01 Jan 1973

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Recommended Citation

B. Park, "On The Method Of Stationary Phase," *Proceedings of the IEEE*, vol. 61, no. 8, p. 1165, Institute of Electrical and Electronics Engineers, Jan 1973.
The definitive version is available at https://doi.org/10.1109/PROC.1973.9237

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On the Method of Stationary Phase

B. K. PARK

Abstract—**The method of stationary phase (MSP), which is an asymptotic method of integration, nevertheless yields general (nonasymptotic) results when applied to certain integrals involving the spherical function. In such cases one may drop the requirement of far-field or high frequency from the results.**

The purpose of this letter is to clarify some ambiguities which seem to exist over the region of validity of the method of stationary phase (MSP) when applied to certain types of integrals. The MSP is an asymptotic method [1]–[3] and applicable to integrals of the form

$$f(x) = \int_a^b e^{jxh(t)} g(t)\, dt \tag{1}$$

where x is large (x is called the asymptotic parameter) and there is at least a point where the phase function $xh(t)$ is stationary between the finite or infinite limits a and b.

The particular types of integrals considered here involve the spherical function e^{jkR}/R (of course, this is also the familiar free-space Green's function). Let us consider the two-dimensional Fourier transform integral of it [3], [4]:

$$\frac{e^{jkR}}{R} = \frac{j}{2\pi} \int_{-\infty}^{\infty}\int \frac{e^{j\boldsymbol{k}\cdot\boldsymbol{R}}}{k_z}\, dk_x dk_y \tag{2}$$

where $\boldsymbol{k}=k_x\boldsymbol{a}_x+k_y\boldsymbol{a}_y+k_z\boldsymbol{a}_z$, $k_z=\sqrt{k^2-k_x^2-k_y^2}$, and $\boldsymbol{R}=x\boldsymbol{a}_x+y\boldsymbol{a}_y+|z|\boldsymbol{a}_z$. Equation (2) can be proved by the standard techniques of complex integration. Nevertheless, if one attempts to evaluate the integral using the MSP, assuming large parameter kR (the high-frequency and/or the far-field case), he finds that the point of stationary phase is given by

$$\boldsymbol{k}_0 = \left(k\frac{x}{R}, k\frac{y}{R}, k\frac{|z|}{R}\right) \tag{3}$$

and that the asymptotic result is none other than the exact spherical function e^{jkR}/R. A question then arises as to how an asymptotic method yields an exact result, more specifically, how the combination of the phase function expanded up to second order (and no more) and the denominator up to zeroth order about the stationary point makes the value of the integral *exactly* unchanged. Apparently, the error introduced in approximating the phase function $\boldsymbol{k}\cdot\boldsymbol{R}$ is exactly made up for as the denominator k_z is replaced by its stationary value $k|z|/R$ over the infinite limits.

The above result indicates that if an integral is of the form

$$\int_{-\infty}^{\infty}\int \frac{e^{j\boldsymbol{k}\cdot\boldsymbol{R}}}{k_z} F(k_x, k_y)\, dk_x dk_y \tag{4}$$

the MSP will still give a general (nonasymptotic) result, provided $F(k_x, k_y)$ is approximately independent of its arguments. This is quite a stringent condition. This condition on F can be relaxed at the expense of requiring kR to be accordingly large. However, the condition on kR may not have to be as restrictive as heretofore believed for the MSP to be valid.

The MSP also produces the exact identity when directly applied to Sommerfeld's integral representation [3]:

$$\frac{e^{jkR}}{R} = \int_0^\infty \frac{J_0(\lambda r)\exp\left[\sqrt{\lambda^2-k^2}\,|z|\right]\lambda\, d\lambda}{\sqrt{\lambda^2-k^2}} \tag{5}$$

where $r=\sqrt{x^2+y^2}$.

Another exact identity which can be proved by the MSP is

$$\frac{e^{jk|z|}}{k} = \frac{1}{2\pi j}\int_{-\infty}^{\infty}\int \frac{e^{jkR}}{R}\, dxdy. \tag{6}$$

Normally, the use of the MSP here would indicate the high-frequency approximation (k is large, but R need not be large here). To extend, we can approximately evaluate integrals of the form

$$\int_{-\infty}^{\infty}\int F(x, y)\frac{e^{jkR}}{R}\, dxdy \tag{7}$$

by the MSP, and the restriction on $F(x, y)$ in connection with the value of k is similar to that in the previous example.

In order to get a better idea of how the behavior of F is related to the requirement on the asymptotic parameter in the MSP, let us examine a few examples of F by way of integral (4). First, any F which has zero value at the stationary point will give zero MSP result regardless of the behavior of F outside the stationary point. Assuming this is not the case, if F is a straight line with slope A, then the MSP result is a valid result only if $kR \gg 1$, regardless of the slope A. For example, if $F=Ak_x$, we can integrate (4) both ways (exact and the MSP), noting for exact integration the identity

$$\int_{-\infty}^{\infty}\int Ak_x\frac{e^{j\boldsymbol{k}\cdot\boldsymbol{R}}}{k_z}\, dk_x dk_y = \frac{2\pi A}{j}\frac{\partial}{\partial x}\frac{e^{jkR}}{R}. \tag{8}$$

Comparing the two results yields the condition $kR \gg 1$. If F is a parabola, e.g., $F\sim k_x^2$, it can be shown that the MSP requires $kx^2/R \gg 1$. We can also find conditions for some other F's such as k_x^3, k_xk_y, etc. By comparing the actual F with these cases, we may be able to find in some cases more specific conditions on the parameter for the MSP.

For some functions F, the MSP may not be applicable even for an extremely large parameter value. Such cases may correspond, in terms of electromagnetic propagation, to either the space wave no longer dominating and thus the surface wave also being considered, or simply the MSP failing to yield a good approximation even for the space wave alone. It is well known that the method of steepest descent [1], [3], [4] is commonly used for such cases.

The fact that the MSP may give general results for the aforementioned types of integrals apparently has been overlooked in the past.

Manuscript received March 1, 1973; revised April 18, 1973.
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Acoustic Surface-Wave Matched Filters Using a MOSFET Array

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Abstract—**An acoustic surface-wave matched filter which combines a planar ZnO overlay transducer to generate acoustic surface waves on silicon and a MOSFET array to detect the acoustic surface-wave energy has been developed, thus providing a signal-processing technique which will be compatible with MOS/LSI technology.**

Manuscript received November 2, 1972; revised January 30, 1973. This work was supported by the Rome Air Development Center under Contract F30602-71-C-0037.
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