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Reconstruction of Finite Duration Signals

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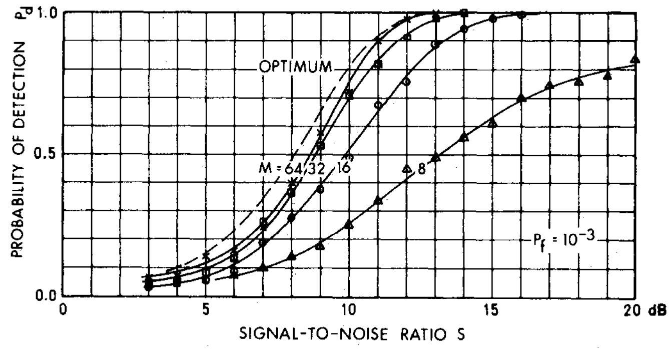

Fig. 2. Detection performance of Carlyle narrow-band Wilcoxon detector for $P_f = 10^{-3}$ with M as parameter. Broken-line curve—optimum parametric detector.

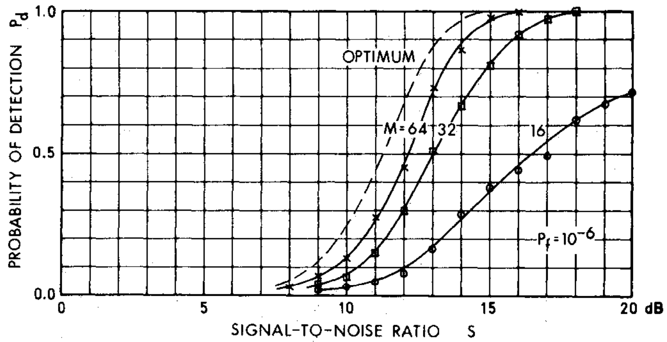

Fig. 3. Detection performance of Carlyle narrow-band Wilcoxon detector for $P_f = 10^{-6}$ with M as parameter. Broken-line curve—optimum parametric detector.

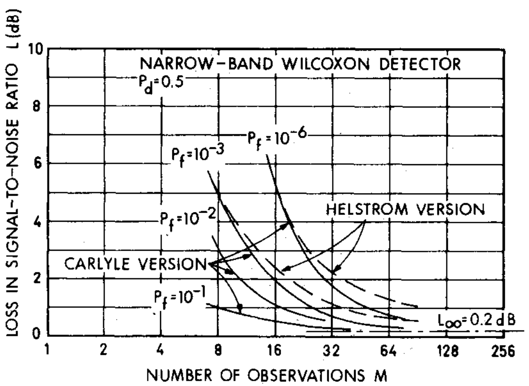

Fig. 4. Solid line—equivalent loss in signal-to-noise ratio for $P_d = 0.5$ from Figs. 2 and 3. Dash-dotted line—asymptotic loss of 0.2 dB. Broken line—corresponding results for the Helstrom version of the detector. Additional results for $P_f = 10^{-1}$ and 10^{-2} are also shown for the Carlyle version of the detector.

The asymptotic loss in this case depends on the particular values of P_d and P_f assumed. For large M the detector given by (13), (14), and (10) essentially adds a bias to the absolute value of each of the components t_{cW}' and t_{sW}' before squaring and summing. The resulting effect will be similar to the loss resulting from approximating an envelope detector by the sum of the absolute values of the two quadrature components. This loss has been found in recent work by Nathanson and Luke [7] to be about 0.5 dB for $P_f = 10^{-3}$ and about 0.7 dB for $P_f = 10^{-6}$. Adding the asymptotic loss of 0.2 dB for the Wilcoxon detector to these values, the asymptote of the curves in Fig. 4 for the Helstrom version of the detector are estimated to be 0.7 and 0.9 dB, respectively.

V. Conclusions

The detection performance for a coherent train of narrow-band signal pulses in stationary Gaussian noise has been determined for two adaptations of the nonparametric Wilcoxon test. It has been shown that the equivalent loss in signal-to-noise ratio for the best of the two detectors is asymptotically within 0.2 dB of the optimum. When the number of observations is $M = 50$, the loss for $P_d = 0.5$ and $P_f = 10^{-6}$ is about 1 dB, and it increases rapidly as M decreases further. Thus, these results demonstrate the inadequacy of asymptotic results in predicting detector performance for a small number of observations.

The major complication in a practical implementation of the Wilcoxon detector is the requirement for a ranking of the absolute values of the quadrature observations. This requirement is avoided in the narrow-band adaptation of the sign detector at the cost of an additional loss of $1\frac{1}{2} - 2$ dB.[2]

[2] These detailed results will be presented in a subsequent paper.

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Reconstruction of Finite Duration Signals

JOHN A. STULLER

Abstract—**The problem of reconstructing a finite-duration finite-energy signal that has been band limited and sampled is considered. An interpolation formula is derived that, in principle, permits perfect signal reconstruction in the noiseless case provided only that the sampling frequency exceeds the cutoff frequency of the band-limiting filter. The degradation introduced by measurement noise on the samples is evaluated.**

In recent years much attention has been given to the problem of reconstructing a finite-duration finite-energy (FDFE) signal that is observed through an ideal low-pass filter. The original analytical work in this subject is apparently due to Slepian and Pollak [1], who showed that this reconstruction could be performed without error by expanding the time-limited signal into a series of prolate spheroidal wave functions. Thus: 1) the prolate spheroidal wave functions form a complete set in the class of FDFE functions; 2) they are the eigenfunctions of the transformation described by band limiting the time-limited signal; and 3) each eigenfunction is associated with a nonzero eigenvalue. Hence, by representing the observed band-limited waveform by a prolate spheroidal wave-function expansion

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(which is also complete in the class of band-limited functions), the filter input signal may be obtained by dividing each coefficient in the expansion by the corresponding eigenvalue and time limiting the result. Barnes [2] described how this technique applies to the problem of object restoration in a one-dimensional diffraction-limited imaging system. Rushforth and Harris [3] extended the analysis to include both diffraction and noise. An alternative method of reconstructing the finite-duration signal is described by Harris [4].

The purpose of this correspondence is to point out that by combining the series described in [1]–[3] with certain time-domain sampling arguments, one may obtain an interesting interpolation rule for reconstruction of an FDFE signal from equally spaced samples of the observed band-limited waveform. We show that in the noiseless case perfect reconstruction of the FDFE signal can be obtained when the sampling rate exceeds one-half the minimum rate specified by the Shannon–Whittaker interpolation formula [5]. The limitations imposed by measurement noise are also described.

Assume that an FDFE signal $x(t)$ having duration T is put through an ideal low-pass filter having cutoff frequency W hertz; i.e.,

$$x(t) = x(t) \operatorname{rect}(t/T) \tag{1}$$

with

$$\int_{-T/2}^{+T/2} x^2(t)\, dt = E < \infty \tag{2}$$

and

$$y(t) = \int_{-T/2}^{+T/2} x(\tau) 2W \operatorname{sinc}[2W(t-\tau)]\, d\tau. \tag{3}$$

The observed quantity considered here is not $y(t)$ as discussed in [1]–[3], but rather noisy samples $r(mt_0)$ of $y(t)$:

$$r(mt_0) = y(mt_0) + n(mt_0), \qquad m = 0, \pm 1, \pm 2, \cdots. \tag{4}$$

The sampling rate in samples per second is

$$f_0 = 1/t_0. \tag{5}$$

Although the measurement noise $n(t)$ may have its physical origin in the detector that samples $y(t)$ and therefore may be defined only for $t = mt_0$, we extend the definition of $n(t)$ to include all values of its argument by setting

$$n(t) \equiv \sum_{m=-\infty}^{+\infty} n(mt_0) \operatorname{sinc}\left[\frac{t - mt_0}{t_0}\right]. \tag{6}$$

Similarly we set

$$z(t) = \sum_{m=-\infty}^{+\infty} y(mt_0) \operatorname{sinc}\left[\frac{t - mt_0}{t_0}\right] \tag{7}$$

and

$$r(t) = \sum_{m=-\infty}^{+\infty} r(mt_0) \operatorname{sinc}\left[\frac{t - mt_0}{t_0}\right] = z(t) + n(t). \tag{8}$$

Note that if f_0 exceeds $2W$ then the Shannon sampling theorem [5] is satisfied for perfect reconstruction of $y(t)$ from the $\{y(mt_0)\}$. Thus

$$z(t) = y(t), \qquad f_0 > 2W. \tag{9}$$

We do not so restrict f_0 however, but only require that f_0 exceed W,

$$f_0 > W. \tag{10}$$

If $W < f_0 < 2W$ then, of course, the spectral components in $z(t)$ will be aliased for frequencies exceeding $f_0 - W$. Components of $z(t)$ associated with frequencies less than $f_0 - W$ are identical, however, to those of $y(t)$ in this range. We now consider the problem of reconstructing $x(t)$ from the samples $\{r(mt_0)\}$. For the moment we consider the noiseless case. Then we generalize the result to include the effects of noise.

Noiseless Data

In the noiseless case, $z(t)$ and $r(t)$ are equal. We form a new function

$$z'(t) = \int_{-\infty}^{+\infty} z(\tau) 2f_1 \operatorname{sinc}[2f_1(t-\tau)]\, d\tau, \tag{11}$$

where

$$f_1 \equiv \begin{cases} f_0 - W, & W < f_0 < 2W \\ W, & 2W \le f_0. \end{cases} \tag{12}$$

Note that this step amounts to passing $z(t)$ through an ideal low-pass filter having cutoff frequency f_1. Equation (12) defines f_1 in such a way that only those spectral components in $z(t)$ that are identical to those of $y(t)$ will appear in $z'(t)$. Since $y(t)$ is itself the result of ideal low-pass filtering of $x(t)$, we have

$$z'(t) = \int_{-T/2}^{+T/2} x(\tau) 2f_1 \operatorname{sinc}[2f_1(t-\tau)]\, d\tau. \tag{13}$$

In addition, it follows from (7) and (11) that

$$z'(t) = \sum_{m=-\infty}^{+\infty} y(mt_0) 2f_1 t_0 \operatorname{sinc}[2f_1(t - mt_0)]. \tag{14}$$

Now form an approximation $x_N(t)$ to $x(t)$ given by

$$x_N(t) = \sum_{i=0}^{N} x_i \phi_i(t), \qquad |t| < T/2 \tag{15}$$

with

$$x_i = \int_{-T/2}^{+T/2} x(t)\phi_i(t)\, dt \tag{16}$$

such that

$$\lim_{N\to\infty} x_N(t) = x(t), \qquad |t| < T/2. \tag{17}$$

The ϕ_i of (15) are chosen to be a complete orthonormal set over the interval $|t| < T/2$. In particular we choose the ϕ_i to be the solutions to the integral equation

$$\int_{-T/2}^{+T/2} \phi_i(\tau) 2f_1 \operatorname{sinc}[2f_1(t-\tau)]\, d\tau = \lambda_i \phi_i(t). \tag{18}$$

As discussed by Rushforth [3], the solutions to (18) are given by

$$\phi_i(c,x) = [S_{0i}(c,t/\tfrac{1}{2}T)/u_i(c)], \tag{19}$$

where

$$[u_i(c)]^2 = \int_{-T/2}^{+T/2} [S_{0i}(c,t/\tfrac{1}{2}T)]^2\, dt. \tag{20}$$

$S_{0i}(c,t/\alpha)$ is the angular prolate spheroidal wave function in the notation of Flammer [6] and $c = \pi f_1 T$. By this normalization the ϕ_i have the properties

$$\int_{-T/2}^{+T/2} \phi_i(t)\phi_j(t)\, dt = \delta_{ij} \tag{21}$$

$$\int_{-\infty}^{+\infty} \phi_i(t)\phi_j(t)\, dt = \delta_{ij}/\lambda_i \tag{22}$$

with

$$\lambda_i(c) = (2c/\pi)[R_{0i}^{(1)}(c,1)]^2, \quad (23)$$

where $R_{0i}^{(1)}(c,1)$ is the radial prolate spheroidal wave function in Flammer's notation. Substitution of (15) into (13) and use of (18) then yields an approximation to $z'(t)$,

$$z_N'(t) = \sum_{i=0}^{N} z_i'\phi_i(t), \quad (24)$$

where

$$z_i' = x_i\lambda_i. \quad (25)$$

Hence

$$x_i = \frac{z_i'}{\lambda_i} = \frac{1}{\lambda_i}\int_{-T/2}^{+T/2} z'(t)\phi_i(t)\, dt. \quad (26)$$

To express the x_i in terms of the observed samples $y(mt_0)$ we introduce (14) into (26) and again simplify by (18). The result is

$$x_i = \sum_{m=-\infty}^{+\infty} y(mt_0)\phi_i(mt_0)t_0. \quad (27)$$

Substitution of (27) into (15) gives the final result

$$x_N(t) = t_0 \sum_{i=0}^{N}\sum_{m=-\infty}^{+\infty} y(mt_0)\phi_i(mt_0)\phi_i(t), \qquad |t| < T/2, \quad (28)$$

where f_0 satisfies (10). In the limit, (17) applies:

$$x(t) = t_0 \sum_{i=0}^{\infty}\sum_{m=-\infty}^{+\infty} y(mt_0)\phi_i(mt_0)\phi_i(t), \qquad |t| < T/2. \quad (29)$$

Equation (29) states that $x(t)$ may be reconstructed exactly from knowledge of noiseless samples of its band-limited version $y(t)$ (3) provided that the samples occur at a rate f_0 exceeding the highest frequency component in $y(t)$. This result is in sharp contrast with the well-known Shannon rate [5] and is of course, due to our consideration of the restricted set of FDFE signals. For finite N, the orders of summation may be interchanged in (28), giving the interesting result

$$x_N(t) = \sum_{m=-\infty}^{+\infty} y(mt_0)h_N(t,mt_0), \qquad |t| < T/2, \quad (30)$$

where

$$h_N(t,mt_0) = t_0 \sum_{i=0}^{N} \phi_i(t)\phi_i(mt_0). \quad (31)$$

For infinite N, the sum over m must be taken first as in (29).

Noisy Data

If now our measured data are not $\{y(mt_0)\}$ but rather the $\{r(mt_0)\}$ of (4), then use of (28) [with $r(mt_0)$ substituted for $y(mt_0)$] gives a noisy estimate $\hat{x}_N(t)$ of $x_N(t)$

$$\hat{x}_N(t) = x_N(t) + e_N(t), \qquad |t| < T/2, \quad (32)$$

where

$$e_N(t) = t_0 \sum_{i=0}^{N}\sum_{m=-\infty}^{+\infty} n(mt_0)\phi_i(mt_0)\phi_i(t), \qquad |t| < T/2. \quad (33)$$

We assume that the noise samples $n(mt_0)$ are zero-mean uncorrelated random variables with variance σ^2. The variance of $e_N(t)$ averaged over the interval $|t| < T/2$ is then easily found from (33) and (21)

$$\frac{1}{T}\int_{-T/2}^{+T/2} E\{e_N^2(t)\}\, dt = \frac{t_0^2\sigma^2}{T}\sum_{i=0}^{N}\sum_{m=-\infty}^{\infty} \phi_i^2(mt_0). \quad (34)$$

This expression may be simplified by noting that the ϕ_i have no frequency components beyond f_1. Furthermore, (12) implies that $f_0 \geq 2f_1$. Hence

$$\phi_i(t) = \sum_{m=-\infty}^{+\infty} \phi_i(mt_0)\, \text{sinc}\left[\frac{t - mt_0}{t_0}\right]. \quad (35)$$

Substitution of (35) into (22) then yields the identity

$$t_0 \sum_{m=-\infty}^{+\infty} \phi_i(mt_0)\phi_j(mt_0) = \frac{\delta_{ij}}{\lambda_i} \quad (36)$$

from which (34) becomes

$$\frac{1}{T}\int_{-T/2}^{+T/2} E\{e_N^2(t)\}\, dt = \frac{t_0\sigma^2}{T}\sum_{i=0}^{N}\frac{1}{\lambda_i(c)}, \quad (37)$$

where the $\lambda_i(c)$ are given by (23).

Slepian and Pollak have shown that for a fixed value of c, the λ_i approach zero very rapidly for $i > 2f_1T$. Therefore the largest N we might expect to use in practice without introducing excessive noise error will be on the order of

$$N \approx 2f_1T. \quad (38)$$

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An Error Bound for Lagrange Interpolation of Low-Pass Functions

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Abstract—The well-known error formula for Lagrange interpolation is used to derive an expression for a truncation error bound in terms of the sampling rate and Nyquist frequency for regular samples and central interpolation. The proof is restricted to pulse-type functions possessing a Fourier transform. The formula finds application to the estimation of convergence rate in iterative interpolation, thus providing a criterion for the choice of sampling rate to achieve a specified truncation error level in a given number of steps. The formula can also be used as a guide when the samples are not regular but fairly evenly distributed.

I. Introduction

Whittaker [1] established the equivalence between Lagrange interpolation in the limit for equispaced tabular values,

$$\hat{g}_L(t) = \sum_{k=-\infty}^{\infty} g(kh) \prod_{\substack{p=-\infty\\ p\neq k}}^{\infty} \frac{(t - ph)}{(t_k - ph)} \quad (1)$$

and the cardinal function (or Shannon–Kotelnikov [2], [3]

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