A Balancing Method For Reduced Order Modelling Of Unstable Systems

Rajiva Prakash

Vittal S. Rao
Missouri University of Science and Technology

Mathl Comput. Modelling, Vol. 14, pp. 418–423, 1990
Printed in Great Britain

0895-7177/90 $3.00 + 0.00
Pergamon Press plc

A BALANCING METHOD FOR REDUCED ORDER MODELLING OF UNSTABLE SYSTEMS

Rajiva Prakash and S. Vittal Rao
Department of Electrical Engineering
University of Missouri - Rolla
Rolla, Mo 65401

Abstract. *Moore's balancing method of model reduction has many attractive properties and is used widely in practice. However, this method suffers from the drawback that it is not applicable for unstable systems. This paper introduces a balancing method of model reduction to overcome this problem. The proposed technique obtains the reduced order model by balancing and truncating frequency domain controllability and observability Gramians. Analytical treatment is given for associated properties of the Gramians, and the balancing and truncation steps of model reduction. A numerical example is used to demonstrate the new approach.*

Keywords. *Reduced Order Modelling; Frequency domain; Gramians; Balancing; Unstable systems.*

1. Introduction

Many practical systems are complex in nature and the orders of their mathematical models are too large. Control system design with such large order systems is computationally burdensome and the designed controller may be impractical for implementation. For these reasons, it is desirable to obtain a reduced order model for the plant, that represents the plant with sufficient fidelity. Among various techniques of reduced order modelling available, Moore's balancing technique [1,2] has gained much recognition and application. This method has many attractive features, but it has a shortcoming that it is applicable only for stable systems.

Several different methods have been suggested for applying the balancing procedure for model reduction of unstable systems. Enns [3] separates the unstable modes of the transfer function and applies Moore's balancing technique to the stable part. An eigenvalue shifting scheme is suggested by Santiago and Jamshidi [4]. Another method called Fractional Balanced Reduction [5] applies balance-and-truncate to a special representation of the graph operator of the plant.

In this paper, a different balancing technique is proposed for model reduction, which is applicable to unstable systems. It is formulated along the same rationale as the original Moore's technique, but the difference is that it uses different types of Gramians. In Moore's method, the Gramians involve integration over time, but the Gramians introduced and employed in the present work, involve integration over frequency. In the former, of course, the integrand involves time-domain impulse response, but in the latter, frequency response functions are used. The proposed method of model reduction is in two steps. The first step is to obtain a balanced realization of the original system, wherein, the frequency domain controllability and observability Gramians are equal and diagonal. The final step is to truncate this representation off the "weak" subsystem to yield the desired reduced model.

In Section 2, a review of Moore's balancing technique is given. In Section 3, the definitions of Frequency Domain Controllability and Observability Gramians are given and various properties are derived. The balancing similarity transformation is constructed in Section 4 and the model reduction by truncation is discussed in Section 5. Numerical example of an unstable system is furnished in Section 6, which illustrates the proposed model reduction algorithm. The suggested technique does fall short in several respects; such criticism alongwith some recommendations for further research can be found in the last section.

2. Moore's Balancing Method

In this section, we give a brief description of Moore's "balance and truncate" technique of model reduction. Consider an n^{th} order, stable and minimal, linear, time invariant, and multivariable system denoted by S(A,B,C,D). The state space equations of S are:

$$\dot{x}(t) = A\,x(t) + B\,u(t), \tag{2.1}$$

and,
$$y(t) = Cx(t) + D\,u(t). \tag{2.2}$$

The coefficient matrices A, B, C and D can be real or complex. The controllabillity Gramian ($\hat{P}$) and the observability Gramian ($\hat{Q}$) are defined as

$$\hat{P} = \int_0^\infty e^{At} B B^H e^{A^H t}\, dt, \tag{2.3}$$

and,
$$\hat{Q} = \int_0^\infty e^{A^H t} C^H C e^{At}\, dt. \tag{2.4}$$

These Gramians are Hermition and positive definite and can be calculated from the Lyapunov equations:

$$A\hat{P} + \hat{P}A^H = -BB^H, \tag{2.5}$$

$$A^H\hat{Q} + \hat{Q}A = -C^H C. \tag{2.6}$$

The internally balanced realization of S represents a set of coordinates, in which the controllability and observability Gramians are equal and diagonal [1,2]. It is known [1,2,3] that a similarity transformation can be performed to arrive at the balanced representation. We will denote the balanced representation by $\hat{S}_b$ ($\hat{A}_b$, $\hat{B}_b$, $\hat{C}_b$, $\hat{D}_b$) and its controllability and observability Gramians by

$$\hat{\Sigma}=\mathrm{diag}(\hat{\sigma}_1, \hat{\sigma}_2,\ldots,\hat{\sigma}_n), \qquad (2.7)$$

with

$$\hat{\sigma}_1 \geq \hat{\sigma}_2 \geq \ldots \hat{\sigma}_n, \qquad (2.8)$$

where, $\hat{\sigma}_i$ (i=1, ..., n) are called second order modes of the system S. It is assumed that the states of the balanced realization are arranged in accordance with (2.8). The following Lyapunov equations hold for the balanced realization $\hat{S}_b$:

$$\hat{A}_b\hat{\Sigma} + \hat{\Sigma}\hat{A}_b^H = -\hat{B}_b\hat{B}_b^H, \qquad (2.9)$$

and,

$$\hat{A}_b^H\hat{\Sigma} + \hat{\Sigma}\hat{A}_b = -\hat{C}_b^H\hat{C}_b. \qquad (2.10)$$

To obtain an r^{th} order (r<n) reduced order model of S, Moore [1] proposed partitioning $\hat{S}_b$ as follows:

$$\left[\begin{array}{c|c} \hat{A}_b & \hat{B}_b \\ \hline \hat{C}_b & \hat{D}_b \end{array}\right] = \left[\begin{array}{c:c|c} \hat{A}_{11} & \hat{A}_{12} & \hat{B}_1 \\ \hdashline \hat{A}_{21} & \hat{A}_{22} & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{C}_2 & \hat{D}_1 \end{array}\right], \qquad (2.11)$$

where, $A_{11}$and A_{22} are of dimensions r×r and (n-r)×(n-r) respectively. The reduced order model $\hat{S}_r(\hat{A}_r, \hat{B}_r, \hat{C}_r, \hat{D}_r)$ results by truncating last (n-r) states, the least controllable and observable ones. The coefficient matrices are:

$$\hat{A}_r=\hat{A}_{11}, \quad \hat{B}_r=\hat{B}_1, \quad \hat{C}_r=\hat{C}_1 \text{ and, } \hat{D}_r=\hat{D}_1. \qquad (2.12)$$

It has been proved that the reduced model $\hat{S}_r$ (for any order r) is stable, minimal and balanced with the Gramians equal and diagonal. Denoting the controllability and observability Gramians as $\hat{P}_r$ and $\hat{Q}_r$ respectively, we write

$$\hat{P}_r=\hat{Q}_r=\hat{\Sigma}_r=\mathrm{diag}\ (\hat{\sigma}_1, \ldots ,\hat{\sigma}_r). \qquad (2.13)$$

If the coefficient matrices A, B, C, D of the original system S are real, then the Gramians of S, and the coefficient matrices and Gramians of both of $\hat{S}_b$ and $\hat{S}_r$ will be real. In such a case, the Hermitian operator "*H*" can be replaced by transposition in the Lyapunov equations (2.5), (2.6), (2.9) and (2.10).

In order to make clear distinction with the proposed model reduction method, we redefine various terms that were used above in the description of Moore's model reduction method. The Gramians $\hat{P}$ and $\hat{Q}$ will hereafter be called as *time domain* controllability and observability Gramians respectively. The realization (or system), in which these Gramians are balanced, will be termed internally balanced in *time domain.* Moreover, the second order modes referred above will be redefined as *time domain* second order modes.

3. Frequency Domain Gramians

In this paper we introduce two Gramians, the Frequency Domain Controllability Gramian (FDCG) and the Frequency Domain Observability Gramian (FDOG). A detailed discussion of these Gramians is the subject of this section.

The full order system is assumed to be an n^{th} order, minimal, linear, time invariant and multivariable system denoted by S(A,B,C,D) and defined by state equations (2.1) and (2.2). However, here we will pose a restriction that the system is complex-valued (i.e., each of the outputs y(t) is complex), either by virtue of complex inputs u(t) or because of complex coefficient matrices A, B, C and D. We will, of course, make no limitation that S is a stable system.

3.1 Definitions:

DEFINITION 3.1:

The Frequency Domain Controllability Gramian (FDCG) for the system S is defined as,

$$P=\int_0^{\infty} F_c(j\omega)F_c^H(j\omega)\, d\omega, \qquad (3.1)$$

where,

$$F_c(j\omega)=[\mathcal{L}(e^{At}B)]|_{s=j\omega}=(j\omega I-A)^{-1}B \qquad (3.2)$$

As a result,

$$P=\int_0^{\infty}(j\omega I-A)^{-1}BB^H(-j\omega I-A^H)^{-1}d\omega \qquad (3.3)$$

DEFINITION 3.2:

The Frequency Domain Observability Gramian (FDOG) for the system S is defined as,

$$Q=\int_0^{\infty} F_o^H(j\omega)F_o(j\omega)\, d\omega, \qquad (3.4)$$

where,

$$F_o(j\omega)=[\mathcal{L}(Ce^{At})]|_{s=j\omega}=C(j\omega I-A)^{-1}. \qquad (3.5)$$

Thus,

$$Q=\int_0^{\infty}(-j\omega I-A^H)^{-1}C^H C(j\omega I-A)^{-1}\, d\omega \qquad (3.6)$$

3.2 Existence, Uniqueness, and Significance:

The Gramians defined above incorporate improper integrals. The integrals of both the Gramians exist if A has no eigenvalue at origin. When the integrals exist, they are unique. In other words, whenever the system S as no pole at origin, FDCG and FDOG are defined and have unique values.

It can be observed from the definition that FDCG inherently indicates how strongly the states are affected by the inputs, i.e., the degree of controllability of the system. Similarly FDOG indicates how strongly are the outputs affected by the states. In essence, FDOG is a measure of observability.

3.3 Computation:

Except for very trivial cases, no closed form expression for the Gramians is possible. Falling short of any algebraic matrix equations, that may be satisfied by these Gramians, the improper integrals can be evaluated using numerical techniques. It may be noted that the integrand in each of the integrals diminishes in magnitude as ω becomes large. In the present work, the integration is performed by a simple technique of segmentation into small stripes. The following Control-C program is written for computation of P:

```
j=sqrt(-1); P=0+j*0;
//n , deltaw, and wfinal are predefined.
for w=0:deltaw:wfinal,...
      Fc=inv(j*w*eye(n)-A)*B;...
      integrand=Fc*Fc';...
      P=P+integrand*deltaw;...
end;
```

3.4 Properties :

1. P and Q are square (n×n) matrices. If S has complex coefficient matrices, P and Q are also complex.

2. The elements on the principal diagonal of P and Q are nonegative real numbers as,

$$(P)_{i,i}=\sum_{k=1}^{r}\int_{o}^{\infty}|(F_c)_{i,k}(j\omega)|^2\, d\omega,\ (i=1,\ \dots\ ,n), \quad (3.7)$$

and, $$(Q)_{i,i}=\sum_{k=1}^{r}\int_{o}^{\infty}|(F_o)_{k,i}(j\omega)|^2\, d\omega,\ (i=1,\ \dots\ ,n). \quad (3.8)$$

3. P and Q are Hermitian matrices. Following is the proof for P; the proof for Q is analogous.

$$\begin{aligned} P^H &= [\int_o^{\infty} F_c F_c^H\, d\omega]^H \\ &= \int_o^{\infty}[F_c F_c^H]^H d\omega \\ &= \int_o^{\infty} F_c F_c^H\, d\omega = P. \end{aligned}$$

4. <u>CONJECTURE 3.1</u>: P and Q are positive definite matrices.

This has been observed numerically in several examples, but is not proved.

5. Now we will investigate the effect of a similarity transformation on the Gramians.

<u>THEOREM 3.1</u>: Suppose that the system S with state vector x(t) described by (2.1) and (2.2) has FDCG and FDOG as P and Q respectively. The new system with state vector x′(t) obtained by similarity transformation,

$$x(t)=Tx'(t) \quad (3.9)$$

has FDCG given by
$$P'=T^{-1}PT^{-H} \quad (3.10)$$

and FDOG given by
$$Q'=T^{H}QT \quad (3.11)$$

<u>Proof</u>:

$$\begin{aligned} P' &= \int_o^{\infty}[j\omega I-T^{-1}AT]^{-1}[T^{-1}B][T^{-1}B]^H[-j\omega I-(T^{-1}AT)^H]^{-1}d\omega \\ &= \int_o^{\infty}T^{-1}(j\omega I-A)^{-1}TT^{-1}BB^H\,T^{-H}T^H(-j\omega I-A^H)^{-1}T^{-H}\,d\omega \\ &= T^{-1}[\int_o^{\infty}(j\omega I-A)^{-1}\,BB^H(-j\omega I-A^H)^{-1}d\omega]T^{-H} \\ &= T^{-1}PT^{-H} \end{aligned}$$

$$\begin{aligned} Q' &= \int_o^{\infty}[-j\omega I-(T^{-1}AT)^H]^{-1}[CT]^H[CT][j\omega I-T^{-1}AT]^{-1}d\omega \\ &= \int_o^{\infty}[T^H(-j\omega I-A^H)T^{-H}]^{-1}[CT]^H[CT][T^{-1}(j\omega I-A)T]^{-1}d\omega \\ &= T^H[\int_o^{\infty}(-j\omega I-A^H)^{-1}(T^{-H}T^H)\,C^HC\,(TT^{-1})\,(j\omega I-A)^{-1}d\omega]T \\ &= T^HQT \end{aligned}$$

□

4. Balancing Transformation

In the last section, the effect of a similarity transformation on frequency-domain Gramians was explored. Now we formulate a procedure to find the transformation matix, which renders the Gramians to be equal and diagonal. The (frequency domain) balancing transformation matrix, T_b, will be constructed in a step-by-step manner in the following. To start with, we assume that the Gramians P and Q of the original system S have been computed.

1. If P is positive definite, it has a Cholesky factorization of the following form

$$P=R_c^H\,R_c, \quad (4.1)$$

where, R_c is n×n.

2. Now we construct a matrix Z_o defined as follows:

$$Z_o \triangleq R_c\,Q\,R_c^H \quad (4.2)$$

Since Z_o is Hermition ($Z_o^H=Z_o$), it has an orthonormal set of eigenvectors and all the eigenvalues are nonnegative. The modal decomposition is:

$$Z_o=V\Lambda\,V^{-1} \quad (4.3)$$

i.e., $$\Lambda=V^{-1}\,Z_o\,V, \quad (4.4)$$

where, $$V\,V^H=I, \quad (4.5)$$

and, Λ, expressed below, is a diagonal matrix containing the eigenvalues of Z_o on its diagonal.

$$\Lambda=\mathrm{diag}\ (\sigma_1^2,\ \sigma_2^2,\ \dots,\ \sigma_n^2\ ) \quad (4.6)$$

with $$\sigma_1\geq\ \sigma_2\ \geq\ \dots\ \geq\ \sigma_n. \quad (4.7)$$

Thus we have assumed that the eigenvectors of Z_o are so arranged in the matrix V that the eigenvalves $\{\sigma_i^2\}$ (i= 1, 2, ..., n) are in descending order . This sequencing is not necessary for balancing but will be required in the model - reduction step.

3. The desired frequency domain balancing transformation matrix, T_b, can be calculated with the expression

$$T_b \triangleq R_c^H\ V\Lambda^{-1/4} \quad (4.8)$$

□

The new realization with the state vector

$$x_b(t) \triangleq T_b^{-1}\,x(t) \quad (4.9)$$

is the following:

$$\dot{x}_b(t)=A_b\,x_b(t)+B_b u(t) \quad (4.10)$$

$$y(t)=C_b x_b(t)+D_b u(t), \quad (4.11)$$

where, $$A_b=T_b^{-1}A\,T_b, \quad (4.12)$$

$$B_b=T_b^{-1}B, \quad (4.13)$$

$$C_b=C\,T_b, \quad (4.14)$$

and, $$D_b=D. \quad (4.15)$$

The frequency domain Gramians of the system in the new coordinates are:

$$FDCG=P_b=\int_o^{\infty}(j\omega I-A_b)^{-1}B_bB_b^H(-j\omega I-A_b^H)^{-1}d\omega \quad (4.16)$$

$$FDOG=Q_b=\int_o^{\infty}(-j\omega I-A_b^H)^{-1}C_b^H\,C_b\,(j\omega I-A_b)^{-1}d\omega \quad (4.17)$$

The following theorem establishes that the transformed system (4.10) and (4.11) is balanced in frequency domain.

THEOREM 4.1: FDCG and FDOG given by (4.16) and (4.17) respectively are equal and diagonal.

$$P_b = Q_b = \Lambda^{1/2} = \text{diag}\ (\sigma_1,\ \sigma_2,\ \ldots,\sigma_n). \tag{4.18}$$

Proof:

$$
\begin{aligned}
P_b &= T_b^{-1} P T_b^{-H} \\
&= (\Lambda^{1/4} V^{-1} R_c^{-H})\ (R_c^H R_c)\ (\Lambda^{1/4} V^{-1} R_c^{-H})^H \\
&= \Lambda^{1/4} V^{-1} (R_c^{-H} R_c^H)\ (R_c R_c^{-1})\ V^{-H} \Lambda^{1/4} \\
&= \Lambda^{1/4} (\ V^H V)^{-1} \Lambda^{1/4} \\
&= \Lambda^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
Q_b &= T_b^H Q T_b \\
&= \Lambda^{-1/4} V^H R_c Q R_c^H V \Lambda^{-1/4} \\
&= \Lambda^{-1/4} V^H Z_o V \Lambda^{-1/4} \\
&= \Lambda^{-1/4} (V^H V) \Lambda (V^{-1} V) \Lambda^{-1/4} \\
&= \Lambda^{1/2}
\end{aligned}
$$

□

5. Reduction by Truncation

Since the frequency domain Gramians are equal and diagonal, the realization given in (4.10) and (4.11) is a balanced realization in frequency domain. The subscript "b" in the state vector (x_b), transformation matrix (T_b), coefficient matrices (A_b, B_b, C_b, D_b) stands for the property of being balanced in frequency domain. The proposed model reduction algorithm involves truncation of this balanced realization to remove the "weak subsystem" .

5.1 Truncation of the Balanced Representation:

As we noted earlier, we will assume that the diagonal elements of Λ are in descending order. Rewriting (4.6) and (4.7),

$$\Lambda^{1/2} = \text{diag}\ (\sigma_1, \sigma_2, \ldots,\ \sigma_n) \tag{5.1}$$

with

$$\sigma_1 \geq \sigma_2 \geq\ \ldots\ \geq \sigma_n. \tag{5.2}$$

$\sigma_i (i=1,2,\ \ldots\ ,n)$ will be called "frequency- domain second order modes". The magnitude of σ_i indicates the contribution of the i^{th} state of the frequency domain balanced realization to the frequency response.

PROPOSITION 5.1: Select an appropriate order r such that $\sigma_r \gg \sigma_{r+1}$ and partition the frequency domain balanced system given in (4.10) and (4.11) with the first r and the last (n−r) states as,

$$\left[\begin{array}{c|c} A_b & B_b \\ \hline C_b & D_b \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D_1 \end{array}\right] \tag{5.3}$$

where, $A_{11} \in \mathbb{C}^{r \times r}$ and $A_{22} \in \mathbb{C}^{(n-r)\times(n-r)}$. The subsystem corresponding to the first n and last (n−r) states will be termed "strong" and "weak" subsystems. The reduced order model $S_r(A_r,\ B_r,\ C_r,\ D_r)$ results by truncating the "weak" subsystem. The coefficient matrices are:

$$A_r = A_{11},\quad B_r = B_1,\quad C_r = C_1 \text{ and, } \quad D_r = D_1. \tag{5.4}$$

In state space differential equation form the reduced model is:

$$\dot{x}_r(t) = A_r x_r(t) + B_r u(t), \tag{5.5}$$

and,

$$y_r(t) = C_r x_r(t) + D_r u(t), \tag{5.6}$$

where, $x_r(t)$ is the state vector and $y_r(t)$ is the approximation of output.

□

5.2 Gramians of the Reduced Model:

We will call FDCG and FDOG of the reduced order model as P_r and Q_r respectively. It has been observed numerically in several different examples (but not analytically) that the diagonal elements of P_r are exactly equal to the corresponding diagonal elements of Q_r and they are close to the dominant frequency domain second order modes ($\sigma_1,\ \sigma_2,\ \ldots.,\ \sigma_r$). The off-diagonal elements are small complex numbers and they are not the same in the two Gramians. For clarity, we write the form of Gramians for a third order reduced model as

$$P_r = \begin{bmatrix} \mu_1 & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \mu_2 & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \mu_3 \end{bmatrix}, \tag{5.7}$$

and,

$$Q_r = \begin{bmatrix} \mu_1 & \delta_{12} & \delta_{13} \\ \delta_{12} & \mu_2 & \delta_{23} \\ \delta_{13} & \delta_{23} & \mu_3 \end{bmatrix}, \tag{5.8}$$

where, μ_1, μ_2 and μ_3 are positive real numbers and are quite close to σ_1, σ_2 and σ_3 respectively. The off-diagonal elements ϵ's and δ's are small numbers, which are complex in general.

6. Numerical Example

Now we illustrate the proposed model reduction scheme by a numerical example. The given system is of 3rd order, SISO and has the state- space description as in (2.1). The coefficient matrices are given below:

$$A = \begin{bmatrix} 1 & -j & 0 \\ 0 & -0.1+j0.02 & 0 \\ j0.1 & 0 & -0.2-j0.05 \end{bmatrix},\ B = \begin{bmatrix} 5-j0.2 \\ -5 \\ 1+j0.2 \end{bmatrix},$$

$$C = \begin{bmatrix} 5 & 2+j0.1 & -3 \end{bmatrix}, \quad \text{and } D=0.$$

The eigenvalues of the A matrix are

$$1,\quad -0.1+j0.02 \text{ and } -0.2-j0.05.$$

FDCG (P) and FDOG (Q) are given below:

$$P = \begin{bmatrix} 555.2 & -57.9+j468.9 & 105.4-j198.5 \\ -57.9-468.9 & 442.8 & -190-j84 \\ 105.4+j198.5 & -190+j84 & 102.56 \end{bmatrix}$$

$$Q = \begin{bmatrix} 34.22 & -50.1-j12.97 & -1.14-j24.89 \\ -50.1+j12.97 & 427.6 & -56.4+j129.1 \\ -1.14+j24.89 & -56.4-j129.1 & 59.6 \end{bmatrix}$$

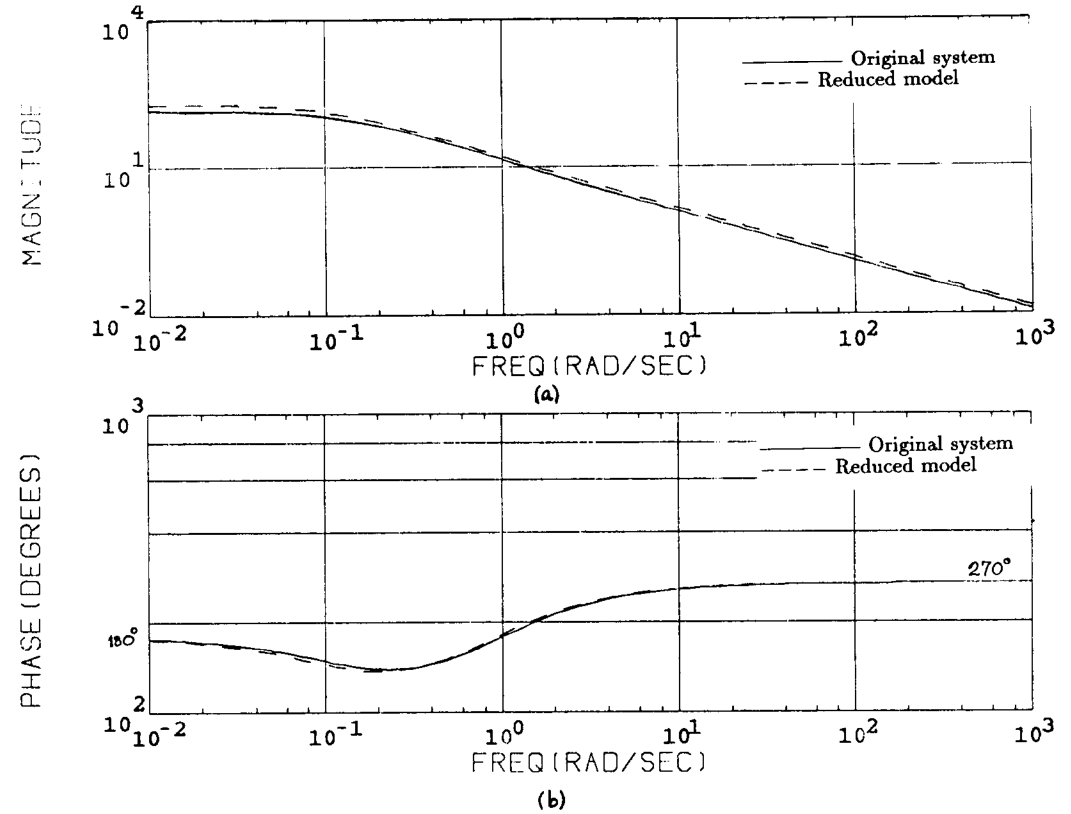

Figure 1 Frequency Responses for the Example

For balancing the transformation matrix, T_b, is computed to be:

$$T_b=\begin{bmatrix} 1.06 & 0.23-j0.918 & 0.321+j0.415 \\ -0.128-j0.966 & 0.188-j0.098 & 0.197-j0.197 \\ 0.233+j0.393 & 0.106+j0.298 & 0.838+j0.115 \end{bmatrix}$$

The frequency domain balanced representation given by equations (4.10) and (4.11) has the following coefficient matrices:

$$A_b=\begin{bmatrix} -0.06-j0.002 & -0.128-j0.139 & 0.054+j0.045 \\ -0.057+j0.18 & 0.909-j0.024 & -0.21+j0.176 \\ 0.06+j0.037 & -0.086-j0.26 & -0.148-j0.004 \end{bmatrix},$$

$$B_b=\begin{bmatrix} 2.08-j5.0 \\ -3.74+j4.42 \\ 0.5+j1.35 \end{bmatrix},$$

$$C_b=\begin{bmatrix} 4.43-j3.12 & 1.22-j5.66 & -0.49+j1.36 \end{bmatrix},$$

and, $D_b=0.$

The balanced Gramians are:

$$P_b=Q_b=\begin{bmatrix} 464.7 & 0 & 0 \\ 0 & 39.09 & 0 \\ 0 & 0 & 2.61 \end{bmatrix}.$$

Thus the frequency domain second order modes are :

$$\sigma_1=464.7,\quad \sigma_2=39.09 \quad \text{and,} \quad \sigma_3=2.61.$$

$\sigma_2 \gg \sigma_3$ and so we will obtain a second order reduced model. Partitioning and truncating A_b, B_b and C_b yield the following coefficient matrices for the reduced model:

$$A_r=\begin{bmatrix} -0.06-j0.002 & -0.128-j0.139 \\ -0.057+j0.18 & 0.909-j0.024 \end{bmatrix},$$

$$B_r=\begin{bmatrix} 2.08-j5.0 \\ -3.74+j4.42 \end{bmatrix},$$

$$C_r=\begin{bmatrix} 4.43-j3.12 & 1.22-j5.66 \end{bmatrix},$$

and, $D_r=0.$

The eigenvalues of A_r are:

$$-0.094+j0.012 \quad \text{and} \quad 0.942-j0.038$$

The Gramians are calculated to be :

$$P_r=\begin{bmatrix} 470.0 & -0.12-j0.16 \\ -0.12+j0.16 & 39.56 \end{bmatrix}$$

$$Q_r=\begin{bmatrix} 470.0 & -0.08-j0.19 \\ -0.08+j0.19 & 39.5 \end{bmatrix}$$

The diagonal element of these Gramians are equal and are close to the dominant frequency domain second order modes, σ_1 and σ_2. The off- diagonal terms are quite small and may be ignored. We may say that the reduced order model $S(A_r, B_r, C_r, D_r)$ has retained the dominant frequency domain second order modes and hence is an approximation of the original system $S(A, B, C, D)$.

In order to make a comparison with the original system, the responses were computed. The magnitude and phase frequency responses are given in Fig 1a and 1b, which display a good match between the original and reduced order systems.

7. Concluding Remarks

In this paper, a balancing method of model reduction is presented for linear time invariant multi-input muti-output systems, which may be stable or unstable. The approach adopted is to balance and truncate the frequency domain Gramians. The basic philosophy of model reduction in the proposed method is the same as in the conventional Moore's technique, but with the difference that the two techniques are based on two different types of Gramians.

If the coefficient matrices of the full order system are all real, the coefficient matrices of the reduced model obtained by the proposed technique turn out to be complex. This is not an acceptable situation. For this reason, the proposed technique can be used only for complex-valued systems, i.e., those, in which the outputs are complex-valued. This translates to the requirement that the inputs and/or the coefficient matrices must be complex. The outputs of the reduced model will also be complex and will approximate the (complex) outputs of the original system. It may be noted that there is no requirement that the states be complex-valued. This is because the method approximates the outputs, and not the states per se. Furthermore, there is another limitation in using the proposed method that the system must not have a pole at origin.

The frequency domain Gramians introduced in this paper need to be calculated using numerical methods directly from the definitions. The time-domain Gramians used in Moore's method can be obtained by solving Algebraic matrix equations (Lyapunov equations). No such equation has yet been found for the frequency domain Gramians.

Even though the (frequency domain) Gramians are, in general, complex valued, their diagonal elements are always positive real numbers. The respective diagonal elements of the controllability and observability Gramians of the reduced model are equal and they are close to the dominant frequency domain second order modes of the original system. It was observed numerically that the Gramians of the reduced order model are not balanced in the proposed technique, as there are (small) off-diagonal elements present. In contrast, the time-domain Gramians of the reduced model are balanced in Moore's technique.

A possible topic for further research is to find the equations (if any), whose solutions are the frequency domain controllability and observability Gramians. These equations should be instrumental in proving (or disproving) the conjecture about positive definiteness of the Gramians (Conjecture 3.1). They may also be helpful in exploring the characteristics of the Gramians of the reduced model.

Another direction for research would be to take the Gramians as diagonal matrices containing diagonal elements of FDCG and FDOG and formulate possible balancing and truncation phases. Since the Gramians would necessarily be real, perhaps it would be possible to obtain reduced order model with real coefficient matrices, if starting with original system of real coefficient matrices.

References

[1] B. C. Moore, "Principal Component Analysis in Linear Systems: Controllability, Observability and Model Reduction", IEEE Transactions on Automatic Control, Vol. AC-26, Feb 1981, pp 17-32.

[2] Keith Glover, "All Optimal Hankel Norm Approximations of Linear Multivariable Systems and their L^∞ Error Bounds", Int. J. Control, 1984, Vol. AC-39, No.6, pp1115-1193.

[3] D. F. Enns, "Model Reduction for Control System Design", Ph. D. Dissertation, Stanford University, 1984.

[4] J.M. Santiago and M. Jamshidi, "On the Extensions of the Balanced Approach for Model Reduction", Control-Theory and Advanced Technology, Vol. 2, No.2, June 1986, pp 207-226.

[5] David G. Meyer, "A Fractional Approach to Model Reduction", Proceedings of 1988 American Control Conference, pp 1041 - 1047.