Numerical Calculation Of Distributed-Network Transfer Matrices

Robert C. Peirson
Missouri University of Science and Technology

Edward C. Bertnolli
Missouri University of Science and Technology

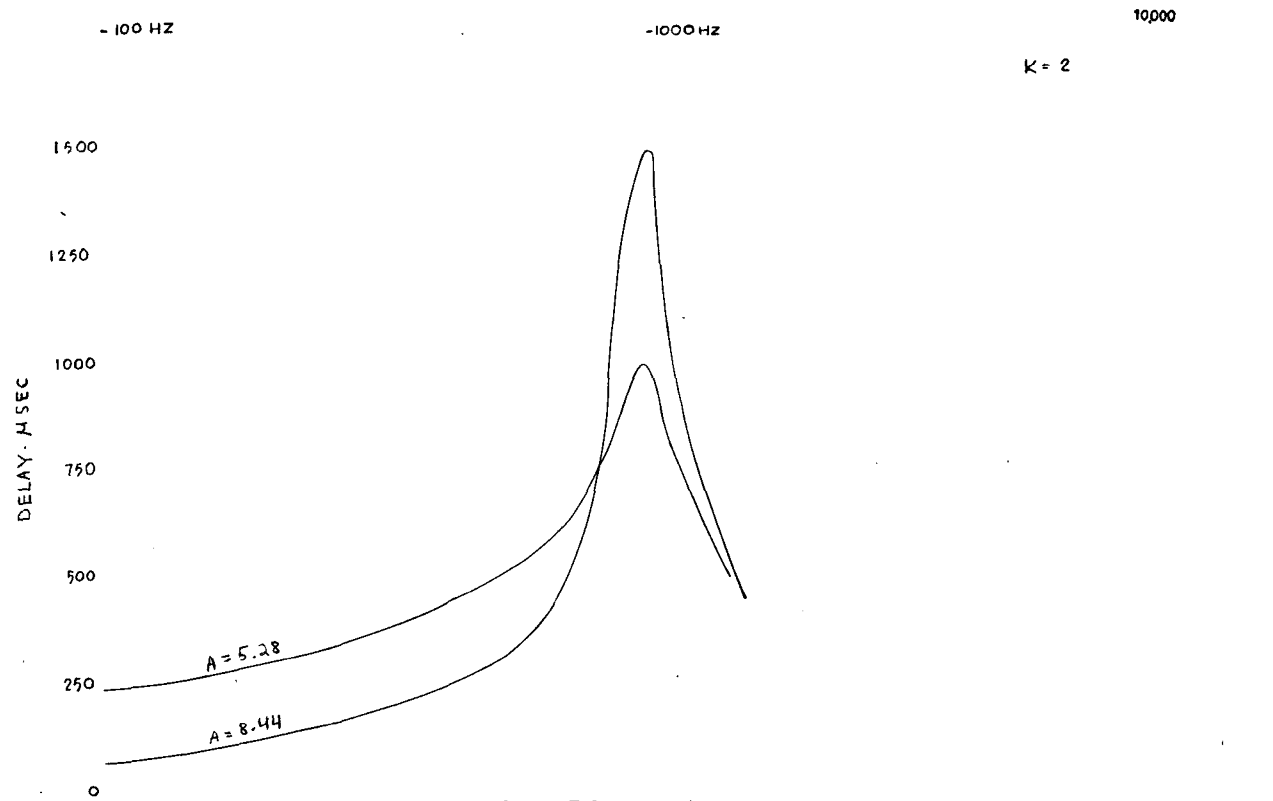

Fig. 3. Delay versus A.

Figs. 2 and 3 contain measured amplitude and delay characteristics.

ARTHUR B. WILLIAMS
Tele-Signal Corp.
Woodbury, N. Y. 11797

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INTRODUCTION

Exact closed-form solutions for arbitrarily tapered multilayered distributed parameter networks are often impossible to obtain. The steady-state analysis of these lines rests most often on numerical methods with little or no assurance of solution accuracy. Bertnolli and Halijak [1] developed numerical methods for solving the multiterminal variable parameter $\overline{RC}$ network problem.

This correspondence presents an improved numerical method based on [1], for determining the transfer matrix of an arbitrarily tapered distributed network and an expression for the error of calculation. This expression provides a bound on the solution error that enables the analyst to select his solution accuracy prior to the calculation of the actual transfer matrix.

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Determining the transfer matrix of a multilayered network is desirable because it is independent of any excitation and loading conditions.

THE TRANSFER MATRIX SOLUTION

Fig. 1 depicts a tapered multilayered distributed parameter $\overline{RC}$ network. (The following method will use the $\overline{RC}$ network as an example, but the method is certainly not limited to this specific network.) Bertnolli and Halijak [1] demonstrated that the network's voltage and current law equations may be concisely written in the "state–space" formulation as

$$\frac{\partial}{\partial x}\begin{bmatrix} V(x, s) \\ I(x, s) \end{bmatrix} = \begin{bmatrix} 0_n & R(x) \\ sC(x) & 0_n \end{bmatrix}\begin{bmatrix} V(x, s) \\ I(x, s) \end{bmatrix} = K(x)\begin{bmatrix} V(x, s) \\ I(x, s) \end{bmatrix} \quad (1)$$

or

$$\dot{\xi}(x) = K(x)\xi(x). \quad (2)$$

Here, the functional notation for s has been suppressed since (1) is a differential equation in x. The transfer matrix solution at frequency $s = 0 + j\omega$ for a network of length d has been shown [1] to be the matrizant of $K(x)$,

$$\xi(d) = \Omega_0^d[K(x)]\cdot\xi(0). \quad (3)$$

This matrizant solution is difficult to calculate in most cases.

Subdivide the network into n equal subnetworks of length $\Delta x = d/n$. Corresponding to this subdivision, $\xi(x)$ can be written in product form [2],

$$\xi(x) = \{\Omega_{(n-1)\Delta x}^{d}[K(x)] \cdots \Omega_{i\Delta x}^{(i+1)\Delta x}[K(x)] \cdots \Omega_0^{\Delta x}[K(x)]\}\xi(0)$$

or

$$[T_n \cdots T_1]\xi(0). \quad (4)$$

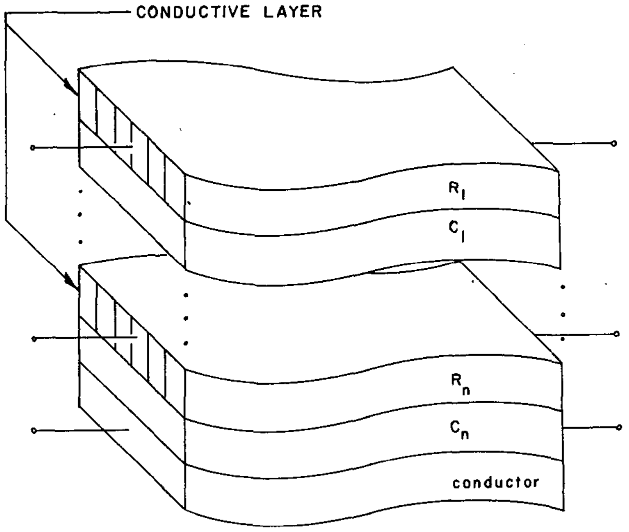

Fig. 1. The $2n + 2$ terminal tapered $\overline{RC}$ microcircuit.

The Incremental Transfer Matrix

Consider the incremental transfer matrix $[T_i]$, $\Omega_{i\Delta x}^{(i+1)\Delta x}[K(x)]$, which relates the network variable vector at position $(i+1)\,\Delta x$ to the network variable vector at position $i\Delta x$,

$$\xi[(i+1)\,\Delta x] = \{\Omega_{i\Delta x}^{(i+1)\Delta x}[K(x)]\}\cdot\xi(i\,\Delta x) = [T_i]\xi(i\,\Delta x). \quad (5)$$

The transfer matrix relating $\xi[(i+1)\Delta x]$ and $\xi(i\Delta x)$ may also be expanded in a Taylor's series in x at the point $x = i\Delta x$ [3].

$$T_i = I + T_i^{(1)}(i\,\Delta x)\,\Delta x + T_i^{(2)}(i\,\Delta x)\,\frac{\Delta x^2}{2!} + \cdots \quad (6)$$

where

$$T_i = \Omega_{i\Delta x}^{(i+1)\Delta x}[K(x)] \quad (7)$$

and $T_i^{(r)}(i\Delta x)$ indicates $(\partial^r/\partial x^r)[T_i(x)]$ evaluated at $x = i\Delta x$.

The terms of this Taylor's series may be generated by repeated differentiation of (2). When this is done, (6) becomes

$$T_i = I + \Delta x\; K(i\,\Delta x) + [\dot{K}(i\,\Delta x) + K^2(i\,\Delta x)]\,\frac{\Delta x^2}{2!}$$
$$+ [\ddot{K}(i\,\Delta x) + K(i\,\Delta)\dot{K}(i\,\Delta x)$$
$$+ 2\dot{K}(i\,\Delta x)K(i\,\Delta x) + K^3(i\,\Delta x)]\,\frac{\Delta x^3}{3!} + \cdots. \quad (8)$$

The Transfer Matrix Solution and Solution Error

The desired transfer matrix from (3) is

$$[T] = \Omega_0^d[K(x)]. \quad (9)$$

This transfer matrix is given by the ordered product of the incremental transfer matrices $[T_i]$

$$[T] = \prod_{i=1}^{n}\,[T_i] = T_n\cdot T_{n+1}\cdots T_2\cdot T_1. \quad (10)$$

The error in the transfer matrix $[T]$ due to any errors in the incremental transfer matrices $[T_i]$ is found by taking the differential of $[T]$,

$$\delta[T] = \delta T_n T_{n-1}\cdots T_1 + T_n\,\delta T_{n-1}T_{n-2}\cdots T_1$$
$$+ T_n\cdots T_{i+1}\,\delta T_i T_{i-1}\cdots T_1 + T_n\cdots T_2\,\delta T_1. \quad (11)$$

Using $|A + B| \le |A| + |B|$ and $|AB| \le |A|\,|B|$ from [4], the Euclidean norm of $\delta[T]$ is

$$|\delta[T]| \le |\delta T_n|\,|T_{n-1}\cdots T_1|$$
$$+ |T_n|\,|\delta T_{n-1}|\,|T_{n-2}\cdots T_1| + \cdots$$
$$+ |T_n\cdots T_{i+1}|\,|\delta T_i|\,|T_{i-1}\cdots T_1| + \cdots$$
$$+ |T_n\cdots T_2|\,|\delta T_1|. \quad (12)$$

The matrizant for a section of uniform line extending from $i = \alpha$ to β is

$$\Omega_{\alpha\Delta x}^{\beta\Delta x}[K_0] = \exp\,[K_0(\beta-\alpha)\,\Delta x] = T_{\beta\Delta x}\cdots T_{\alpha\Delta x} \quad (13)$$

since K_0 is independent of x.

For a tapered line, a constant matrix K_0 can always be selected such that

$$|\Omega_{\alpha\Delta x}^{\beta\Delta x}[K(x)]| \le |\exp\,[K_0(\beta-\alpha)\,\Delta x]|, \qquad 1 \le \alpha < \beta \le n \quad (14)$$

by selecting K_0 such that $|K_0| \ge |K(x)|_{\max}$ over $(0, d)$.

Then, from (12), the error in the network's transfer matrix will be bounded by the inequality

$$|\delta T| \le |\delta T_n|\,|\exp\,[K_0(n-1)\,\Delta x]|$$
$$+ |\exp\,[K_0\,\Delta x]|\,|\delta T_{n-1}|\,|\exp\,[K_0(n-2)\,\Delta x]| + \cdots$$
$$+ |\exp\,[K_0(n-i)\,\Delta x]|\,|\delta T_i|$$
$$\cdot|\exp\,[K_0(i-1)\,\Delta x]| + \cdots$$
$$+ |\exp\,[K_0(n-1)\,\Delta x]|\,|\delta T_1|. \quad (15)$$

Expanding $\exp\,[K_0 x]$ in its Maclaurin series and using properties of Euclidean norms [4], it can be shown that

$$|\exp\,[K_0 x]| \le \exp\,[|K_0|\,x]. \quad (16)$$

The above results can now be used on (15) to place a bound on $|\delta[T]|$.

$$|\delta[T]| \le |\delta T_n|\,\exp\,[|K_0|\,(n-1)\,\Delta x]$$
$$+ \exp\,[|K_0|\,\Delta x]\,|\delta T_{n-1}|$$
$$\cdot\exp\,[|K_0|\,(n-2)\,\Delta x] + \cdots$$
$$+ \exp\,[|K_0|\,(n-i)\,\Delta x]\,|\delta T_i|$$
$$\cdot\exp\,[|K_0|\,(i-1)\,\Delta x] + \cdots$$
$$+ \exp\,[|K_0|\,(n-1)\,\Delta x]\,|\delta T_1| \quad (17)$$

or

$$|\delta[T]| \le \exp\,[|K_0|\,(n-1)x]$$
$$\cdot[|\delta T_n| + |\delta T_{n-1}| + \cdots + |\delta T_1|] \quad (18)$$
$$\le \exp\,[|K_0|\,d]n\,|\delta T|_{\max} \quad (19)$$

since

$$\exp\,[|K_0|\,(n-1)\,\Delta x] < \exp\,[|K_0|\,d] \quad (20)$$

and $|\delta T|_{\max}$ is the largest of the normed errors encountered when calculating the incremental transfer matrices T_i.

If the incremental transfer matrix series (6) is truncated

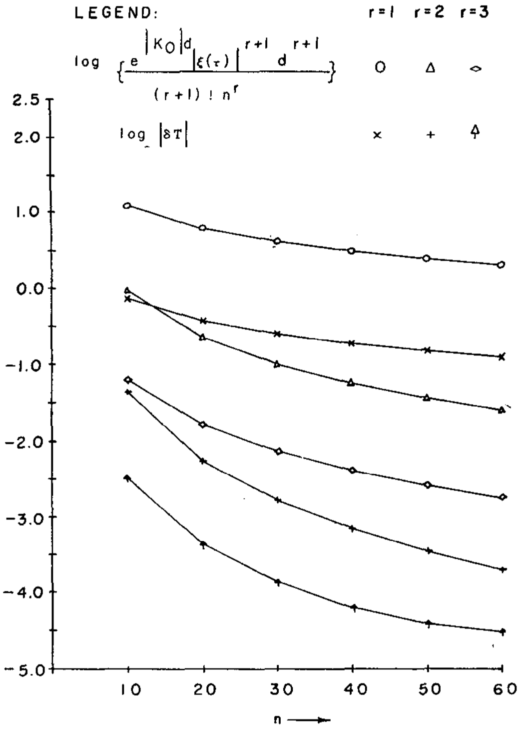

Fig. 2. Convergence data for the exponential transmission line.

after r terms, the resulting error is given by [6]

$$R_{r+1} = \frac{T_i^{(r+1)}(\tau)(\Delta x)^{r+1}}{(r+1)!}, \qquad (21)$$

and taking the Euclidean norm gives

$$|R_{r+1}| = \left|\frac{T_i^{(r+1)}(\tau)(\Delta x)^{r+1}}{(r+1)!}\right| \le \frac{|T_i(\tau)^{(r+1)}|\,(\Delta x)^{r+1}}{(r+1)!} \qquad (22)$$

where τ is between $(i+1)\Delta x$ and $i\Delta x$.

Since $\Delta x = d/n$, (22) can be written

$$|R_{r+1}| \le \frac{|T_i(\tau)^{(r+1)}|\, d^{r+1}}{(r+1)!\, n^{r+1}} \le \frac{|T_{i\,\max}^{(r+1)}|\, d^{r+1}}{(r+1)!\, n^{r+1}}. \qquad (23)$$

The total error from (19) is

$$|\delta[T]| \le \frac{\exp\,[|K_0|\, d] T_{i\,\max}^{(r+1)} d^{r+1}}{(r+1)!\, n^r}. \qquad (24)$$

From the above inequality, it can be seen that the calculation error bound for this method is a function of n and r, the number of time increments, and the order of the Taylor's series, respectively. This demonstrates thc convergence of the method and that the solution error decreases approximately in proportion to $(r+1)!n^r$, since $|T_{i\max}{}^{(r+1)}|$ is a fixed quantity. Computer time depends directly on n, the number of increments, while the amount of work needed to program this method depends on r, the order of the Taylor's series. Therefore, one must trade off programming time versus computer run time. Equation (24) gives the relationship between these two calculation parameters and the calculation accuracy to aid in selecting n and r.

To illustrate convergence, the transfer matrix for an exponentially tapered three-wire transmission line was determined. This solution was found for r of 1, 2, and 3, and for n varying from 10 to 60. Both the log of the bound on the normed error predicted by (24) and the log of the actual error (the left-hand side of (24), which is available since the exact solution is known [3]), are plotted versus n in Fig. 2. By inspection, the information in Fig. 2 satisfies the inequality (24). Using an IBM 360/50 computer system, 114 seconds were required to compile and execute evaluation of the approximate transfer matrix for $r = 3$ and $n = 60$.

Conclusion

A numerical method for calculating the transfer matrix of a multilayered distributed parameter network with an arbitrary taper has been presented. This presentation also includes an expression for solution error.

Other authors [5] have presented various methods of solving tapered distributed parameter network problems. Most of these methods rely on an integration process to obtain the solution. Indeed, the matrizant solution mentioned herein is an integral solution. The authors [5] presenting these "integral methods" make the point that their analysis applies to lines whose immittance distributions along the structure are piecewise, continuous, and bounded functions of position, i.e., integrable functions.

With the method presented here, the immittance function, i.e., the taper function $w(x)$, must be differentiable. This apparent restriction does not preclude physically realizable network tapers. Discontinuously tapered lines can be handled by appropriate network subdivision (piecewise-continuous methods).

Robert C. Peirson
Edward C. Bertnolli
Dept. of Elect. Engrg.
University of Missouri
Rolla, Mo. 65401

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Corrections

The Synthesis of Narrow-Band Crystal Band Elimination Filters[1]

Dr. H. Matthes of the Siemens Research Laboratories in Munich, Germany, called my attention to the fact that Fig. 6 of my paper is incorrect. In fact, a conventional image parameter design to meet the stopband requirements can be obtained by the method described in [1] of my paper.

I also wish to correct the following errors that slipped into the paper. In the two equations of Section V, the subscripts of X_0 and X_i should be interchanged and in the following numerical example, the correct value is $X_0 = -0.25$.

G. Szentirmai
School of Elec. Engrg.
Cornell University
Ithaca, N. Y.

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