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Generalized Running Discrete Transforms

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carefully selected. Temporal interpolation of the data may at times be beneficial, e.g., 2 ms data interpolated to 1 ms data will allow treatment of velocities between $0.25\ \Delta s/0.001$ and $0.50\ \Delta s/0.001$, but twice the number of channels may be required and there will still be variation in filter performance as a function of dip. This author has not seen the odd-length filter presented before. This latter formulation is better in certain situations because there is neither temporal nor spatial shifts in the output data.

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Generalized Running Discrete Transforms

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***Abstract*–This paper introduces a generalized running discrete transform with respect to arbitrary transform bases, and relates the generalized transform to the running discrete Fourier *z* and short-time discrete Fourier transforms. Concepts associated with the running and short-time discrete Fourier transforms such as 1) filter bank implementation, 2) synthesis of the original sequence by summation of the filter bank outputs, 3) frequency sampling, and 4) recursive implementations are all extended to the generalized transform case. A formula is obtained for computing the transform coefficients of a segment of data at time *n* recursively from the transform coefficients of the segment of data at time *n* - 1. The computational efficiency of this formula is studied, and the class of transforms requiring the minimum possible number of arithmetic operations per coefficient is described.**

I. Introduction

THIS PAPER defines a generalized running discrete transform with respect to arbitrary transform bases. The running or sliding discrete Fourier transform (running DFT) [1], [2] is a special case of this generalized transform. The formulas presented in this paper also have as special cases, for example, running Hadamard, Haar, and slant transforms.

One realization of the generalized running discrete transform consists of a bank of N FIR filters whose inputs are the sequence to be transformed and whose outputs are the running discrete transform coefficients. As will be shown, the original sequence can often be synthesized by simply summing the filter outputs. This result generalizes the so called "filter bank summation method" [3]-[5] of running Fourier synthesis to arbitrary discrete transforms. A second realization of the generalized running discrete transform consists of a recursive system in which output coefficients are used to recursively update each other in the course of running transformation of the input sequence. This realization generalizes the recursive form of the DFT [1], [2]. The recursive realization of the generalized running transform is also shown to be related to the frequency sampling implementation [1], [7]-[9] of each of the FIR filters of the filter bank realization.

The number of computations performed by the general recursive system in updating transform coefficients depends

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upon the particular transform type in question. This paper derives the class of transforms that require the minimum possible number of computations per coefficient update, and shows that the running DFT is a special member of this class having coefficient update equations that are uncoupled.

Section II of this paper defines the generalized running discrete transform and derives the formula for recursively updating coefficients. Examples are given of running Hadamard and Haar transforms. Section III develops the filter bank and frequency sampling implementations, and relates the generalized transform to the running DFT, running z [1], [2], and short-time DFT [1], [3]-[5], [10], [11] transforms. Section IV derives the class of transforms that result in the minimum possible number of computations for updating coefficients of consecutive data blocks.

II. Definitions and Basic Recursions

We define the *running discrete transform* of an arbitrary sequence $f(n)$ with respect to a nonsingular but otherwise arbitrary $N \times N$ matrix $T = [t_{pk}]$ as

$$F(p, n) = \sum_{k=0}^{N-1} f(n-k)\, t_{pk}; \quad 0 \leqslant p \leqslant N-1 \tag{2.1}$$

where t_{pk} is the pkth element of T. Throughout this paper, rows and columns of matrices are numbered from 0 to $N-1$. Let $\boldsymbol{t}_k = (t_{0k}, t_{1k}, \cdots, t_{(N-1)k})'$ denote the kth column of T, $0 \leqslant k \leqslant N-1$, where the prime denotes vector or matrix transposition, and let $\boldsymbol{F}(n) = (F(0,n), F(1,n), \cdots, F(N-1,n))'$ be the vector of transform coefficients. Then (2.1) is equivalent to

$$\boldsymbol{F}(n) = \sum_{k=0}^{N-1} f(n-k)\, \boldsymbol{t}_k. \tag{2.2}$$

We wish to obtain a recursive algorithm for computing $\boldsymbol{F}(n)$ from $\boldsymbol{F}(n-1)$, and for this purpose introduce a nonsingular matrix A that cyclically advances the column vectors $\boldsymbol{t}_k$ as follows:

$$A\boldsymbol{t}_{((k))} = \boldsymbol{t}_{((k+1))}, \quad k = 0, \pm 1, \pm 2, \cdots \tag{2.3}$$

where $((i))$ denotes i modulo N. In subsequent developments, A will be referred to as the *circular advance* matrix associated with transform matrix T. Using (2.3) it follows that $\boldsymbol{t}_k = A\boldsymbol{t}_{((k-1))}$, $0 \leqslant k \leqslant N-1$, so that (2.2) can be written as

$$\begin{aligned} \boldsymbol{F}(n) &= A \sum_{k=0}^{N-1} f(n-k)\, \boldsymbol{t}_{((k-1))} \\ &= A \Bigg[\sum_{k=1}^{N} f(n-k)\, \boldsymbol{t}_{((k-1))} + f(n)\, \boldsymbol{t}_{((-1))} \\ &\quad - f(n-N)\, \boldsymbol{t}_{((N-1))} \Bigg]. \end{aligned} \tag{2.4}$$

Substituting $k = r + 1$ into (2.4), and using (2.3), yields the forward recursion

$$\begin{aligned} \boldsymbol{F}(n) &= A \sum_{r=0}^{N-1} f(n-1-r)\, \boldsymbol{t}_r + [f(n) - f(n-N)]\, \boldsymbol{t}_0 \\ &= A\boldsymbol{F}(n-1) + [f(n) - f(n-N)]\, \boldsymbol{t}_0. \end{aligned} \tag{2.5}$$

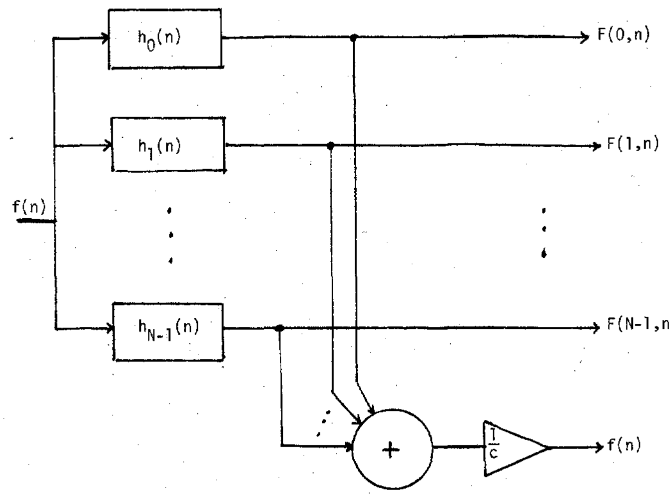


Fig. 1. Filter bank interpretation of generalized running discrete transform. Synthesis of the original sequence $f(n)$ by summing FIR filter bank outputs as shown is possible if and only if (2.7) of the text is satisfied.

The backward recursion follows directly from (2.5) and is

$$\boldsymbol{F}(n-1) = D\boldsymbol{F}(n) + [f(n-N) - f(n)]\, \boldsymbol{t}_{N-1} \tag{2.6}$$

where $D \equiv A^{-1}$ is the *circular delay* matrix associated with transform matrix T.

Computation of the original sequence $f(n)$ from its running transform $\boldsymbol{F}(n)$ (2.2) is simple when only the zeroth column of T contains a nonzero dc component, i.e., when

$$\sum_{p=0}^{N-1} t_{pk} = \begin{cases} c \neq 0; & k = 0 \\ 0; & \text{otherwise.} \end{cases} \tag{2.7}$$

It follows from (2.2) and (2.7) that

$$f(n) = \frac{1}{c} \sum_{p=0}^{N-1} F(p, n). \tag{2.8}$$

Equation (2.8) is related to the filter bank summation method of short-time Fourier-synthesis discussed, for example, in [3]-[5]. This can be seen upon identifying the pth row of T with an impulse response $h_p(n)$ of an FIR filter

$$h_p(n) = \begin{cases} t_{pn}; & 0 \leqslant n \leqslant N-1 \\ 0; & \text{otherwise.} \end{cases} \tag{2.9}$$

The transform coefficient $F(p, n)$ of (2.1) is then recognized as the output of FIR filter $h_p(n)$ when the input sequence is $\{f(n)\}$. There are N such filters $h_p(n)$, $p = 0, 1, \cdots, N-1$, which constitute a filter bank with outputs $F_p(n)$, $p = 0, 1, \cdots, N-1$. The sum of filter bank outputs as in (2.8) is the synthesized original sequence $f(n)$. This interpretation of (2.1) is summarized in Fig. 1. Because of its physical appeal, we will continue the identification of (2.9) for the remainder of this paper.

Direct evaluation of (2.2) requires N^2 complex multiplica-

1.000	1.000
1.000	-1.000

(a)

1.000	0.000
0.000	-1.000

(b)

Fig. 2. (a) 2 × 2 Hadamard matrix. (b) Associated circular advance matrix.

1.000	1.000	1.000	1.000
1.000	-1.000	1.000	-1.000
1.000	1.000	-1.000	-1.000
1.000	-1.000	-1.000	1.000

(a)

1.000	0.000	0.000	0.000
0.000	-1.000	0.000	0.000
0.000	0.000	0.000	1.000
0.000	0.000	-1.000	0.000

(b)

Fig. 3. (a) 4 × 4 Hadamard matrix. (b) Associated circular advance matrix.

1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.000	-1.000	1.000	-1.000	1.000	-1.000	1.000	-1.000
1.000	1.000	-1.000	-1.000	1.000	1.000	-1.000	-1.000
1.000	-1.000	-1.000	1.000	1.000	-1.000	-1.000	1.000
1.000	1.000	1.000	1.000	-1.000	-1.000	-1.000	-1.000
1.000	-1.000	1.000	-1.000	-1.000	1.000	-1.000	1.000
1.000	1.000	-1.000	-1.000	-1.000	-1.000	1.000	1.000
1.000	-1.000	-1.000	1.000	-1.000	1.000	1.000	-1.000

(a)

1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.000	-1.000	0.000	0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000
0.000	0.000	-1.000	0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.500	0.500	0.500	-0.500
0.000	0.000	0.000	0.000	-0.500	-0.500	0.500	-0.500
0.000	0.000	0.000	0.000	-0.500	0.500	0.500	0.500
0.000	0.000	0.000	0.000	-0.500	0.500	-0.500	-0.500

(b)

Fig. 4. (a) 8 × 8 Hadamard matrix. (b) Associated circular advance matrix.

1.000	1.000	1.000	1.000
1.000	1.000	-1.000	-1.000
1.414	-1.414	0.000	0.000
0.000	0.000	1.414	-1.414

(a)

1.000	0.000	0.000	0.000
0.000	0.000	0.707	-0.707
0.000	-0.707	-0.500	-0.500
0.000	0.707	-0.500	-0.500

(b)

Fig. 5. (a) 4 × 4 Haar matrix. (b) Associated circular advance matrix.

tions and $N^2 - N$ additions while direct evaluation of (2.5) requires $N^2 + N$ complex multiplications and $N^2 + 1$ complex additions. On this basis there is no advantage in using (2.5) to compute $\boldsymbol{F}(n)$. However, if circular advance matrix A is sufficiently sparse then the opposite conclusion can be true. The sparsest possible circular advance matrix has exactly one nonzero entry in each row and column. In Section IV of this paper, we shall refer to such matrices as *maximally sparse* and shall derive the class of transform matrices having maximally sparse circular advance matrices. Evaluation of $\boldsymbol{F}(n)$ from $\boldsymbol{F}(n-1)$ in (2.5) when matrix A is maximally sparse requires only $2N$ complex multiplications and $N + 1$ complex additions.

In the formulation (2.1) the analysis window is absorbed into transform matrix T. An example is given in (3.17). In most practical systems, running transforms are sampled at a rate commensurate with the analysis window—a rate considerably lower than the sampling rate of the data. Applications do exist [6], however, which require efficient recursive computation of transform samples at the sampled data rate. These applications provide some practical motivation for our work. The primary motivation, however, has been to develop greater insight into the structure of running discrete Fourier, running z, and short-time discrete Fourier transforms by showing that these transforms are special members of a larger class of discrete running transforms.

Examples of the circular advance matrix for various Hadamard and Harr transforms are given in Figs. 2–5. The circular advance matrices in these figures were computed using (3.22) of this paper and are numerically correct to three decimal places. Note that the circular advance matrices for the 2 × 2 and the 4 × 4 Hadamard transforms are maximally sparse, but this is not the case for any other example shown.

As will be shown in Section III, the transform matrix T associated with the discrete Fourier transform has a particularly simple circular advance matrix. In this case A is diagonal with entries equaling the N roots of unity. Section IV of this paper shows that the DFT is the only possible transform having a *diagonal* circular advance matrix.

III. Properties of Circular Advance Matrices

In this section we show that matrix A of (2.3) is similar to a certain circulant matrix B (3.1) and use this fact to obtain four properties of A. These concern the characteristic decomposition of A, the explicit form for the entries of $A^p, p = 0, \pm 1, \pm 2, \cdots$ when T is unitary, and the effect upon A of permuting row vectors of T.

Let B be the $N \times N$ circulant matrix having mnth entry $b_{mn} = \delta_{((m-n-1))}$, $0 \leqslant m, n \leqslant N - 1$, where δ is the Kronecker delta function. That is, let

$$B = \left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \\ \hline & & & 0 \\ & I & & \vdots \\ & & & 0 \end{array}\right] \tag{3.1}$$

where I is the identity matrix. Then

$$TB = [\boldsymbol{t}_1 \boldsymbol{t}_2 \cdots \boldsymbol{t}_{N-1} \boldsymbol{t}_0] \tag{3.2}$$

so that by (2.3),

$$TB = AT. \tag{3.3}$$

It follows that A is similar to B and is in fact given by

$$A = TBT^{-1}. \tag{3.4}$$

As is well known, similar matrices have the same characteristic

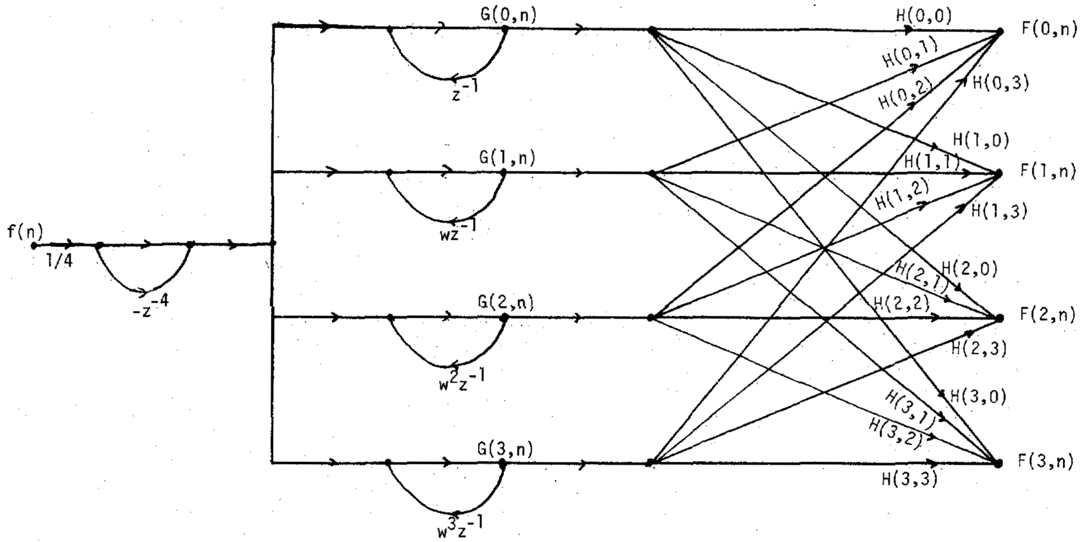


Fig. 6. Frequency sampling structure implementation of generalized running discrete transform. Filter $h_p(n)$ of Fig. 1 is implemented by its frequency sampling structure implementation, $p = 0, 1, 2, 3$. Coefficients $G(p,n)$ are the running DFT coefficients for rectangularly windowed $f(n)$. Illustration assumes $N = 4$ for simplicity.

polynomial. The characteristic polynomial, $p(\lambda) = \det[\lambda I - B]$, of B can be obtained easily from (3.1). This results in the following property.

Property 1: A circular advance matrix has characteristic polynomial

$$p(\lambda) = \det[\lambda I - A] = \lambda^N - 1. \tag{3.5}$$

The characteristic values of a circular advance matrix are, accordingly, the N roots of unity, $\lambda_i = w^i$, $0 \leqslant i \leqslant N-1$, where

$$w \equiv \exp[j(2\pi/N)]. \tag{3.6}$$

Note that since the characteristic values of A lie on the unit circle, recursion (2.5) is marginally stable. To have a stable recursion, one must perturb the characteristic values of A so that they are slightly within the unit circle. Similar comments apply to recursion (2.6).

Knowledge of the characteristic values of A leads to its diagonal form. The result is stated as Property 2.

Property 2: The circular advance matrix A is given by

$$A = H\Omega H^{-1} \tag{3.7}$$

where

$$\Omega = \begin{bmatrix} 1 & & & & 0 \\ & w & & & \\ & & w^2 & & \\ & & & \ddots & \\ & & & & w^{N-1} \end{bmatrix} \tag{3.8}$$

and where the pth row of H, $H_p(0), H_p(1), \cdots, H_p(N-1)$, *is the DFT of the pth row of T, i.e.,*

$$H_p(k) = \sum_{n=0}^{N-1} h_p(n) w^{-nk}; \quad 0 \leqslant k \leqslant N-1. \tag{3.9}$$

Proof: Since B is circulant, its characteristic vectors are given by

$$\gamma_i = (1, w^{-i}, w^{-2i}, \cdots, w^{-(N-1)i})' \tag{3.10}$$

which is easy to verify using (3.1) and the rule $B\gamma_i = \lambda_i \gamma_i$, $0 \leqslant i \leqslant N-1$. It follows that

$$B = \Gamma\Omega\Gamma^{-1} \tag{3.11}$$

where Ω is given in (3.8) and where Γ is the discrete Fourier basis matrix

$$\Gamma = [\gamma_{mn}] = [w^{-mn}]. \tag{3.12}$$

Letting $*$ denote conjugate transposition, it follows from (3.12) that

$$\Gamma^{-1} = \frac{1}{N}\Gamma^*. \tag{3.13}$$

Equation (3.7) follows directly upon substituting (3.11) into (3.4) and identifying

$$H = T\Gamma. \tag{3.14}$$

Q.E.D.

It is instructive to consider (2.5) in light of Property 2. Substituting (3.7) into (2.5) yields, after a little algebra,

$$G(n) = \Omega G(n-1) + \frac{1}{N}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}[f(n) - f(n-N)] \tag{3.15a}$$

where

$$F(n) = HG(n). \tag{3.15b}$$

The system described by (3.15) is illustrated in Fig. 6 and can be seen to involve the frequency-sampling structure [1], [7]-[9] for implementing each of the N FIR filters $h_p(n)$ of Fig. 1. In this connection we see from (3.14) that if $T = \Gamma^* = N\Gamma^{-1}$ then $H = NI$ so that system (3.15) reduces to the N uncoupled first-order systems

$$F(p,n) = w^p F(p, n-1) + f(n) - f(n-N); \quad p = 0, 1, \cdots, N-1. \tag{3.16}$$

Recursions (3.16) are a special form that arises in the computation of the running discrete Fourier transform [1], [2]. In general, the running discrete Fourier transform can be defined as[1]

$$F(p,n) = \sum_{k=0}^{N-1} f(n-k)\, v(k)\, w^{kp} \tag{3.17}$$

where $v(n)$, $0 \leqslant n \leqslant N-1$, is some chosen window sequence. Using (3.4) and the association

$$h_p(n) = \begin{cases} v(n)\, w^{np}; & 0 \leqslant n \leqslant N-1 \\ 0; & \text{elsewhere} \end{cases} \tag{3.18}$$

it is straightforward to compute the corresponding circular advance matrix. The mnth entry of the result is

$$a_{mn} = \left[\frac{1}{N} \sum_{k=0}^{N-1} v(k)\, v^{-1}((k-1))\, w^{k(m-n)}\right] w^n. \tag{3.19}$$

One can see from (3.19) that, for a general window, the circular advance matrix used in (2.5) will not be sparse and therefore the computation of coefficient $F(p,n)$, $0 \leqslant p \leqslant N-1$, will be coupled to many or all previous coefficients $F(q, n-1)$ $0 \leqslant q \leqslant N-1$. For the special case of a rectangular window $v(n) = 1/N$, $0 \leqslant n \leqslant N-1$, however, A is the diagonal $A = \Omega$ and the uncoupled recursions (3.16) result.

Another window of particular interest is the exponential window $v(k) = \exp(\sigma/N)$. For this window (3.17) becomes the running z transform [1], [2] of $f(n)$ evaluated at points $z = \exp[(-j2\pi/N - \sigma)p]$, $p = 0, 1, \cdots, N-1$. Evaluation of (3.19) for the exponential window yields a circular advance matrix whose elements are $a_{mn} = \{[\exp(-\sigma(N-1)/N) - \exp(-\sigma/N)]/N + \exp(\sigma/N)\, \delta((m-n))\}\, w^n$. Although this matrix is not sparse, it has a form that leads [in (2.5)] to the uncoupled recursive formula

$$F(n,p) = w^p \exp(\sigma/N) F(n-1, p) + f(n) - \exp(\sigma) f(n-N). \tag{3.20}$$

Equation (3.20) is related to a recursion introduced by Weinstein [10] in reference to the computation of the short-time exponentially windowed DFT.

The entries of A have a particularly simple form when T is unitary to within a constant α such that

$$T^{-1} = \alpha T^*. \tag{3.21}$$

Property 3 describes the circular advance matrix corresponding to any unitary matrix.

Property 3: If T is unitary to within a constant α as in (3.21), then A is strictly unitary: $A^{-1} = A^*$. Furthermore, the mnth entry of A^p, say $a_{mn}^{(p)}$, $0 \leqslant m, n \leqslant N-1$, where p is any integer, $p = 0, \pm 1, \pm 2, \cdots$, is

$$a_{mn}^{(p)} = \alpha \sum_{k=0}^{N-1} h_m(k)\, h_n^*((k-p)). \tag{3.22}$$

Note that the sum on the right side of (3.22) is the circular cross correlation function between FIR sequences $h_m(k)$ and $h_n(k)$.

Proof: Using (3.4) and (3.21) one has

$$A^{-1} = \alpha T B^{-1} T^*; \tag{3.23}$$

but again from (3.4) and (3.16)

$$A^* = \alpha T B^* T^*. \tag{3.24}$$

It is easily confirmed from (3.1) that $B^* = B^{-1}$ which by (3.23)–(3.24) shows that $A^{-1} = A^*$. To establish (3.22), use (3.21) in (3.4) to obtain

$$A^p = \alpha T B^p T^* \tag{3.25}$$

where B^p has mnth entry $b_{mn}^{(p)} = \delta_{((m-n-p))}$. Then

$$\begin{aligned} a_{mn}^{(p)} &= \alpha \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} h_m(u)\, b_{uv}^{(p)}\, h_n^*(v) \\ &= \alpha \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} h_m(u)\, \delta((u-v-p))\, h_n^*(v) \\ &= \alpha \sum_{u=0}^{N-1} h_m(u)\, h_n^*((u-p)) \qquad \text{Q.E.D.} \end{aligned} \tag{3.26}$$

Another property of circular advance matrices concerns the effect of a permutation among the row vectors of T. Since by (2.2) $\boldsymbol{F}(n)$ is a linear combination of the column vectors of T, the effect of such a permutation is to similarly permute the elements $F(p,n)$ of $\boldsymbol{F}(n)$. For example, the effect of interchanging the $p = 0$th and $p = N-1$st row of T is to interchange $F(0,n)$ with $F(N-1, n)$. Such operations do not essentially alter the transformation in question, but amount to only a rearrangement of transform coefficients. Therefore, we will call transform matrices that differ only by row permutations *equivalent* transform matrices. Property 4 concerns the circular advance matrices of equivalent transform matrices.

Property 4: The effect of interchanging rows i and j of T is to interchange both the ith and jth rows and the ith and jth columns of A.

Proof: Let P_{ij} be the matrix obtained by interchanging the ith and jth rows of an $N \times N$ identity matrix, and note that P_{ij} is its own inverse. The operation of interchanging the ith and jth rows of T can be performed by premultiplying T by P_{ij}. Replacing T by $P_{ij}T$ in (3.4) then gives the resulting circular advance matrix

$$P_{ij} T B T^{-1} P_{ij} = P_{ij} A P_{ij}. \tag{3.27}$$

Property 2 then follows from the fact that postmultiplication of a matrix by P_{ij} interchanges columns i and j of that matrix. Q.E.D.

IV. Minimal Transform Matrices

This section develops the class of transform matrices that are associated with maximally sparse circular advance matrices. Since these transform matrices lead to the minimum computational requirements for calculating $\boldsymbol{F}(n)$ from $\boldsymbol{F}(n-1)$ in (2.5), they will be called *minimal transform matrices*. We begin the development by showing that minimal transform matrices can be put into a certain canonic form T_c, and that associated with

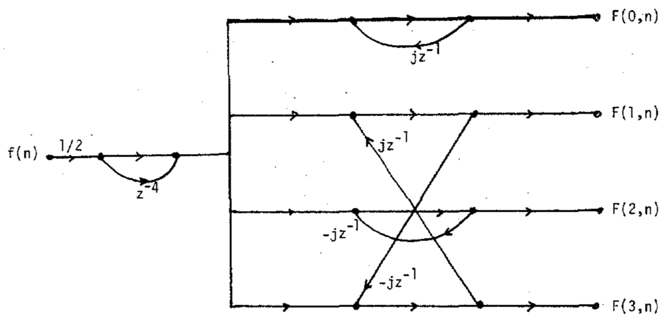
[1]The running discrete Fourier transform (3.16) is related to the short-time discrete Fourier transform $X(p,n)$ [1], [3]–[5], [10], [11] by $X(p,n) = w^{-np} F(p,n)$.

$$\frac{1}{2}\begin{bmatrix} 1 & j & -1 & -j \\ 1 & j & 1 & j \\ 1 & -j & -1 & j \\ 1 & -j & 1 & -j \end{bmatrix}$$

(a)

$$\begin{bmatrix} j & 0 & 0 & 0 \\ 0 & 0 & 0 & -j \\ 0 & 0 & -j & 0 \\ 0 & j & 0 & 0 \end{bmatrix}$$

(b)



(c)

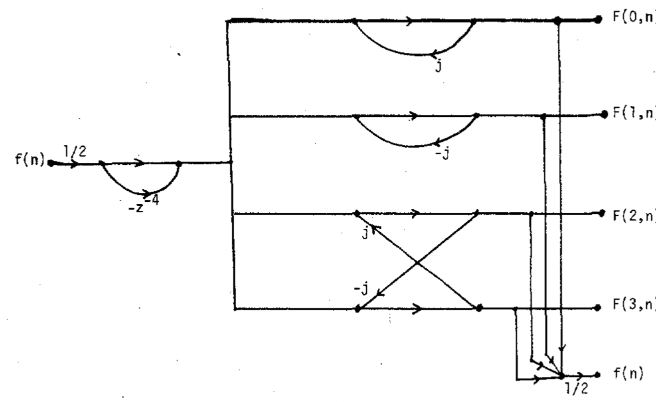
Fig. 7. Example of noncanonic minimal transforms. (a) Transform matrix T. The row vectors of T satisfy $\boldsymbol{h}_0 = j\mathfrak{D}\{\boldsymbol{h}_0\}$, $\boldsymbol{h}_1 = j\mathfrak{D}\{\boldsymbol{h}_3\}$. $\boldsymbol{h}_2 = -j\mathfrak{D}\{\boldsymbol{h}_2\}$ and $\boldsymbol{h}_3 = -j\mathfrak{D}\{\boldsymbol{h}_1\}$. (b) Circular advance matrix A associated with T. (c) Recursive realization. Note that there are two first-order subsystems. In deriving the canonic form of T shown as T_c in Fig. 8, we associate the top row of T in (a) above with S_0 (4.6), the next and the bottom row with S_2, and the remaining row with S_1. By definition (4.6) $S_0 = \{\boldsymbol{h}_0^{(0)}\}$, $S_1 = \{\boldsymbol{h}_0^{(1)}\}$, $S_2 = \{\boldsymbol{h}_0^{(2)}, \boldsymbol{h}_1^{(2)}\}$.

$$\frac{1}{2}\begin{bmatrix} 1 & j & -1 & -j \\ 1 & -j & -1 & j \\ 1 & j & 1 & j \\ 1 & -j & 1 & -j \end{bmatrix}$$

(a)

$$\begin{bmatrix} j & 0 & 0 & 0 \\ 0 & -j & 0 & 0 \\ 0 & 0 & 0 & j \\ 0 & 0 & -j & 0 \end{bmatrix}$$

(b)



(c)

Fig. 8. Canonic form of transform of Fig. 7. (a) T_c. (b) A_c. (c) Recursive realization with synthesis of $f(n)$ by coefficient summation (2.8). Matrix T_c of (a) illustrates example 3-iv) of Section IV.

this form is a corresponding canonic form $A_c = T_c B T_c^{-1}$ for the maximally sparse circular advance matrix.

Assume that A is an arbitrary maximally sparse matrix. Denote the nonzero entry the pth column of A by x_p, $0 \leqslant p \leqslant N-1$. This element is located in some qth row of A where q, $0 \leqslant q \leqslant N-1$, is related to p by a one-to-one mapping $q = \beta(p)$. Consequently, equating the pth row vector of AT to that of TB in (3.3) yields

$$x_p \boldsymbol{h}_p = \mathfrak{A}\{\boldsymbol{h}_{q=\beta(p)}\} \tag{4.1}$$

where $\boldsymbol{h}_p$ is the pth *row* vector of T

$$\boldsymbol{h}_p \triangleq (h_p(0)\, h_p(1) \cdots h_p(N-1)) \tag{4.2}$$

and $\mathfrak{A}$ is the *circular advance operator*

$$\mathfrak{A}\{\boldsymbol{h}_q\} = (h_q(1)\, h_q(2) \cdots h_q(M-1)\, h_q(0)). \tag{4.3}$$

In subsequent developments it will be more convenient to write (4.1) in terms of the *circular delay* operator $\mathfrak{D} = \mathfrak{A}^{-1}$

$$\mathfrak{D}\{\boldsymbol{h}_p\} = (h_p(N-1)\, h_p(0)\, h_p(1) \cdots h_p(N-2)) \tag{4.4}$$

as

$$\boldsymbol{h}_q = x_p \mathfrak{D}\{\boldsymbol{h}_p\}. \tag{4.5}$$

Equation (4.5) states that for maximally sparse A, the $q = \beta(p)$th row of T is a scaled circular delay of the pth row of T, where β is the mapping relating row and column numbers, q and p, respectively, of the nonzero elements in A. It similarly follows that the rth row of T, $r = \beta(q)$, is a scaled circular delay of the qth row. By continuing this process, $p \overset{\beta}{\to} q \overset{\beta}{\to} r \cdots$, one must eventually arrive back at row p after which the row index sequence will repeat $\cdots p, q, r, \cdots p, q, r \cdots$. An example is shown in Fig. 7 in which three such sequences occur, namely $\cdots 0, 0, 0, \cdots$; $\cdots 1, 3, 1, 3, \cdots$; and $\cdots 2, 2, 2, \cdots$. In general, there will be I periodic sequences of row indexes. The row vectors associated with the ith sequence will constitute a subset S_i of the row vectors of T. We will denote the number of row vectors in S_i by l_i, and use the superscript (i) to denote the row vectors themselves, i.e.,

$$S_i = \{\boldsymbol{h}_m^{(i)}: 0 \leqslant m \leqslant l_i - 1\}. \tag{4.6}$$

We will choose the index i so that $l_0 \leqslant l_1 \leqslant \cdots \leqslant l_{I-1}$. These conventions are illustrated in Fig. 7. It should be clear from Fig. 7 that I equals the number of unconnected subsystems represented by T. It should also be clear that one can permute the row vectors in T to yield an equivalent and canonic transform matrix T_c in which row vectors from S_i occupy consecutive rows as illustrated in Fig. 8. The general form of the canonic transform matrix is

$$T_c = \begin{bmatrix} T_0 \\ T_1 \\ \vdots \\ T_{I-1} \end{bmatrix} \tag{4.7}$$

where submatrix T_i contains the l_i row vectors in S_i:

$$T_i = \begin{bmatrix} h_0^{(i)}(0)\, h_0^{(i)}(1) & \cdots h_0^{(i)}(N-1) \\ h_1^{(i)}(0)\, h_1^{(i)}(1) & \cdots h_1^{(i)}(N-1) \\ \vdots & \\ h_{l_i-1}^{(i)}(0)\, h_{l_i-1}^{(i)}(1) & \cdots h_{l_i-1}^{(i)}(N-1) \end{bmatrix} \tag{4.8}$$

and where the row vectors in T_i are related by

$$\begin{aligned} \boldsymbol{h}_1^{(i)} &= x_0^{(i)}\, \mathfrak{D}\{\boldsymbol{h}_0^{(i)}\} \\ \boldsymbol{h}_2^{(i)} &= x_1^{(i)}\, \mathfrak{D}\{\boldsymbol{h}_1^{(i)}\} \\ &\vdots \\ \boldsymbol{h}_0^{(i)} &= x_{l_i-1}^{(i)}\, \mathfrak{D}\{\boldsymbol{h}_{l_i-1}^{(i)}\}. \end{aligned} \tag{4.9}$$

Associated with T_c of (4.7) is the maximally sparse matrix:

$$A_c = \begin{bmatrix} B_0 & & & \\ & B_1 & & 0 \\ & & \ddots & \\ 0 & & & B_{I-1} \end{bmatrix} \tag{4.10}$$

where

$$B_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & x_{l_i-1}^{(i)} \\ x_0^{(i)} & 0 & \cdots & 0 & 0 \\ 0 & x_1^{(i)} & \cdots & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & x_{l_i-2}^{(i)} & 0 \end{bmatrix}, \quad l_i \neq 1; \tag{4.11a}$$

and

$$B_i = x_0^{(i)}, \quad l_i = 1. \tag{4.11b}$$

Since T and T_c are related by row permutations, the nonzero entries of A and A_c are related by row and column permutations described by Property 4.

The structure of A_c in (4.10)-(4.11) implies that for a minimal transform T_c, recursion (2.5) is implemented by I uncoupled and parallel subsystems represented by submatrices T_i of (4.8), where l_i is the order of the ith subsystem $0 \leqslant i \leqslant I-1$. The distribution poles among the I subsystems are required by Property 1 to be a partition of the N roots of unity. To identify this partition further, we evaluate the characteristic polynomial of A_c, $p_c(\lambda)$. Direct evaluation of det $[\lambda I - A_c]$ yields

$$p_c(\lambda) = \prod_{i=0}^{I-1} (\lambda^{l_i} - p_i) \tag{4.12}$$

where

$$p_i = |p_i| \exp(j\phi_i) \triangleq \prod_{n=0}^{l_i-1} x_n^{(i)}. \tag{4.13}$$

The factor $\lambda^{l_i} - p_i$ has l_i roots

$$z_{im} = |p_i|^{1/l_i} \exp[j(\phi_i + 2\pi m)/l_i], \qquad m = 0, 1, \cdots, l_i - 1 \tag{4.14}$$

which are the poles of the ith subsystem represented by T_i. It is easy to show that for given i these poles are distinct Nth roots of unity iff $|p_i| = 1$, $\phi = 2\pi k_i$ for some integer k_i, and l_i divides N, say

$$N/l_i = \eta_i \tag{4.15}$$

where η_i is an integer. Substituting these conditions into (4.13) and (4.14) gives, respectively,

$$\prod_{n=0}^{l_i-1} x_n^{(i)} = w^{l_i k_i} \tag{4.16}$$

and

$$z_{im} = w^{k_i + m\eta_i}, \qquad m = 0, 1, \cdots, l_i - 1. \tag{4.17}$$

Viewed as a function of k_i, the right-hand side of (4.16) has period n_i. We can therefore choose k_i to lie in the range $0 \leqslant k_i \leqslant \eta_i - 1$ without loss in generality. Equation (4.17) states that the l_i poles of the ith subsystem are evenly spaced around the unit circle of the z plane starting at an angle $k_i 2\pi/N$. Integers k_i, $i = 0, 1, \cdots, I-1$, are constrained by the requirement of Property 1 that the set of z_{im}, $0 \leqslant m \leqslant l_i - 1$, $0 \leqslant i \leqslant I-1$, be the set of Nth roots of unity. We summarize these findings in the following lemmas.

Lemma 1: The number of rows, l_i, in T_i of (4.8) is a divisor of N as in (4.15).

Lemma 2: Quantities $x_n^{(i)}$ in (4.9) and (4.11) satisfy (4.16) for $i = 0, 1, \cdots, I-1$. Quantities k_i in (4.16) are integers in the range $0 \leqslant k_i \leqslant \eta_i - 1$ and are such that the z_{im} of (4.17), $0 \leqslant m \leqslant l_i - 1$, $0 \leqslant i \leqslant I-1$, are the N roots of unity. Equivalently,

$$\{k_i + m\eta_i: 0 \leqslant k_i \leqslant \eta_i - 1, 0 \leqslant m \leqslant l_i - 1, \\ 0 \leqslant i \leqslant I-1\} = \{n: 0 \leqslant n \leqslant N-1\}. \tag{4.18}$$

We are now in a position to give an explicit description of the row vectors in T_i. Since row vectors $\boldsymbol{h}_1^{(i)}, \boldsymbol{h}_2^{(i)}, \cdots, \boldsymbol{h}_{l_i-1}^{(i)}$ are related to $\boldsymbol{h}_0^{(i)}$ by the scaled circular delays (4.9), it suffices to state the explicit form of $\boldsymbol{h}_0^{(i)}$. This is done in the following theorem.

Theorem 1: The upper row vector in T_i, $\boldsymbol{h}_0^{(i)}$, has the form

$$h_0^{(i)}(n) = v^{(i)}(n|l_i) w^{nk_i}, \qquad 0 \leqslant n \leqslant N-1 \tag{4.19}$$

where $v^{(i)}(n|l_i)$ is a periodic sequence having period l_i, and where k_i satisfies (4.18). Equivalently, transfer function $H_0^{(i)}(k)$ is nonzero only for $k = k_i + m\eta_i$, $0 \leqslant m \leqslant l_i - 1$.

Proof: Repeated substitutions in system (4.9) gives

$$\boldsymbol{h}_0^{(i)} = \left[\prod_{n=0}^{l_i-1} x_n^{(i)}\right] \mathfrak{D}^{l_i} \{\boldsymbol{h}_0^{(i)}\} \tag{4.20}$$

where $\mathfrak{D}^{l_i}\{\boldsymbol{h}_0^{(i)}\}$ denotes l_i consecutive circular delay operations on $\boldsymbol{h}_0^{(i)}$. Let $H^{(i)}(k)$ denote the DFT of $h_0^{(i)}(n)$:

$$H_0^{(i)}(k) = \sum_{n=0}^{N-1} h_0^{(i)}(n) w^{-nk}, \qquad 0 \leqslant k \leqslant N-1. \tag{4.21}$$

Using (4.20) it follows from the circular shift property of the DFT that

$$H_0^{(i)}(k) = \left[\prod_{n=0}^{l_i-1} x_n^{(i)}\right] w^{-l_i k} H_0^{(i)}(k), \qquad 0 \leqslant k \leqslant N-1 \tag{4.22}$$

which is, from (4.16):

$$H_0^{(i)}(k) = w^{l_i(k_i-k)} H_0^{(i)}(k), \qquad 0 \leqslant k \leqslant N-1. \tag{4.23}$$

The factor $w^{l_i(k_i-k)}$ is unity at points

$$k = k_i + m\eta_i, \qquad 0 \leqslant m \leqslant l_i - 1 \tag{4.24}$$

which correspond to the l_i roots of unity of (4.17). Clearly, $H_0^{(i)}(k)$ of (4.23) can be nonzero only at these values of k. Substituting (4.24) into (4.21) yields

$$h_0^{(i)}(n) = w^{nk_i} \frac{1}{N} \sum_{m=0}^{l_i-1} H_0^{(i)}(k_i + m\eta_i) w^{m\eta_i n}. \tag{4.25}$$

The sum in (4.25) is a periodic function of n having period l_i so that $h_0^i(n)$ has the form (4.19). Q.E.D.

We conclude from the preceding development that, in general, a minimal transform represents I parallel subtransforms T_i, $i = 0, 1, \cdots, I-1$, where I is an integer from 1 to N. Subtransform T_i represents a system having a single input $f(n)$ and $l_i = N/\eta_i$ outputs which form a subset of the transform coefficients $F(p, n)$ of (2.1). The zeroth row vector of T_i, namely $h_0^{(i)}(n)$, represents an FIR filter whose transfer function $H_0^{(i)}(k)$ is zero at harmonics $k \neq k_i + m\eta_i$, $0 \leqslant m \leqslant l_i - 1$. Subsequent row vectors of T_i are, by (4.9), scaled circular delays of $h_0^{(i)}(n)$ and as such, reject these same harmonics. By (4.18) the harmonics passed by the row vectors of T_i, namely $k_i + m\eta_i$, $0 \leqslant m \leqslant l_i - 1$, are different from those contained in row vectors of T_j, $j \neq i$. It follows that the row vectors in T_i are orthogonal to those not in T_i. The question of the orthogonality of the row vectors within T_i is answered by the following lemma.

Lemma 3: The row vectors in T_i are

a) *linearly independent iff* $H_0^{(i)}(k_i + m\eta_i) \neq 0$ *for* $m = 0, 1, \cdots, l_i - 1$;

b) *orthogonal iff* $|H_0^{(i)}(k_i + m\eta_i)| = const \neq 0$ *for* $m = 0, 1, \cdots, l_i - 1$;

c) *orthonormal iff* $|H_0^{(i)}(k_i + m\eta_i)|^2 = \eta_i$ and $|x_m^{(i)}| = 1$ *for* $m = 0, 1, \cdots, l_i - 1$.

Proof: The proof is long but straightforward and is outlined in the Appendix.

Examples of Minimal Transforms

1) Diagonal A_c: Selecting $I = N$ in (4.10) leads to N uncoupled subsystems, each with order $l_i = 1$, $0 \leqslant i \leqslant N-1$. Equation (4.16) yields $x_0^{(i)} = w^{k_i}$ with $\{k_i: 0 \leqslant i \leqslant N-1\} = \{n: 0 \leqslant n \leqslant N-1\}$. Hence from (4.10) and (4.11), $A_c = \mathrm{diag}\,[w^{k_i}]$. The ith row vector of T_c is by (4.19) the discrete Fourier sequence $h_0^{(i)}(n) = s_i w^{nk_i}$ where s_i is any nonzero constant. We conclude that, except for minor variations, the running discrete Fourier transform is the only transform having a diagonal circular advance matrix.

2) Circulant T_c: Selecting $I = 1$ in (4.10) leads to one system whose order is $l_0 = N$. In this event, $\prod_{n=0}^{N-1} x_n^{(0)} = 1$ and $h_0^{(0)}(n) = s^{(0)}(n|N)$. The N rows of T_c consist of weighted circular delays of $h_0^{(0)}(n)$ as in (4.9) with $i = 0$ and $l_0 = N$. When the weights are equal, say $x_n^{(0)} = 1$, $0 \leqslant n \leqslant N-1$, then T_c is circulant, and $A_c = B$ of (3.1).

The above two examples have assumed either $l_i = 1$ or $l_i = N$. Clearly, if N is prime then its only divisors are 1 and N and the above represent the only possible forms for T_c and A_c. If N is composite, there will be several possible canonic minimal transform matrices.

3) $N = 4$: There are five ways to partition the four roots of unity consistent with (4.18) and five corresponding canonic minimal transforms. These are characterized by the following five choices for the "l"'s and "k"'s:

i) $I = 4$: $l_0 = l_1 = l_2 = l_3 = 1$ as in example 1)

ii) $I = 1$: $l_0 = 4; k_0 = 0$ as in example 2)

iii) $I = 2$: $l_0 = l_1 = 2; k_0 = 0, k_1 = 1$

iv) $I = 3$: $l_0 = 2, l_1 = l_2 = 1; k_0 = 0, k_1 = 1, k_2 = 3$

v) $I = 3$: $l_0 = 2, l_1 = l_2 = 1; k_0 = 1, k_1 = 0, k_2 = 2$.

It can be verified that the matrices of Fig. 8 exemplify choice iv). Note that matrix T_c of Fig. 8 satisfies invertibility condition (2.7). The developments in Section IV did not require that (2.7) be satisfied, and it is easy to find examples which cannot satisfy (2.7) (example 2 above). The determination of the class of minimal transforms that satisfy (2.7) is a possible area for future study.

Conclusions

This paper has defined a running discrete transform with respect to an arbitrary transform matrix T. It has shown that the running discrete transform satisfies a recursion relation that, in certain cases, leads to computational savings in obtaining transform coefficients of displaced data. The paper has derived the class of transforms (called minimal transforms) having the minimum possible number of complex operations in running transform computation.

Appendix
Outline of Proof of Lemma 3

The pth row vector in T_i is, from (4.9) and (4.25)

$$h_p^{(i)}(n) = \phi^{(i)}(p) \sum_{q=0}^{l_i-1} H_0^{(i)}(k_i + q\eta_i)\, w^{(n-p)(k_i+q\eta_i)} \tag{A1}$$

where

$$\phi^{(i)}(p) = \begin{cases} 1/N, & p = 0 \\ \dfrac{1}{N} \displaystyle\prod_{m=0}^{p-1} x_m^{(i)}, & 1 \leqslant p \leqslant l_i - 1. \end{cases} \tag{A2}$$

It follows from (4.8) and (A1) that

$$T_i = \Phi M_1 M_2 M_3 \tag{A3}$$

where

i) Φ is an $l_i \times l_i$ nonsingular diagonal matrix whose pth diagonal element is $\phi^{(i)}(p)$;

ii) M_1 is an $l_i \times l_i$ nonsingular matrix whose mnth entry is $w^{-m(k_i+n\eta_i)}$;

iii) M_2 is an $l_i \times l_i$ diagonal matrix whose pth diagonal element is $H_0^{(i)}(k_i + p\eta_i)$; and

iv) M_3 is an N column matrix of l_i linearly independent rows having mnth entry $w^{n(k_i+m\eta_i)}$, $0 \leqslant m \leqslant l_i - 1$, $0 \leqslant n \leqslant N-1$.

The linear independence of the row vectors in M_1 and in M_3 in ii) and iv) follows from the fact that these matrices can be obtained by deleting rows and columns from Γ or Γ^* of (3.12). Since the l_i rows of M_3 are linearly independent, the l_i rows of T_i will be linearly independent if and only if $\Phi M_2 M_3$ is nonsingular. The necessary and sufficient condition for this is that $H_0^{(i)}(k_i + p\eta_i)$ be nonzero for $p = 0, 1, \cdots, l_i - 1$. This establishes part a) of Lemma 3.

Using (A1) and Parseval's theorem, one can show that the inner product $\boldsymbol{h}_m^{(i)}\boldsymbol{h}_n^{(i)*}$ is given by

$$\sum_{l=0}^{N-1} h_m^{(i)}(l) h_n^{(i)*}(l) = N\phi^{(i)}(m)\phi^{(i)*}(n) w^{(n-m)k_i} \cdot \sum_{q=0}^{l_i-1} |H_0^{(i)}(k_i + q\eta_i)|^2 w^{q\eta_i(n-m)}. \tag{A4}$$

The sum on the right-hand side of (A4) is zero for $m \neq n$ if and only if $|H_0^{(i)}(k_i + q\eta_i)|$ is constant for $0 \leqslant q \leqslant l_i - 1$. This establishes part b) of Lemma 3.

Suppose we take $|H_0^{(i)}(k_i + q\eta_i)|^2 = \beta_i$, $0 \leqslant q \leqslant l_i - 1$, and evaluate (A4) with $m = n$. The result is

$$\sum_{l=0}^{N-1} |h_n^{(i)}(l)|^2 = N|\phi^{(i)}(n)|^2 \beta_i l_i. \tag{A5}$$

From (A5) and (A2) it follows that the row vectors $\boldsymbol{h}_n^{(i)}$, $0 \leqslant n \leqslant l_i - 1$, are normalized if and only if $\beta_i = N/l_i$ and $|x_n^{(i)}| = 1$, $0 \leqslant n \leqslant l_i - 1$. (A trivial exception occurs for $l_i = 1$ in which case one must have only $\beta_i = N/l_i$). This establishes part c) of Lemma 3.

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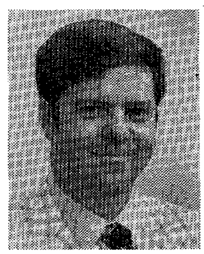

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