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Abstract

Curves and equations are presented from which the exact performance of truncated sequential tests can be determined for one important case: the biased square-law detector for the detection of rapidly fading targets. The method of generating functions is used to derive probability distributions for sample size. It is shown how these probability distributions can be used to determine truncation errors and the effects of multiple-resolution elements. Sample calculations are performed to determine the effects of a particular truncation procedure.

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Introduction

Most search radars scan the angular search region uniformly and process a fixed number of samples of the received waveform to determine the presence or absence of a target. Sequential detection is an alternate radar detection technique in which a variable number of samples is processed in each angular position. There are ways of taking advantage of sequential detection with constant angular scanning systems, but the most straightforward and generally the most efficient use of sequential detection is in conjunction with electronically scanned antenna systems. For this case, a sequential detection test proceeds as follows.

On the occurrence of the ith sample of the received waveform, a cumulative test variable x_i is formed and compared to two thresholds A and B. If x_i is greater than A, a target detection is announced; if x_i is less than B, a no-target decision is made and a new angular position is investigated; if x_i is between A and B, another sample is taken in the same angular position and the test is repeated.

The number of samples tested in each angular position is now a random variable, and if multiple parallel tests are being performed (e.g., to investigate several range or Doppler resolution cells) the time spent in one angular position may be quite large. Thus, the tests are usually forced to end (truncated) after some maximum number of samples.

Wald, an early pioneer in the theory of sequential hypothesis testing, proposed that the test variable be the likelihood ratio.[1] This type of test is known as the sequential probability ratio test. In radar terminology, Wald and Wolfowitz have shown that the sequential probability ratio test will require fewer samples, on the average, than a fixed sample test with the same false-alarm and detection probabilities.[2] Thus, if the required beam scanning versatility is available, there are two ways to take advantage of the benefits of sequential detection:

1) for the same false-alarm probability, detection probability, and signal-to-noise ratio as with a fixed sample detector, the sequential detector will allow a smaller average time to search a given angular region;
2) for the same false-alarm and detection probabilities, and with an average search time equal to the search time of a uniformly scanning antenna, the sequential detector will require less signal-to-noise ratio than a fixed sample detector.

A number of investigators have shown, using the theory developed by Wald, that sequential detection offers considerable improvement over fixed sample detectors. Though no attempt is made to give a complete listing of the work done, Bussgang and Middleton,[3] Preston,[4] and Blasbalg[5] are representative. The benefits of sequential detection are accrued at the expense of an increase in signal processing complexity. In many cases this increase in signal processing complexity is a small price to pay for

the increase in system sensitivity, but a proper trade-off must be made for each situation. Before a proper trade-off can be made, exact solutions to three problems must be obtained:

1) the effect of the test variables on false-alarm probability, detection probability, and average sample size;
2) the effects of truncation on the error probabilities and average sample size;
3) the effect of performing multiple tests in each angular position on the average sample size.

Using some results of Albert,[8] Kendall developed exact equations for average sample size and false-alarm probability for the biased square-law detector.[6] With proper interpretation, Kendall's equations also yield the exact detection probability for a class of rapidly fading targets.[7] In this paper, the method of generating functions is used to extend Kendall's results and determine exact conditional probability distributions for sample size. These probability distributions are used to analyze truncation errors and the effects of multiple-resolution elements. In order to facilitate sequential test design, curves depicting test performance are provided for a variety of test parameters.

Nontruncated Tests

By assuming that the test ends exactly at the boundaries, Wald derived the following approximations to the false-alarm and detection probabilities:[1]

$$P_f \cong \frac{1 - e^B}{e^A - e^B} \tag{1}$$

$$P_d \cong \frac{e^A(1 - e^B)}{e^A - e^B}, \tag{2}$$

where A and B are the upper and lower test boundaries, respectively. Equations (1) and (2) make the design of a sequential test quite simple. All one need do is select the desired value of P_f and P_d, and from (1) and (2) determine the test boundaries that will yield these values. No sampling distributions need be derived, and although the sample size is now a random variable, the average sample size can be approximated by another of Wald's formulas.[1] However, sampled data tests do not end exactly at the boundaries (with nonzero probability), and the error in Wald's approximations due to the "excess over the boundaries" may be considerable for many important radar situations.

Albert[8] developed integral equations for the study of a much more general sequential test than Wald's, with Wald's test included as a special case. These integral equations yield exact results when they are solvable. Kendall has analyzed the important case of the biased square-law detector with signal absent.[6] There are several reasons for studying the square-law detector. First of all it can be shown that the square-law detector is the likelihood ratio detector for constant cross-section radar targets when the signal-to-noise ratio is low, and most important fixed sample radar detection analyses assume a square-law detector.[7],[9],[10] More important to the present study, however, is the fact that the square-law detector is the likelihood ratio detector for a common class of rapidly fading signals [7, case 2] at all signal-to-noise ratios. Rapidly fading signals are important because virtually all targets of tactical interest have constantly varying backscattering cross sections, and slowly fading targets can be converted to rapidly fading targets by proper use of frequency agility.[11]

The sequential test variable for rapidly fading targets[2] assumes the form (see the Appendix)

$$x_i = \sum_{j=1}^{i} \left[\frac{\nu y_j^2}{2(1 + \nu)} - b \right], \tag{3}$$

where ν is the design signal-to-noise ratio, y_j is a sample of the envelope of the received waveform divided by the rms value of the noise, and

$$b = \log_e (1 + \nu).$$

In this case, the probability density function of y_j^2 is exponential both for Gaussian noise and signal plus Gaussian noise. Therefore, with proper interpretation of the parameters, Kendall's equations can be used to determine average sample size, false-alarm probability, and detection probability. The relevent equations are (38) and (42) in the Appendix.

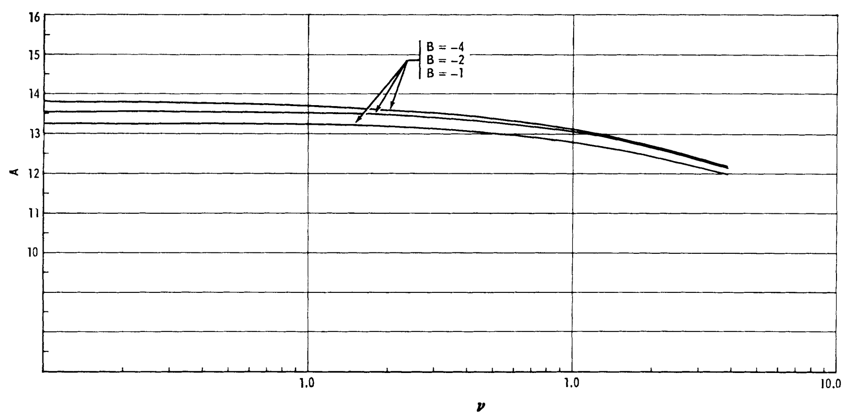
It is well known that the average sample size is a function of the correspondence between the actual signal level and the design signal level.[3] However, in typical search radar situations the vast majority of the search region is unoccupied by targets, and the average search time is controlled by the zero-signal average sample size $m_1(0)$. Thus, the test design and the radar detection performance will revolve around $m_1(0)$. The evaluation of (28) and (42) is difficult. Therefore, in order to facilitate the test design for control of false-alarm probability and $m_1(0)$, Figs. 1 and 2 are plotted for a variety of the test parameters. It is noted that values of the upper boundary A are only plotted for $P_f = 10^{-6}$. Values of A for other false-alarm probabilities can be easily found from the relationship

$$\Delta A = - \log_e f_m,$$

where ΔA is the amount by which the value of A from Fig. 1 is modified and f_m is the factor by which 10^{-6} is

[1] Equations (1) and (2) assume that the test variable is the logarithm of the likelihood ratio.

[2] In this paper, the phrase "rapidly fading target" always refers to the class of radar target conforming to the Swerling model [7, case 2].



ig. 1. Plot of upper boundary A versus signal-to-noise ratio ν for P_f—10^{-6}.

Fig. 2. Plot of average sample size versus signal-to-noise ratio for $P_f \leq 10^{-4}$.

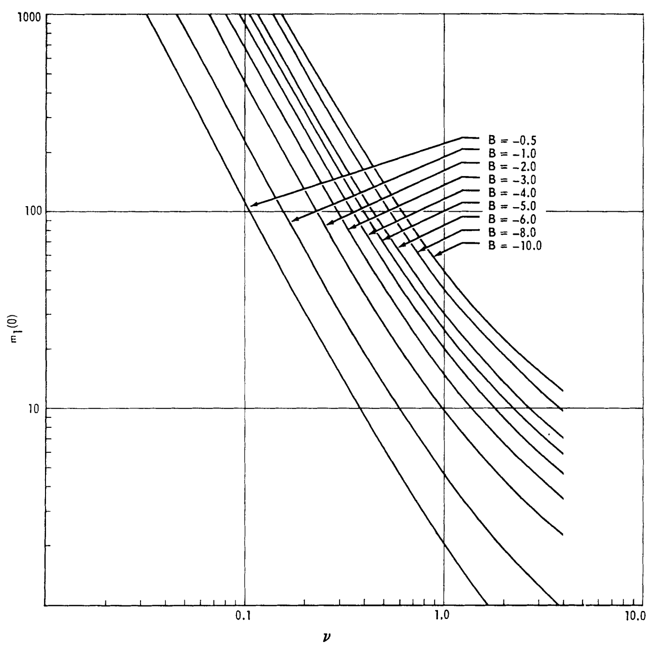
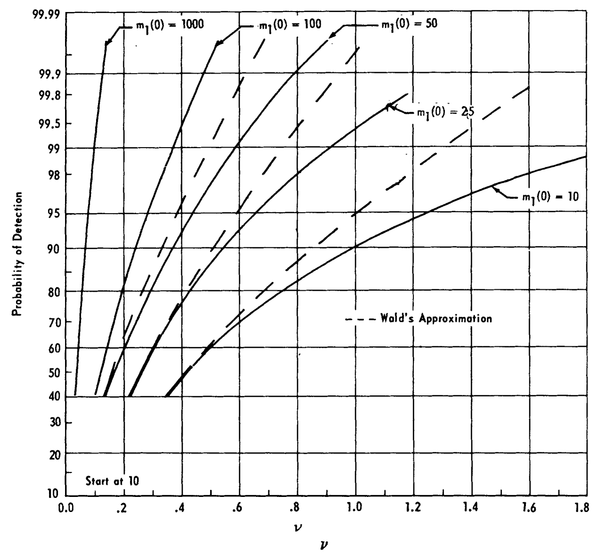


Fig. 3. Probability of detection versus signal-to-noise ratio.



multiplied. For example, A must be increased by 2.3 to obtain a false-alarm probability of 10^{-7}.

A is not shown as a parameter in Fig. 2, because for the false-alarm probabilities being considered ($P_f \leqq 10^{-4}$), $m_1(0)$ is independent of the upper boundary A. This is intuitively satisfying since the test rarely ends by crossing the upper boundary.

In the Appendix it is shown that (28) can be adapted to determine detection probabilities for rapidly fading targets. The format used in Fig. 3 for presenting the detection probabilities seems to be best in this case. A value of the lower boundary B is implicit for each value of $m_1(0)$ and signal-to-noise ratio. For the values of P_f being considered, the detection probability is independent of the false-alarm probability. Selected curves of detection probability using Wald's approximations are presented in Fig. 3 for comparison purposes. It can be seen that the difference between the actual detection probability and Wald's approximation is significant for high detection probabilities or low values of $m_1(0)$.

Truncated Tests

Truncation refers to the procedure of forcing a decision after a fixed number of samples are processed. Truncation is necessary when sequential detection is used with a search radar because it is possible to spend an exorbitant amount of time in any one angular position. This effect is most apparent when signals are present with a marginal signal-to-noise ratio between zero and the design value, although it can also happen with signal absent.

Truncation is achieved by choosing a level T between the test boundaries A and B. After a fixed number of trials k a target decision is made if $x_k > T$ and a no-target decision is made if $x_k \leqq T$. The truncated false-alarm probability can be written as

$$P_{f,t} = P_{f,k} + \epsilon_{f,k}, \tag{4}$$

where $P_{f,k}$ is the probability of a false-alarm by exceeding the boundary A at a trial $\leqq k$ and $\epsilon_{f,k}$ is the probability that on the kth trial the following conditions are satisfied:

1) $B < x_i < A \qquad i = 1, 2, 3, \cdots, k;$
2) $T < x_k < A.$

The probability of occurrence of the second condition depends on the probability of occurrence of the first, since the first condition implies that $B < x_k < A$. Therefore, the two conditions listed above are simultaneously satisfied if a decision would have been made at a trial greater than k with a nontruncated test, and $T < x_k < A$ given that $B < x_k < A$. Let q_k be the probability of making a decision at a trial $> k$, and $f_0(x_k)$ be the probability density function of x_k with signal absent. Then $\epsilon_{f,k}$ can be written as

$$\epsilon_{f,k} = \frac{q_k \int_T^A f_0(x_k) dx_k}{\int_B^A f_0(x_k) dx_k}. \tag{5}$$

It can be seen that an exact analysis of the truncation problem requires knowledge of the probability distribution of sample size. Procedures for determining sample-size probability distributions for quantized signals have been obtained by Wald[1] and by Proakis,[12] but the only available results for nonquantized signals are approximations based on large-sample theory.[1],[13],[16] In the Appendix, exact probability distributions for sample size are obtained by means of generating functions and Albert's integral equations. Fig. 4 shows graphs of q_k versus $k/m_1(0)$ for various values of B, which were obtained by means of (40). The graphs are independent of A for values of $P_f = 10^{-4}$.

The probability of detection with a truncated test may be obtained in exactly the same fashion as in (4) and (5). Let $\epsilon_{d,k}$ be the probability that conditions 1) and 2) listed above are satisfied when signal is present, and $P_{d,k}$ be the probability of detection at a trial $\leqq k$. Then the truncated detection probability becomes

$$P_{d,t} = P_{d,k} + \epsilon_{d,k}, \tag{6}$$

where

$$\epsilon_{d,k} = \frac{r_k \int_T^A f_s(x_k) dx_k}{\int_B^A f_s(x_k) dx_k}, \tag{7}$$

$f_s(x_k)$ is the probability density function for x_k, and r_k is the probability of making a decision at a trial $> k$ when signal is present. It is shown in the Appendix that (40) may again be used for calculating r_k, if the parameters in the equation are properly interpreted. Figs. 5 and 6 show

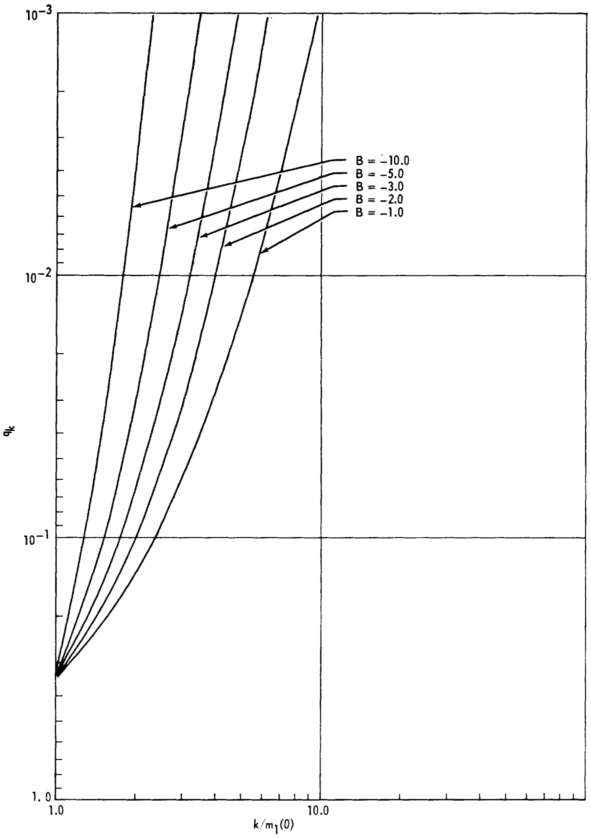


Fig. 4. Probability of making a decision beyond k when signal is absent versus the ratio of k to zero-signal average sample size.

graphs of r_k versus $m_1(0)$. In this case, the graphs depend both on A and B. The values for A and B in Figs. and 6 cover the range of values that would typically be used.

$P_{f,k}$ and $P_{d,k}$ can be determined from (28). However, it is usually not necessary to calculate these quantities. For reasonable truncation points, $P_{f,k}$ will be very nearly equal to the design value of P_f, and $P_{d,k}$ is very nearly equal to $P_d - r_k$. Thus, the increase in P_f due to truncation will essentially be $\epsilon_{f,k}$ and the decrease in P_d due to truncation will essentially be $r_k - \epsilon_{d,k}$. The values of the integrals in (5) and (7) are the last remaining pieces of information necessary to determine completely $P_{f,t}$ and $P_{d,t}$. Useful approximate procedures for evaluating these integrals are given in Swerling[7] and Marcum.[9]

The zero-signal truncated average sample size $m_{1,t}(0)$ will be smaller than $m_1(0)$. Nontruncated tests, which would have lasted longer than the truncation stage k, will yield a sample size k. Thus, $m_{1,t}(0)$ can be written as

$$m_{1,t}(0) = \sum_{j=1}^{k-1} j p_j + k q_k, \tag{8}$$

where p_j is the probability of making a decision at trial j. Alternatively, by (41) it can be shown that

$$m_{1,t}(0) = \sum_{j=0}^{k} q_j. \tag{9}$$

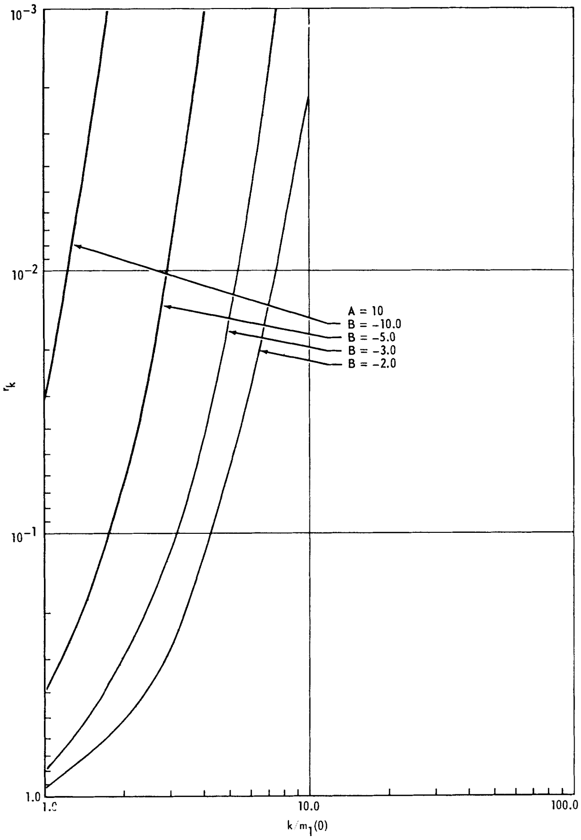


Fig. 5. Probability of making a decision beyond trial k when signal is present versus the ratio of k to zero-signal average sample size.

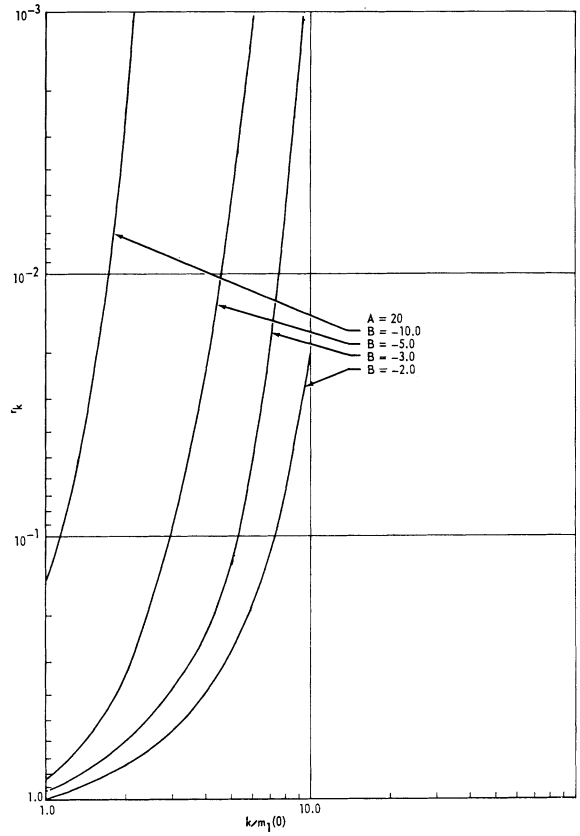
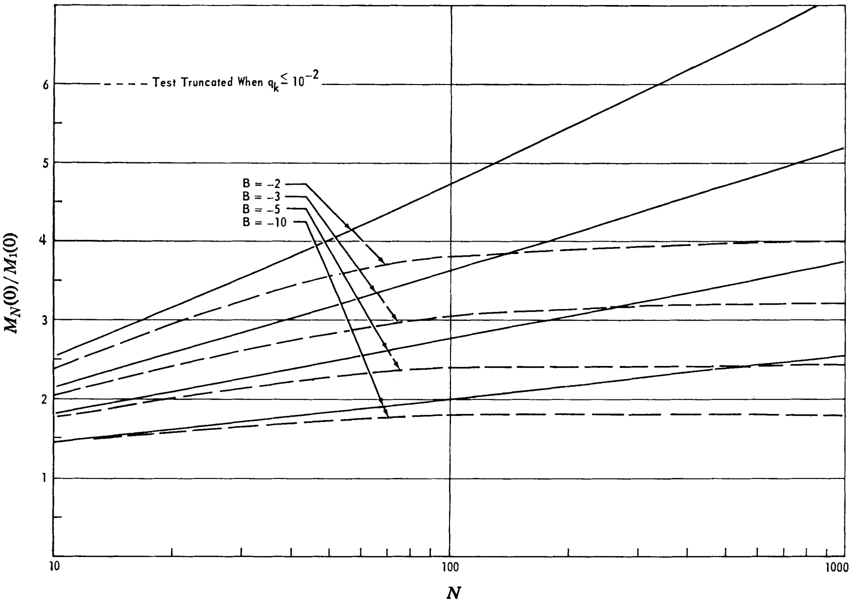


Fig. 6. Probability of making a decision beyond trial k when signal is present versus the ratio of k to zero-signal average sample size.

Fig. 7. Ratio of zero-signal average sample size for N parallel tests to the average sample size for one test.



Equation (9) may be evaluated by means of (40), and is a more convenient way of obtaining $m_{1,t}(0)$ than (8).

An "optimum" truncation procedure might be defined as the procedure that allows truncation at the smallest possible test stage without significantly changing the average sample-size, false-alarm, or detection probability. A truncation point chosen too small could result in a test no more efficient than a fixed sample test. On the other hand, a truncation point chosen too large would not efficiently accomplish the desired purpose of limiting the time spent in one angular position. Finding the optimum procedure involves a selection of the best combination of A, B, T, and truncation point k. The primary intent of this paper is to present equations and curves from which the effects of a variety of truncation procedures can be analyzed, rather than determine the optimum procedure. For illustrative purposes, however, the following comments suggest a procedure that, although not optimum, is believed to be realistic.

The suggested truncation procedure is based on the practical point of view that one would usually like to truncate the test at a point no longer than 3 or 4 times $m_1(0)$. From Fig. 4, it can be seen that if k is chosen so that $q_k = 10^{-2}$, $k/m_1(0)$ will be less than 4 for $B \leqq -2$. The most frequently used values of B will be less than -2, so this is a reasonable choice for k.

If A is chosen so that the nontruncated value of P_f is the desired value, and T is chosen so that $\epsilon_{f,k}$ is negligible, the detection probability will be drastically reduced. It is much better to increase A and reduce the value of T toward the lower boundary. This cannot be carried to an extreme, however, because A affects r_k (see Figs. 5 to 7), and $T=B$ would seriously affect the false-alarm probability (5). After some trial and error, the best compromise found was to select A and T so that $P_{f,k}$ and $\epsilon_{f,k}$ are each equal to one-half the desired value of false-alarm probability. For this purpose the approximation previously mentioned for evaluating the integrals in (5) is quite useful [9, page 166]. By use of this approximation, a good first choice for T was found to be 0.75 to 0.8 times A.

Sample calculations have shown that $m_1(0)$ is essentially unaffected by the choice of truncation point. Thus, the procedure for design of truncated tests, which has been suggested by the preceding remarks, may be summarized as follows:

1) for a given value of $m_1(0)$, determine the required signal-to-noise ratio for the desired P_d (Fig. 3) and determine the required lower boundary B (Fig. 2);
2) select A so that $P_{f,k}$ is one-half the desired false-alarm probability (Fig. 1);
3) select the truncation point so that $q_k = 10^{-2}$;
4) select T so that $\epsilon_{f,k}$ is one-half the desired false-alarm probability (0.75 to 0.8 times A is usually satisfactory);
5) calculate $P_{d,t}$ and if there is a significant difference from the desired value of P_d, increase B and repeat the design procedure.

Table I shows selected calculations based on the given truncation procedure. Truncation points as well as comparisons of P_d and $P_{d,t}$ are shown. It can be seen that the effect of this truncation procedure on detection probability is significant. This, of course, is a result of trying to truncate so close to $m_1(0)$.

TABLE

Sample Truncation Results

A	B	T	ν	$m_1(0)$	k	P_d	$P_{d,t}$	P_f
14.0	-3	11.2	1.38	10	32	0.965	0.846	10^{-6}
14.5	-3	11.6	0.29	100	320	0.955	0.80	10^{-6}
14.0	-2	11.2	0.975	10	40	0.90	0.73	10^{-6}
14.5	-2	11.6	0.23	100	400	0.89	0.70	10^{-6}
18.6	-3	14.8	1.38	10	32	0.965	0.768	10^{-8}
19.1	-3	15.25	0.29	100	320	0.955	0.71	10^{-8}

Although one might find a truncation procedure that would allow truncation at slightly closer points without loss of sensitivity, it is safe to say that any significant reduction of the truncation points in Table I would cause a further loss in sensitivity. Obviously, truncation cannot be carried to an extreme or it will result in nothing more than a complicated fixed sample test. It is not possible to discuss all of the reduced sensitivity tests that could be designed. However, should tests other than the one described here be desired, the equations and curves in this paper will provide the information necessary to analyze their performance.

Multiple Range Gates

Since $1-q_j$ is the probability that a single test ends at a trial less than or equal to j, the probability that N independent tests end on or before the jth trial is $(1-q_j)^N$. Thus, by (41) the average sample size for N independent tests when signal is absent becomes

$$m_N(0) = \sum_{j=0}^{\infty} [1 - (1 - q_j)^N]. \qquad (10)$$

Equation (10) yields the average sample size for the range-gated sequential test proposed by Helstrom.[14] This is not the most efficient multiple-hypothesis test that could be used, but it is the most straightforward to analyze and simplest to implement.

Equation (40) in conjunction with (10) provides the information necessary to calculate $m_N(0)$. Fig. 7 shows plots of $m_N(0)/m_1(0)$ for various values of B. Again, for the values of P_f considered in this paper, the curves are independent of A.

In a manner analogous to the single test case, the zero-signal average sample size for N independent tests truncated at trial k can be written

$$m_{N,t}(0) = \sum_{j=0}^{k} [1 - (1 - q_j)^N]. \qquad (11)$$

Curves of $m_{N,t}(0)$ are shown in Fig. 7 for comparison. It can be seen that the value of $m_{N,t}(0)$ rapidly approaches the value of the truncation point k as N is increased. This is indicative of the fact that when the number of parallel

tests is large, the probability that at least one test truncates is large. No test can last longer than k, and since the multiple test lasts until all parallel tests have ended, the average sample size approaches k. It might seem that the penalty for performing a large number of parallel tests is not as great with truncated tests as it is with nontruncated tests. This is not true, as we will see from the following examples.

Examples and Comparisons to Fixed Sample Detection

Assume that a truncated sequential test is to be designed to detect a rapidly fading target, and that the pertinent design parameters have been determined from system studies to be

$$m_1(0) = 100$$
$$P_d = 0.8$$
$$P_f = 10^{-6}.$$

The sequential test will be compared to a fixed sample test, which is designed to process 100 samples.

After applying the truncation procedure, the required values of B and signal-to-noise ratio are -3 and 0.29, respectively. From Fig. 1 (with extrapolation) the value of A is found to be 14.5.

From Fig. 7 in Swerling[7] the required signal-to-noise ratio of a fixed sample test is 0.555. Thus, the improvement in sensitivity by use of the sequential detector is 2.82 dB.

Now suppose it is desired to design parallel sequential tests for a system with the following characteristics:

$$m_N(0) = 100$$
$$P_d = 0.8$$
$$P_f = 10^{-8}$$
$$N = 100.$$

$P_f = 10^{-8}$ is required in this case so that the system false-alarm probability is still approximately 10^{-6}. After applying the truncation procedure for multiple tests, the values of the test parameters are found to be $m_1(0)=37$, $\nu=0.64$, $B=-4$, $A=10.1$. From Fig. 8 in Swerling,[7] a fixed sample detector would require a signal-to-noise ratio of 0.692. Thus, the sensitivity improvement by use of a sequential detector is only 0.34 dB and the use of 100 parallel detectors has cost 2.48 dB in sensitivity. It can be seen that we have come very close to designing a complicated fixed sample detector. More sensitivity could have been retained had we been willing to truncate at a later stage.

The preceding examples are only given to illustrate use of the curves. Other choices for design parameters could be made that would show either more or less favorable sequential detection than the examples given.

Appendix

Sample-Size Probability Distributions

We will assume that the rapidly fading radar target has the following characteristics:

1) the signal return from the target is uncorrelated from pulse to pulse;
2) the probability density function of the target's back-scattering cross section is exponential.

Using this model, the probability density function of the signal plus Gaussian noise is given by (see Swerling[7])

$$f_1(y) = \begin{cases} \dfrac{y \exp(-y^2/2) \exp[\nu y^2/2(1+\nu)]}{(1+\nu)} & y \geq 0 \\ 0 & \text{elsewhere,} \end{cases} \tag{12}$$

where y is the signal-plus-noise envelope divided by the rms value of the noise and ν is the average signal-to-noise ratio. The probability density function for noise alone is given by (12) with $\nu=0$.

$$f_0(y) = \begin{cases} y \exp(-y^2/2) & y \geq 0 \\ 0 & \text{elsewhere.} \end{cases} \tag{13}$$

By use of (12) and (13), the likelihood ratio λ_j for independent samples is obtained:

$$\lambda_j = \prod_{i=1}^{j} \frac{\exp[\nu y_j^2/2(1+\nu)]}{1+\nu}. \tag{14}$$

The natural logarithm of the likelihood ratio, which we will denote by x_j, becomes

$$x_j = \sum_{i=1}^{j} [\nu y_j^2/2(1+\nu) - \log_e(1+\nu)]. \tag{15}$$

It can be seen that a biased square-law envelope detector is the likelihood ratio detector for all values of signal-to-noise ratio.

Let $z_j = x_j - x_{j-1}$. Then by (13) and (14) the probability density function when signal is absent is obtained:

$$f_0(z_j) = \begin{cases} [(1+\nu)/\nu] \exp[-(1+\nu)(z_j+b)/\nu] & z_j \geq -b \\ 0 & \text{elsewhere.} \end{cases} \tag{16}$$

By use of (12) and (14), we have for signal present,

$$f_1(z_j) = \begin{cases} (1/\nu) \exp[-(z_j+b)/\nu] & z_j \geq -b \\ 0 & \text{elsewhere.} \end{cases} \tag{17}$$

Both (16) and (17) are of exponential form. If we write a general probability density function for z_j as

$$f(z_j) = \begin{cases} (2/a^2) \exp[-(2/a^2)(z_j+b)] & z_j \geq -b \\ 0 & \text{elsewhere,} \end{cases} \tag{18}$$

then the notation conforms to Kendall's.[6] By properly relating the parameters in (18) to those in (16) and (17), Kendell's results can be directly used to obtain average sample size, false-alarm probability, and detection probability. The intent of this Appendix is to show how the generating function approach can be used to obtain single-

stage false-alarm and detection probabilities, as well as probability distributions for sample size and related results.

We will use the general notation of (18) but for simplicity of discussion we will first assume that signal is absent, and obtain results related to the false-alarm probability and zero-signal average sample size.

Let $p_{f,k}$ be the probability of a false alarm at the kth trial of the test, and $p_{c,k}$ be the probability of making a no-target decision at the kth trial. The probability of making either decision at the kth trial is simply $p_k = p_{f,k} + p_{c,k}$. The generating function for p_k is given by (see Feller[15])

$$P(z) = \sum_{k=0}^{\infty} p_k z^k \tag{19}$$

with generating functions for $p_{f,k}$ and $p_{c,k}$ similarly defined. Since $P(1)=1$, $P(z)$ converges at least for $-1<z<1$.

Let $p_{f,k}(x_0)$ be the probability that a test starting with some general value x_0 ends with a false alarm at the kth trial. We will specialize to $x_0=0$ later, and when we write $p_{f,k}$ it is assumed that the test starts with $x_0=0$. Albert[8] has shown that

$$p_{f,0}(x_0) = \pi_A(x_0) \tag{20}$$

$$p_{f,k}(x_0) = \pi_0(x_0) \int_{-\infty}^{\infty} p_{f,k-1}(y) dF(y \mid x_0), \tag{21}$$

where $F(x_i \mid x_{i-1})$ is the probability distribution function governing transitions of the test variable from x_i to x_{i-1} and

$$\pi_A(x) = \begin{cases} 1 & A \leqq x \\ 0 & \text{elsewhere} \end{cases} \tag{22}$$

$$\pi_0(x_0) = \begin{cases} 1 & B < x_0 < A \\ 0 & \text{elsewhere.} \end{cases} \tag{23}$$

Thus

$$\begin{aligned} P_f(x_0; z) &= z \int_A^{\infty} \left(\frac{2}{a^2}\right) \exp\left[-2(y - x_0 + b)/a^2\right] dy \\ &\quad + z \int_{\max(x_0-b,B)}^{A} \left(\frac{2}{a^2}\right) P_f(y; z) \\ &\quad \cdot \exp\left[-2(y - x_0 + b)/a^2\right] dy \\ &= z \exp\left[-2(A - B + b)/a^2\right] \\ &\quad + z \int_{\max(x_0-b,B)}^{A} \left(\frac{2}{a^2}\right) P_f(y; z) \\ &\quad \cdot \exp\left[-2(y - x_0 + b)/a^2\right] dy \\ &\qquad\qquad A < x_0 < B, \end{aligned} \tag{24}$$

where

$$\max(x_0 - b, B) = \begin{cases} x_0 - b & B \leqq x_0 - b \\ B & B \geqq x_0 - b. \end{cases} \tag{25}$$

By a procedure similar to Kendall's[6] or by use of Laplace transforms, the solution of (24) is obtained.

$$P_f(x_0; z) = \frac{z \exp\left[-2(A - x_0 + b)/a^2\right] g\left[D(x_0 - B); z; Db\right]}{g\left[D(A - B + b); z; Db\right]}, \tag{26}$$

where

$$g(x; z; Db) = \begin{cases} 1 & 0 < x \leqq Db \\ 1 + \sum_{j=1}^{n} z^j \dfrac{(jDb - x)^j}{j!} & Db \leqq nDb \leqq x \leqq (n+1)Db. \end{cases} \tag{27}$$

By letting $x_0 = 0$ we obtain the generating function for $\{p_{f,k}\}$.

$$P_f(z) = \frac{z \exp\left[-2(A + b)/a^2\right] g\left[-DB; z; Db\right]}{g\left[D(A - B + b); z; Db\right]}. \tag{28}$$

Note that $P_f(1)$ yields the result previously obtained by Kendall for false-alarm probability. Of course, we must let

$$a^2/2 = \nu/(1 + \nu)$$

to get the proper result. If we let $a^2/2 = \nu$ then $P_f(1)$ yields the detection probability. A procedure for obtaining coefficients of z^k from (28) will be given later.

In an exactly similar manner, the generating function for $\{p_{c,k}(x_0)\}$ is given by a solution of

$$\begin{aligned} P_c(x_0; z) &= \pi_B(x_0) + z\pi_0(x_0) \int_{x_0-b}^{\infty} (2/a^2) P_c(y; z) \\ &\quad \cdot \exp\left[-2(y - x_0 + b)/a^2\right] dy, \end{aligned} \tag{29}$$

where

$$\pi_B(x) = \begin{cases} 1 & x \leqq B \\ 0 & \text{elsewhere.} \end{cases} \tag{30}$$

It is clear that

$$P_c(x_0; z) = \begin{cases} 1 & x_0 \leqq B \\ 0 & x_0 \geqq A \end{cases} \tag{31}$$

$$\begin{aligned} P_c(x_0; z) &= z \int_{\max(x_0-b,B)}^{A} (2/a^2) P_c(y; z) \\ &\quad \cdot \exp\left[-2(y - x_0 + b)/a^2\right] dy \\ &\quad + z \int_{\min(x_0-b,B)}^{B} (2/a^2) \\ &\quad \cdot \exp\left[-2(y - x_0 + b)/a^2\right] dy \\ &\qquad\qquad B < x_0 < A, \end{aligned} \tag{32}$$

where

$$\min(x_0 - b, B) = \begin{cases} x_0 - b & x_0 - b \leqq B \\ B & x_0 - b \geqq B. \end{cases} \tag{33}$$

By means of Laplace transforms, the solution to (32) is obtained:

$$P_c(0; z) = 1 + (1 - z)e^{-2B/a^2} \cdot \frac{g[-DB; z; Db]w[D(A - B + b); z; Db]}{g[D(A - B + b); z; Db]} + (z - 1)e^{-2B/a^2}w[-DB; z; Db] - \frac{\exp[-2(A + b)/a^2]g[-DB; z; Db]}{g[D(A - B + b); z; Db]}, \tag{34}$$

where x_0 has been set equal to zero, and

$$w(x; z; Db) = \begin{cases} \sum_{i=0}^{n} z^i \exp(-2x/a^2D) - \sum_{j=1}^{n}\sum_{i=0}^{j-1} z^j \frac{(jDb - x)^i}{i!(2D/a^2)^{j-i}} & Db \leqq nDb \leqq x \leqq (n + 1)Db \\ \exp(-2x/a^2D) & 0 < x \leqq Db. \end{cases} \tag{35}$$

The generating function for sample size is obtained by combining (28) and (34).

$$P(z) = P_f(z) + P_c(z) = (1 - z)e^{-2B/a^2} \cdot \frac{g[-DB; z; Db]w[D(A - B + b); z; Db]}{g[D(A - B + b); z; Db]} + (z - 1)e^{-2B/a^2} \cdot w[-DB; z; Db] + (z - 1) \cdot \frac{\exp[2(A + b)/a^2]g[-DB; z; Db]}{g[D(A - B + b); z; Db]} + 1. \tag{36}$$

Let q_k be the probability that a test with signal absent ends at a stage greater than k. Then

$$q_k = \sum_{j=k+1}^{\infty} p_k \tag{37}$$

and the generating function for $\{q_k\}$ is given by

$$Q(z) = \sum_{k=0}^{\infty} q_k z^k. \tag{38}$$

$Q(z)$ can be obtained from $P(z)$ by the relation (see Feller[15])

$$Q(z) = \frac{1 - P(z)}{1 - z}. \tag{39}$$

Thus, from (36)

$$Q(z) = e^{-2B/a^2}w(-DB; z; Db) + \frac{\exp[-2(A + b)/a^2]g(-DB; z; Db)}{g[D(A - B + b); z; Db]} - e^{-2B/a^2} \cdot \frac{g(-DB; z; Db)w[D(A - B + b); z; Db]}{g[D(A - B + b); z; Db]}. \tag{40}$$

It can be seen by differentiating both sides of

$$(1 - z)Q(z) = 1 - P(z)$$

that

$$P'(1) = Q(1) = \sum_{k=0}^{\infty} q_k. \tag{41}$$

But $P'(1)$ is equal to the average sample size $m_1(0)$. Thus we have

$$m_1(0) = e^{-2B/a^2}w[-DB; Db] + P_f\{1 - \exp[2(A - B + b)/a^2] \cdot w[D(A - B + b); Db]\}, \tag{42}$$

which again agrees with the result previously obtained by Kendall when $a^2/2$ is properly interpreted.

It is hopeless to try and obtain explicit expressions for the coefficients of z_k from (28) and (40). However, recursive relations for $p_{f,k}$ and q_k can be obtained by multiplying both sides of the equations by $g[D(A-B+b); z; Db]$ and equating coefficients of z^k.

Applying this procedure to (28) we obtain $s_0=0$, $s_1=1$, and

$$s_k = \begin{cases} u_{k-1} - \sum_{i=1}^{k-1} s_i v_{k-i} & 2 \leqq k \leqq m + 1 \\ -\sum_{i=1}^{k-1} s_i v_{k-i} & m + 2 \leqq k \leqq n \\ -\sum_{i=l}^{k-1} s_i v_{k-i} & k = n + l;\ l = 1, 2, 3, \cdots, \end{cases} \tag{43}$$

where

$$s_k = p_{f,k} \exp[2(A + b)/a^2] \tag{44}$$

$$u_j = \frac{(jb + B)^j}{j!}(2/a^2)^j e^{-j2b/a^2} \tag{45}$$

$$v_j = \frac{[jb - (A - B + b)]^j}{j!}(2/a^2)^j e^{-j2b/a^2} \tag{46}$$

and m and n are determined by

$$0 \leqq mb \leqq -B \leqq (m + 1)b \tag{47}$$

$$0 \leqq nb \leqq (A - B + b) \leqq (n + 1)b. \tag{48}$$

Similarly, from (40) we obtain $q_0=1$, and

$$q_k = \begin{cases} 1 - f_k - \sum_{i=1}^{k} p_{f,i} + \sum_{i=0}^{k-1} s_{i+1} g_{k-i} \\ 1 \leqq k \leqq m \\ -\sum_{i=0}^{k} p_{f,i} + \sum_{i=0}^{k-1} s_{i+1} g_{k-i} \\ m+1 \leqq k \leqq n \\ \sum_{i=l}^{k-1} s_{i+1} g_{k-i} - \sum_{i=l+1}^{k} p_{f,i} \\ k = n + l;\ l = 1, 2, 3, \cdots, \end{cases} \tag{49}$$

where m and n are determined by (47) and (48), $f_0 = g_0 = 0$, and

$$f_j = e^{-2B/a^2} \sum_{i=0}^{j-1} \frac{(jb+B)^i}{i!} (2/a^2)^i e^{-j2b/a^2} \qquad j > 0 \tag{50}$$

$$g_j = e^{-2B/a^2} \sum_{i=0}^{j-1} \frac{[jb - (A - B + b)]^i}{i!} (2/a^2)^i e^{-j2b/a^2} \qquad j > 0. \tag{51}$$

It is again noted that (49) can be used for r_k (defined in the body of the paper) if $a^2/2$ is set equal to ν. For signal absent, $a^2/2$ is interpreted as $\nu/(1+\nu)$. Equations (43) through (51) were used to generate the curves of Figs. 4 through 7.

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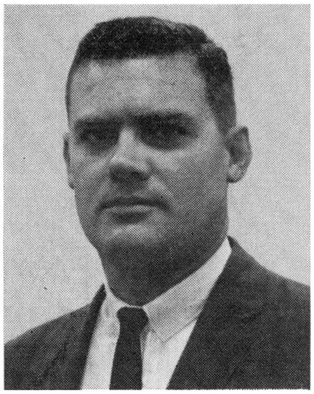

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