01 Jan 1986

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Recommended Citation

G. E. Carlson, "ALTERNATE PRESENTATION OF FOURIER TRANSFORM DEVELOPMENT FROM FOURIER SERIES.," *IEEE Transactions on Education*, vol. E thru 29, no. 4, pp. 201 - 203, Institute of Electrical and Electronics Engineers, Jan 1986.
The definitive version is available at https://doi.org/10.1109/TE.1986.5570566

An Alternate Presentation of Fourier Transform Development from Fourier Series

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Abstract—**Students often have difficulty understanding the Fourier transform concept and its relation to Fourier series concepts and signal spectrum concepts. Conceptual problems occur for the student in understanding the transition from the line spectra obtained with Fourier series to the continuous frequency spectra obtained with Fourier transforms and in understanding the reason that the resulting amplitude spectrum is in fact an amplitude density spectrum. An alternate presentation of the development of the Fourier transform from the complex exponential Fourier series is presented. This presentation has been found to improve students' understanding of the nature of the spectral information obtained for an aperiodic signal with the Fourier transform.**

I. Introduction

An important part of continuous-time system analysis courses is the introduction of students to the concept of the representation of a signal by its frequency content. Thus, methods of obtaining the amplitude spectrum and phase spectrum of a signal are presented. This usually takes the form of first developing the trigonometric and complex exponential Fourier series representations of a periodic signal to identify the characteristics of the frequency components in the signal. The trigonometric Fourier series is used to identify the single-sided amplitude and phase spectra of the signal as line spectra indicating the amplitude and phase of each frequency component of the signal. This provides a tie with earlier sinusoidal signal analysis performed by the students and gives them an intuitive feel for the concepts of amplitude and phase spectra. The line spectra plotted from the complex exponential Fourier series representation of the periodic signal introduce the students to the concept of double-sided amplitude and phase spectra and shows them that the same signal frequency content information is contained in this spectral representation.

The amplitude and phase spectra of an aperiodic signal is usually found by first obtaining a complex exponential Fourier series representation of the periodic signal with period T which matches the aperiodic signal in the time interval $-T/2 < t < T/2$ and then taking the limit as T approaches ∞. Examples of various methods of presenting this limiting approach are given in [1]–[4]. One of the main conceptual problems for students with these methods is in the transition from the line spectra to the continuous frequency spectra.

Manuscript received December 4, 1985; revised May 20, 1986.
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IEEE Log Number 8610104.

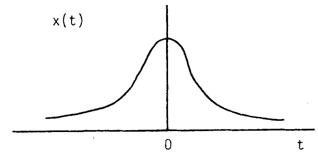

Fig. 1. Example aperiodic signal $x(t)$.

The same basic limiting approach as used in other presentation methods is used in the alternate presentation method discussed here. The new feature of the method is the use of spectral concepts and redefined spectral sketches in conjunction with the mathematical developments to illustrate more clearly the concept and nature of the continuous frequency amplitude density and phase spectra. Mathematical detail included here is of the same level as normally used in introductory continuous-time systems courses.

II. Alternate Presentation Method

An aperiodic signal $x(t)$ as shown in Fig. 1 is given. It is assumed that $x(t)$ satisfies the Dirchlet conditions, that is 1) $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$, 2) $x(t)$ has a finite number of maxima and minima in any finite interval, and 3) $x(t)$ has a finite number of finite discontinuities in any finite interval.

Construct a complex exponential Fourier series which correctly represents $x(t)$ over the interval $-T/2 < t < T/2$. The above Dirchlet conditions assure that this can be done.

The resulting complex exponential Fourier series is

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_o t} \tag{1}$$

where

$$f_o = \frac{1}{T} = \text{fundamental frequency} \tag{2}$$

and

$$X_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, e^{-j2\pi n f_o t}\, dt. \tag{3}$$

This series correctly matches $x(t)$ over the interval $-T/2 < t < T/2$ but repeats outside this interval. Thus,

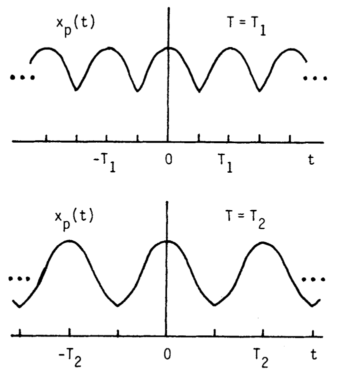

Fig. 2. Periodic extension of portion of $x(t)$.

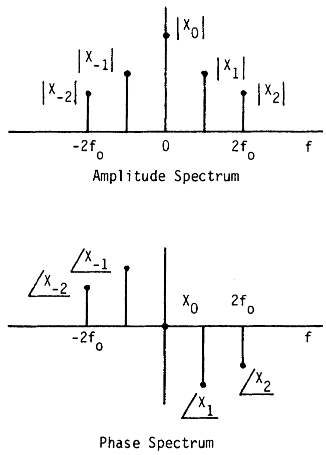

Fig. 3. Amplitude and phase spectra for $x_p(t)$.

the resulting complex exponential Fourier series actually is a representation of the periodic repetition of the portion of $x(t)$ in the time interval $-T/2 < t < T/2$. This is referred to as the periodic extension $x_p(t)$ of the portion of $x(t)$ and is illustrated in Fig. 2 for two different values of T.

The Fourier series coefficients are in general complex and thus may be written as $X_n = |X_n| \exp [\underline{/X_n}]$ where $|X_n|$ is the magnitude of the complex coefficient and $\underline{/X_n}$ is the angle associated with the complex coefficient. These give the amplitude and phase information for the signal $x_p(t)$. The lower frequency terms of the double-sided amplitude and phase spectra of $x_p(t)$ are used to illustrate a portion of these spectra and are assumed to be as shown in Fig. 3.

To obtain the amplitude and phase spectra for the original signal $x(t)$, take the limit as $T \to \infty$ so $x_p(t) = x(t)$ for $-\infty < t < \infty$. As T becomes larger, the lines in the spectra become closer together since $f_o = 1/T$. The amplitude spectrum lines also become smaller since the Dirchlet conditions assure that

$$\lim_{T\to\infty} \int_{-T/2}^{T/2} x(t)\, e^{-j2\pi n f_o t}\, dt < \infty \tag{4}$$

which means that

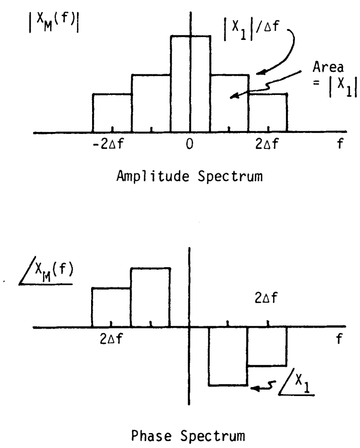

Fig. 4. Redefined amplitude and phase spectra for $x_p(t)$.

$$X_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, e^{-j2\pi n f_o t}\, dt \to 0 \qquad \text{as } T \to \infty. \tag{5}$$

Therefore, as $T \to \infty$, an infinite number of infinitesimally small amplitude spectra lines occur in any finite region of the frequency axis.

To circumvent the above problem and illustrate more clearly the nature of the continuous frequency amplitude density and phase spectra which result, it is convenient to redefine the amplitude and phase spectra for $x_p(t)$ for a given T to be sets of adjacent rectangles of widths f_o centered at the locations of the spectral lines of $x_p(t)$. These redefined spectral plots are shown in Fig. 4 where $\Delta f = f_o$ = fundamental frequency.

The redefined phase spectra is

$$\underline{/X_M(f)} = \sum_{n=-\infty}^{\infty} \underline{/X_n}\, \Pi\left(\frac{f - n\Delta f}{\Delta f}\right) \tag{6}$$

where $\Pi[(f - f_1)/f_2]$ is a rectangular pulse of height 1 and width f_2 centered at $f = f_1$. Thus, the height of the rectangles is made equal to the phase of the spectral component within the rectangle width for the phase spectrum.

The redefined amplitude spectrum is

$$|X_M(f)| = \sum_{n=-\infty}^{\infty} \frac{|X_n|}{\Delta f}\, \Pi\left(\frac{f - n\Delta f}{\Delta f}\right). \tag{7}$$

Thus, the area under the rectangles is made equal to the amplitude of the spectral component within the rectangle width. Note that the height of the rectangle (i.e., $|X_n|/\Delta f$) has units of amplitude per unit frequency. The redefined amplitude spectrum is therefore an amplitude density spectra showing the amplitude contribution per unit frequency as a function of frequency.

With the above redefined spectra, the limiting operation can proceed. The redefined spectrum is obtained from the complex quantities

$$X_M(f) = |X_M(f)| \exp\, (\underline{/X_M(f)}). \tag{8}$$

At $f = n\Delta f$,

$$X_M(n\Delta f) = |X_M(n\Delta f)| \exp \underline{/X_M(n\Delta f)} \tag{9}$$

where

$$|X_M(n\Delta f)| = \frac{|X_n|}{\Delta f} = T|X_n| \tag{10}$$

and

$$\underline{/X_M(n\Delta f)} = \underline{/X_n}. \tag{11}$$

Now let $T \to \infty$ so $x_p(t) = x(t)$ for $-\infty < t < \infty$ so the redefined amplitude and phase spectra are the spectra of $x(t)$. In order to compute these continuous frequency (i.e., defined for all frequencies) spectra from the continuous frequency spectra $|X_M(f)|$ and $\underline{/X_M(f)}$ for a particular arbitrary frequency f_a, let $n \to \infty$ as $T \to \infty$ so that $n\Delta f = f_a$. Then define

$$X(f_a) = \lim_{T\to\infty} X_M(f_a) = \lim_{\substack{T\to\infty \\ n\Delta f = f_a}} X_M(n\Delta f). \tag{12}$$

Therefore,

$$\begin{aligned} X(f_a) &= \lim_{\substack{T\to\infty \\ n\Delta f = f_a}} |X_M(n\Delta f)| \exp\,[\underline{/X_M(n\Delta f)}] \\ &= \lim_{\substack{T\to\infty \\ n\Delta f = f_a}} T|X_n| \exp\,[\underline{/X_n}] \\ &= \lim_{\substack{T\to\infty \\ n\Delta f = f_a}} TX_n \\ &= \lim_{\substack{T\to\infty \\ n\Delta f = f_a}} \int_{-T/2}^{T/2} x(t)\, e^{-j2\pi n\Delta f t}\, dt \\ &= \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f_a t}\, dt. \end{aligned} \tag{13}$$

Since f_a is arbitrary, this can be written in terms of the general frequency f as $X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$. The function $X(f)$ is called the Fourier transform of $x(t)$. Note that its magnitude $|X(f)|$ is the double-sided amplitude density spectra of $x(t)$ and that its angle $\underline{/X(f)}$ is the phase spectrum of $x(t)$. $X(f)$ is referred to as the frequency spectrum of $x(t)$ and is a continuous frequency function since $X_M(f)$ is a continuous frequency function.

Next investigate the limit as $T \to \infty$ of the Fourier series representation of $x_p(t)$. Since the Fourier series representation of $x_p(t)$ matches $x(t)$ for $-T/2 < t < T/2$, then it will match $x(t)$ for all t (i.e., $-\infty < t < \infty$) when $T \to \infty$. Since $X_M(n\Delta f) = TX_n$, then $X_n = X_M(n\Delta f)\Delta f$ and

$$\begin{aligned} x_p(t) = \hat{x}(t) &= \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_o t} \\ &= \sum_{n=-\infty}^{\infty} X_M(n\Delta f)\, e^{j2\pi n f_o t}\, \Delta f. \end{aligned} \tag{14}$$

Taking the limit as $T \to \infty$, or equivalently as $\Delta f \to 0$,

$$\begin{aligned} x(t) &= \lim_{T\to\infty} x_p(t) \\ &= \lim_{\Delta f \to 0} \sum_{n=-\infty}^{\infty} X_M(n\Delta f)\, e^{j2\pi n\Delta f t}\, \Delta f. \end{aligned} \tag{15}$$

The values of $X_M(n\Delta f)\, e^{j2\pi n\Delta f t}$ are the values of $X_M(f)\, e^{j2\pi f t}$ at $f = n\Delta f$ which are the values at regularly spaced Δf intervals and thus occur within adjacent intervals of Δf along the f coordinate. Therefore, the limit is a Riemann integral and can be written as

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi t}\, dt \tag{16}$$

since

$$\lim_{T\to\infty} X_M(f) = X(f). \tag{17}$$

This is referred to as the inverse Fourier transform and provides a method for determining a signal from its Fourier transform.

III. Conclusion

An alternate method of presenting the transition from the Fourier series to the Fourier transform has been presented. It uses spectral concepts to aid the mathematical development and a redefined spectral form for the spectral information obtained with the Fourier series. It has been used a number of times in the classroom and been found to aid the students' understanding of the Fourier transform and the spectra of an aperiodic signal.

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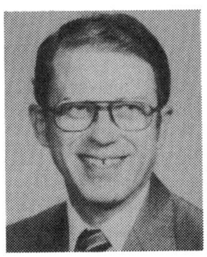

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