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# Analysis Of The N-Wire Exponential Line

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## Analysis of the $n$-Wire Exponential Line

*Abstract*—**A new approach to distributed network analysis is presented with the analysis of the $n$-wire exponential line as example. This approach provides the distributed network's transfer matrix in a form useful for both transient and steady-state network calculations.**

This letter presents a new approach to distributed network analysis using the analysis of the $n$-wire exponential line as example.

An $n$-wire variable parameter transmission line is shown in Fig. 1. Figure 2 shows a portion of an equivalent circuit for the incremental length $\Delta x$ of the line located at position $x$ from the right line boundary. Functions $z_i(x)=r_i(x)+sL_i(x)$ specify respective series line impedances per meter, and functions $y_{ij}(x)=g_{ij}(x)+sC_{ij}(x)$ specify shunt admittances per meter between lines $i$ and $j$.

Defining column vectors and matrices

$$V(x) \equiv \operatorname{col}(V_i(x)), \qquad I(x) \equiv \operatorname{col}(I_i(x))$$

$$Z(x) \equiv \operatorname{diag.}(z_i(x)), \; Y(x) \equiv [y_{ij}(x)]$$

$$i, j = 1, 2, \cdots, n,$$

permits writing Kirchhoff voltage and current relations for the equivalent circuit as

$$V(x+\Delta x) = \Delta X \cdot Z(x) I(x+\Delta x) + V(x),$$

$$I(x+\Delta x) = \Delta x Y(x) V(x) \; WI(x).$$

Dividing the above difference equation by $\Delta x$ and letting $\Delta x \to 0$ gives the generalized telegraphist's equations

$$\frac{\partial V(x)}{\partial x} = Z(x) I(x)$$

$$\frac{\partial I(x)}{\partial x} = Y(x) V(x),$$

or

$$\frac{\partial}{\partial x}\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} = \begin{bmatrix} 0 & Z(x) \\ Y(x) & 0 \end{bmatrix}\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} = K(x)\begin{bmatrix} V(x) \\ I(x) \end{bmatrix}. \tag{1}$$

The $n$-wire *exponential* line is characterized by differential equation (1) wherein

$$Z(x) = Ze^{2ax} \quad \text{and} \quad Y(x) = Ye^{-2ax},$$

$Y$ and $Z$ being constant matrices. This generalized exponential line problem was solved recently[1] by expanding col $(V(x), I(x))$ in its Taylor series about $x=0$ and summing the series to the closed form that will be developed in a simpler fashion as follows.

A mathematical statement of the exponential line problem is

$$\frac{\partial}{\partial x}\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} = \begin{bmatrix} 0 & Ze^{2ax} \\ Ye^{-2ax} & 0 \end{bmatrix}\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} = K(x)\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} \tag{2}$$

plus

$$\begin{bmatrix} V(0) \\ I(0) \end{bmatrix} = \begin{bmatrix} V_2 \\ I_2 \end{bmatrix},$$

the variables at the right terminals of the network. Transforming from solution variable, col $(V(x), I(x))$, in (2) to position column vector $P(x)$ via the transformation

$$\operatorname{col}(V(x), I(x)) = \begin{bmatrix} Ie^{ax} & 0 \\ 0 & Ie^{-ax} \end{bmatrix} P(x) = J(x) P(x), \tag{3}$$

transforms (2) into

$$\frac{\partial P}{\partial x} = [J_{(x)}^{-1} K(x) J(x) - J_{(x)}^{-1} \dot{J}(x)] P(x) \tag{4}$$

or

$$\frac{\partial P}{\partial x} = \begin{bmatrix} -aI & Z \\ Y & aI \end{bmatrix} P(x) = QP(x). \tag{5}$$

$(\;)^T$, $(\;)^{-1}$, and $(\dot{\;})$ denote transposition, inversion, and differentiation, respectively.

Since $Q$ is independent of $x$, the solution of (5) is

$$P(x) = \exp(Qx) \cdot P(0), \tag{6}$$

and recalling transformation (3), the desired solution is

$$\operatorname{col}(V(x), I(x)) = [J(x) \cdot \exp(Qx) \cdot J(0)] \operatorname{col}(V(0), I(0))$$

$$= [J(x) \cdot \exp(Qx)] \operatorname{col}(V_2, I_2), \tag{7}$$

or

$$\operatorname{col}(V(x), I(x)) \equiv T(x) \cdot \operatorname{col}(V_2, I_2). \tag{8}$$

$T(x)$ is the transfer matrix of the $n$-wire exponential line. Straightforward calculation[1] yields

$$T(x) = \begin{bmatrix} Ie^{ax} & 0 \\ 0 & Ie^{-ax} \end{bmatrix} \begin{bmatrix} \cosh \Gamma x - a\Gamma^{-1} \sinh \Gamma x & Z(\Gamma^T)^{-1} \sinh \Gamma^T x \\ Y\Gamma^{-1} \sinh \Gamma x & \cosh \Gamma^T x + a(\Gamma^T)^{-1} \sinh \Gamma^T x \end{bmatrix} \tag{9}$$

where $\Gamma^2 = a^2 I + ZY$.

This solution, (9), may be used in transient or steady-state exponential line analysis or design problems. Many interesting line components should now be conceived and developed from this solution. Specialized solutions for the lossless and diffusion types of multiwire exponential lines should provide a number of interesting new components.

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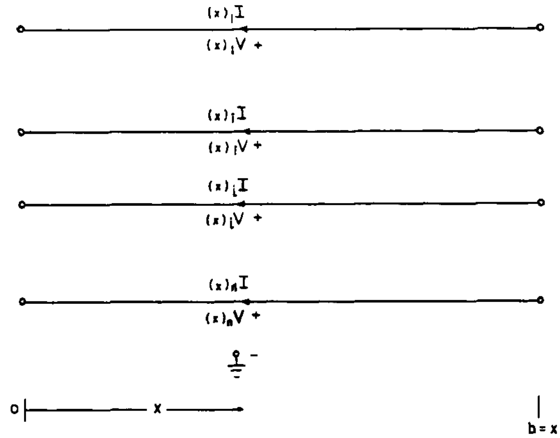

Fig. 1. $n$-wire line.

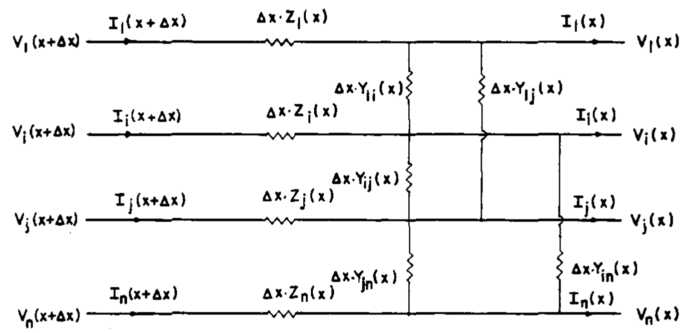


Fig. 2. Equivalent circuit.

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[1] C. A. Vandivort and E. C. Bertnolli, "Determining the transfer matrix of tapered multiware transmission lines," presented at the 10th Midwest Symp. on Circuit Theory, Purdue University, May 1967.