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for the MIMO LQ stochastic regulators," *IEEE Trans. Automat. Contr.*, vol. 35, no. 3, pp. 320–322, 1990.
[3] M. J. Grimble, "Single versus double diophantine equation debate: Comments on 'A polynomial approach to Wiener filtering'," *Int. J. Contr.*, vol. 48, no. 5, pp. 2161–2165, 1988.
[4] V. Kucera, *Discrete Linear Control.* Chichester: Wiley, 1979.
[5] V. Peterka, "Predictor-based self-tuning control," *Automatica*, vol. 20, pp. 39–50, 1984.

Linear Interpolation Lattice

Cameron K. Coursey and John A. Stuller

Abstract—**The well-known analysis and synthesis filters of linear prediction theory are extended here to include linear interpolation.**

I. Introduction

Lattice filters have been used in linear prediction and joint process estimation for several years [1], [2]. To our knowledge, however, there is no previous reference to using a lattice structure as an interpolation filter. In this correspondence we extend the well-known lattice of linear prediction theory to include linear interpolation. The result provides insight into the interrelation of prediction and interpolation, and of one- and two-sided representations of one-dimensional Markov sequences [3].

II. Mathematical Preliminaries

Let $\{Y_n\}$ be a real valued wide-sense stationary random process indexed on the set of integers. In pth order linear prediction, we assume that Y_n can be modeled as an autoregressive process, and forwardly "predict" Y_n from the p "previous" data, viz.,

$$\hat{Y}_p(n) = -\sum_{i=1}^{p} a_{p,i} Y_{n-i} \tag{1}$$

where $\hat{Y}_p(n)$ denotes the predicted value of Y_n, and where the coefficients $a_{p,i}$, $1 \le i \le p$, are pth order prediction coefficients. The Levinson–Durbin algorithm [2], [4], [5] provides a means of determining the minimum-mean-square (MMS) error pth order prediction coefficients and the pth order MMS prediction error, $P_n = E\{(Y_n - \hat{Y}_p(n))^2\}$, from the MMS error p-first order prediction coefficients and P_{n-1}. The Levinson–Durbin algorithm leads to analysis and synthesis lattice structures in which the process Y_n is related to its forward and backward prediction errors of orders $p = 0, 1, 2, \cdots$ [1], [2].

In pth order linear interpolation, [6], [7], we linearly estimate Y_n from p "previous" and p "future" data, viz.,

$$\hat{Y}_p(n) = -\sum_{\substack{i=-p \\ i \ne 0}}^{p} b_{p,i} Y_{n+i}. \tag{2}$$

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The pth order interpolation error is

$$u_p(n) = Y_n - \hat{Y}_p(n) = Y_n + \sum_{\substack{i=-p \\ i \ne 0}}^{p} b_{p,i} Y_{n+i} = \boldsymbol{Y}_p^T \boldsymbol{B}_p \tag{3}$$

where $\boldsymbol{Y}_p \triangleq (Y_{n-p}, Y_{n-p+1}, \cdots, Y_n, \cdots, Y_{n+p-1}, Y_{n+p})^T$ and $\boldsymbol{B}_p \triangleq (b_{p,p}, b_{p,p-1}, \cdots, 1, \cdots, b_{p,-p+1}, b_{p,-p})^T$. (Superscript T denotes the transpose of a matrix.) The interpolation coefficients are chosen to minimize the mean-squared interpolation error $E\{u_p^2(n)\}$. The optimum interpolation vector $\boldsymbol{B}_p$ is the solution to the normal matrix equation

$$\boldsymbol{R}_{2p} \boldsymbol{B}_p = \boldsymbol{U}_p \tag{4}$$

where $\boldsymbol{U}_p \triangleq (0, \cdots, 0, U_p, 0, \cdots, 0)^T$, U_p is the MMS interpolation error, and $\boldsymbol{R}_{2p} \triangleq E\{\boldsymbol{Y}_p \boldsymbol{Y}_p^T\}$ is the $(2p+1) \times (2p+1)$ autocorrelation matrix of $\boldsymbol{Y}_p$. Since $\{Y_n\}$ is a stationary process, the autocorrelation matrix, $\boldsymbol{R}_{2p}$, is a Toeplitz matrix whose inverse is symmetric about both its major and minor diagonals. Further, $\boldsymbol{U}_p$ represents the unit-sample response of a linear phase FIR filter with even symmetry, so $b_{p,i} = b_{p,-i}$ [9], [10].

III. Recursive Solution for $\boldsymbol{B}_p$

The substitution $p \to p+1$ in (4) yields

$$\boldsymbol{R}_{2p+2} \boldsymbol{B}_{p+1} = \boldsymbol{U}_{p+1}. \tag{5}$$

We can derive the $(p+1)$st order linear interpolator from the pth order linear interpolator by taking advantage of the fact that $\boldsymbol{R}_{2p+2}$ contains $\boldsymbol{R}_{2p}$ as a submatrix:

$$\boldsymbol{R}_{2p+2} = \begin{bmatrix} r_0 & r_1 \cdots r_{2p+1} & r_{2p+2} \\ r_1 & & r_{2p+1} \\ \vdots & \boldsymbol{R}_{2p} & \vdots \\ r_{2p+1} & & r_1 \\ r_{2p+2} & \cdots r_1 & r_0 \end{bmatrix}. \tag{6}$$

Where $r_k = E\{Y_n Y_{n-k}\}$. We try a solution to (5) of the form

$$\boldsymbol{B}_{p+1} = \beta_{p+1} \begin{bmatrix} 0 \\ \boldsymbol{B}_p \\ 0 \end{bmatrix} + \boldsymbol{C}_{p+1} \tag{7}$$

where β_{p+1} is a constant and $\boldsymbol{C}_{p+1}$ is a vector. The substitution of (7) into (5) yields, with the aid of (4) and (6)

$$\beta_{p+1} \boldsymbol{R}_{2p+2} \begin{bmatrix} 0 \\ \boldsymbol{B}_p \\ 0 \end{bmatrix} + \boldsymbol{R}_{2p+2} \boldsymbol{C}_{p+1} = \beta_{p+1} \begin{bmatrix} h_p \\ \boldsymbol{U}_p \\ h_p \end{bmatrix} + \boldsymbol{R}_{2p+2} \boldsymbol{C}_{p+1} = \boldsymbol{U}_{p+1} \tag{8}$$

where

$$h_p \triangleq \sum_{i=-p}^{p} b_{p,|i|} r_{p+1-i}. \tag{9}$$

We choose

$$\beta_{p+1} = \frac{U_{p+1}}{U_p} \tag{10}$$

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so that (8) becomes

$$R_{2p+2}C_{p+1} = \begin{bmatrix} -\beta_{p+1}h_p \\ \mathbf{0} \\ -\beta_{p+1}h_p \end{bmatrix}. \tag{11}$$

To obtain a recursive solution for vector C_{p+1}, we recall that the MMSE $(2p + 2)$nd order linear predictor coefficients satisfy [2], [4], [5]:

$$R_{2p+2}A_{2p+2} = P_{2p+2} \tag{12}$$

and

$$R_{2p+2}A^R_{2p+2} = P^R_{2p+2} \tag{13}$$

where $A_{2p+2} \triangleq (1, a_{2p+2,1}, a_{2p+2,2}, \cdots, a_{2p+2,2p+2})^T$, $P_{2p+2} \triangleq (0, 0, \cdots, 0, P_{2p+2})^T$ and superscript R denotes the reverse of a vector, i.e., $x^R = (x_N, x_{N-1}, \cdots, x_2, x_1)^T$ for $x = (x_1, x_2, \cdots, x_N)^T$. Therefore, we try a solution to (11) of the form

$$C_{p+1} = -\alpha_{p+1}\beta_{p+1}(A_{2p+2} + A^R_{2p+2}) \tag{14}$$

where α_{p+1} is a constant to be determined. The substitution of (14) into (11) yields, with the aid of (12), (13)

$$-\alpha_{p+1}\beta_{p+1}\begin{bmatrix} P_{2p+2} \\ \mathbf{0} \\ P_{2p+2} \end{bmatrix} = \begin{bmatrix} -\beta_{p+1}h_p \\ \mathbf{0} \\ -\beta_{p+1}h_p \end{bmatrix}. \tag{15}$$

Therefore,

$$\alpha_{p+1} = \frac{h_p}{P_{2p+2}} = \frac{1}{P_{2p+2}}\sum_{i=-p}^{p} b_{p,|i|}r_{p+1-i}. \tag{16}$$

Thus, by (7) and (14), we have found that

$$B_{p+1} = \beta_{p+1}\left\{\begin{bmatrix} 0 \\ B_p \\ 0 \end{bmatrix} - \alpha_{p+1}(A_{2p+2} + A^R_{2p+2})\right\} \tag{17}$$

where β_{p+1} and α_{p+1} satisfy (10) and (16), respectively. Note from (10) that β_{p+1} is the ratio of the $p + 1$st order MMS interpolation error to the pth order MMS interpolation error. Clearly, therefore, $0 \le \beta_{p+1} \le 1$. We can obtain another expression for β_{p+1} by rearranging the middle row of (17) to yield

$$\beta_{p+1} = \frac{1}{1 - 2\alpha_{p+1}a_{2p+2,p+1}}. \tag{18}$$

A recursion for the interpolation coefficients can be written directly from (17)

$$b_{p+1,k} = \beta_{p+1}\left\{b_{p,k} - \alpha_{p+1}[a_{2p+2,p+1+k} + a_{2p+2,p+1-k}]\right\}, \quad 0 \le k \le p$$
$$b_{p+1,p+1} = -\alpha_{p+1}\beta_{p+1}(1 + a_{2p+2,2p+2}). \tag{19}$$

The recursion is initialized by setting $p = 0$, $a_{0,0} = b_{0,0} = 1$, $P_0 = U_0 = r_0$, and $h_0 = r_1$. Two iterations of the Levinson–Durbin algorithm then yields the prediction coefficients, $a_{2,k}$, $k = 1, 2$, and the second-order MMS prediction error P_2. Constants α_1 and β_1 are then given by (16) and (18), respectively. These, in turn, permit generation of the first-order interpolation coefficients $b_{1,k}$, $k = 1$, from (19). The first-order MMS interpolation error U_1 is then given by (10), and h_1 is computed using (9). Returning to the Levinson–Durbin algorithm with $p = 1$, one similarly obtains the second-order interpolation coefficients, $b_{2,k}$, $k = 1, 2$, and the second-order MMS interpolation error U_2, etc.

The Levinson algorithm is computationally less efficient than the split-Levinson algorithm [2], [10], which takes advantage of the centro-symmetry of R_{2p} in a way that reduces the number of multiplications by a factor of two. Our recursion (19) is indirectly related to the split-Levinson algorithm and we could easily instrument interpolation as a special case of linear-phase filtering using the split-Levinson algorithm. However, our goal is not to obtain the most efficient algorithm, but to relate the analysis and synthesis lattices of linear prediction theory to linear interpolation.

IV. Lattice

A noncausal lattice filter for linear interpolation is obtained from the linear interpolation recursion as follows: Let $p \to p + 1$ in (3) and use (17) to write

$$u_{p+1}(n) = Y^T_{p+1}B_{p+1} = \beta_{p+1}\left\{Y^T_{p+1}\begin{bmatrix} 0 \\ B_p \\ 0 \end{bmatrix} - \alpha_{p+1}(Y^T_{p+1}A_{2p+2} + Y^T_{p+1}A^R_{2p+2})\right\}. \tag{20}$$

Where

$$Y^T_{p+1}\begin{bmatrix} 0 \\ B_p \\ 0 \end{bmatrix} = u_p(n) \tag{21}$$

$$Y^T_{p+1}A_{2p+2} = \sum_{i=0}^{2p+2} a_{2p+2,i}Y_{n-(p+1)+i} \tag{22}$$

and

$$Y^T_{p+1}A^R_{2p+2} = \sum_{i=0}^{2p+2} a_{2p+2,i}Y_{n+(p+1)-i}. \tag{23}$$

It is well known that [2]

$$e^+_{2p+2}(n) = \sum_{i=0}^{2p+2} a_{2p+2,i}Y_{n-i} \tag{24}$$

and

$$e^-_{2p+2}(n) = \sum_{i=0}^{2p+2} a_{2p+2,i}Y_{n-(2p+2)+i} \tag{25}$$

are the forward and backward $(2p + 2)$th order MMS prediction errors. Thus, the substitution of (21)–(25) into (20) yields the following relation between interpolation error and prediction error:

$$u_{p+1}(n) = \beta_{p+1}\left\{u_p(n) - \alpha_{p+1}[e^+_{2p+2}(n + p + 1) + e^-_{2p+2}(n + p + 1)]\right\} \tag{26}$$

where $p = 0, 1, 2, \cdots$. The forward and backward prediction errors can be obtained using the lattice analysis filter enclosed by dashes in Fig. 1 [2]. Here the $\gamma_p \equiv -a_{p,p}$ are the reflection or partial correlation coefficients and z^{-1} denotes unit delay. The reflection coefficient γ_p relates the MMS prediction error of order p, P_p, to that of order $p - 1$ by $P_p = (1 - \gamma_p^2)P_{p-1}$ [2]. We see from (26) that the interpolation errors may also be obtained from the lattice of Fig. 1 by extending the lattice to include the noncausal operations added outside the dashes. By rearranging (26) we obtain

$$u_p(n) = \frac{1}{\beta_{p+1}}u_{p+1}(n) - \alpha_{p+1}[e^+_{2p+2}(n + p + 1) + e^-_{2p+2}(n + p + 1)]. \tag{27}$$

Equation (27) can be instrumented by the noncausal lattice filter of Fig. 2. This is the synthesis lattice which generates the process

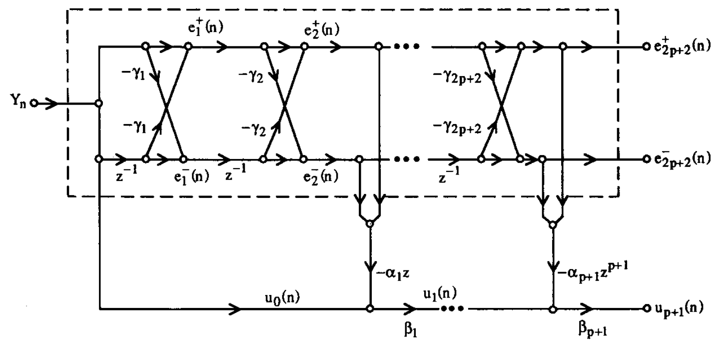

Fig. 1. Analysis lattice.

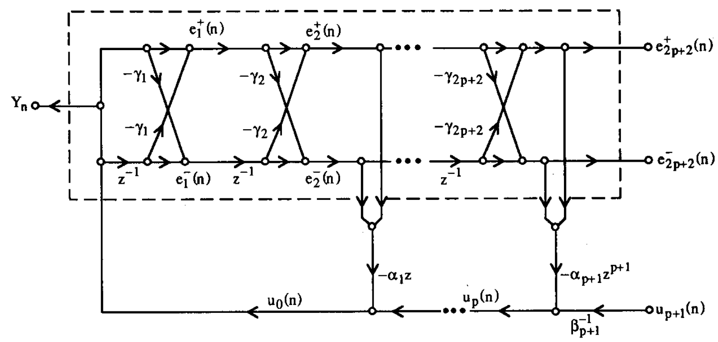

Fig. 2. Synthesis lattice.

$\{Y_n\}$ and the forward and backward prediction errors from the interpolation error process.

References

[1] B. Friedlander, "Lattice methods for spectral estimation," *Proc. IEEE*, pp. 990-1017, Sept. 1982.
[2] S. J. Orfanidis, *Optimum Signal Processing*, 2nd ed. New York: McGraw-Hill, 1988.
[3] J. W. Woods, "Two-dimensional discrete Markovian fields," *IEEE Trans. Inform. Theory*, vol. IT-18, pp. 232-240, Mar. 1972.
[4] N. Levinson, "The Wiener RMS (root mean square) error criterion in filter design and prediction," *J. Math Phys.*, vol. 25, pp. 261-278, 1947.
[5] J. Durbin, "Efficient estimation of parameters in moving-average models," *Biometrike*, vol. 46, pts. 1 and 2, pp. 306-316, 1959.
[6] S. Kay, "Some results in linear interpolation theory," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-31, no. 3, pp. 746-749, June 1983.
[7] B. Picinbono and J. Kerilis, "Some properties of prediction and interpolation errors," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 36, no. 4, pp. 525-531, Apr. 1988.
[8] D. Heisey and L. Griffiths, "Linear estimation filters in spectral analysis," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing* (Philadelphia, PA), 1976, pp. 493-496.
[9] J. Bellegarda and D. Farden, "A new structure for adaptive linear-phase filtering," *IEEE Trans. Circuits Syst.*, vol. CAS-34, no. 7, pp. 712-720, July 1987.
[10] P. Delsarte and Y. Genin, "The split Levinson algorithm," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-34, pp. 470-478, 1986.

The Use of Block Truncation Coding in DPCM Image Coding

Edward J. Delp and O. Robert Mitchell

Abstract—**In this correspondence we present the results of using block truncation coding (BTC) in predictive differential pulse code modulation (DPCM) coding of images. BTC is based on the use of a moment preserving (MP) quantizer that is designed such that the quantizer preserves statistical moments of the input and output. We show that while it is theoretically impossible to preserve moments in the reconstructed image, as is normally done in BTC, the DPCM/BTC system works quite well at data rates of 1.18 b/pixel. The performance of the DPCM/BTC system is compared to classical mimimum mean-square error quantizers. These methods are also compared in the presence of channel errors.**

I. Introduction

Differential pulse code modulation (DPCM) has been widely used in image bandwidth compression [1]-[3]. In DPCM the difference

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