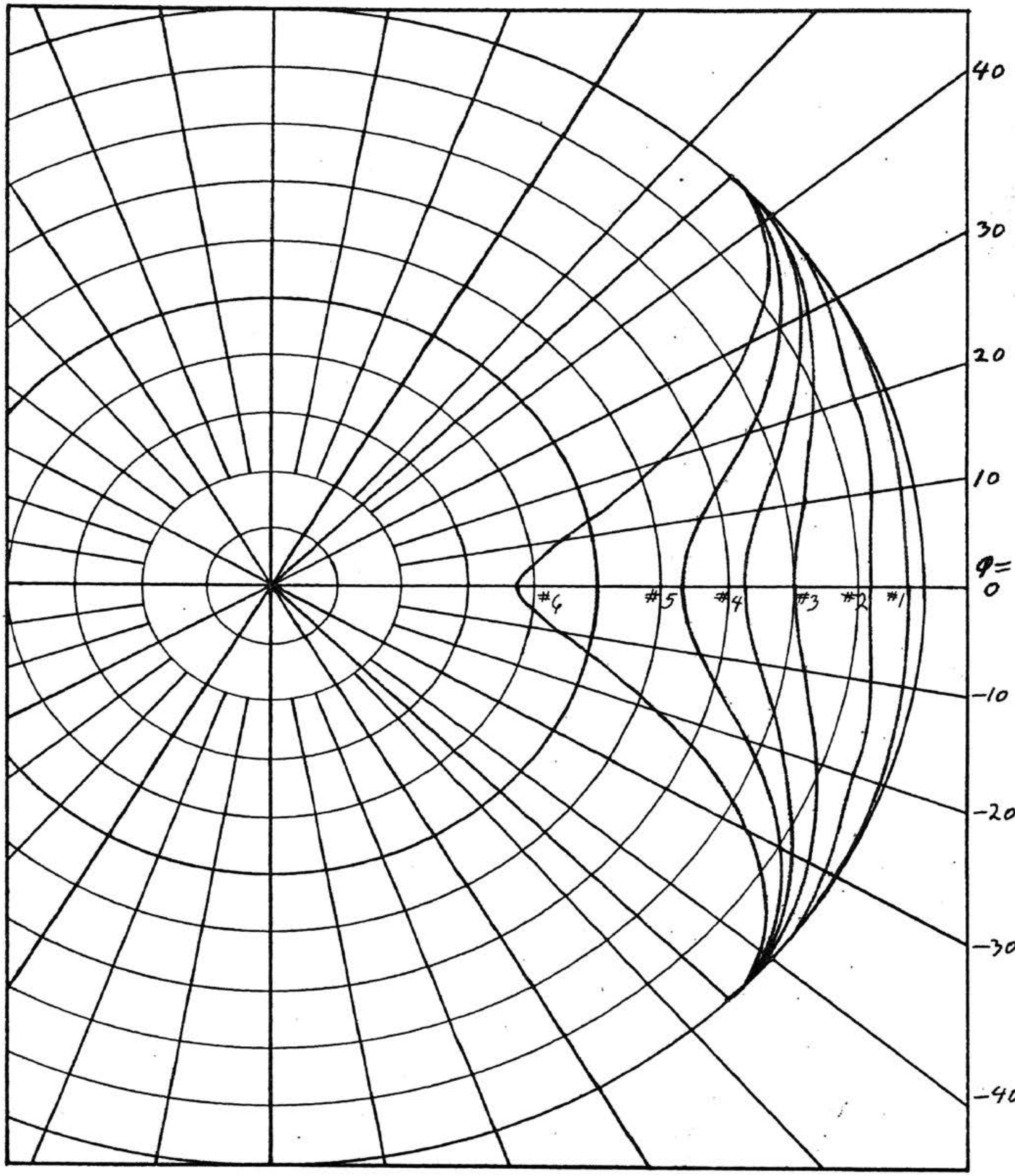

Figure 14. Field Patterns: Branch 2, Horizontal Plane, E_ϕ

#1	$\beta = 15°$	dr = .092λ	#4	$\beta = 52.7°$	dr = .309λ
#2	$\beta = 30°$	dr = .181λ	#5	$\beta = 60°$	dr = .348λ
#3	$\beta = 45°$	dr = .267λ	#6	$\beta = 75°$	dr = .425λ

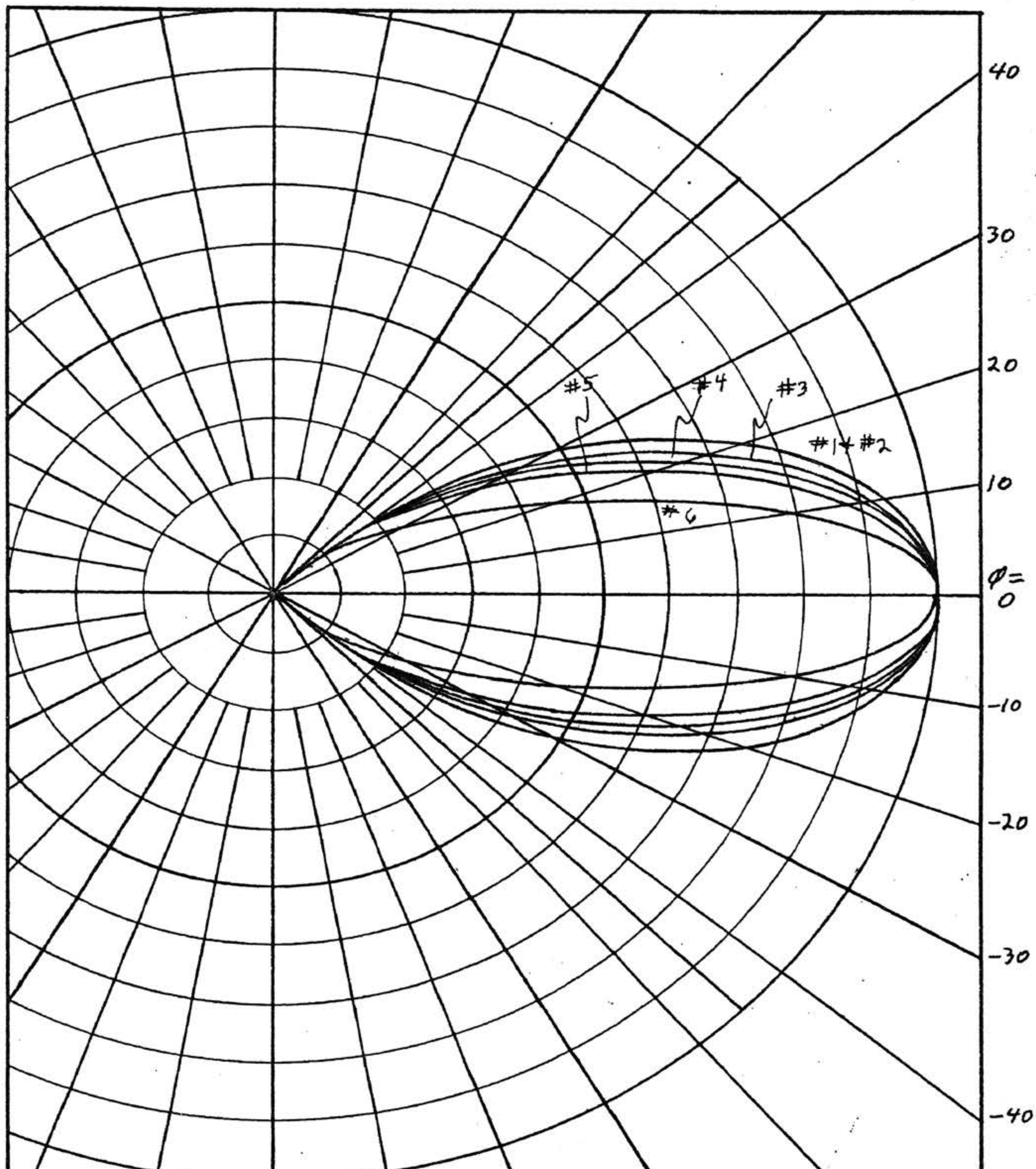

Figure 15. Field Patterns: Branch 2, Horizontal Plane, AR

#1	$\beta = 15°$	dr = .092λ	#4	$\beta = 52.7°$	dr = .309λ
#2	$\beta = 30°$	dr = .181λ	#5	$\beta = 60°$	dr = .348λ
#3	$\beta = 45°$	dr = .267λ	#6	$\beta = 75°$	dr = .425λ

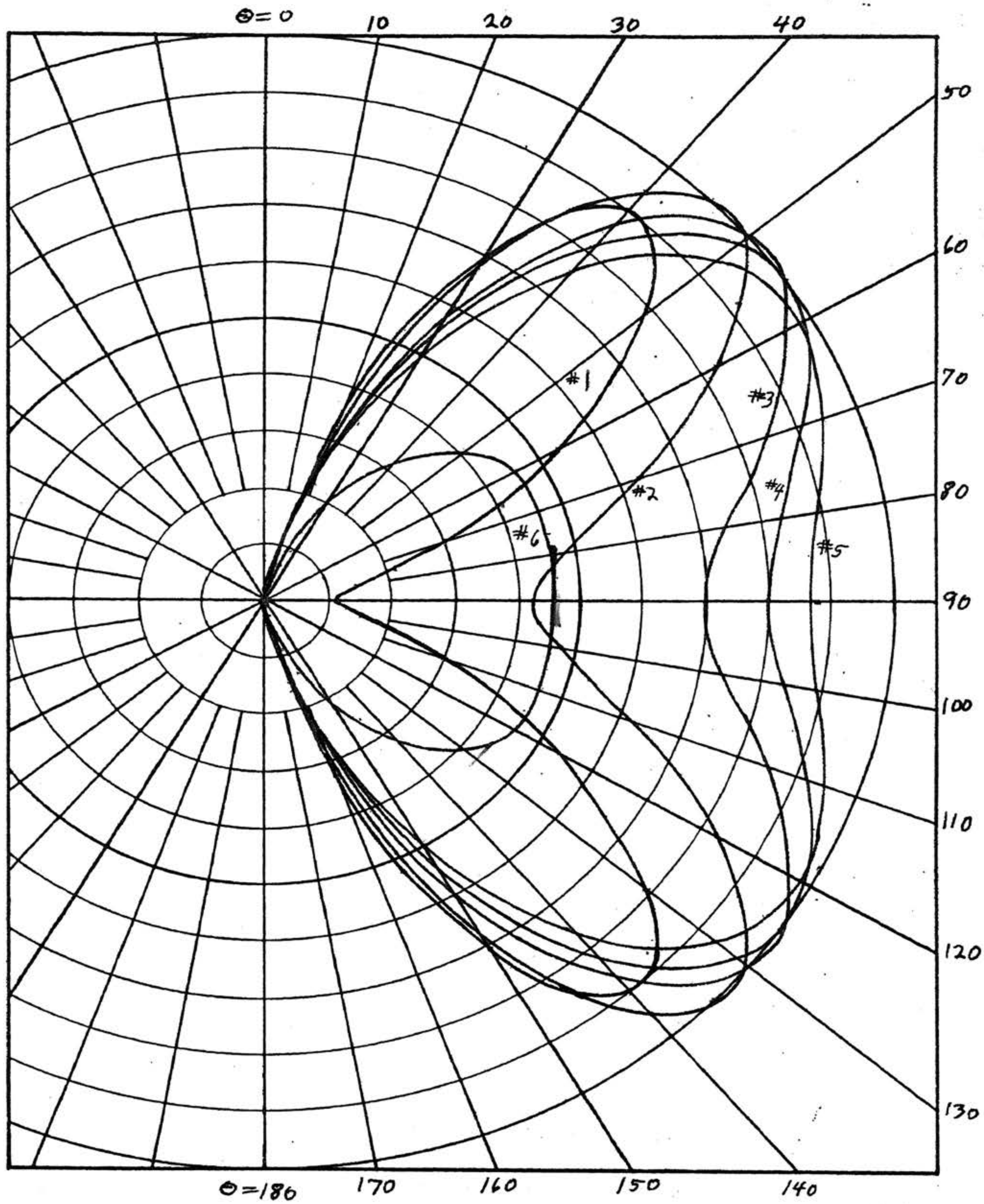

Figure 16. Field Patterns: Branch 3, Vertical Plane, E_θ

#1	$\beta = 15°$	dr = .908λ	#4	$\beta = 52.7°$	dr = .691λ
#2	$\beta = 30°$	dr = .819λ	#5	$\beta = 60°$	dr = .652λ
#3	$\beta = 45°$	dr = .733λ	#6	$\beta = 75°$	dr = .575λ

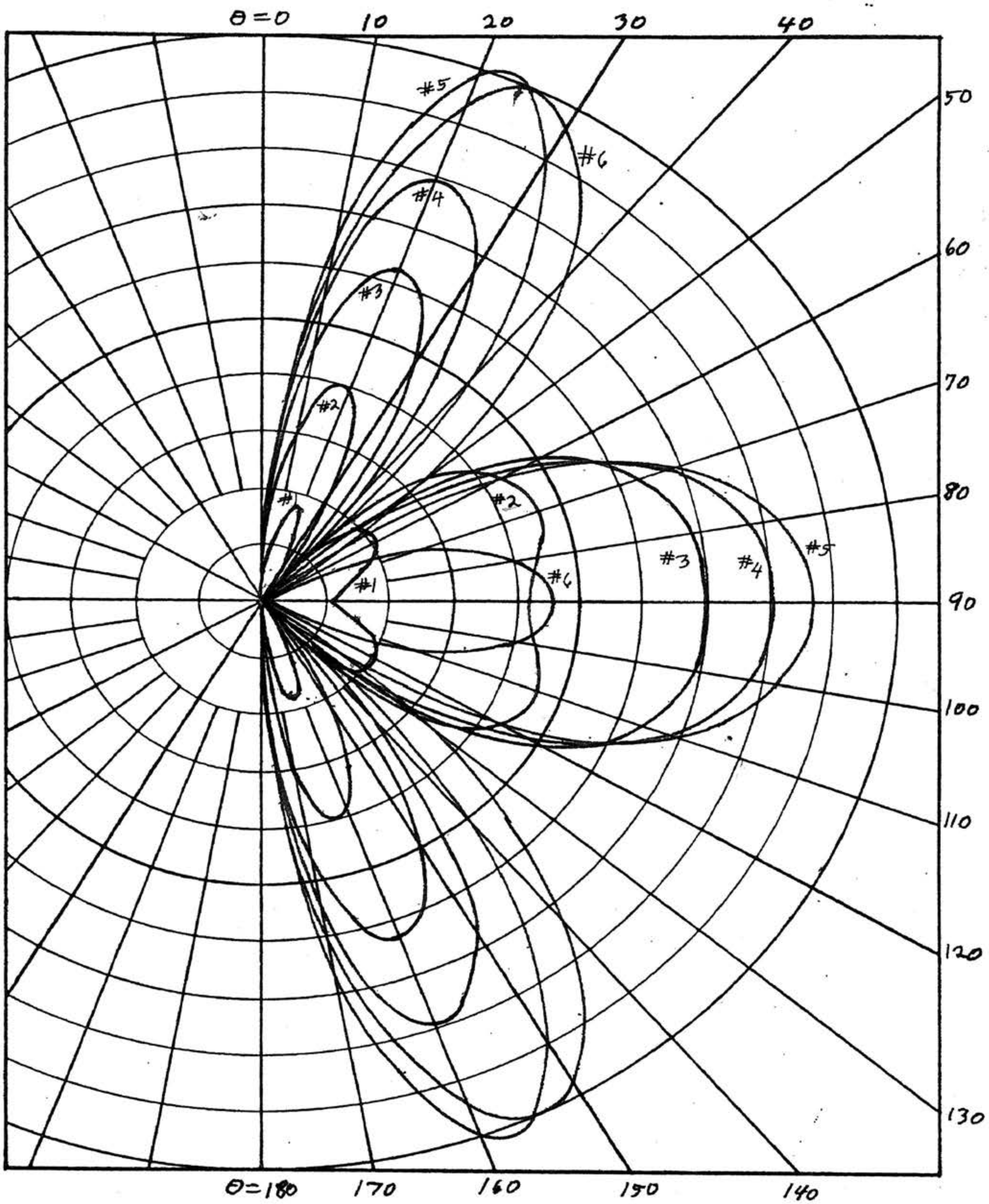

Figure 17. Field Patterns: Branch 3, Vertical Plane, E_ϕ

#1	$\beta = 15°$	$dr = .908\lambda$	#4	$\beta = 52.7°$	$dr = .691\lambda$
#2	$\beta = 30°$	$dr = .819\lambda$	#5	$\beta = 60°$	$dr = .652\lambda$
#3	$\beta = 45°$	$dr = .733\lambda$	#6	$\beta = 75°$	$dr = .575\lambda$

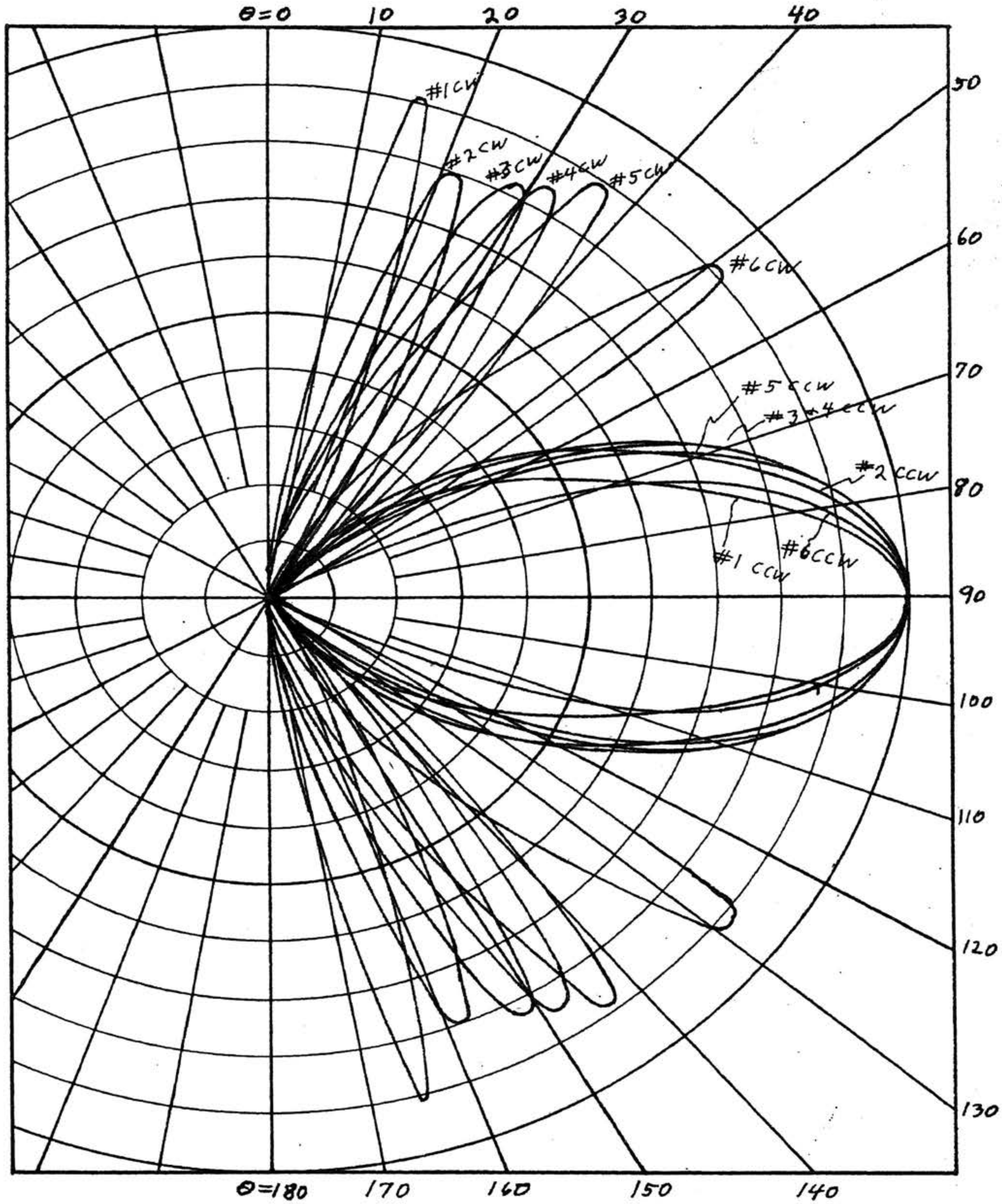

Figure 18. Field Patterns: Branch 3, Vertical Plane, AR

#1	β = 15°	dr = .908λ	#4	β = 52.7°	dr = .691λ
#2	β = 30°	dr = .819λ	#5	β = 60°	dr = .652λ
#3	β = 45°	dr = .733λ	#6	β = 75°	dr = .575λ

cw = clockwise
ccw = counterclockwise

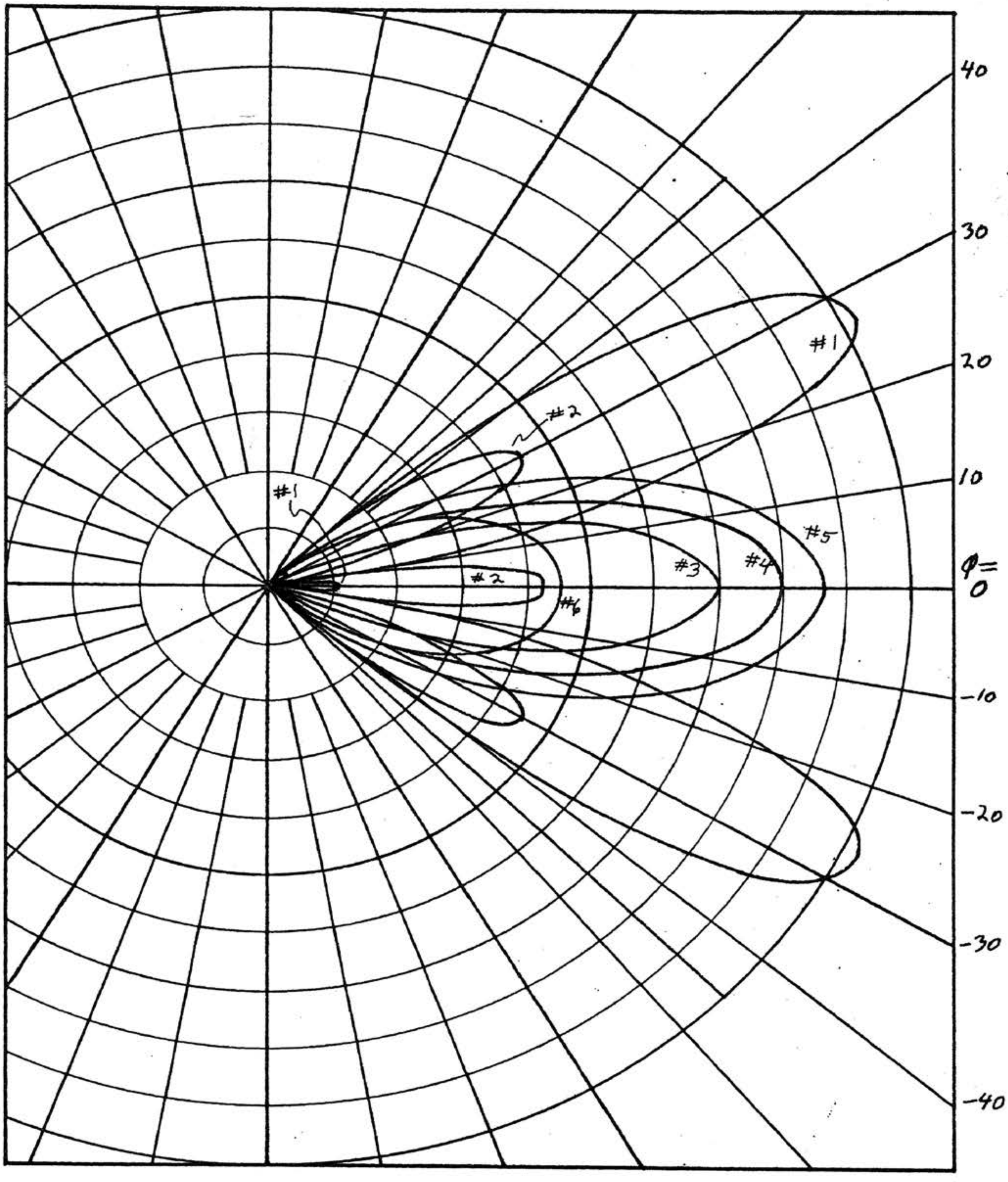

Figure 19. Field Patterns: Branch 3, Horizontal Plane, E_θ

#1	$\beta = 15°$	dr = .908λ	#4	$\beta = 52.7°$	dr = .691λ
#2	$\beta = 30°$	dr = .819λ	#5	$\beta = 60°$	dr = .652λ
#3	$\beta = 45°$	dr = .733λ	#6	$\beta = 75°$	dr = .575λ

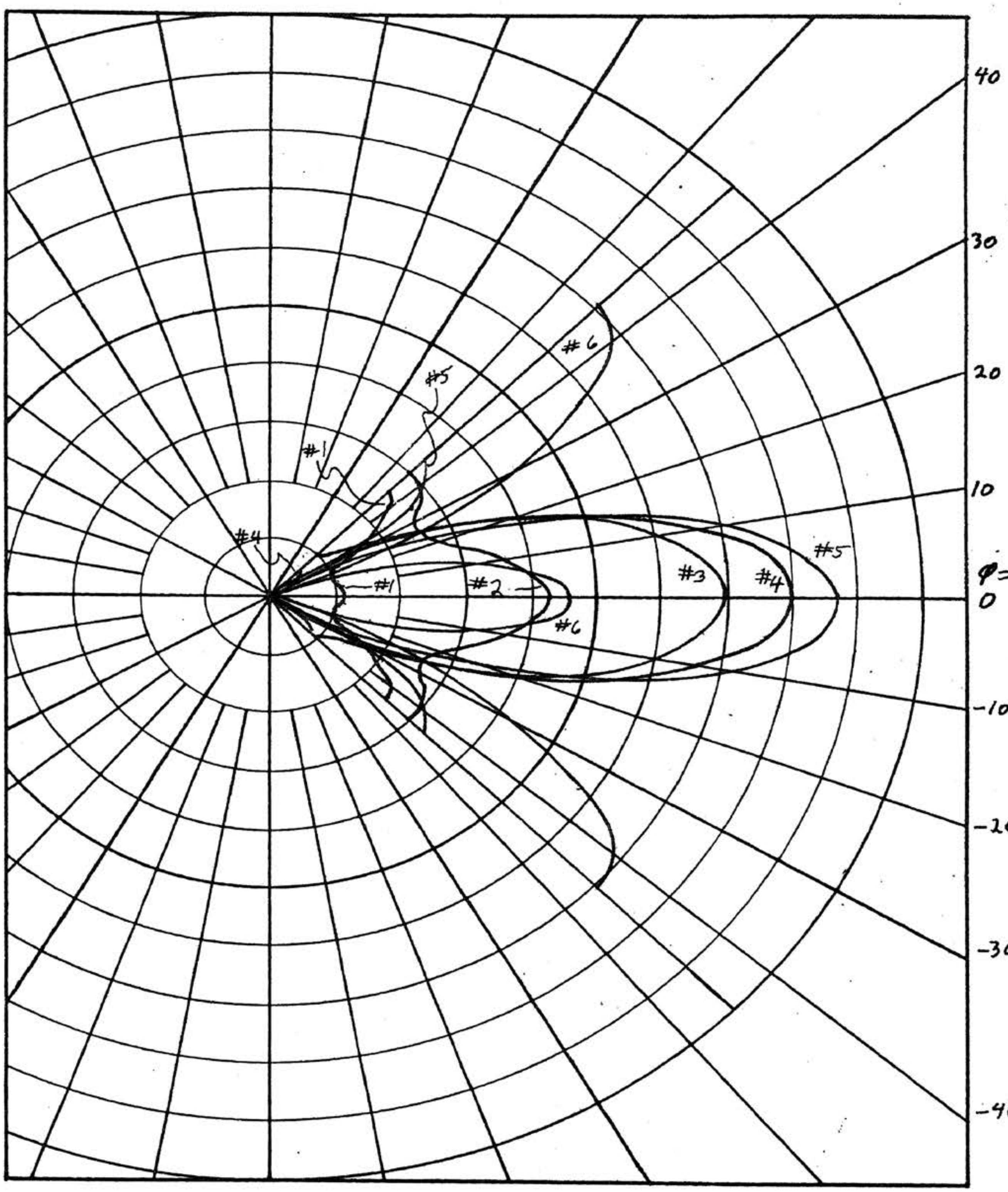

Figure 20. Field Patterns: Branch 3, Horizontal Plane, E_ϕ

#1	$\beta = 15°$	$dr = .908\lambda$	#4	$\beta = 52.7°$	$dr = .691\lambda$
#2	$\beta = 30°$	$dr = .819\lambda$	#5	$\beta = 60°$	$dr = :652\lambda$
#3	$\beta = 45°$	$dr = .733\lambda$	#6	$\beta = 75°$	$dr = .575\lambda$

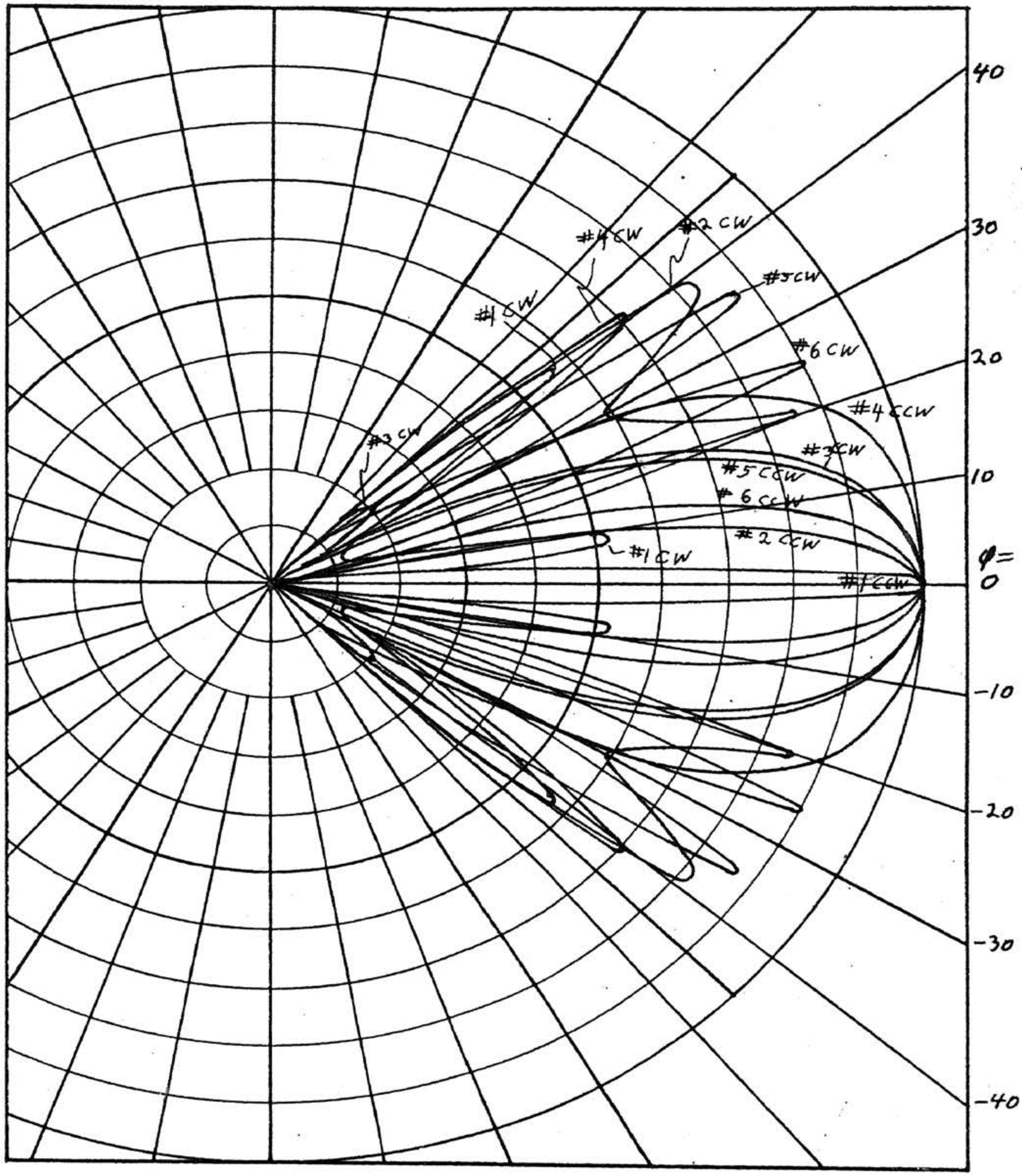

Figure 21. Field Patterns: Branch 3, Horizontal Plane, AR

#1	$\beta = 15°$	dr = .908λ	#4	$\beta = 52.7°$	dr = .691λ
#2	$\beta = 30°$	dr = .819λ	#5	$\beta = 60°$	dr = .652λ
#3	$\beta = 45°$	dr = .733λ	#6	$\beta = 75°$	dr = .575λ

cw = clockwise
ccw = counterclockwise

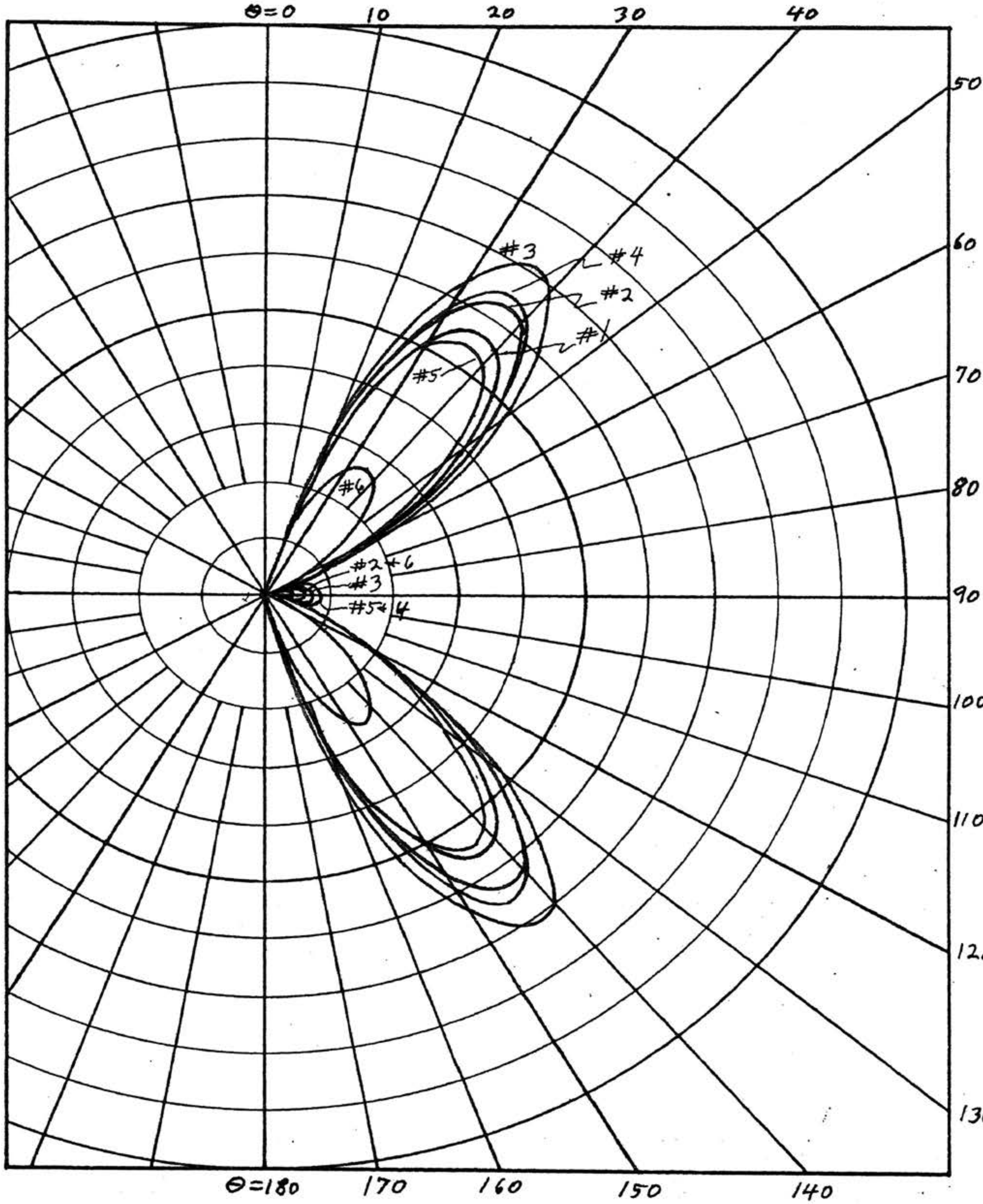

Figure 22. Field Patterns: Branch 4, Vertical Plane, E_θ

#1	$\beta = 15°$	dr = .992λ	#4	$\beta = 54.9°$	dr = .984λ
#2	$\beta = 30°$	dr = .985λ	#5	$\beta = 60°$	dr = .985λ
#3	$\beta = 45°$	dr = .983λ	#6	$\beta = 75°$	dr = .991λ

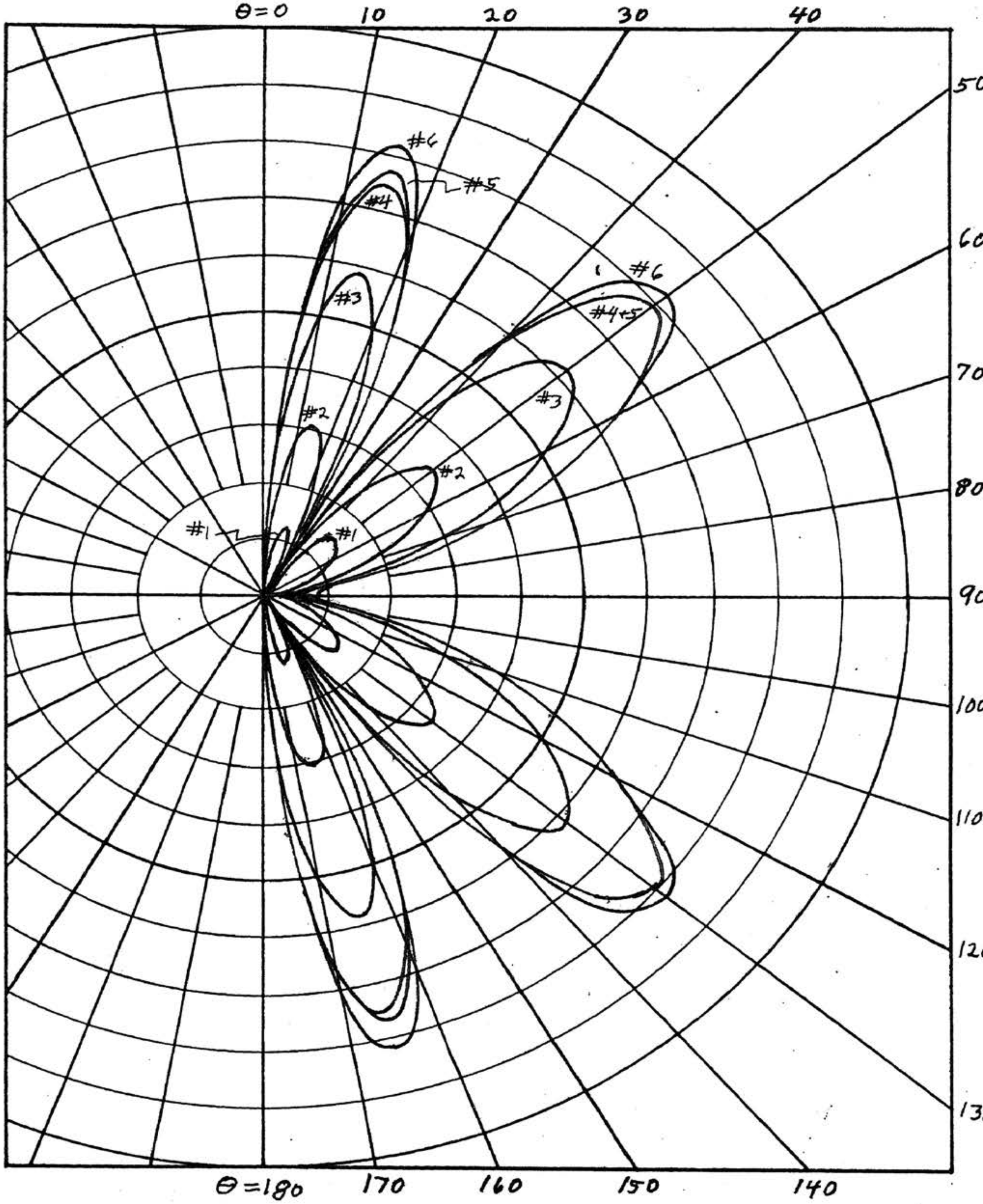

Figure 23. Field Patterns: Branch 4, Vertical Plane, E_ϕ

#1	$\beta = 15°$	dr = .992λ	#4	$\beta = 54.9°$	dr = .984λ
#2	$\beta = 30°$	dr = .985λ	#5	$\beta = 60°$	dr = .985λ
#3	$\beta = 45°$	dr = .983λ	#6	$\beta = 75°$	dr = .991λ

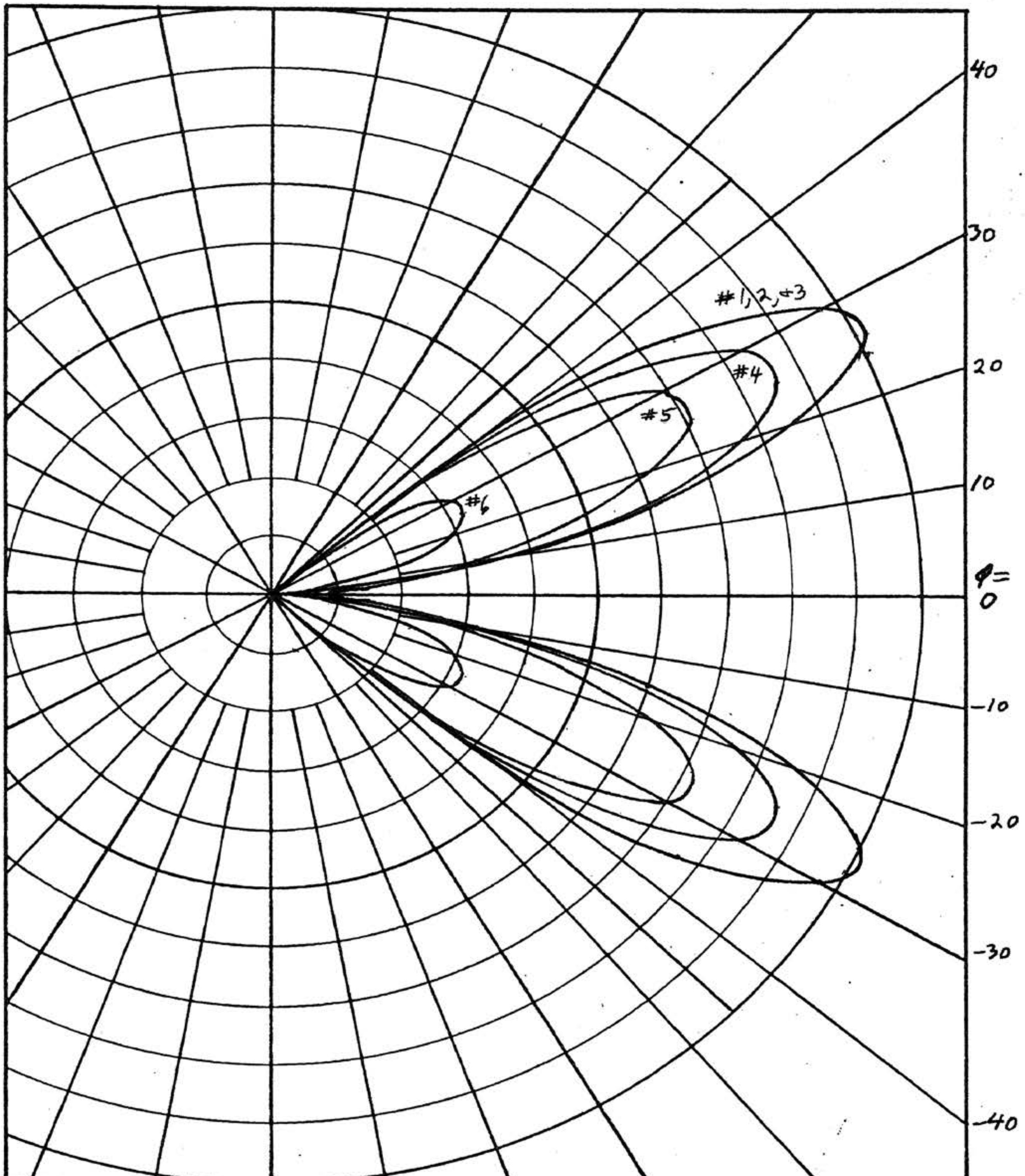

Figure 24. Field Patterns: Branch 4, Horizontal Plane, E_θ

#1	β = 15°	dr = .992λ	#4	β = 54.9°	dr = .984λ
#2	β = 30°	dr = .985λ	#5	β = 60°	dr = .985λ
#3	β = 45°	dr = .983λ	#6	β = 75°	dr = .991λ

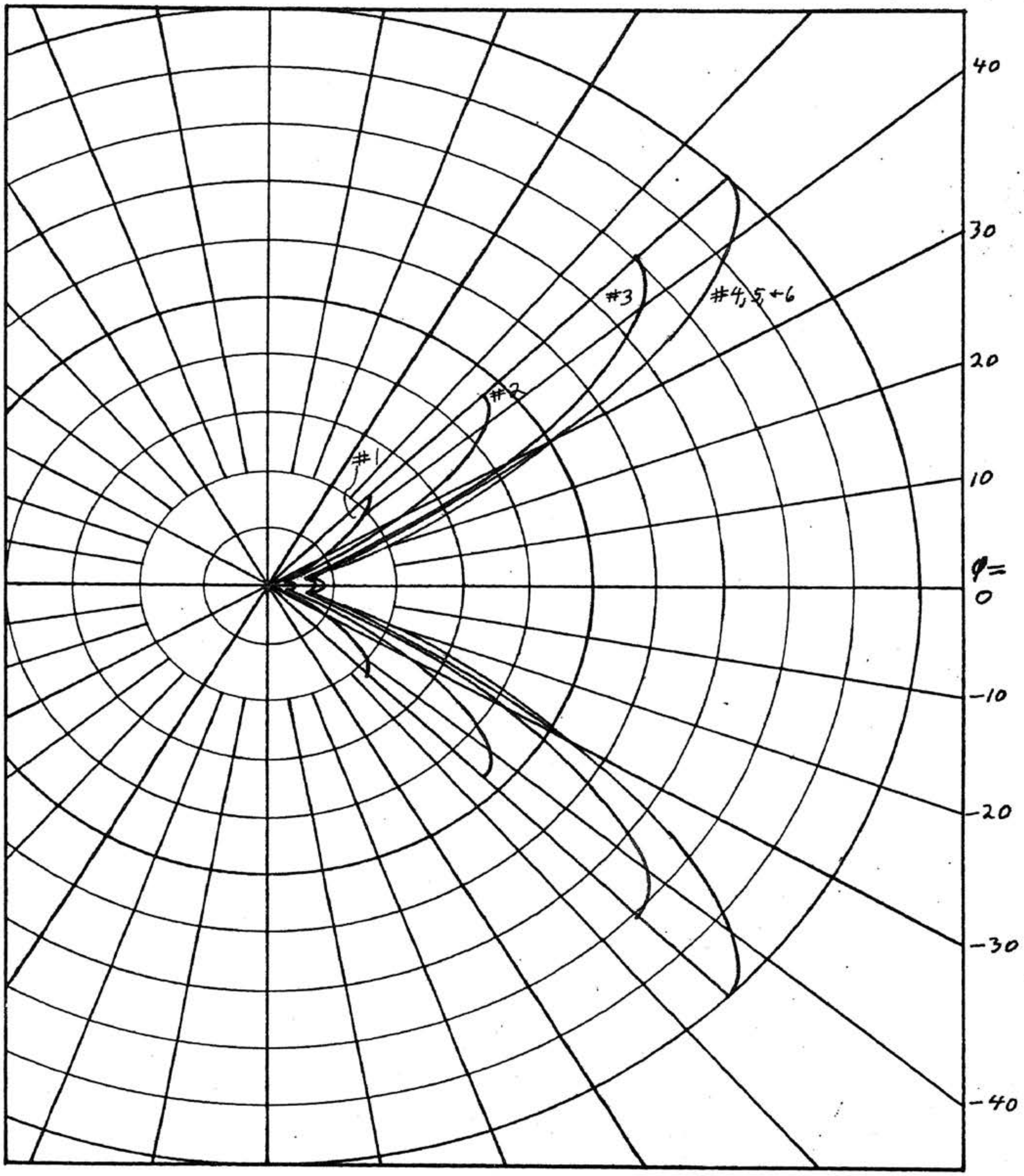

Figure 25. Field Patterns: Branch 4, Horizontal Plane, E_ϕ

#1	$\beta = 15°$	dr = .992λ	#4	$\beta = 54.9°$	dr = .984λ
#2	$\beta = 30°$	dr = .985λ	#5	$\beta = 60°$	dr = .985λ
#3	$\beta = 45°$	dr = .983λ	#6	$\beta = 75°$	dr = .991λ

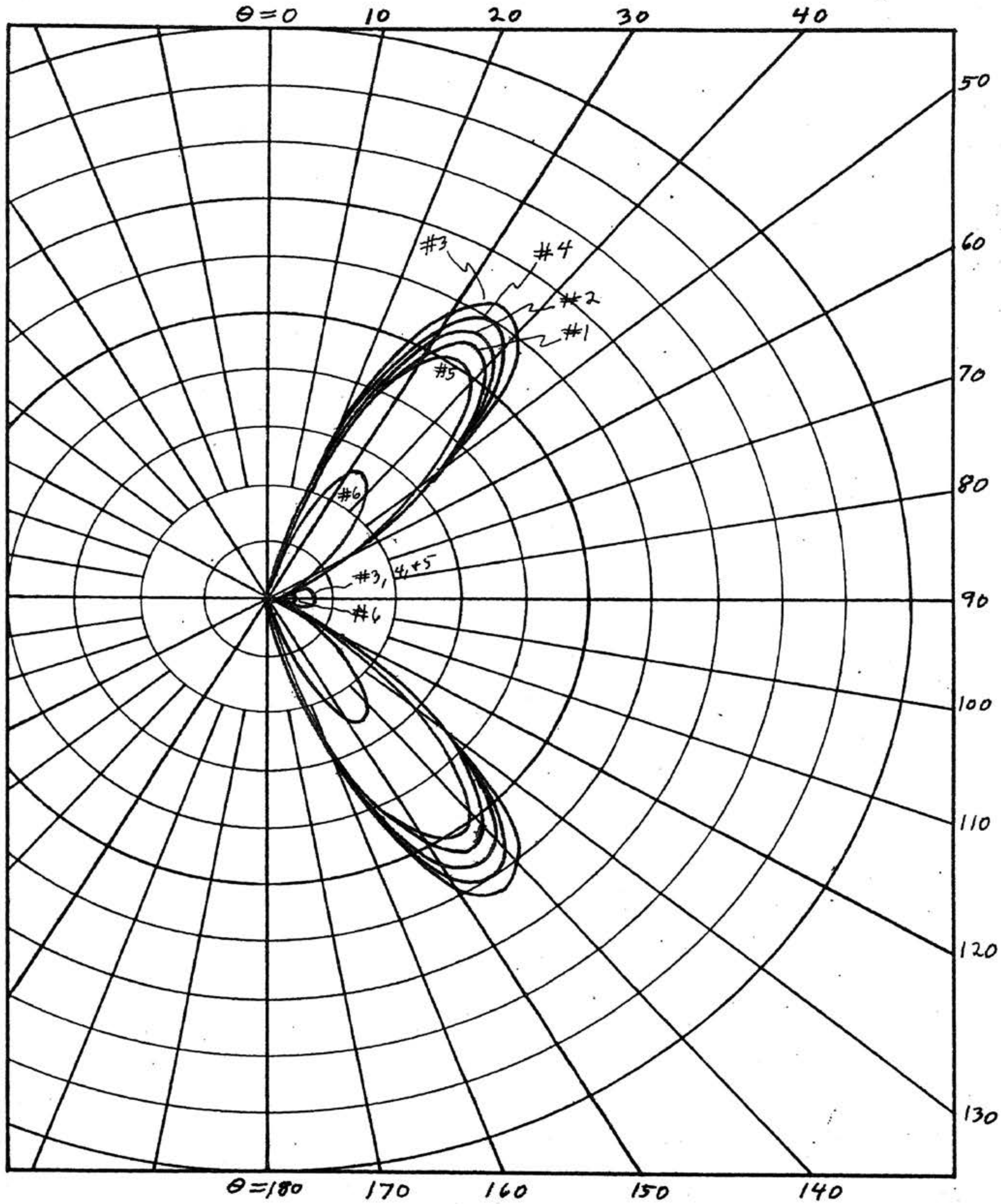

Figure 26. Field Patterns: Branch 5, Vertical Plane, E_θ

#1	β = 15°	dr = 1.008λ	#4	β = 54.9°	dr = 1.016λ
#2	β = 30°	dr = 1.015λ	#5	β = 60°	dr = 1.015λ
#3	β = 45°	dr = 1.017λ	#6	β = 75°	dr = 1.009λ

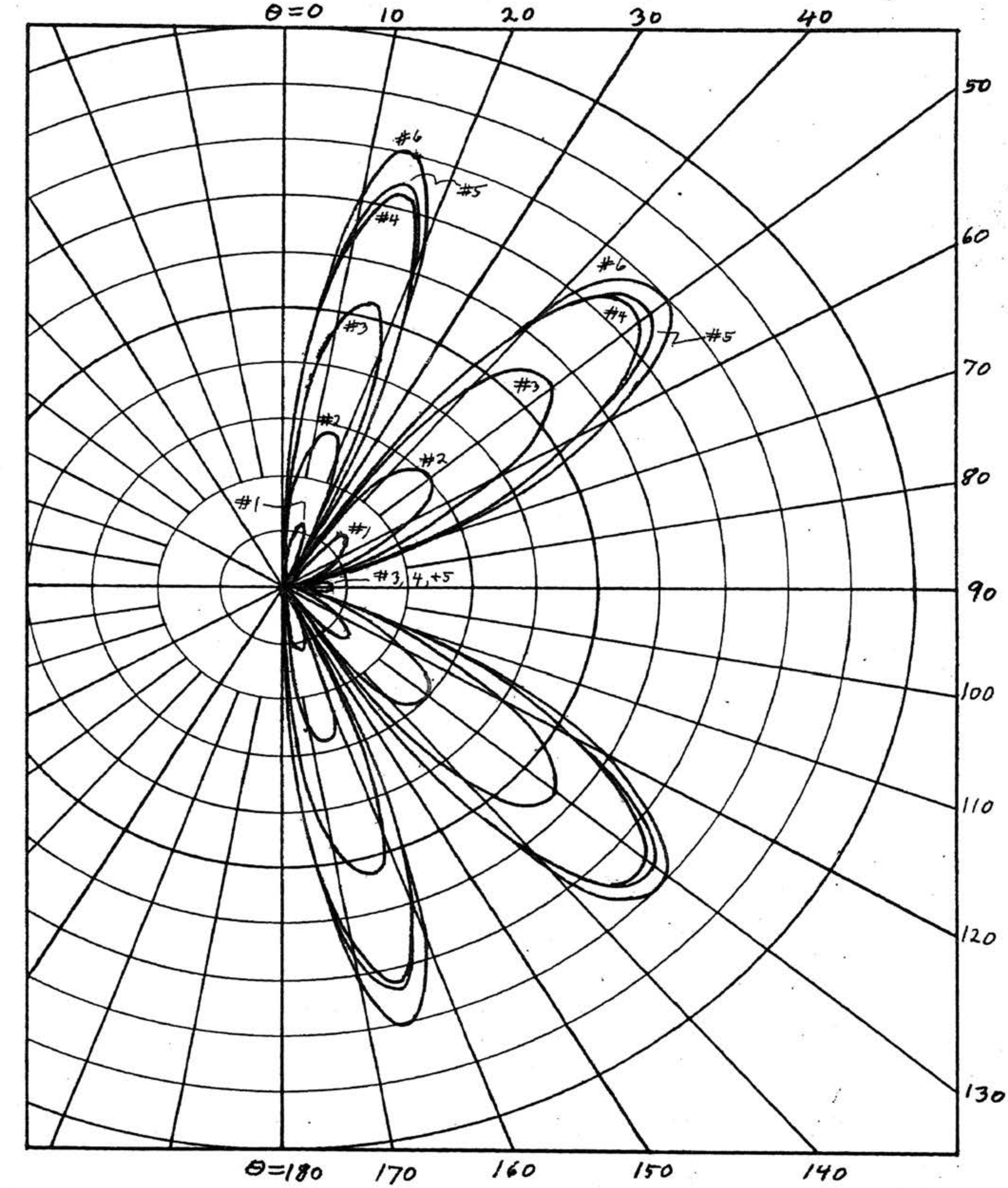

Figure 27. Field Patterns: Branch 5, Vertical Plane, E_ϕ

#1	$\beta = 15°$	$dr = 1.008\lambda$	#4	$\beta = 54.9°$	$dr = 1.016\lambda$
#2	$\beta = 30°$	$dr = 1.015\lambda$	#5	$\beta = 60°$	$dr = 1.015\lambda$
#3	$\beta = 45°$	$dr = 1.017\lambda$	#6	$\beta = 75°$	$dr = 1.009\lambda$

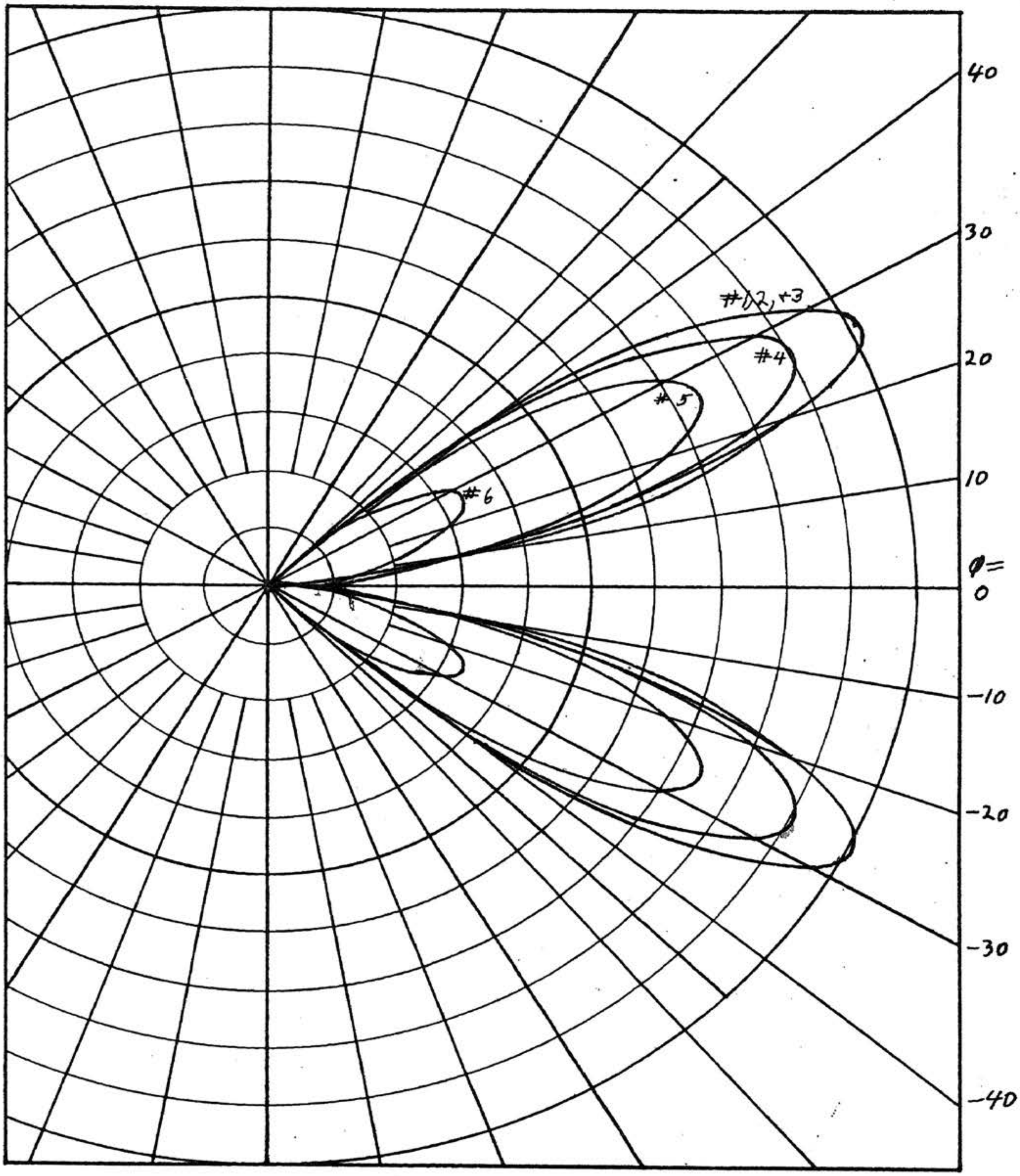

Figure 28. Field Patterns: Branch 5, Horizontal Plane, E_θ

#1	$\beta = 15°$	$dr = 1.008\lambda$	#4	$\beta = 54.9°$	$dr = 1.016\lambda$
#2	$\beta = 30°$	$dr = 1.015\lambda$	#5	$\beta = 60°$	$dr = 1.015\lambda$
#3	$\beta = 45°$	$dr = 1.017\lambda$	#6	$\beta = 75°$	$dr = 1.009\lambda$

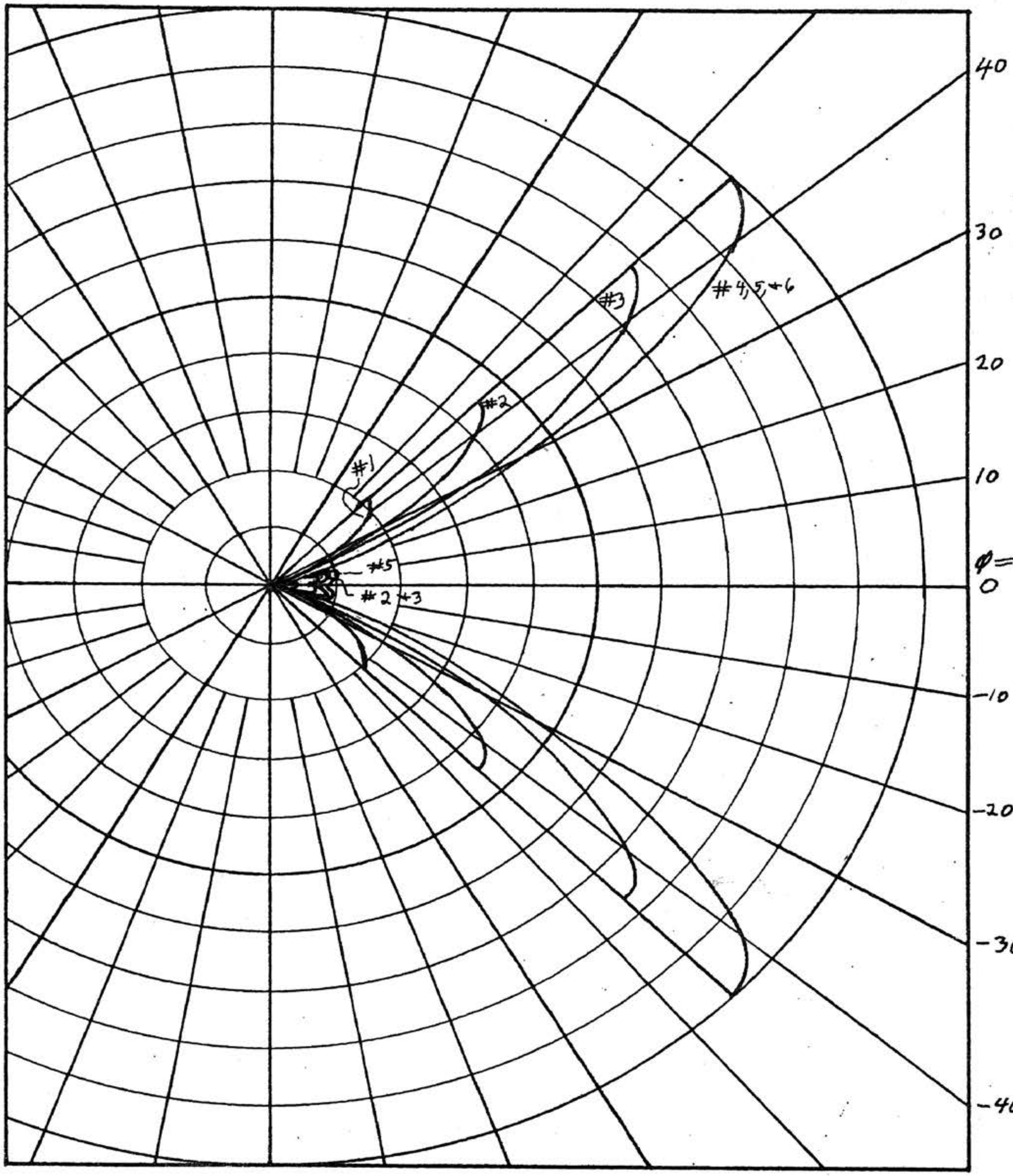

Figure 29. Field Patterns: Branch 5, Horizontal Plane, E_ϕ

#1	β = 15°	dr = 1.008λ	#4	β = 54.9°	dr = 1.016λ
#2	β = 30°	dr = 1.015λ	#5	β = 60°	dr = 1.015λ
#3	β = 45°	dr = 1.017λ	#6	β = 75°	dr = 1.009λ

VI. CONCLUSIONS

A. Results

Image theory was used to derive the far field equations for a ninety degree corner reflector antenna. The equations were in terms of the spherical coordinates of the point of observation, the angle of tilt of the dipole element with respect to the axis of the corner, the distance of the element from the corner, and the length of the dipole. The instantaneous forms of these equations were solved simultaneously to derive the equation of an ellipse. The axial ratio and the magnitude of the normal linear components of the elliptical wave were obtained from the elliptical equation.

These equations were used to investigate the far field of a corner reflector antenna with a dipole length equal to a half-wave. It was found that for any given angle of tilt, circular polarization may be obtained at four different distances between zero and one wavelength of the dipole from the corner. A plot of these distances is symmetric about the distance of one half wavelength and recurs with a period of one wavelength.

The fields in the vertical and horizontal planes through the bore of the antenna were investigated for four dipole distances at fifteen degree intervals of dipole tilt. It was found that the best patterns occurred when the dipole element was near the corner. As the dipole was moved away, high side lobes developed and completely dominated the main lobe.

B. Suggestions for Further Development

There are several important areas of work on the corner reflector antenna that remain to be done. An investigation of the far field patterns for dipole lengths greater than one half wavelength might give interesting results. The radiation resistance and input impedance are other important quantities that should be known. A comparison of this antenna with other circularly polarized antennas would show advantages and disadvantages of this method of obtaining circular polarization. Thus gain plots relative to, for example, a turnstile antenna or an isotropic source would be useful. Then lastly, a theory that takes into account the effects of the finite size reflector of the actual physical situation rather than the ideal infinite reflector would be helpful.

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APPENDIX A

DERIVATION OF THE FIELD EQUATIONS FOR A CORNER REFLECTOR ANTENNA

On page 11 it was shown that the far field of the corner reflector antenna is equal to the sum of the field from the real element and the three images, or

$$\dot{\bar{E}} = \sum_{n=1}^{4} \dot{\bar{E}}_n \tag{1A}$$

where $\dot{\bar{E}}_n = \dot{\bar{E}}_n(\theta, \phi, \beta, \bar{a}_\theta, \bar{a}_\phi) = \dot{E}_{\theta n}(\theta_n)\,\bar{a}_{\theta_n}$

The pattern factor for the nth dipole in a coordinate system in which the θ=0 axis is along the dipole is

$$E_n = \left[\frac{\cos\left(\frac{L}{2}\cos\theta_n\right) - \cos\left(\frac{L}{2}\right)}{\sin\theta_n}\right]\bar{a}_{\theta_n} \tag{2A}$$

This must be rewritten in terms of the array coordinates as shown in figure A-1.

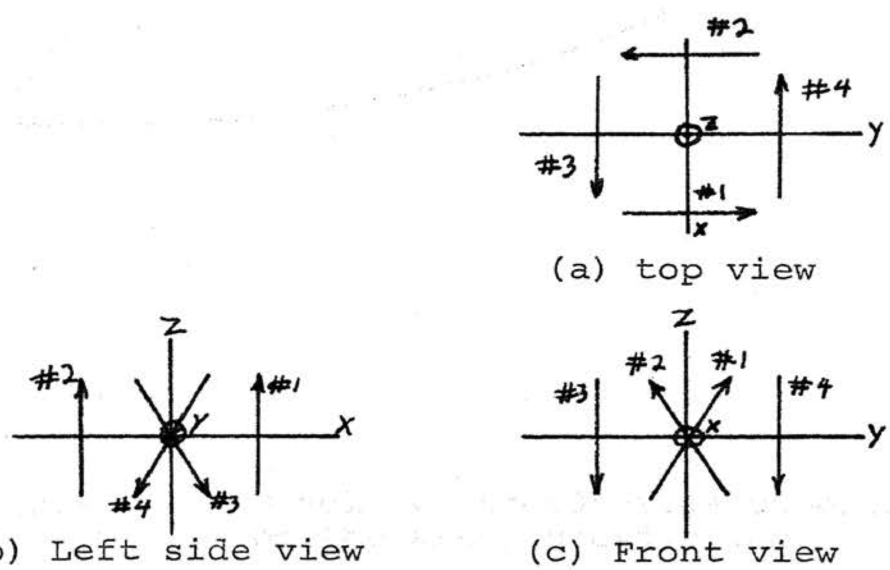


Figure A-1. Front, top, and side views of the array

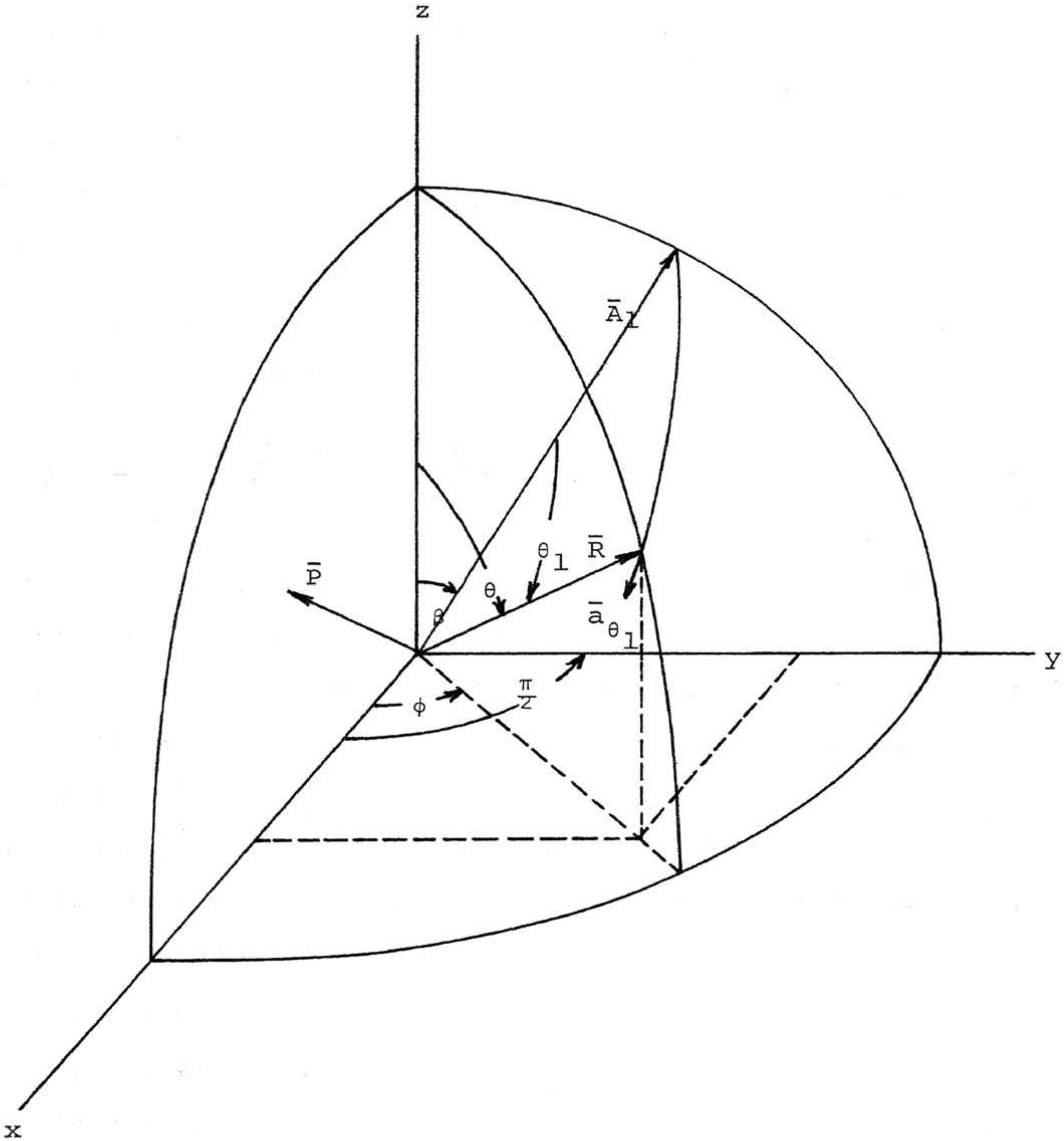


Figure A-2. Dipole #1 coordinates superimposed on the array coordinates

The first step in determining the general field equation is to determine the complex field of the real dipole. First the field of the element will be determined as if it were at the origin, then the effect of the displacement of the element from the origin on the complex field will be considered.

The electric field for dipole #1 in terms of its own coordinate system is

$$E_{\theta_1} = \left[\frac{\cos\left(\frac{L}{2}\cos\theta_1\right) - \left(\frac{L}{2}\right)}{\sin\theta_1} \right] = \bar{E}_1(\theta, \phi, \beta, \bar{a}_\theta, \bar{a}_\phi) \tag{3A}$$

This system together with the array coordinates is shown in figure A-2. $\bar{R}$ is the position vector to some point in space, $\bar{A}_1$ is in the direction of the dipole current and has the same length as $\bar{R}$, and $\bar{P}$ is a vector perpendicular to the plane of $\bar{A}_1$ and $\bar{R}$. The quantities $\bar{a}_{\theta_1}$ and θ_1 must be found in terms of θ, ϕ, β, $\bar{a}_\theta$, and $\bar{a}_\phi$. The following transformation equations from rectangular to spherical coordinates will be useful.

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

$$\begin{aligned} \bar{a}_x &= \bar{a}_r \sin\theta\cos\phi + \bar{a}_\theta \cos\theta\cos\phi - \bar{a}_\phi \sin\phi \\ \bar{a}_y &= \bar{a}_r \sin\theta\sin\phi + \bar{a}_\theta \cos\theta\sin\phi + \bar{a}_\phi \cos\phi \\ \bar{a}_z &= \bar{a}_r \cos\theta - \bar{a}_\theta \sin\theta \end{aligned} \tag{4A}$$

The vector $\bar{R}$ may be written

$$\bar{R} = \bar{a}_r r = \bar{a}_x x + \bar{a}_y y + \bar{a}_z z$$

$$= \bar{a}_x r \sin\theta \cos\phi + \bar{a}_y r \sin\theta \sin\phi + \bar{a}_z r \cos\theta \tag{5A}$$

Likewise vector $\bar{A}_1$ may be written

$$\bar{A}_1 = \bar{a}_r r = \bar{a}_x x + \bar{a}_y y + \bar{a}_z z$$

$$= \bar{a}_x r \sin\theta \cos\phi + \bar{a}_y r \sin\theta \sin\phi + \bar{a}_z r \cos\theta \tag{6A}$$

But $\bar{A}_1$ is a specific vector with coordinates $\theta=\beta$ and $\phi=\frac{\pi}{2}$. Thus

$$\bar{A}_1 = \bar{a}_x r \sin\beta \cos\tfrac{\pi}{2} + \bar{a}_y r \sin\beta \sin\tfrac{\pi}{2} + \bar{a}_z r \cos\beta$$

$$= \bar{a}_y r \sin\beta + \bar{a}_z r \cos\beta \tag{7A}$$

Now the vector dot product of $\bar{R}$ and $\bar{A}_1$ is

$$\bar{R}\cdot\bar{A}_1 = R A_1 \cos\theta_1$$

or $$\cos\theta_1 = \frac{\bar{R}\cdot\bar{A}_1}{RA_1} = \frac{r^2 \sin\theta \sin\phi \sin\beta + r^2 \cos\theta \cos\beta}{r^2}$$

$$= \sin\theta \sin\phi \sin\beta + \cos\theta \cos\beta \tag{8A}$$

And since $\sin^2\theta_1 + \cos^2\theta_1 = 1$

$$\sin\theta_1 = \left[1 - (\sin\theta \sin\phi \sin\beta + \cos\theta \cos\beta)^2\right]^{\frac{1}{2}} \tag{9A}$$

Let vector $\bar{P}$ be the vector cross product of $\bar{R}$ and $\bar{A}_1$, that is, $\bar{P} = \bar{R} \times \bar{A}_1$. Then $\bar{P}$ is normal to the plane of $\bar{R}$ and $\bar{A}_1$ and thus perpendicular to every vector in that plane. The unit vector $\bar{a}_{\theta_1}$ is normal to $\bar{R}$, but since it is in the plane of $\bar{R}$ and $\bar{A}_1$ it is also normal to $\bar{P}$. Hence

$$\bar{R} \times \bar{P} = RP \sin\text{ (angle between } \bar{R} \text{ and } \bar{P}) \ \bar{a}_{\theta_1} \tag{10A}$$

And since the angle between $\bar{R}$ and $\bar{P}$ is $\frac{\pi}{2}$

$$\bar{a}_{\theta_1} = \frac{\bar{R} \times \bar{P}}{R\,P} = \frac{\bar{R} \times (\bar{R} \times \bar{A}_1)}{R\,(\bar{R} \times \bar{A}_1)} \tag{11A}$$

Making use of the vector identity

$$\bar{A} \times (\bar{B} \times \bar{C}) = (\bar{A} \cdot \bar{C})\bar{B} - (\bar{A} \cdot \bar{B})\bar{C} \tag{12A}$$

results in

$$\bar{a}_{\theta_1} = \frac{(\bar{R} \cdot \bar{A}_1)\bar{R} - (\bar{R} \cdot \bar{R})\bar{A}_1}{R\,R\,A_1 \sin\theta_1}$$

$$= \frac{(r^2 \sin\theta \sin\phi \sin\beta + r^2 \cos\theta \cos\beta) r\,\bar{a}_r - r^2(\bar{a}_y r \sin\beta + \bar{a}_z r \cos\beta)}{r^3 \sin\theta_1}$$

Substitution of $\bar{a}_y$ and $\bar{a}_z$ from equations (4A) and $\sin\theta_1$ from (9A) gives

$$\bar{a}_{\theta_1} = \frac{\bar{a}_\theta (\sin\theta \cos\beta - \cos\theta \sin\phi \sin\beta) - \bar{a}_\phi \cos\phi \sin\beta}{[1 - (\sin\theta \sin\phi \sin\beta + \cos\theta \cos\beta)^2]^{\frac{1}{2}}} \tag{13A}$$

Equations (8A), (9A), and (13A) are the desired results.

The complex electric field is a function of time t, frequency ω, and phase Ψ. If Ψ_1 is the difference in phase between the dipole and a reference at the origin, the complex field is

$$\dot{E}_1 = E_{\theta_1} \bar{a}_{\theta_1} e^{j(\omega t + \Psi_1)} \tag{14A}$$

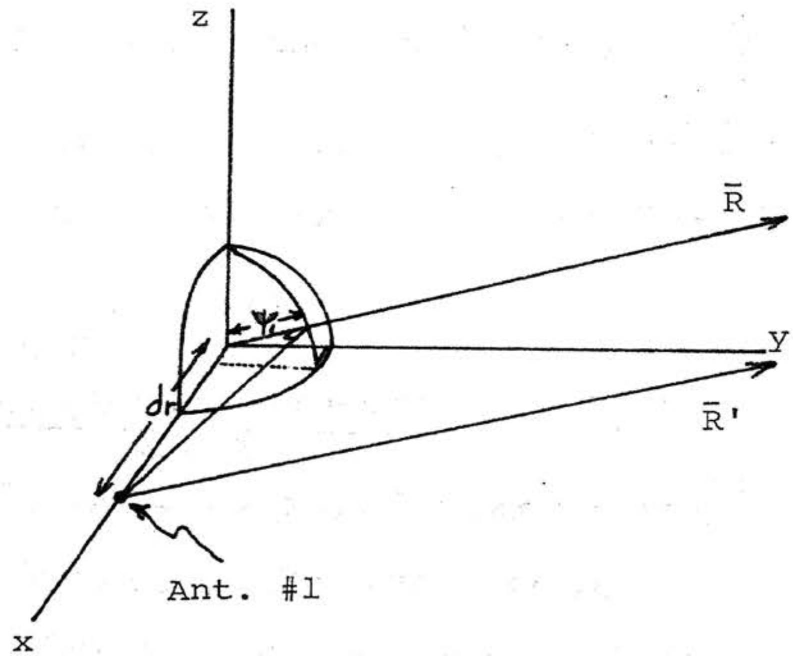


Figure A-3. Difference in path between the origin and the center of dipole #1 to a point in space

In figure A-3, R and R' are the distances from the origin and dipole #1 respectively to a distant point in space. The point is assumed to be far enough away that $\bar{R}$ and $\bar{R}'$ may be considered parallel. Taking the origin as reference, the path from the dipole to the distant point is less than the path from the origin by an amount Ψ_1. Figure A-3 shows that Ψ_1 is the length of the projection of the directed line segment $\bar{d}r$ from the origin onto the path $\bar{R}$. Both dr and Ψ_1 are in terms of radians, or distance divided by wavelengths.

$$\Psi_1 = \frac{\bar{d}_r \cdot \bar{R}}{R} = \frac{d_r \bar{a}_x \cdot (\bar{a}_x X + \bar{a}_y y + \bar{a}_z Z)}{r}$$

$$= \frac{d_r X}{r} = d_r \sin\theta \cos\phi \qquad (15A)$$

The phase leads the reference and so is positive.

Equations 3A, 8A, 9A, 13A, 14A, and 15A together give the far field pattern of dipole #1. The complex field is

$$\dot{\bar{E}}_1 = E_{\theta_1} \bar{a}_{\theta_1} e^{j(\omega t + \psi_1)} \tag{16A}$$

where $E_{\theta_1} = \dfrac{\cos\left(\frac{L}{2}\cos\theta_1\right) - \cos\left(\frac{L}{2}\right)}{\sin\ \theta_1}$

$$\sin\theta_1 = \left[1 - (\sin\theta\ \sin\phi\ \sin\beta + \cos\theta\ \cos\beta)^2\right]^{\frac{1}{2}}$$

$$\cos\theta_1 = (\sin\theta\ \sin\phi\ \sin\beta + \cos\theta\ \cos\beta)$$

$$\bar{a}_{\theta_1} = \frac{\bar{a}_\theta (\sin\theta\ \cos\beta - \cos\theta\ \sin\phi\ \sin\beta) - \bar{a}_\phi \cos\phi\ \sin\beta}{\left[1 - (\sin\theta\ \sin\phi\ \sin\beta + \cos\theta\ \cos\beta)^2\right]^{\frac{1}{2}}}$$

$$\psi_1 = d_r\ \sin\theta\ \cos\phi$$

Combining equations (16A) results in

$$\dot{\bar{E}} = \frac{\cos\left[\frac{L}{2}(\sin\theta\sin\phi\ \sin\beta + \cos\theta\ \cos\beta)\right] - \cos\left(\frac{L}{2}\right)}{1 - (\sin\theta\ \sin\phi\ \sin\beta + \cos\phi\ \cos\beta)^2}$$

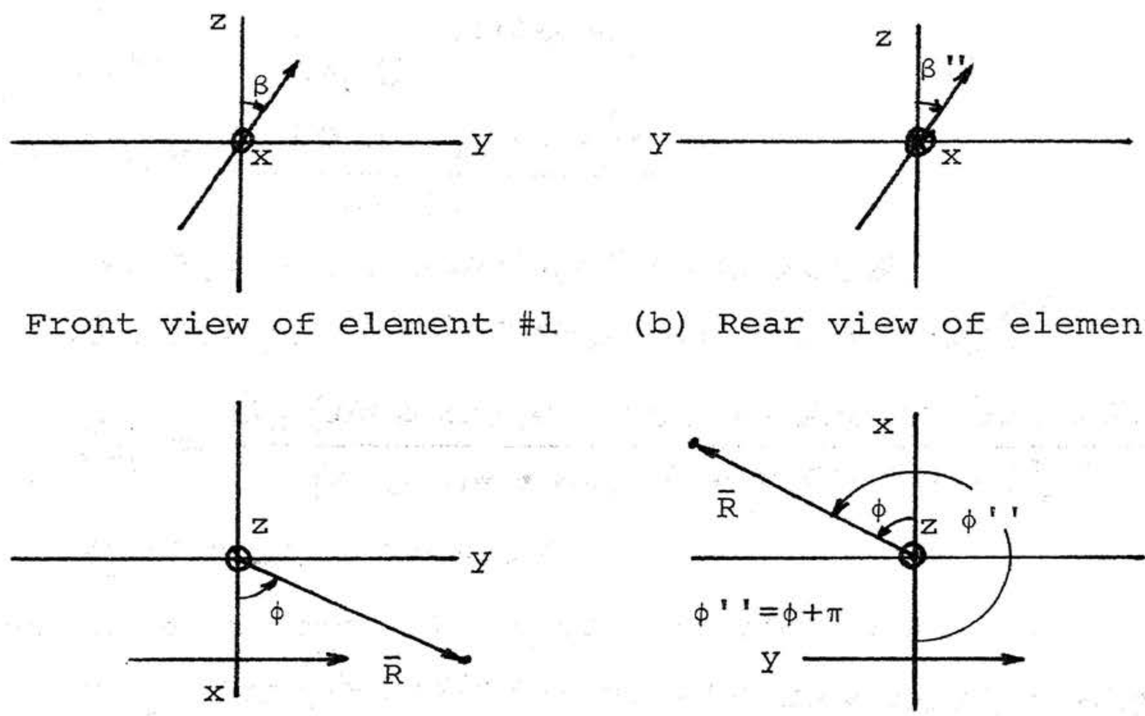
$$\times \left[\bar{a}_\theta (\sin\theta\ \cos\beta - \cos\theta\ \sin\phi\ \sin\beta\right.$$

$$\left. - \bar{a}_\phi \cos\phi\ \sin\beta\right] e^{j(\omega t + d_r \sin\theta\ \cos\phi} \tag{17A}$$

The field equation of the real element may be used to obtain the fields of the image elements if a suitable change in variables is made.

The development of the field equation for element #2 is aided by comparing sketches of various views of elements #1 and #2. Denote the variables θ, ϕ, and β for element #2 by a double prime.

Figure 4 shows that the rear and top vies of element #2 are identical to the front and top views of element #1.



(a) Front view of element #1 (b) Rear view of element #2

(c) Top view of element #1 (d) Top view of element #2

Figure A-4. Sketches comparing views of elements #1 and #2

Thus equations (16A) and (17A) hold for element #2 if the zero azimuth angle ($\phi''=0$) is taken to be the negative x-axis. Part (d) of figure A-4 shows that to change to the coordinate system of the real element, merely set $\phi''=\phi+\pi$. Then the following relationships are substituted into equations (16A):

$$\sin \phi'' = \sin (\phi+\pi) = - \sin \phi$$

$$\cos \phi'' = \cos (\phi+\pi) = - \cos \phi$$

$$\beta'' = \beta$$

$$\theta'' = \theta \tag{18A}$$

These substitutions together with the identiy cos $(-\theta)$ $= \cos\theta$ give the results

$$\dot{\bar{E}} = E_{\theta_2}\,\bar{a}_{\theta_2}\,e^{j(\omega t + \Psi_2)} \tag{19A}$$

where $E_{\theta_2} = \dfrac{\cos\left(\frac{L}{2}\cos\theta_2\right) - \cos\left(\frac{L}{2}\right)}{\sin\theta_2}$

$$\cos\theta_2 = -\sin\theta\sin\phi\sin\beta + \cos\theta\cos\beta$$

$$\sin\theta_2 = \left[1 - (-\sin\theta\sin\phi\sin\beta + \cos\theta\cos\beta)^2\right]^{\frac{1}{2}}$$

$$\bar{a}_{\theta_2} = \frac{\bar{a}_\theta(\sin\theta\cos\beta + \cos\theta\sin\phi\sin\beta) + \bar{a}_\phi\cos\phi\sin\beta}{\left[1-(-\sin\theta\sin\phi\sin\beta + \cos\theta\cos\beta)^2\right]^{\frac{1}{2}}}$$

$$\Psi_2 = -dr\sin\theta\cos\phi$$

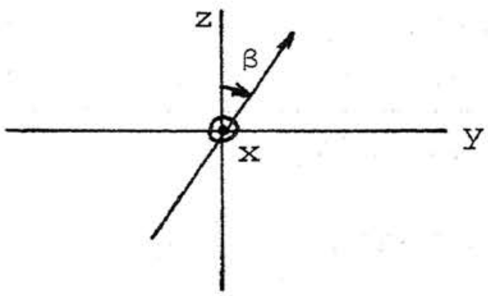
Combining equations (19A) gives the field of element #2 as

$$\dot{\bar{E}}_2 = \frac{\cos\left[\frac{L}{2}(-\sin\theta\sin\phi\sin\beta + \cos\theta\cos\beta)\right] - \cos\left(\frac{L}{2}\right)}{1-(-\sin\theta\sin\phi\sin\beta + \cos\theta\cos\beta)^2}$$
$$\times\left[\bar{a}_\theta(\sin\theta\cos\beta + \cos\theta\sin\phi\sin\beta\right.$$
$$\left. + \bar{a}_\phi\cos\phi\sin\beta\right]e^{j(\omega t - dr\sin\theta\cos\phi)} \tag{20A}$$

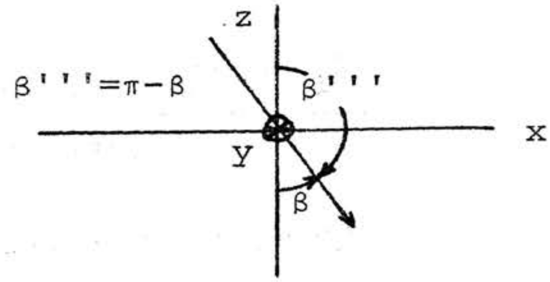
The fields of the third element may be found in a manner similar to the method used to find $\dot{\bar{E}}_2$. Figure 5 shows that the top and left side views of element #3 are similar to the top and front view of element #1. If θ, ϕ, and β are denoted by a triple prime, the equations (16A) and (17A) will apply to element #3. To express these equations in terms of the same parameters as in $\dot{\bar{E}}_1$ set $\beta''' = \pi-\beta$, $\phi''' = \phi+\frac{\pi}{2}$, and $\theta''' = \theta$.

Then $\sin\beta''' = \sin(\pi-\beta) = \sin\beta$

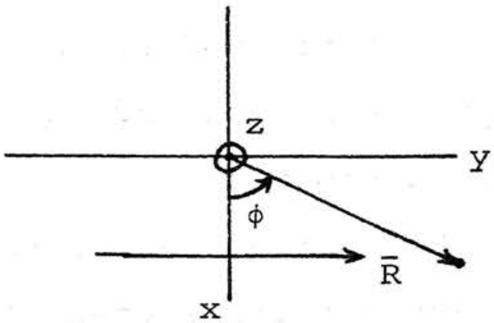
$\cos\beta''' = \cos(\pi-\beta) = -\cos\beta$



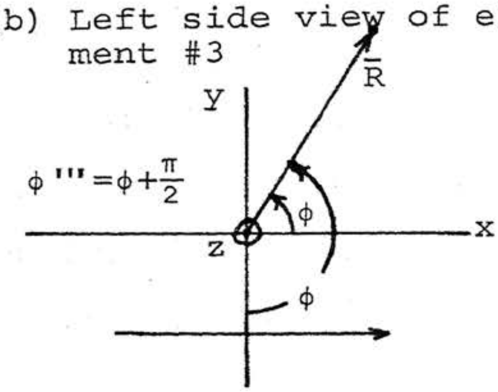
(a) Front view of element #1



(b) Left side view of element #3



(c) Top view of element #1



(d) Top view of element #3

Figure A-5. Sketches comparing views of elements #1 and #3

$$\begin{aligned} \sin \phi''' &= \sin (\phi+\tfrac{\pi}{2}) = \cos \phi \\ \cos \phi''' &= \cos (\phi+\tfrac{\pi}{2}) = - \sin \phi \end{aligned} \qquad (21A)$$

Substituting equations (21A) into equations (16A) results in

$$\dot{\bar{E}}_3 = E_{\theta_3} \bar{a}_{\theta_3} e^{j(\omega t + \Psi_3)}$$

$$\text{where } E_{\theta_3} = \frac{\cos (\frac{L}{2} \cos \theta_3) - \cos (\frac{L}{2})}{\sin \theta_3} \qquad (22A)$$

$$\cos \theta_3 = \sin \theta \cos \phi - \cos \theta \cos \beta$$

$$\sin \theta_3 = [1 - (\sin \theta \cos \phi \sin \beta - \cos \theta \cos \beta)^2]^{1/2}$$

$$\bar{a}_{\theta_3} = \frac{\bar{a}_\theta (-\sin \theta \cos \beta - \cos \theta \cos \phi \sin \beta) + \bar{a}_\phi \sin \phi \sin \beta}{[1 - (\sin \theta \cos \phi \sin \beta - \cos \theta \cos \beta)^2]^{1/2}}$$

$$\Psi_3 = - dr \sin \theta \sin$$

Combining these equations gives

$$\dot{\bar{E}}_3 = \frac{\cos\left[\frac{L}{2}(\sin\theta\cos\phi\sin\beta - \cos\theta\cos\beta)\right] - \cos\left(\frac{L}{2}\right)}{1 - (\sin\theta\cos\phi\sin\beta - \cos\theta\cos\beta)^2}$$

$$\times\left[-\bar{a}_\theta(\sin\theta\cos\beta + \cos\theta\cos\phi\sin\beta) + \bar{a}_\phi\sin\phi\sin\beta\right]e^{j(\omega t - d_r\sin\theta\sin\phi)} \qquad (23A)$$

A similar procedure will show that the equations for the fourth element are the same as those for the third except that ϕ is replaced by $\phi+\pi$ and thus sin ϕ and cos ϕ are replaced by - sin ϕ and - cos ϕ respectively. So equations (22A) become for element #4

$$\dot{\bar{E}}_4 = E_{\theta 4}\,\bar{a}_{\theta 4}\,e^{j(\omega t + \psi_4)} \qquad (24A)$$

where $E_{\theta 4} = \dfrac{\cos\left(\frac{L}{2}\cos\theta_4\right) - \cos\left(\frac{L}{2}\right)}{\sin\theta_4}$

$$\sin\theta_4 = \left[1 - (-\sin\theta\cos\phi\sin\beta - \cos\theta\cos\beta)^2\right]^{\frac{1}{2}}$$

$$\cos\theta_4 = -\sin\theta\cos\phi\sin\beta - \cos\theta\cos\beta$$

$$\bar{a}_{\theta 4} = \frac{\bar{a}_\theta(-\sin\theta\cos\beta + \cos\theta\cos\phi\sin\beta) - \bar{a}_\phi\sin\phi\sin\beta}{\left[1 - (-\sin\theta\cos\phi\sin\beta - \cos\theta\cos\beta)^2\right]^{\frac{1}{2}}}$$

$$\psi_4 = d_r\sin\theta\sin\phi$$

Combining these equations results in

$$\dot{\bar{E}}_4 = \frac{\cos\left[\frac{L}{2}(\sin\theta\cos\phi\sin\beta + \cos\theta\cos\beta)\right] - \cos\left(\frac{L}{2}\right)}{1 - (\sin\theta\cos\phi\sin\beta + \cos\theta\cos\beta)^2}$$

$$\times\left[\bar{a}_\theta(-\sin\theta\cos\beta + \cos\theta\cos\phi\sin\beta) - \bar{a}_\phi\sin\phi\sin\beta\right]e^{j(\omega t + d_r\sin\theta\sin\phi)} \qquad (25A)$$

The total field is equal to the sum of the fields from the individual four dipoles.

$$\dot{\bar{E}} = \dot{\bar{E}}_1 + \dot{\bar{E}}_2 + \dot{\bar{E}}_3 + \dot{\bar{E}}_4$$

where $\dot{\bar{E}}_1$, $\dot{\bar{E}}_2$, $\dot{\bar{E}}_3$, and $\dot{\bar{E}}_4$ are given in equations (17A), (20A), (23A), and (25A). To simplify the addition make the following substitutions:

Let
$$M_1 = \frac{\cos\left[\frac{L}{2}(\sin\theta\sin\phi\sin\beta + \cos\theta\cos\beta)\right] - \cos\left(\frac{L}{2}\right)}{1-(\sin\theta\sin\phi\sin\beta+\cos\theta\cos\beta)^2}$$

$$M_2 = \frac{\cos\left[\frac{L}{2}(-\sin\theta\sin\phi\sin\beta + \cos\theta\cos\beta)\right] - \cos\left(\frac{L}{2}\right)}{1-(-\sin\theta\sin\phi\sin\beta+\cos\theta\cos\beta)^2}$$

$$M_3 = \frac{\cos\left[\frac{L}{2}(\sin\theta\cos\phi\sin\beta - \cos\theta\cos\beta)\right] - \cos\left(\frac{L}{2}\right)}{1-(\sin\theta\cos\phi\sin\beta-\cos\theta\cos\beta)^2} \tag{26A}$$

$$M_4 = \frac{\cos\left[\frac{L}{2}(\sin\theta\cos\phi\sin\beta + \cos\theta\cos\beta)\right] - \cos\left(\frac{L}{2}\right)}{1-(\sin\theta\cos\phi\sin\beta+\cos\theta\cos\beta)^2}$$

Then
$$\begin{aligned}\dot{\bar{E}} = {} & M_1\left[\bar{a}_\theta(\sin\theta\cos\beta - \cos\theta\sin\phi\sin\beta) - \bar{a}_\phi\cos\phi\sin\beta\right] \\ & \times e^{j(\omega t + d_r\sin\theta\cos\phi)} + M_2\left[\bar{a}_\theta\sin\theta\cos\beta + \cos\theta\sin\phi\sin\beta\right) \\ & + \bar{a}_\phi\sin\phi\sin\beta\Big]e^{j(\omega t - d_r\sin\theta\cos\phi)} - M_3\left[\sin\theta\cos\beta\right. \\ & + \cos\theta\cos\phi\sin\beta) - \bar{a}_\phi\sin\phi\sin\beta\Big]e^{j(\omega t - d_r\sin\theta\sin\phi)} \\ & - M_4\left[\bar{a}_\theta(\sin\theta\cos\beta - \cos\theta\cos\phi\sin\beta)\right. \\ & + \bar{a}_\phi\sin\phi\sin\beta\Big]e^{j(\omega t + d_r\sin\theta\sin\phi)}\end{aligned} \tag{27A}$$

Make the further substitutions

$$\begin{aligned} a &= \sin\theta\cos\beta \\ b &= \cos\theta\sin\phi\sin\beta \\ c &= \cos\theta\cos\phi\sin\beta \\ d &= \cos\phi\sin\beta \\ \not{e} &= \sin\phi\sin\beta \end{aligned} \tag{28A}$$

and recall that

$$\Psi_1 = dr \sin\theta \cos\phi \qquad (16A)$$

$$\Psi_4 = dr \sin\theta \sin\phi \qquad (24A)$$

Then
$$\begin{aligned}\dot{E} = {} & M_1\left[\bar{a}_\theta\,(a-b) - \bar{a}_\phi\, d\right] e^{j(\omega t+\psi_1)} \\ & + M_2\left[\bar{a}_\theta\,(a+b) + \bar{a}_\phi\, d\right] e^{j(\omega t-\psi_1)} \\ & - M_3\left[\bar{a}_\theta\,(a+c) - \bar{a}_\phi\, c\right] e^{j(\omega t-\psi_4)} \\ & - M_3\left[\bar{a}_\theta\,(a-b) + \bar{a}_\phi\, c\right] e^{j(\omega t+\psi_4)}\end{aligned}$$

$$\begin{aligned} = {} & \bar{a}_\theta\left[M_1(a-b)e^{j(\omega t+\psi_1)} + M_2(a+b)e^{j(\omega t-\psi_1)}\right. \\ & \left. - M_3(a+c)e^{j(\omega t-\psi_4)} - M_4(a-c)e^{j(\omega t+\psi_4)}\right] \\ & + \bar{a}_\phi\left[-M_1 d\, e^{j(\omega t+\psi_1)} + M_2 d\, e^{j(\omega t-\psi_1)}\right. \\ & \left. + M_3\, c\, e^{j(\omega t-\psi_4)} - M_4\, c\, e^{j(\omega t+\psi_4)}\right]\end{aligned}$$

$$= \bar{a}_\theta\,\dot{E} + \bar{a}_\phi\,\dot{E} \qquad (29A)$$

To change the complex values $\dot{E}_\theta$ and $\dot{E}_\phi$ to instantaneous values E_θ and E_ϕ, the real part of the respective field must be taken. Thus

$$\mathcal{E}_\theta = \mathrm{Re}\,\dot{\mathcal{E}}_\theta = M_1(a-b)\cos(\omega t+\psi_1) + M_2(a+b)\cos(\omega t-\psi_1)$$

$$- M_3(a+c)\cos(\omega t-\psi_4) - M_4(a-c)\cos(\omega t+\psi_4)$$

$$= M_1(a-b)\cos\omega t\cos\psi_1 - M_1(a-b)\sin\omega t\sin\psi_1$$

$$+ M_2(a+b)\cos\omega t\cos\psi_1 + M_2(a+b)\sin\omega t\sin\psi_1$$

$$- M_3(a+c)\cos\omega t\cos\psi_4 - M_3(a+c)\sin\omega t\sin\psi_4$$

$$- M_4(a-c)\cos\omega t\cos\psi_4 + M_4(a-c)\sin\omega t\sin\psi_4$$

$$= \Big\{[M_1(a-b)+M_2(a+b)]\sin\psi_1 - [M_3(a+c)$$

$$+ M_4(a-c)]\cos\psi_4\Big\}\cos\omega t + \Big\{[-M_1(a-b)+M_2(a+b)]\sin\psi_1$$

$$- [M_3(a+c) - M_4(a-c)]\sin\psi_4\Big\}\sin\omega t \qquad (30A)$$

$$\mathcal{E}_\phi = \mathrm{Re}\,\dot{\mathcal{E}}_\phi = -M_1 d\cos(\omega t+\psi_1) + M_2 d\cos(\omega t-\psi_1)$$

$$+ M_3 \not{c}\cos(\omega t-\psi_4) - M_4 \not{c}\cos(\omega t+\psi_4)$$

$$= -M_1 d\cos\omega t\cos\psi_1 + M_1 d\sin\omega t\sin\psi_1$$

$$+ M_2 d\cos\omega t\cos\psi_1 + M_2 d\sin\omega t\sin\psi_1 + M_3\not{c}\cos\omega t\cos\psi_4$$

$$+ M_3\not{c}\sin\omega t\sin\psi_4 - M_4\not{c}\cos\omega t\cos\psi_4 + M_4\not{c}\sin\omega t\sin\psi_4$$

$$= [(M_2-M_1)d\cos\psi_1 + (M_3-M_4)\not{c}\cos\psi_4]\cos\omega t$$

$$+ [(M_2+M_1)d\sin\psi_1 + (M_3+M_4)\not{c}\sin\psi_4]\sin\omega t \qquad (31A)$$

The final results are stated in equations (26A), (28A), (30A) and (31A). It should be kept in mind that these results apply only to the quarter space $0 \leq \theta \leq \pi$ and $-\frac{\pi}{4} \leq \phi \leq \frac{\pi}{4}$. These limits are the boundaries of the reflector. The field is zero everywhere else. The E_θ and E_ϕ components of the far field of the array are listed below.

$$
\begin{aligned}
E_\theta &= A \cos \omega t + B \sin \omega t \\
E_\phi &= C \cos \omega t + D \sin \omega t
\end{aligned}
\qquad (32A)
$$

where

$$A = [M_1 \ (a-b) + M_2 \ (a+b)] \cos \Psi_1 - [M_3 \ (a+c) + M_4 (a-c)] \cos \Psi_4$$

$$B = [-M_1 (a-b) + M_2 (a+b)] \sin \Psi_1 - [M_3 (a+c) - M_4 (a-c)] \sin \Psi_4$$

$$C = (M_2 - M_1) \ d \cos \Psi_1 + (M_3 - M_4) \ \not{e} \cos \Psi_4$$

$$D = (M_2 + M_1) \ d \sin \Psi_1 + (M_3 + M_4) \ \not{e} \sin \Psi_4$$

$$M_1 = \frac{\cos\left[\frac{L}{2}(\sin\theta \sin\phi \sin\beta + \cos\theta \cos\beta)\right] - \cos\left(\frac{L}{2}\right)}{1 - (\sin\theta \sin\phi \sin\beta + \cos\theta \cos\beta)^2}$$

$$M_2 = \frac{\cos\left[\frac{L}{2}(\sin\theta \sin\phi \sin\beta - \cos\theta \cos\beta)\right] - \cos\left(\frac{L}{2}\right)}{1 - (\sin\theta \sin\phi \sin\beta - \cos\theta \cos\beta)^2}$$

$$M_3 = \frac{\cos\left[\frac{L}{2}(\sin\theta \cos\phi \sin\beta - \cos\theta \cos\beta)\right] - \cos\left(\frac{L}{2}\right)}{1 - (\sin\theta \cos\phi \sin\beta - \cos\theta \cos\beta)^2}$$

$$M_4 = \frac{\cos\left[\frac{L}{2}(\sin\theta \cos\phi \sin\beta + \cos\theta \cos\beta)\right] - \cos\left(\frac{L}{2}\right)}{1 - (\sin\theta \cos\phi \sin\beta + \cos\theta \cos\beta)^2}$$

$$a = \sin\theta\cos\beta$$

$$b = \cos\theta\sin\phi\sin\beta$$

$$c = \cos\theta\cos\phi\sin\beta$$

$$d = \cos\phi\sin\beta$$

$$\not{e} = \sin\phi\sin\beta$$

$$\Psi_1 = dr\sin\theta\cos\phi$$

$$\Psi_4 = dr\sin\theta\sin\phi$$

APPENDIX B

DETERMINATION OF THE POLARIZATION EQUATIONS FOR THE GENERAL CASE

The results of Appendix A (equation (32A)) state that the instantaneous far fields are

$$\begin{aligned} E_\theta &= A \cos \omega t + B \sin \omega t \\ E_\phi &= C \cos \omega t + D \sin \omega t \end{aligned} \qquad (1B)$$

Cramer's rule may be used to solve these equations simultaneously for cos ωt. Thus

$$\cos \omega t = \frac{\begin{vmatrix} E_\theta & B \\ E_\phi & D \end{vmatrix}}{\begin{vmatrix} A & B \\ C & D \end{vmatrix}} = \frac{E_\theta D - E_\phi B}{AD - BC} \qquad (2B)$$

Since $\cos^2 \omega t + \sin^2 \omega t = 1$

$$\sin^2 \omega t = 1 - \cos^2 \omega t = 1 - \left(\frac{E_\theta D - E_\phi B}{AD - BC}\right)^2 \qquad (3B)$$

The first of equations (1B) may be written

$$E_\theta - A \cos \omega t = B \sin \omega t$$

Squaring gives

$$E_\theta^2 - 2\, E_\theta\, A \cos \omega t + A^2 \cos^2 \omega t = B^2 \sin^2 \omega t \qquad (4B)$$

Substitute equations (2B) and (3B) into (4B)

$$E_\theta^2 - 2E_\theta A\left(\frac{E_\theta D - E_\phi A}{AD - BC}\right) + A^2\left(\frac{E_\theta D - E_\phi B}{AD - BC}\right)^2 = B^2 - B^2\left(\frac{E_\theta D - E_\phi B}{AD - BC}\right)^2 \qquad (5B)$$

Clearing the fractions, expanding, and collecting terms yields the result

$$E_\theta^{\,2}\ (C^2+D^2)\ -\ 2E_\theta E_\phi (AC+BD)\ +\ E_\phi^2\ (A^2+B^2)\ =\ (AD-BC)^2 \qquad (6B)$$

This has the form of an equation of an ellipse with arbitrary orientation. To simplify the notation, let

$$\begin{aligned} Q &= C^2 + D^2 \\ R &= AC + BD \\ S &= A^2 + B^2 \\ T &= (AD - BC)^2 \end{aligned} \qquad (7B)$$

Then $E^{\,2}Q - 2\ E_\theta\, E_\phi\ R + E_\phi^{\,2}\ S = T$ (8B)

To get this into the canonical form, that is to eliminate the cross product term, (8B) is written as a matrix equation.

$$T = \begin{bmatrix} E_\theta & E_\phi \end{bmatrix} \begin{bmatrix} Q & -R \\ -R & S \end{bmatrix} \begin{bmatrix} E_\theta \\ E_\phi \end{bmatrix} = \underline{E}^T\ \underline{W}\ \underline{E} \qquad (9B)$$

where $\underline{W}$ is the matrix $\begin{bmatrix} Q & -R \\ -R & S \end{bmatrix}$ and $\underline{E}$ is the matrix vector

$\begin{bmatrix} E_\theta \\ E_\phi \end{bmatrix}$. According to matrix theory the canonical form may be realized by using the characteristic values (or eigenvalues) as the coefficients of new variables, thus

$$T = \lambda_1\ E_\theta{}'^{\,2} + \lambda_2\ E_\phi{}'^{\,2} \qquad (10B)$$

Now, the characteristic values λ must be found.

$$\left|\underline{P} - \lambda \underline{I}\right| = 0 = \begin{vmatrix} Q-\lambda & -R \\ -R & S-\lambda \end{vmatrix} = QS - (Q+S)\lambda + \lambda^2 - R^2$$

$$\lambda^2 - \lambda(Q+S) + (QS - R^2) = 0 \tag{11B}$$

Using the quadratic formula

$$\lambda = \frac{Q+S \pm \sqrt{(Q+S)^2 - 4(QS - R^2)}}{2}$$

$$= \frac{Q+S \pm \sqrt{(Q-S)^2 + 4R^2}}{2}$$

Then $\lambda_1 = \dfrac{Q+S \pm \sqrt{(Q-S)^2 + 4R^2}}{2}$

$$\lambda_2 = \frac{Q+S \mp \sqrt{(Q-S)^2 + R^2}}{2} \tag{12B}$$

Substituting (12B) into (10B) and dividing through by T yields

$$\frac{E_\theta'^2}{2T \Big/ \left(Q+S \pm \sqrt{(Q-S)^2+4R^2}\right)} + \frac{E_\phi'^2}{2T \Big/ \left(Q+S \mp \sqrt{(Q-S)^2+4R^2}\right)} = 1 \tag{13B}$$

This is the equation of an ellipse with the major and minor axes in line with the coordinates E_θ' and E_ϕ'. The peak values $\hat{E}_\theta'$ and $\hat{E}_\phi'$ are $\sqrt{T/\lambda_1}$ and $\sqrt{T/\lambda_2}$ respectively. If R=0 the matrix $\underline{P}$ is a diagonal matrix and equation (8B) is already in canonical form and $E'=E$.

The boundary conditions of the original problem, the corner reflector antenna, require that $\hat{E}_\theta=0$ at the conducting planes. Thus if the $\hat{E}'_\theta$ and $\hat{E}'_\phi$ fields are evaluated at, for example, $\theta=\frac{\pi}{2}$, $\phi=\frac{\pi}{4}$, and $L=\frac{\pi}{2}$, it will be seen that $\hat{E}'_\theta$ and $\hat{E}'_\phi$ go into $\hat{E}_\theta$ and $\hat{E}_\phi$ respectively, and the plus sign must be chosen for λ_1 to make E_θ vanish at that point. This forces the choice of the minus sign for λ_2. Equation (13B) then becomes

$$\frac{E_\theta'^2}{2T \Big/ Q+S+\sqrt{(Q-S)^2+4R^2}} + \frac{E_\phi'^2}{2T \Big/ Q+S-\sqrt{(Q-S)^2+4R^2}} = 1 \tag{14B}$$

Equation (14B) is the final result. The λ's are always positive for a real symmetric matrix and $T = (AD - BC)^2$ is also positive since A, B, C, and D are real. Thus the denominators of (14B) are always equal to or greater than zero, and so the peak values $\hat{E}'_\theta$ and $\hat{E}'_\phi$ are real.

APPENDIX C

POLARIZATION EQUATIONS FOR A HALF-WAVELENGTH ELEMENT

For the special case of a half-wavelength element ($L=\pi$) the equations of Appendices A and B may be simplified. To obtain the polarization equation broadside, set $\theta=\frac{\pi}{2}$, $L=\pi$, and $\phi=0$ in equations (32A), (7B), and (14B). Thus

$$a = \cos \beta$$

$$d = \sin \beta$$

$$b = c = \phi = 0$$

$$\Psi_1 = dr$$

$$\Psi_4 = 0$$

$$M_1 = M_2 = 1$$

$$M_3 = M_4 = \frac{\cos (\frac{\pi}{2} \sin \beta)}{\cos^2 \beta}$$

$$A = 2 \left[\cos \beta \cos dr - \frac{\cos (\frac{\pi}{2} \sin \beta)}{\cos \beta}\right]$$

$$B = C = 0$$

$$D = 2 \sin \beta \sin dr$$

$$Q = 4 \sin^2 \beta \sin^2 dr$$

$$R = 0$$

$$S = 4 \left[\cos \beta \cos dr - \frac{\cos (\frac{\pi}{2} \sin \beta)}{\cos \beta}\right]^2$$

$$T = 16 \left[\cos \beta \cos dr - \frac{\cos (\frac{\pi}{2} \sin \beta)}{\cos \beta}\right]^2 \sin^2 \beta \sin^2 dr$$

Since R = 0, $E'_\theta \equiv E_\theta$ and $E'_\phi \equiv E_\phi$. Then

$$\hat{E}_\theta^{\,2} = \frac{2T}{Q + S + \sqrt{(Q-S)^2}} = \frac{2T}{Q + S + Q - S} = \frac{T}{Q}$$

$$= 4\ [\cos\ \beta\ \cos\ dr - \frac{\cos\ (\frac{\pi}{2}\ \sin\ \beta)}{\cos\ \beta}]^2 \tag{1C}$$

$$\hat{E}_\phi^{\,2} = \frac{2T}{Q + S - \sqrt{(Q-S)^2}} = \frac{2T}{Q + S - Q + S} = \frac{T}{S}$$

$$= 4\ \sin^2\ \beta\ \sin^2\ dr \tag{2C}$$

The polarization equation then becomes

$$\frac{E_\theta^{\,2}}{4\ [\cos\ \beta\ \cos\ dr - \frac{\cos\ (\frac{\pi}{2}\ \sin\ \beta)}{\cos\ \beta}]^2} + \frac{E_\phi^{\,2}}{4\ \sin^2\ \beta\ \sin^2\ dr} = 1 \tag{3C}$$

When $\hat{E}_\theta^{\,2} = \hat{E}_\phi^{\,2}$ equation (3C) become the equation of a circle of radius $|E_\theta| = |E_\phi|$. Thus circular polarization occurs when

$$4\ [\cos\ \beta\ \cos\ dr - \frac{\cos\ (\frac{\pi}{2}\ \sin\ \beta)}{\cos\ \beta}]^2 = 4\ \sin^2\ \beta\ \sin^2\ dr$$

$$\cos^2\ \beta\ \cos^2\ dr - 2\ \cos\ dr\ \cos\ (\tfrac{\pi}{2}\ \sin\ \beta)$$

$$+ \frac{\cos^2\ (\frac{\pi}{2}\ \sin\ \beta)}{\cos^2\ \beta} = \sin^2\ \beta\ \sin^2\ dr \tag{4C}$$

But $\sin^2 dr = 1 - \cos^2\ dr$ (5C)

Substitute (5C) into (4C).

$$\cos^2\beta \cos^2 dr - 2\cos dr \cos\left(\tfrac{\pi}{2}\sin\beta\right) + \frac{\cos^2\left(\frac{\pi}{2}\sin\beta\right)}{\cos^2\beta} = \sin^2\beta - \sin^2\beta\cos^2 dr$$

$$\cos^2 dr - 2\cos dr \cos\left(\tfrac{\pi}{2}\sin\beta\right) + \frac{\cos^2\left(\frac{\pi}{2}\sin\beta\right)}{\cos^2\beta} - \sin^2\beta = 0 \qquad (6C)$$

Solve (6C) for cos dr using the quadratic formula.

$$\cos dr = \frac{2\cos\left(\frac{\pi}{2}\sin\beta\right) \pm \sqrt{4\cos^2\left(\frac{\pi}{2}\sin\beta\right) - \frac{4\cos^2\left(\frac{\pi}{2}\sin\beta\right)}{\cos^2\beta} + 4\sin^2\beta}}{2}$$

$$= \cos\left(\tfrac{\pi}{2}\sin\beta\right) \pm \sin\beta\sqrt{1 - \frac{\cos^2\left(\frac{\pi}{2}\sin\beta\right)}{\cos^2\beta}} \qquad (7C)$$

Or $dr = \cos^{-1}\left[\cos\left(\tfrac{\pi}{2}\sin\beta\right) \pm \sin\beta\sqrt{1 - \frac{\cos^2\left(\frac{\pi}{2}\sin\beta\right)}{\cos^2\beta}}\right]$

To arrive at the equations for the electric fields in the vertical plane for the half-wavelength element substitute $L = \pi$ and $\phi = 0$ into the equations (32A). Thus

$a = \sin\ \theta\ \cos\ \beta$

$c = \cos\ \theta\ \sin\ \beta$

$d = \sin\ \beta$

$b = \cancel{e} = 0$

$\Psi_1 = dr \sin\theta$

$\Psi_4 = 0$

$$M_1 = M_2 = \frac{\cos\left(\frac{\pi}{2}\cos\theta\cos\beta\right)}{1-\cos^2\theta\cos^2\beta}$$

$$M_3 = \frac{\cos\left[\frac{\pi}{2}\cos(\theta+\beta)\right]}{\sin^2(\theta+\beta)}$$

$$M_4 = \frac{\cos\left[\frac{\pi}{2}\cos(\theta-\beta)\right]}{\sin^2(\theta-\beta)}$$

$$A = \frac{2\cos\left(\frac{\pi}{2}\cos\theta\cos\beta\right)}{1-\cos^2\theta\cos^2\beta}\sin\theta\cos\beta\cos(dr\sin\theta)$$

$$-\frac{\cos\left[\frac{\pi}{2}\cos(\theta+\beta)\right]}{\sin(\theta+\beta)} - \frac{\cos\left[\frac{\pi}{2}\cos(\theta-\beta)\right]}{\sin(\theta-\beta)}$$

$$B = C = 0$$

$$D = \frac{2\cos\left(\frac{\pi}{2}\cos\theta\cos\beta\right)}{1-\cos^2\theta\cos^2\beta}\sin\beta\sin(dr\sin\theta)$$

Then

$$E_\theta = \left\{\frac{2\cos\left(\frac{\pi}{2}\cos\theta\cos\beta\right)}{1-\cos^2\theta\cos^2\beta}\sin\theta\cos\beta(dr\sin\theta)\right.$$

$$\left.-\frac{\cos\left[\frac{\pi}{2}\cos(\theta+\beta)\right]}{\sin(\theta+\beta)} - \frac{\cos\left[\frac{\pi}{2}\cos(\theta-\beta)\right]}{\sin(\theta-\beta)}\right\}\cos\omega t \qquad (8C)$$

$$E_\phi = \left[\frac{2\cos\left(\frac{\pi}{2}\cos\theta\cos\beta\right)}{1-\cos^2\theta\cos^2\beta}\sin\beta\sin(dr\sin\theta)\right]\sin\omega t$$

To obtain the horizontal field equations substitute $L = \pi$ and $\theta = \frac{\pi}{2}$ into equations (32A).

$a = \cos\beta$

$b = c = 0$

$d = \cos\phi \sin\beta$

$\cancel{e} = \sin\phi \sin\beta$

$\Psi_1 = dr \cos\phi$

$\Psi_4 = dr \sin\phi$

$$M_1 = M_2 = \frac{\cos\left(\frac{\pi}{2}\sin\phi \sin\beta\right)}{1 - \sin^2\phi \sin^2\beta}$$

$$M_3 = M_4 = \frac{\cos\left(\frac{\pi}{2}\cos\phi \sin\beta\right)}{1 - \cos^2\phi \sin^2\beta}$$

$$A = 2\cos\beta\left[\frac{\cos\left(\frac{\pi}{2}\sin\phi \sin\beta\right)}{1 - \sin^2\phi \sin^2\beta}\cos(d_r \cos\phi) - \frac{\cos\left(\frac{\pi}{2}\cos\phi \sin\beta\right)}{1 - \cos^2\phi \sin^2\beta}\cos(d_r \sin\phi)\right]$$

$$B = C = 0$$

$$D = 2\sin\beta\left[\frac{\cos\left(\frac{\pi}{2}\sin\phi \sin\beta\right)}{1 - \sin^2\phi \sin^2\beta}\cos\phi \sin(d_r \cos\phi) + \frac{\cos\left(\frac{\pi}{2}\cos\phi \sin\beta\right)}{1 - \cos^2\phi \sin^2\beta}\sin\phi \sin(d_r \sin\phi)\right]$$

Then

$$E_\theta = 2\cos\beta\left[\frac{\cos\left(\frac{\pi}{2}\sin\phi\sin\beta\right)}{1-\sin^2\phi\sin^2\beta}\cos(d_r\cos\phi)\right.$$

$$\left.-\frac{\cos\left(\frac{\pi}{2}\cos\phi\sin\beta\right)}{1-\cos^2\phi\sin^2\beta}\cos(d_r\sin\phi)\right]\cos\omega t$$

$$E_\phi = 2\sin\beta\left[\frac{\cos\left(\frac{\pi}{2}\sin\phi\sin\beta\right)}{1-\sin^2\phi\sin^2\beta}\cos\phi\sin(d_r\cos\phi)\right.$$

$$\left.+\frac{\cos\left(\frac{\pi}{2}\cos\phi\sin\beta\right)}{1-\cos^2\phi\sin^2\beta}\sin\phi\sin(d_r\sin\phi)\right]\sin\omega t \qquad (9C)$$

Inspection of equations (8C) and (9C) shows that in both cases E_θ is a function of cos ωt and E_ϕ is a function of sin ωt. This means that E_θ and E_ϕ are in phase quadrature in both the vertical and horizontal planes. Thus the fields are elliptically polarized with the major and minor axes parallel to the coordinate axes. The axial ratio is then simply $(E_\Theta/E_\Phi)^{\pm 1}$ with the sign of the exponent chosen so $AR \leq 1$.

VITA

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