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On the Relation between Triangular Matrix Decomposition and Linear Interpolation

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where $D(\boldsymbol{z})$ is an n-dimensional polynomial in the complex variables $\boldsymbol{z} = (z_1, \cdots, z_n)$. The test is carried out by checking relatively simple one-dimensional conditions, e.g. [1], [2]

$$D(z, \cdots, z) \neq 0 \text{ in } |z| \leq 1 \tag{2}$$

in addition to a multidimensional condition on the distinguished boundary

$$D(\boldsymbol{z}) \neq 0 \text{ in } \bigcap_{i=1}^{n} |z_i| = 1 \tag{3}$$

which is the main computational burden, although simpler than the original condition (1).

In [3], a set of simple sufficient conditions has been given for the multidimensional condition (3). Although these conditions are correct[1] and easy to check, it turns out that these very same conditions are sufficient for the violation of (2). Hence, they also constitute a set of simple sufficient conditions for instability of the system, which joins recently presented instability sufficient conditions [4].

The Sufficient Conditions for Instability

Let

$$D(\boldsymbol{z}) = \sum_{i=1}^{m} \alpha_i z_1^{k_{i1}} \cdots z_n^{k_{in}} + \alpha_0 \tag{4}$$

be the denominator of the transfer function of an n-dimensional discrete causal system, and assume that the numerator and denominator are mutually prime. In [3] it has been shown that the following $m + 1$ conditions are each sufficient to ensure (3):

$$|\alpha_j| > \sum_{\substack{i=0 \\ i \neq j}}^{m} |\alpha_i|, \qquad j \in \{0, 1, \cdots, m\}. \tag{5}$$

We show here that except for the condition with $j = 0$, the other m conditions in (5) are each sufficient to violate (2) and therefore to violate (1), i.e., they are sufficient conditions for structural instability of the system. To do this we use a result due to Cohn and to Thoma, and described in [4]:

Let $P(z)$ be a polynomial in z

$$P(z) = \sum_{i=0}^{q} \beta_i z^i. \tag{6}$$

If

$$|\beta_j| > \sum_{\substack{i=0 \\ i \neq j}}^{q} |\beta_i|, \qquad j \in \{0, \cdots, q\} \tag{7}$$

then $P(z)$ has j zeros in $|z| < 1$ and $q - j$ zeros in $|z| > 1$.

By (4)

$$D(z, \cdots, z) = \sum_{i=1}^{m} \alpha_i z^{\sum_{j=1}^{n} k_{ij}} + \alpha_0. \tag{8}$$

If we denote

$$D(z, \cdots, z) = P(z) \tag{9}$$

where

$$q = \max_i \left(\sum_{j=1}^{n} k_{ij} \right) \tag{10}$$

then to show that conditions (5) for $j \neq 0$ are sufficient for instability, it suffices to show that

$$(5) \Rightarrow (7) \tag{11}$$

and since $P(z)$ must possess j zeros in $|z| < 1$, this means violation of (2) for $j \neq 0$.

Suppose (5) is satisfied for some j, say j_0, and assume without loss of generality that $\alpha_{j_0} > 0$, so that $\alpha_{j_0} = |\alpha_{j_0}|$. Then

$$\exists r \epsilon \beta_r = \alpha_{j_0} + \alpha_{\ell_1} + \cdots + \alpha_{\ell_v} \tag{12}$$

where the set $\{\alpha_{\ell_1}, \cdots, \alpha_{\ell_v}\}$ is part of the set α_i ($i = 0, \cdots, m$), but may, of course, be an empty set. It follows from (5) that $\beta_r > 0$, so that $\beta_r = |\beta_r|$.

Let the set $\{\alpha_{\ell_1}, \cdots, \alpha_{\ell_v}\}$ be partitioned into nonnegative and negative elements, denoted by $\{\alpha_{\ell_1}, \cdots, \alpha_{\ell_u}\}$ and $\{\alpha_{\ell_{u+1}}, \cdots, \alpha_{\ell_v}\}$, respectively. Then, by (5)

$$\alpha_{j_0} = |\alpha_{j_0}| > \sum_{\substack{i=0 \\ i \neq j}}^{m} |\alpha_i| \geq \alpha_{\ell_1} + \cdots + \alpha_{\ell_u} - \alpha_{\ell_{u+1}} - \cdots - \alpha_{\ell_v} + \sum_{\substack{i=0 \\ i \neq r}}^{q} |\beta_i|$$

$$\geq -\alpha_{\ell_{u+1}} - \cdots - \alpha_{\ell_v} + \sum_{\substack{i=0 \\ i \neq r}}^{q} |\beta_i|. \tag{13}$$

Therefore

$$\alpha_{j_0} + \alpha_{\ell_1} + \cdots + \alpha_{\ell_u} > -\alpha_{\ell_{u+1}} - \cdots - \alpha_{\ell_v} + \sum_{\substack{i=0 \\ i \neq r}}^{q} |\beta_i| \tag{14}$$

which implies

$$\beta_r = |\beta_r| > \sum_{\substack{i=0 \\ i \neq r}}^{q} |\beta_i|. \tag{15}$$

This completes the proof.

[1]Equation (11) in [3] is incorrect and should be omitted, without impairing the proof.

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On the Relation Between Triangular Matrix Decomposition and Linear Interpolation

J. A. STULLER

The December 1983 letter by C. W. Therrien concerning the relation between triangular matrix decomposition and linear prediction is extended to include linear interpolation.

The relation between triangular matrix decomposition and linear prediction has been described in a recent tutorial letter by C. W. Therrien [1]. There is an equally interesting relation between triangular matrix decomposition and linear interpolation that also may not be widely recognized [2], [3].

Let $R_x = E\{\boldsymbol{x}\boldsymbol{x}^*\}$ be a positive definite covariance matrix corresponding to a set of observations $\boldsymbol{x} = [x_1 x_2 \cdots x_N]^T$ of a zero-mean stationary random process. Define matrix H having zero principal diagonal

$$H = I - D_u R_x^{-1} \tag{1}$$

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where I is the $N \times N$ identity matrix, and $D_u = \mathrm{diag}[\sigma_{u_1}^2, \sigma_{u_2}^2, \cdots, \sigma_{u_N}^2]$ is a real diagonal matrix. Because H has zero principal diagonal, then

$$\sigma_{u_i}^2 = \frac{1}{\gamma_i}, \qquad i = 1, 2, \cdots, N \tag{2}$$

where γ_i is the ith element on the principal diagonal of R_x^{-1}. Define the random vector $\boldsymbol{u} = [u_1 u_2 \cdots u_N]^T$ by

$$\boldsymbol{u} = [I - H]\boldsymbol{x} \tag{3}$$

so that $\boldsymbol{x}$ can be written as

$$\boldsymbol{x} = H\boldsymbol{x} + \boldsymbol{u}. \tag{4}$$

Because H has zero principal diagonal, (4) represents each observation x_i in terms of a weighted sum of the observations x_j for all $j \neq i$ plus an interpolation error u_i. Using (3) and (1) one finds that the covariance matrix of observations and interpolation errors $R_{xu} = E\{\boldsymbol{xu}^*\}$ is the diagonal matrix D_u

$$R_{xu} = D_u. \tag{5}$$

Consequently, u_i is orthogonal to x_j for all $j \neq i$. Therefore, the ith row of H is the LMMSE interpolator of x_i from the x_j, $j \neq i$, and u_i is the interpolation error. The covariance matrix of interpolation errors $R_u = E\{\boldsymbol{uu}^*\}$ is, from (3) and (1)

$$R_u = D_u R_x^{-1} D_u. \tag{6}$$

The ith element on the principal diagonal of $D_u R_x^{-1} D_u$ is, by (2), $\sigma_{u_i}^2$. Therefore, $\sigma_{u_i}^2$ is the LMMSE interpolation error variance.

As described by Therrien [1], the covariance matrix R_x can be factored as

$$R_x = L D_w L^* \tag{7}$$

where $D_w = \mathrm{diag}[\sigma_{w_1}^2, \sigma_{w_2}^2, \cdots, \sigma_{w_N}^2]$ is a real diagonal matrix, L is lower triangular with 1's on the diagonal, and * represents conjugate transposition. Therrien shows that the rows of $L_p = L^{-1}$ are the coefficients of LMMSE prediction for orders 0 through $N-1$ and the elements of D_w are the corresponding prediction error variances. More specifically, he shows that L_p has the form

$$L_p = \begin{bmatrix} 1 & & & & \\ -\tilde{a}_1^* & 1 & & 0 & \\ -\tilde{a}_2^* & \rightarrow & 1 & & \\ \vdots & & & & \\ \leftarrow & & & -\tilde{a}_{N-1}^* \rightarrow & 1 \end{bmatrix} \tag{8}$$

where $\tilde{\boldsymbol{a}}_k$ represents the reversal of $\boldsymbol{a}_k$ (coefficients written in reverse order), and $\boldsymbol{a}_k$ is the vector of linear prediction coefficients for a predictor of order k.

The relation between triangular matrix decomposition and linear interpolation follows directly from (7) and (1). Because

$$\begin{aligned} R_x^{-1} &= L^{*-1} D_w^{-1} L^{-1} \\ &= L_p^* D_w^{-1} L_p \end{aligned} \tag{9}$$

then

$$H = I - D_u L_p^* D_w^{-1} L_p. \tag{10}$$

The extension of these results to the two-dimensional interpolation problem is straightforward. The special case arising for wide sense Markov processes is the topic of [2], [3].

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An Information-Theoretic Proof of Burg's Maximum Entropy Spectrum

B. S. CHOI AND THOMAS M. COVER

It is known that the maximum entropy stationary Gaussian stochastic process, subject to a finite number of autocorrelation constraints, is the Gauss–Markov process of appropriate order. The associated spectrum is Burg's maximum entropy spectral density. We pose a somewhat broader entropy maximization problem, in which stationarity and normality are not assumed, and shift the burden of proof from the previous focus on the calculus of variations and time series techniques to a string of information-theoretic inequalities. This results in an elementary proof of greater generality.

I. Preliminaries

Let $\{X_i\}_{i=1}^{\infty}$ be a stochastic process specified by its marginal probability density functions $f(x_1, x_2, \cdots, x_n)$, $n = 1, 2, \cdots$. Then the (*differential*) *entropy* of the n-sequence $X_1, X_2, \cdots, X_n$ is defined by

$$h(X_1, X_2, \cdots, X_n) = -\int f(x_1, \cdots, x_n) \cdot \ln f(x_1, \cdots, x_n)\, dx_1\, dx_2 \cdots dx_n = h(f). \tag{1}$$

The stochastic process $\{X_i\}$ will be said to have an *entropy rate*

$$h = \lim_{n \to \infty} \frac{h(X_1, X_2, \cdots, X_n)}{n} \tag{2}$$

if the limit exists. It is known that the limit always exists for stationary processes.

II. History

Previous characterizations of the maximum entropy spectral density assume that the process is stationary and Gaussian. For such a process, the entropy rate is given [1] by

$$h = \frac{1}{2}\ln(2\pi e) + \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln(2\pi S(\lambda))\, d\lambda \tag{3}$$

where the spectral density $S(\lambda)$ is given by

$$S(\lambda) = \frac{1}{2\pi}\sum_{\ell=-\infty}^{\infty} \sigma(\ell) e^{-i\lambda\ell} \tag{4}$$

and $\{\sigma(\ell)\}_{\ell=-\infty}^{\infty}$ is an arbitrary autocovariance function subject to the constraints $\sigma(0) = \alpha_0, \cdots, \sigma(p) = \alpha_p$. Burg [2], [3] was first to find the maximum of (3). Subsequent proofs were exhibited in [4]–[13]. These proofs are deeper than the proof of Theorem 1 that we give in the next section, perhaps because of the complexity of the functional in (3). Such techniques as the calculus of variations, complex integration, and linear prediction theory are used in the proofs. A fuller set of references and further details can be found in [14].

We shall see that the entropy maximization can be captured in the information-theoretic string of inequalities in (6) of the next section.

III. Theorem and Proof

We prove the following theorem.

Theorem 1: The stochastic process $\{X_i\}_{i=1}^{\infty}$ that maximizes the differential entropy rate h subject to the autocorrelation constraints

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