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Singularity Formation in the Geometry of Perturbed Shocks of General Mach Number

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Abstract While planar shock waves are known to be stable to small perturbations in the sense that the perturbation amplitude decays over time, it has also been suggested that plane propagating shocks can develop singularities in some derivative of their geometry (Whitham (1974) Linear and nonlinear waves. Wiley, New York) in a nonlinear, wave reinforcement process. We present a spectral-based analysis of the equations of geometrical shock dynamics that predicts the time to singularity formation in the profile of an initially perturbed planar shock for general shock Mach number. We find that following an initially sinusoidal perturbation, the shock shape remains analytic only up to a finite, critical time that is a monotonically decreasing function of the initial perturbation amplitude. At the critical time, the shock profile ceases to be analytic, corresponding physically to the incipient formation of a "shock-shock" or triple point. We present results for gas-dynamic shocks and discuss the potential for extension to shock dynamics of fast MHD shocks.

1 Introduction

It is well-known that plane shocks are stable in the sense that the amplitude of a small perturbation in the geometry of the shock surface will decay over time [1, 2]. In contrast, shocks moving into converging geometries have been shown to exhibit instability in the sense of relative growth of an initial shape perturbation [1, 3].

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Analysis based on the equations of geometrical shock dynamics [1] indicates that in both plane [4] and converging [1] geometries, initially perturbed shocks lead to the formation of a "shock-shocks" by the mechanism of the overturning of waves propagating on the shock itself. Recent studies based on the numerical solution of the equations of geometrical shock dynamics (GSD) provide numerical evidence that these singularities will develop on a perturbed periodic plane shock in time t_c inversely proportional to the perturbation amplitude ϵ [4]. For general multi-mode initial perturbations on both weak and strong shocks, analytical support for this hypothesis is given by Mostert et al. [5] who utilize a spectral approach motivated in part by Moore's [6] analysis of vortex sheet evolution. This gives analytical forms for constants of proportionality in both the weak- and strong-shock limits as functions of gas properties and flow parameters.

The present study considers the case of gas-dynamic shocks of general Mach number for unimodal perturbations with imaginary Fourier coefficients. The study proceeds as follows. Section 2 derives the leading order governing equation of shock dynamics for general Mach number. In Sect. 3 we apply a Fourier-based method to the governing equation following [5] and determine its asymptotic behavior at large Fourier mode numbers. This provides a time at which the shock surface profile loses analyticity with formation of a singularity in the shock geometry. It is demonstrated that the present results agree with analysis for the weak- and strong-shock limiting cases of [5].

2 Shock Dynamics

In complex coordinates, the motion of a periodic planar shock in extent can be described according to GSD as follows [5]:

$$\frac{\partial Z}{\partial t} = i\ a_0\ M\left(\beta, t\right) s\left(\beta, t\right) \tag{1}$$

where $Z = Z(\beta,t)$ describes the shock position in space, β is a 2π-periodic parameter along the shock, $M(\beta,t)$ is the local Mach number of the shock, and $s(\beta,\ t)$ is the local unit tangent along the shock curve. Equation (1) has been nondimensionalized with respect to a perturbation wavelength L and reference velocity $\sqrt{p_0/\rho_0}$ so that the upstream sound speed is $a_0 = \sqrt{\gamma}$ where $\gamma = 5/3$ is the specific heat ratio. In two dimensions, a horizontal plane shock moving in the imaginary direction has the unperturbed solution $Z_0(\beta, t) = \beta + i\ a_0 M_0 t$, where M_0 is the unperturbed Mach number. Imposing a small perturbation $z(\beta,t)$ we have

$$Z\left(\beta, t\right) = \beta + i\ a_0 M_0 t + z\left(\beta, t\right). \tag{2}$$

In general, this geometric perturbation implies perturbations in the local Mach number and unit tangent. Equation (1) is an exact kinematic statement. For closure and application to shock propagation, the GSD [1] approximation can be used to approximate M as a function of the shock geometry only by introducing the variable A representing the local *ray tube area* of the shock. For this purpose we interpret β as a Lagrangian marker for these ray tubes and $A(\beta,t)$ as the arclength derivative along the shock with respect to β. Thus via the GSD area-Mach number relation,

$$M = q(A) = f^{-1}(M), \tag{3}$$

$$f(M) = \exp\left(-\int M\frac{\lambda(M)}{M^2-1}dM\right), \tag{4}$$

$$\frac{f(M)}{f\,(M_0)} = \frac{A}{A_0}, \tag{5}$$

where $\lambda(M)$ is a known function, the governing equation Eq. (1) can be written in terms of β:

$$\frac{\partial Z}{\partial t} = ia_0M_0\frac{q(A)}{q\,(A_0)}\frac{\partial Z}{\partial \beta}\left(\frac{\partial Z}{\partial \beta}\frac{\partial Z^*}{\partial \beta}\right)^{-1/2}. \tag{6}$$

From (3 and 4) the general inversion relation $q(A)$ cannot be obtained analytically but can be approximated locally by inverting a truncated Taylor series expansion of Eq. (4) around the desired M_0, in the form

$$\frac{A}{A_0}f\,(M_0) = f(M) = \alpha_0 + \alpha_1 M + \alpha_2 M^2 + O\left((M-M_0)^3\right), \tag{7}$$

where the coefficients $\alpha_i = \alpha_i(M_0,\gamma)$, $i = 0, 1, 2$ have been defined appropriately and determined analytically from the Taylor series. It will suffice to truncate at the second order [5]. Next denoting by the operator D differentiation with respect to β,

$$\begin{aligned} A\,(\beta) &= \left[(1+Dz)\left(1+Dz^*\right)\right]^{\frac{1}{2}} A_0 \\ &= A_0\left(1+P^{\left(1/2\right.}(\beta)\right) \end{aligned} \tag{8}$$

where by the binomial theorem (convergent since we assume the perturbation derivative Dz to be small),

$$P^{(\sigma)}\,(\beta) = \sum_k p_k\left(Dz + Dz^* + DzDz^*\right)^k, \quad p_k^{(\sigma)} = \binom{\sigma}{k}, \tag{9}$$

it follows that, taking $A_0 = 1$ without loss of generality,

$$M(\beta) \simeq \frac{1}{2\alpha_2}\left(-\alpha_1 - \sqrt{\alpha_1^2 - 4\alpha_2\left(\alpha_0 - 1 - P^{(\sigma)}(\beta)\right)}\right) = \frac{q(A)}{q(A_0)}. \tag{10}$$

Equation (10) is still cumbersome to substitute directly into Eq. (6), but the square root quantity can itself be expanded in a binomial series after some algebra, truncating at second order:

$$\frac{q(A)}{q(A_0)} = M(\beta) \simeq \frac{\alpha_1}{2\alpha_2} + T\left(1 + p_1^{(1/2)}\, SP^{(1/2)}(\beta) + p_2^{(1/2)}\left(SP^{(1/2)}(\beta)\right)^2\right), \tag{11}$$

where

$$Q = \frac{\alpha_1^2 - 4\alpha_0\alpha_2}{4\alpha_2^2}, \quad R = \frac{\alpha_0 + \alpha_1 M_0 + \alpha_2 M_0^2}{\alpha_2}, \quad S = \frac{R}{Q+R}, \quad T = \sqrt{Q+R}. \tag{12}$$

Equation (11) can now be substituted into Eq. (6). Noting that $P^{(1/2)}(\beta) \ll 1$ and that Eq. (9) is $O(Dz)$, we retain quantities of up to $O((Dz)^2)$, writing

$$\frac{\partial Z}{\partial t} \simeq i a_0 \Big[M_0\left(1 + P^{(-1/2)}\right)(1 + Dz) - ST p_1^{(1/2)} P^{(1/2)}\left(1 + P^{(-1/2)} - Dz\right) - T p_2^{(1/2)}\left(SP^{(1/2)}\right)^2 \Big], \tag{13}$$

where the dependence on β is suppressed for clarity. Equation (13) is now amenable to a spectral treatment since it involves only integer powers of the quantity Dz. The goal is to determine a time at which the Fourier coefficients decay with mode number at a rate slower than any exponential.

3 Fourier Representation

The analysis from here proceeds as in Mostert et al. [5]. Key points are reiterated here for clarity. The derivatives of z have convergent Fourier representations

$$Dz = \sum_{n=-\infty}^{\infty} in\hat{z}_n(\tau)\,e^{in\beta}, \quad Dz^* = \sum_{n=-\infty}^{\infty} in\hat{z}^*_{-n}(\tau)\,e^{in\beta},$$
$$DzDz^* = -\sum_{n=-\infty}^{\infty} e^{in\beta} \sum_{r_1+r_2=m} r_1 r_2 \hat{z}_n(\tau)\,\hat{z}^*_{-n}(\tau), \tag{14}$$

for as long as $z(\beta,t)$ remains analytic in β and where $\hat{z}_m(\tau)$ is the mth Fourier coefficient of z. Thus, Eq. (14) substituted into Eq. (13) yields a set of ordinary differential equations (ODEs) for the $\hat{z}_n(\tau)$. The third of Eq. (14) involves an infinite discrete convolution between all Fourier coefficients of z, and direct substitution of Eq. (14) into Eq. (13) will yield further such convolutions. To address this, we choose a simple initial condition (IC)

$$z(\beta, 0) = \epsilon \sin\beta, \quad \hat{z}_{\pm 1}(0) = \mp\frac{i\epsilon}{2}, \tag{15}$$

and $\hat{z}_n(0) = 0$ for n$\neq \pm 1$. It can be shown that, for this IC, if contributions in (13) of $O((Dz)^3)$ are neglected, then Re $\left(\hat{z}_n(\tau)\right) = 0$ for all n, τ. This implies $\hat{z}^*_{-n} = -\hat{z}_{-n}$. Furthermore, it is plausible to assume that so long as z remains analytic with this IC, the Fourier modes are $O(\epsilon^{|n|})$, so that $\hat{z}_n = \zeta_n \epsilon^{|n|} + O\left(\epsilon^{|n|+1}\right)$, where ζ_n is the leading order coefficient of an expansion of $\hat{z}_n$ in powers of ϵ. Any discrete convolution between the two ζ_r is then $O(\epsilon^{|n|})$ if and only if r_1, $r_2 \geq 1$ or r_1, $r_2 \leq -1$, where $|r_1| + |r_2| = |n|$, so that the leading order part of each convolution is a finite sum for a given n. Performing the necessary algebra thus gives the following set of coupled ODEs, similarly to Mostert et al. [5]:

$$\frac{1}{a_0 M_0}\frac{d\zeta_{\pm n}}{dt} = \mp n\zeta_{\pm n} \mp \mathrm{nb}_1\left(\zeta_{\pm n} - \zeta_{\mp n}\right) + f_{\pm n}, \quad n > 0 \tag{16}$$

where the $f_{\pm n}$ is a known forcing term

$$f_{\pm} = \sum_{r_1+r_2} r_1 r_2 \left[b_1\left\{\zeta_{r_1}\zeta_{-r_2} + \zeta_{r_1}\left(\zeta_{r_2} + \zeta_{-r_2}\right)\right\} + b_2\left(\zeta_{r_1} - \zeta_{-r_1}\right)\left(\zeta_{r_2} - \zeta_{-r_2}\right)\right], \tag{17}$$

which is nonzero for $n > 1$. Since both the positive and negative branches must be accounted for, this is a second-order system, and its homogeneous part is linear. The forcing term is determined by recursive solution for the modes $n = 1,2,\ldots,$ sequentially. The $n = 0$ mode corresponds to the unperturbed shock motion. It can be shown by variation of parameters that this system has a solution

$$\zeta_{\pm n} = 2(a_0 M_0 t)^{n-1} C^n \mid a_{\pm n} \mid \cos\left(m\omega_0 a_0 M_0 t + \phi_{\pm n}\right) + O\left(a_0 M_0 t^{n-2}\right), \tag{18}$$

where the phases $\phi_{\pm n}$ can be determined, and

$$C = \frac{1}{4}\sqrt{1+\omega_0^2}\ , \quad \omega_0 = \sqrt{-(1+2b_1)}. \tag{19}$$

For M_0, $\gamma > 1$, the constant $b_1 < -1/2$ and hence ω_0 are real. The $a_{\pm n}$ are complex coefficients which are determined by the recursion relation

$$a_{\pm n} = \frac{\pm K_\pm}{2(n-1)\omega_0} \sum_{r_1+r_2=m} r_1 r_2 a_{\pm r_1} a_{\pm r_2}, \quad a_1 = 1, \tag{20}$$

where $K_- = K_+^*$ and

$$K_\pm = \frac{2}{b_1}\left(b_1^3 + b_1^2(2+i\omega_0) - (1+i\omega_0)b_2\right). \tag{21}$$

We now have the information required to determine the asymptotic behavior of the leading coefficients $\zeta_{\pm n}$. Following the method of [5, 6], (20) has an asymptotic form as $n \to \infty$

$$|a_{\pm n}| \sim \frac{1}{2\pi}\left|\frac{K_\pm}{\omega_0}\right|^{n+1} e^n n^{-5/2}. \tag{22}$$

Substituting Eq. (22) into Eq. (18) and then using $\hat{z}_{\pm n} = \epsilon^n \zeta_{\pm n} + O\left(\epsilon^{n+1}\right)$ give the desired result

$$\hat{z}_{\pm n}(t) = (eC\epsilon)^n (a_0 M_0 t)^{n-1}\left|\frac{K_\pm}{\omega_0}\right|^{n+1} n^{-5/2}\cos(n\omega_0 a_0 M_0 t + \phi_{\pm n}) + O(a_0 M_0 t)^{n-2} \tag{23}$$

which describes analytic $z(\beta,t)$ for as long as the argument

$$1 + \log\left(a_0 M_0 t\left|\frac{K_\pm}{\omega_0}\right|\epsilon C\right), \tag{24}$$

remains negative. Thus, a singularity forms at the critical time

$$t_c = \frac{e^{-1}\omega_0}{M_0 a_0 |K_\pm| C\epsilon}. \tag{25}$$

This inverse proportionality of t_c to ϵ is identical to the form given in Mostert et al. [5] and is consistent with numerical results [4]. But here the proportionality constant depends in general on both γ and M_0 through the constants $\alpha_0, \alpha_1,$ *and* α_2 arising from the Taylor expansion Eq. (7) around M_0. The singularity forms as the

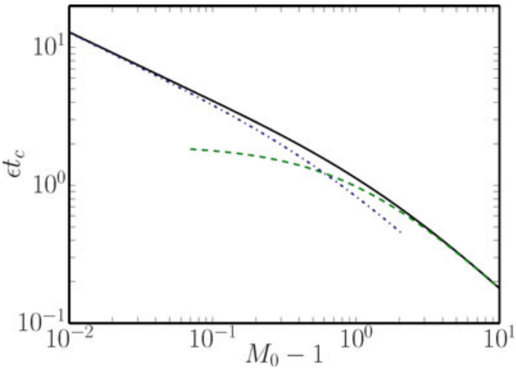

Fig. 1 Behavior of proportionality constant versus M_0-1. $\gamma = 5/3$. Solid: general M case derived in the present study. Dashed: strong-shock case from [5]. Dot-dashed: weak-shock case from [5]

transition from exponential to algebraic decay of coefficients at the critical time with mode exponent $-5/2$. This is the same as that found in [5]. Although a physical interpretation of the singularity is not obvious, we suggest that this is a precursor to and a generator of incipient shock-shock on the shock profile.

Figure 1 shows the behavior of the proportionality constant ϵt_c over M_0 with $\gamma = 5/3$, including the weak- and strong-shock limiting cases derived in [5] for this initial condition. As $M_0 - 1 \rightarrow 0$, clearly the present result tends to the weak-shock case of [5], and for large $M_0 - 1 \sim M_0$, the strong-shock case is reached. Hence the present result is consistent with [5].

4 Conclusion

We derive the time to singularity formation in planar periodic gas-dynamic shocks of general Mach number. The novel result is the generalization proper of [5] to any Mach number. The study finds that a singularity forms in the geometry according to a process corresponding in form to the weak- and strong-shock limiting cases. The singularity is of the same strength, with an exponent of $-5/2$, and appears after a critical time of the same form, as in these limiting cases. The constant of proportionality on the critical time depends numerically on the specific heat ratio γ and initial unperturbed Mach number M_0.

To complete the quantitative evaluation of some observations made in [4], the present result may be extended to other related cases such as for certain plane magnetohydrodynamic shocks or to cylindrical converging shocks.

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