Missouri University of Science and Technology
Scholars' Mine

Electrical and Computer Engineering Faculty Research & Creative Works

Electrical and Computer Engineering

01 Jan 1985

FOURIER TRANSFORMS over WARPED DOMAINS.

Richard E. Dubroff
Missouri University of Science and Technology, red@mst.edu

Follow this and additional works at: https://scholarsmine.mst.edu/ele_comeng_facwork

Part of the Electrical and Computer Engineering Commons

Recommended Citation

R. E. Dubroff, "FOURIER TRANSFORMS over WARPED DOMAINS.," *IEEE Transactions on Geoscience and Remote Sensing*, vol. GE thru 23, no. 3, pp. 349 - 350, Institute of Electrical and Electronics Engineers, Jan 1985.

The definitive version is available at https://doi.org/10.1109/TGRS.1985.289538

This Article - Journal is brought to you for free and open access by Scholars' Mine. It has been accepted for inclusion in Electrical and Computer Engineering Faculty Research & Creative Works by an authorized administrator of Scholars' Mine. This work is protected by U. S. Copyright Law. Unauthorized use including reproduction for redistribution requires the permission of the copyright holder. For more information, please contact scholarsmine@mst.edu.

Letters

Fourier Transforms Over Warped Domains

RICHARD E. DUBROFF

Abstract—**The primary purpose of this letter is to show that there may be an advantage to computing Fourier transforms over warped domains. An example of such an improvement is provided. A secondary purpose is to show how the Fourier transform may be evaluated from data observed over a warped domain.**

I. Introduction—The Moving Observer

A signal $\psi(x, t)$, which varies as a function of both space and time, is presumed to be observed over the finite spatial aperture $-\rho \leqslant x \leqslant \rho$ and over a time interval which is essentially unbounded. The Fourier transform of $\psi(x, t)$, which would be $S(k, \omega)$ if computed from $\psi(x, t)$ over $-\infty < x < +\infty$, suffers a loss of resolution when computed, instead, from observations over the finite aperture (see Jenkins and Watts [1], for example).

Rather than observing $\psi(x, t)$ in the conventional manner, however, let us now consider $\psi(x, t)$ as observed from a moving platform (see Fig. 1)

$$x[\phi, t] = \rho \sin(\Omega t + \phi) \tag{1}$$

where Ω is a constant scanning frequency. The motion of this hypothetical observer has the effect of encoding the temporal and spatial variation of $\psi(x, t)$ into a purely temporal variation of the observed signal $\psi'(\phi, t)$ for each phase angle ϕ between $-\pi$ and $+\pi$

$$\psi'[\phi, t] = (2\pi)^{-2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk\, d\omega\, S(k, \omega) \cdot e^{i[\omega t - k\rho \sin(\Omega t + \phi)]}. \tag{2}$$

The objective then becomes one of determining $S(k, \omega)$ from $\psi'[\phi, t]$. Provided that $S(k, \omega)$ is limited to a bandwidth of $-K_0 \leqslant k \leqslant K_0$ for all ω, a solution of $S(k, \omega)$ evaluated at $\omega = \omega_0$ may be written, at least, formally as [2]

$$S(k, \omega_0) = \sum_p \sum_q \sum_r \left[\frac{(-1)^p J_p(k\rho)}{2\pi}\right] [(\tilde{\tilde{\Gamma}}^{-1})^T]_{pq} [(\tilde{\tilde{\Gamma}}^{-1})]_{qr} \cdot \left[\int_{-\infty}^{\infty} dt\, e^{-i\omega t} \int_{-\pi}^{\pi} d\phi \frac{\psi'[\phi, t]}{2\pi} e^{ir\phi} \bigg|_{\omega = \omega_0 + r\Omega}\right]. \tag{3}$$

The set of functions $J_p(k\rho)$, for $p = 0, 1, 2, 3, \cdots$, consists of the Bessel functions of order p and argument $k\rho$. The matrix $\tilde{\tilde{\Gamma}}$ represents the transformation between the set of functions $[(-1)^p J_p(k\rho)/2\pi]$ and the corresponding Gram-Schmidt (see [3] for example) set of basis functions $U_p(k\rho)$ orthonormalized over $-K_0 \leqslant k \leqslant K_0$. Specifically

$$\left[\frac{(-1)^p}{2\pi} J_p(k\rho)\right] = \sum_{q=0} \tilde{\tilde{\Gamma}}_{pq} U_q(k\rho). \tag{4}$$

Manuscript received November 29, 1982; revised November 15, 1984.
The author is with Phillips Petroleum Company, Research and Development, Bartlesville, OK 74004.

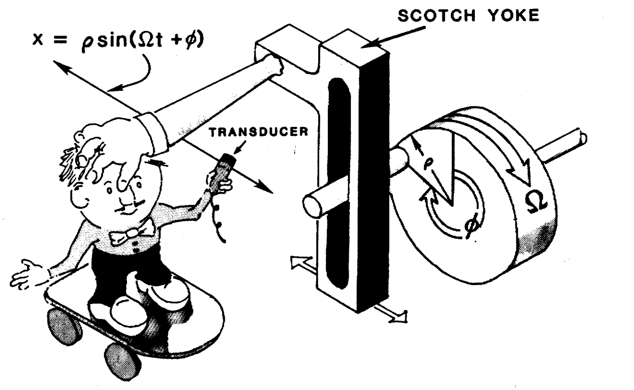

Fig. 1. A hypothetical observer in simple harmonic motion.

The result of the twofold integration on the right side of (3) is a set of numbers G_r which are individually referred to as moments [2]. The upper limits of summation in (3) are all equal to the largest order moment (largest value of r for which G_r is computed). Finally G_r may be shown [2] to be related to $S(k, \omega_0)$ through

$$G_r = (2\pi)^{-1} (-1)^r \int_{-\infty}^{\infty} dk\, S(k, \omega_0)\, J_r(k\rho). \tag{5}$$

II. Functions of One Variable

At this point it should become apparent that a moving observer is not necessary and, in fact, could be replaced by simply recording the function $\psi(x, t)$ over $-\rho \leqslant x \leqslant \rho$ while defining the new function $\psi'(\phi, t)$ as

$$\psi'(\phi, t) = \psi(x, t)\big|_{x = \rho \sin(\Omega t + \phi)}. \tag{6}$$

Certain additional considerations may arise, however, when (6) is used with sampled data [2]. It is now proposed to consider the extension of this method to functions of a single variable.

In particular, one may consider the function $\psi(t)$ as observed over the time interval $-T/2 \leqslant t \leqslant T/2$ where T is the total duration of the observed signal. Defining Ω as $\Omega = \pi/T$, the auxiliary function $\psi'(t)$ may be written as

$$\psi'(t) = \psi(\tau)\big|_{\tau = (T/2)\sin(\Omega t)} = (2\pi)^{-1} \int_{-\infty}^{\infty} d\omega\, S(\omega)\, e^{(i\omega T/2)\sin(\Omega t)} \tag{7}$$

where $S(\omega)$ is the true Fourier transform of $\psi(t)$. A graphical interpretation of the relation between $\psi'(t)$ and $\psi(\tau)$ is shown in Fig. 2 in which the periodicity of $\psi'(t)$ is apparent, suggesting that $\psi'(t)$ may be expanded in a Fourier series

$$\psi'(t) = \sum_{n=-\infty}^{\infty} G_n e^{-in\Omega t}. \tag{8}$$

The fact that G_n is related to $S(\omega)$ through

$$G_n = (2\pi)^{-1} (-1)^n \int_{-\infty}^{\infty} d\omega\, S(\omega)\, J_n(\omega T/2) \tag{9}$$

may be verified through the use of the expansion [4]

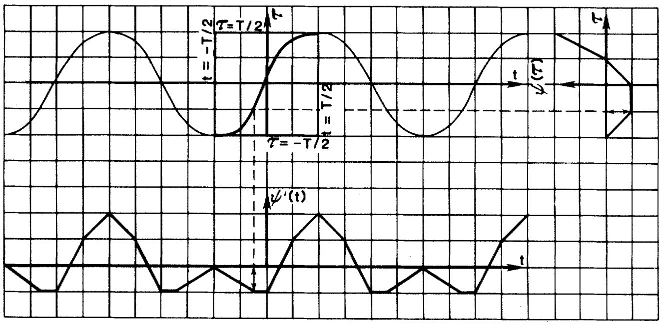

Fig. 2. A graphical interpretation of the relation between $\psi(\tau)$ and $\psi'(t)$. In addition to periodicity, $\psi'(t)$ has symmetry about $t = \pm T/2$.

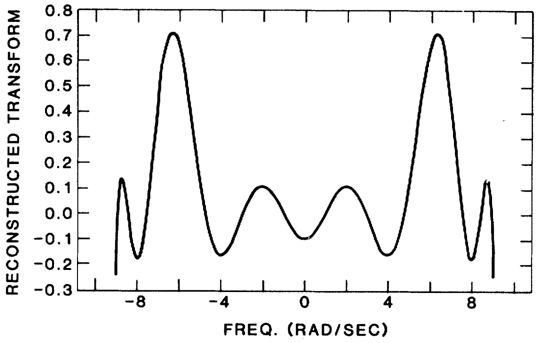

Fig. 3. Reconstructed Fourier transform of a pure cosine.

$$e^{(i\omega T/2)\sin(\Omega t)} = \sum_{n=-\infty}^{\infty} (-1)^n J_n(\omega T/2) e^{-in\Omega t}. \tag{10}$$

The evaluation of the moments G_n from the auxiliary function is simply a matter of finding the Fourier series coefficients. This process can be simplified by using the symmetry of G_n with respect to n, the symmetry of $\psi'(t)$ about $\pm T/2$, and a similar symmetry about $\pm T/2$ of $\cos(n\Omega t)$ for even values of n and $\sin(n\Omega t)$ for odd values of n. With these simplifications

$$G_n = \begin{cases} \displaystyle\int_{-T/2}^{T/2} dt\, \psi'(t) \cos(n\Omega t)/T, & n \text{ even} \\ \displaystyle i\int_{-T/2}^{T/2} dt\, \psi'(t) \sin(n\Omega t)/T, & n \text{ odd.} \end{cases} \tag{11}$$

III. Example and Conclusions

As an illustration of this method, consider the time-varying signal, $\cos(2\pi t)$ as observed from $-0.005 \leqslant t \leqslant +0.005$, a period of observation which is one-hundredth of the wave period. The moments, as evaluated from (9), are

$$G_n = (2\pi)^{-1} (-1)^n J_n(0.01\pi)\, [1 + (-1)^n]/2. \tag{12}$$

Using only the first fourteen moments and orthonormalizing the Bessel functions over $-9 \leqslant \omega \leqslant 9$ results in the reconstructed Fourier transform shown in Fig. 3. Although this specific example illustrates approximately two orders of magnitude improvement in resolution beyond the conventional Fourier transform through the use of only fourteen moments, it is unlikely an improvement of this size will come so cheaply for all signals. In fact, computer simulations and other considerations have suggested that the number of moments generally needs to increase with increasing orthonormalization bandwidth-aperture product in order to conserve resolution. Also, these results have not included the effects of noise.

On the other hand, the central theme of these methods seems to be one warping (transforming) the domains over which the signals are observed. Viewed in this way, the sinusoidal warping presented here may be only one of several alternatives. Finally, the sinusoidal warping can be viewed as an extension of the methods described in [2] to functions of a single variable. In this regard, many of the examples provided in [2] illustrate properties which are relevant to the single variable case as well.

References

[1] G. M. Jenkins and D. G. Watts, *Spectral Analysis and Its Applications*. San Francisco, CA: Holden-Day, 1968, p. 48.

[2] R. E. DuBroff, "Improved wavenumber resolution for small arrays," *IEEE Trans. Geosci. Remote Sensing*, vol. GE-21, pp. 83–93, Jan. 1983.

[3] H. Stark and F. B. Tutuer, *Modern Electrical Communications Theory and Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1979, p. 41.

[4] C. J. Tranter, *Bessel Functions With Some Physical Applications*. New York: Hart, 1968, p. 12.