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A Two-Dimensional Image Model Based on Occlusion and Maximum Entropy

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Abstract

This paper provides new insights into the formation of two-dimensional image autocorrelation functions. We model an image as a maximum-entropy composition of individual occluding object images that have random positions, shapes and intensities. We derive the autocorrelation function of this image model, give an example, and comment on the reasonableness of the frequently-made assumptions of autocovariance separability and isotropy.

1: Introduction

Autocovariance models have basic importance in the areas of image coding and processing [1—8]. In this paper we introduce a two-dimensional image autocovariance model that is based on occlusion and maximum entropy. *Occlusion* is a fundamental physical property that underlies the formation of all images: Since images are created by a perspective transformation of the 3-D world onto the image plane, images of objects in the foreground occlude or partially occlude those of objects in the background. The principle of *maximum entropy* allows us to obtain the autocovariance function of an image composed of occluding object images while being maximally noncommittal with respect to assumptions of object-image intensities and positions.

The development in the present paper is an extension of a previous study [9] in which we obtained the maximum-entropy autocovariance model for a one-dimensional scene image as a function of the object-image width distribution and the average number of object images per unit length. The present paper is the first to investigate the relationship between occlusion and two-dimensional autocovariance, and it yields new insights into the formation of two-dimensional image autocovariance functions. The paper also provides a new look at the issues of autocovariance separability and isotropy.

2. Recursive Scene-Image Generation

We define a *scene image,* $s_n(\boldsymbol{x})$, $\boldsymbol{x} = (x, y) \in \mathcal{DI}(L) = [-L, L]^2$, as a generally nonstationary two-dimensional stochastic process composed of a stochastic *background image,* $b(\boldsymbol{x})$, and n randomly placed stochastic *object images,* $o_i(\boldsymbol{x}-\boldsymbol{p}_i)$, $i = 1, 2, \cdots, n$, where $\boldsymbol{p}_i$ is a random two-dimensional position vector for the i^{th} object image. $\mathcal{DI}(L)$ is the *scene image domain.* We place the following three restrictions on the region of support, $\mathcal{DO}_i$, of $o_i(\boldsymbol{x})$: 1) $\mathcal{DO}_i$ can be drawn physically; 2) $\mathcal{DO}_i$ is specified by a random "domain vector", $\boldsymbol{\nu}_i = (\nu_{i1}, \nu_{i2}, \ldots, \nu_{iM})$, and 3) $\mathcal{DO}_i$ is "centered" so that the bounding rectangle $[-W_i/2, W_i/2] \times [-H_i/2, H_i/2]$ touches the boundary of $\mathcal{DO}_i$ on all four sides as illustrated in Figure 1. $\mathcal{DO}_i$ need not be connected or simply connected.

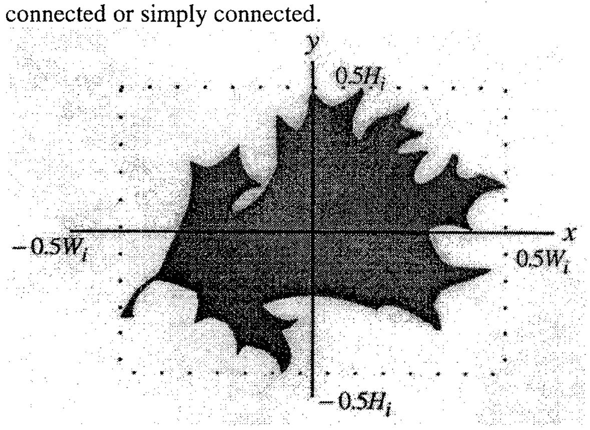

Figure 1. A Possible $\mathcal{DO}_i$ and Bounding Rectangle

$\mathcal{DO}_i$, W_i and H_i depend upon $\boldsymbol{\nu}_i$: $\mathcal{DO}_i = \mathcal{DO}(\boldsymbol{\nu}_i)$, $W_i = W(\boldsymbol{\nu}_i)$ and $H_i = H(\boldsymbol{\nu}_i)$. We define an *indicator function*:

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$$I(\boldsymbol{x};\boldsymbol{v}_i)=\begin{cases}1 & \text{for } \boldsymbol{x}\in\mathcal{DO}(\boldsymbol{v}_i)\\ 0 & \text{otherwise}\end{cases} \tag{1}$$

and its complement

$$I^c(\boldsymbol{x};\boldsymbol{v}_i)=1-I(\boldsymbol{x};\boldsymbol{v}_i) \tag{2}$$

To account for occlusion, we generate the scene image recursively. At the i^{th} step of the recursion, $i=1,2,\cdots,n,$ the scene image is given by

$$s_i(\boldsymbol{x})=\begin{cases}o_i(\boldsymbol{x}-\boldsymbol{p}_i) & \text{if } \boldsymbol{x}\in\mathcal{DO}_i+\boldsymbol{p}_i\\ s_{i-1}(\boldsymbol{x}) & \text{otherwise}\end{cases} \tag{3}$$

where $s_0(\boldsymbol{x})\equiv b(\boldsymbol{x})$ and $\boldsymbol{x}\in\mathcal{DI}(L)$. Thus, at recursion step i, object image $o_i(\boldsymbol{x}-\boldsymbol{p}_i)$, is superimposed on the scene image $s_{i-1}(\boldsymbol{x})$ in such a way such that it occludes $s_{i-1}(\boldsymbol{x})$ for $\boldsymbol{x}\in\mathcal{DO}_i+\boldsymbol{p}_i$. Equation (3) can be written as

$$s_i(\boldsymbol{x}) \;=\; o_i(\boldsymbol{x}-\boldsymbol{p}_i)+s_{i-1}(\boldsymbol{x})I^c(\boldsymbol{x}-\boldsymbol{p}_i;\boldsymbol{v}_i) \tag{4}$$

3. Maximum Entropy Assumptions

To complete the scene image model, we make four assumptions regarding the scene image statistics. In brief, assumptions 1-3 state that object image intensities, shapes and positions are mutually independent. Assumption 4 states that the center point of each object image is uniformly distributed over a domain just large enough so that at least a part of each object appears in the image domain, $\mathcal{DI}(L)$. Specifics follow:

1. The object images are random processes of the form

$$o_i(\boldsymbol{x})=\mathbb{I}_i(\boldsymbol{x})I(\boldsymbol{x};\boldsymbol{v}_i) \tag{5}$$

where "intensity", $\mathbb{I}_i(\boldsymbol{x})$, is a sample function of a wide-sense stationary process, independent of $\boldsymbol{v}_i$.

2. $o_i(\boldsymbol{x})$ and $\boldsymbol{p}_i$ in (4) are statistically independent of $s_j(\boldsymbol{x}), 0\le j<i$, for $i=1,2,\ldots,n$.

3. The domain vectors, $\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \cdots,\ \boldsymbol{v}_i$, are independent and identically distributed (iid). Thus, the probability density function (pdf) of the i tuple of domain vectors, $\underline{\boldsymbol{v}}_i=(\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \cdots,\ \boldsymbol{v}_i)$, has the form

$$f^{(L)}_{\underline{\boldsymbol{v}}_i}(\underline{\boldsymbol{V}}_i)=\prod_{j=1}^{i}f^{(L)}_{\boldsymbol{v}}(\boldsymbol{V}_j) \tag{6}$$

for $i=1,\ 2,\ \cdots n$, where $\underline{\boldsymbol{V}}_i=(\boldsymbol{V}_1,\boldsymbol{V}_2,\ldots,\boldsymbol{V}_i)$. We insist that marginal pdf $f^{(L)}_{\boldsymbol{v}}$ equal some pdf $p_{\boldsymbol{v}}$ of our choice in the limit $L\to\infty$:

$$\lim_{L\to\infty}f^{(L)}_{\boldsymbol{v}}(\boldsymbol{V})\to p_{\boldsymbol{v}}(\boldsymbol{V}) \tag{7}$$

We also insist that $\mathcal{DO}(\boldsymbol{v})$ be completely contained in $\mathcal{DI}(L)$. We accomplish both objectives by setting

$$f^{(L)}_{\boldsymbol{v}}(\boldsymbol{V})=p_{\boldsymbol{v}}\big(\boldsymbol{V}\mid\mathcal{DO}(\boldsymbol{v})\subset\mathcal{DI}(L)\big) \tag{8}$$

which is

$$f^{(L)}_{\boldsymbol{v}}(\boldsymbol{V})=\begin{cases}\frac{p_{\boldsymbol{v}}(\boldsymbol{V})}{\int_{\sigma\in\Psi(L)}p_{\boldsymbol{v}}(\sigma)d\sigma}, & \boldsymbol{V}\in\Psi(L)\\ 0, & \boldsymbol{V}\notin\Psi(L)\end{cases} \tag{9}$$

where $\Psi(L)$ denotes the set of domain-vector values for which $\mathcal{DO}(\boldsymbol{V})\subset\mathcal{DI}(L)$.

4. The conditional pdf of the i-tuple of position vectors $\underline{\boldsymbol{p}}_i=(\boldsymbol{p}_1,\boldsymbol{p}_2,\cdots,\boldsymbol{p}_i)$ given $\underline{\boldsymbol{v}}_i=\underline{\boldsymbol{V}}_i$, is

$$f^{(L)}_{\underline{\boldsymbol{p}}_i|\underline{\boldsymbol{v}}_i}(\underline{\boldsymbol{P}}_i\mid\underline{\boldsymbol{V}}_i)=\prod_{j=1}^{i}f^{(L)}_{\boldsymbol{p}|\boldsymbol{v}}(\boldsymbol{P}_j\mid\boldsymbol{V}_j) \tag{10}$$

for $i=1,2,\cdots,n$ where $\underline{\boldsymbol{P}}_i=(\boldsymbol{P}_1,\boldsymbol{P}_2,\ldots,\boldsymbol{P}_i)$ and

$$f^{(L)}_{\boldsymbol{p}|\boldsymbol{v}}(\boldsymbol{P}\mid\boldsymbol{V})$$

$$=\begin{cases}\left(\frac{1}{2L+W(\boldsymbol{V})}\right)\left(\frac{1}{2L+H(\boldsymbol{V})}\right) & \text{for } \boldsymbol{P}\in\mathcal{DI}^+(\boldsymbol{V})\\ 0 & \text{otherwise}\end{cases} \tag{11}$$

where

$$\mathcal{DI}^+(\boldsymbol{V})=[-L-0.5W(\boldsymbol{V}),L+0.5W(\boldsymbol{V})]$$

$$\times[-L-0.5H(\boldsymbol{V}),\ L+0.5H(\boldsymbol{V})] \tag{12}$$

The above assumptions of independent object-image intensities, shapes, and uniformly-random center points are maximum-entropy assumptions.

4. Image-Model Mean and Autocovariance

In this section we outline the derivation of the mean and autocovariance of $s_n(\boldsymbol{x})$ in the limit $L\to\infty$ holding $n=\lambda(2L)^2$. Details are relegated to the appendices.

4.1 Mean

We denote the mean, $E\{a(\boldsymbol{x})\}$, of a stochastic process $a(\boldsymbol{x})$ by $\eta_a(\boldsymbol{x})$. The following recursion for the conditional mean of $s_i(\boldsymbol{x})$, given $\underline{\boldsymbol{v}}_i=\underline{\boldsymbol{V}}_i$ and $\underline{\boldsymbol{p}}_i=\underline{\boldsymbol{P}}_i$, follows directly from (4) and the assumptions:

$$\eta_{s_i}(\boldsymbol{x}\mid\underline{\boldsymbol{V}}_i,\underline{\boldsymbol{P}}_i)=\eta_o(\boldsymbol{x}-\boldsymbol{P}_i\mid\boldsymbol{V}_i)+$$

$$\eta_{s_{i-1}}(\boldsymbol{x}\mid\underline{\boldsymbol{V}}_{i-1},\underline{\boldsymbol{P}}_{i-1})I^c(\boldsymbol{x}-\boldsymbol{P}_i;\boldsymbol{V}_i) \tag{13}$$

where $\eta_{s_0}(\boldsymbol{x}\mid\boldsymbol{V}_0,\boldsymbol{P}_0)\equiv\eta_{s_0}(\boldsymbol{x})\equiv\eta_b(\boldsymbol{x}),\ \ \boldsymbol{x}\in\mathcal{DI}(L)$, and $i=1,2,\cdots,n$. The conditions in (13) are removed by multiplying both sides by

$$f^{(L)}_{\underline{\boldsymbol{p}}_i,\underline{\boldsymbol{v}}_i}(\underline{\boldsymbol{P}}_i,\underline{\boldsymbol{V}}_i) = f^{(L)}_{\underline{\boldsymbol{p}}_i|\underline{\boldsymbol{v}}_i}(\underline{\boldsymbol{P}}_i \mid \underline{\boldsymbol{V}}_i) f^{(L)}_{\underline{\boldsymbol{v}}_i}(\underline{\boldsymbol{V}}_i) \tag{14}$$

and integrating over all $\underline{\boldsymbol{P}}_i, \underline{\boldsymbol{V}}_i$. This yields (Appendix A):

$$\eta_{s_i}(\boldsymbol{x}) = \eta_{\mathbb{I}}(1-\xi) + \eta_{s_{i-1}}(\boldsymbol{x})\xi \tag{15}$$

for $\boldsymbol{x} \in \mathcal{DI}(L)$, $i = 1, 2, \cdots, n$, where $\eta_{\mathbb{I}} = E\{\mathbb{I}(\boldsymbol{x})\}$ is constant, and ξ is defined by A1. The solution to (15) is

$$\eta_{s_n}(\boldsymbol{x}) = \eta_{\mathbb{I}}(1-\xi^n) + \eta_b(\boldsymbol{x})\xi^n \tag{16}$$

The limiting form of $\eta_{s_n}(\boldsymbol{x})$ for $L \to \infty$ with $n = \lambda(2L)^2$ is (Appendix A)

$$\eta_s(\boldsymbol{x}) = \eta_{\mathbb{I}}\,(1-e^{-\lambda\overline{A}}) + \eta_b(\boldsymbol{x})e^{-\lambda\overline{A}} \tag{17}$$

for $\boldsymbol{x} \in \mathcal{R}^2$ where $\overline{A}$ is the mean object-image area:

$$\overline{A} \triangleq \int_{-\infty}^{\infty} A(\boldsymbol{V})p_{\boldsymbol{v}}(\boldsymbol{V})d\boldsymbol{V} \tag{18}$$

and $A(\boldsymbol{V})$ is the object area when $\boldsymbol{v} = \boldsymbol{V}$

$$A(\boldsymbol{V}) = \int_{\mathcal{DO}(\boldsymbol{V})} d\boldsymbol{x} \tag{19}$$

Also, from probability theory:

$$\boxed{\eta_s(\boldsymbol{x}) = \eta_b(\boldsymbol{x})P_b(0) + \eta_{\mathbb{I}}P_o(0)} \tag{20}$$

where $P_b(0) \equiv \Pr\{B(\boldsymbol{x})\}$ is the probability of the event $B(\boldsymbol{x}) = \{\text{background appears (is not occluded) at } \boldsymbol{x}\}$ and $P_o(0) \equiv \Pr\{O(\boldsymbol{x})\} = 1 - P_b(0)$ is the probability of the event $O(\boldsymbol{x}) = \{\text{an object-image appears at } \boldsymbol{x}\}$. By comparing (17) with (20) we find that

$$P_b(0) = e^{-\lambda\overline{A}} \tag{21}$$

$$P_o(0) = 1 - e^{-\lambda\overline{A}} \tag{22}$$

4.2 Autocovariance for Zero-Mean Image

We denote the autocovariance function of a process $a(\boldsymbol{x})$ by $K_a(\boldsymbol{x}_1, \boldsymbol{x}_2) = R_a(\boldsymbol{x}_1, \boldsymbol{x}_2) - \eta_a(\boldsymbol{x}_1)\eta_a(\boldsymbol{x}_2)$ where $R_a(\boldsymbol{x}_1, \boldsymbol{x}_2) = E\{a(\boldsymbol{x}_1)a(\boldsymbol{x}_2)\}$ is the autocorrelation function of $a(\boldsymbol{x})$. We temporarily assume that $\eta_{\mathbb{I}} = \eta_b(\boldsymbol{x}) = 0$ and use a prime on s of (4) to indicate this assumption. The following recursion for the conditional autocovariance (or autocorrelation) function of $s'_i(\boldsymbol{x})$ is obtained directly from (4) and the assumptions:

$$K_{s'_i}(\boldsymbol{x}_1, \boldsymbol{x}_2 \mid \underline{\boldsymbol{V}}_i, \underline{\boldsymbol{P}}_i) = K_o(\boldsymbol{x}_1 - \boldsymbol{P}_i, \boldsymbol{x}_2 - \boldsymbol{P}_i \mid \boldsymbol{V}_i) +$$

$$K_{s'_{i-1}}(\boldsymbol{x}_1, \boldsymbol{x}_2 \mid \underline{\boldsymbol{V}}_{i-1}, \underline{\boldsymbol{P}}_{i-1})\, I^c(\boldsymbol{x}_1 - \boldsymbol{P}_i; \boldsymbol{V}_i)\, I^c(\boldsymbol{x}_2 - \boldsymbol{P}_i;\, \boldsymbol{V}_i) \tag{23}$$

with $K_{s'_0}(\boldsymbol{x}_1, \boldsymbol{x}_2 \mid \underline{\boldsymbol{V}}_0, \underline{\boldsymbol{P}}_0) \equiv K_{s'_0}(\boldsymbol{x}_1, \boldsymbol{x}_2) \equiv K_b(\boldsymbol{x}_1, \boldsymbol{x}_2)$, $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{DI}(L)$, $i = 1, 2, \cdots, n$, and

$$K_o(\boldsymbol{x}_1 - \boldsymbol{P}_i, \boldsymbol{x}_2 - \boldsymbol{P}_i \mid \boldsymbol{V}_i) =$$

$$K_{\mathbb{I}}(\boldsymbol{\Delta})I(\boldsymbol{x}_1 - \boldsymbol{P}_i; \boldsymbol{V}_i)I(\boldsymbol{x}_2 - \boldsymbol{P}_i; \boldsymbol{V}_i) \tag{24}$$

where $\boldsymbol{\Delta} = \boldsymbol{x}_2 - \boldsymbol{x}_1$. We remove the conditions from (23) in a manner similar to (13). The result is:

$$K_{s'_i}(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{\Delta}) = K_{\mathbb{I}}(\boldsymbol{\Delta})[\phi(\boldsymbol{\Delta}) - \theta(\boldsymbol{\Delta})]$$

$$+ K_{s'_{i-1}}(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{\Delta})\phi(\boldsymbol{\Delta}) \tag{25}$$

for $\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{\Delta} \in \mathcal{DI}(L)$ and $i = 1, 2, \cdots, n$ where $\phi(\boldsymbol{\Delta})$ and $\theta(\boldsymbol{\Delta})$ are defined by B1 and B2. The solution to (25) is

$$K_{s'_n}(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{\Delta}) = K_{\mathbb{I}}(\boldsymbol{\Delta})\frac{\phi(\boldsymbol{\Delta}) - \theta(\boldsymbol{\Delta})}{1-\phi(\boldsymbol{\Delta})}[1-\phi^n(\boldsymbol{\Delta})]$$

$$+ K_b(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{\Delta})\phi^n(\boldsymbol{\Delta}) \tag{26}$$

which becomes, in the limit $L \to \infty$ with $n = \lambda(2L)^2$, (Appendix B):

$$K_{s'}(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{\Delta}) =$$

$$K_{\mathbb{I}}(\boldsymbol{\Delta})\left\{\frac{\overline{C}_n(\boldsymbol{\Delta})}{2-\overline{C}_n(\boldsymbol{\Delta})}\right\}\left\{1-e^{-\lambda\overline{A}[2-\overline{C}_n(\boldsymbol{\Delta})]}\right\}$$

$$+ K_b(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{\Delta})e^{-\lambda\overline{A}[2-\overline{C}_n(\boldsymbol{\Delta})]} \tag{27}$$

for $\boldsymbol{x} \in \mathcal{R}^2$, where $\overline{C}_n(\boldsymbol{\Delta})$ is the normalized mean object-indicator spatial correlation function

$$\overline{C}_n(\boldsymbol{\Delta}) = \overline{C}(\boldsymbol{\Delta})/\overline{C}(\boldsymbol{0}) = \overline{C}(\boldsymbol{\Delta})/\overline{A} \tag{28}$$

with

$$\overline{C}(\boldsymbol{\Delta}) = \int_{-\infty}^{\infty} C(\boldsymbol{\Delta}; \boldsymbol{V})p_{\boldsymbol{v}}(\boldsymbol{V})d\boldsymbol{V} \tag{29}$$

and

$$C(\boldsymbol{\Delta}; \boldsymbol{V}) \triangleq \int_{-\infty}^{\infty} I(\boldsymbol{x}; \boldsymbol{V})I(\boldsymbol{x}+\boldsymbol{\Delta}; \boldsymbol{V})d\boldsymbol{x} \tag{30}$$

is the *spatial correlation function* of $I(\boldsymbol{x}; \boldsymbol{V})$.

Also, from probability theory:

$$K_{s'}(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{\Delta}) = K_{\mathbb{I}}(\boldsymbol{\Delta})P_o(\boldsymbol{\Delta})$$

$$+ K_b(\boldsymbol{x}, \boldsymbol{x}+\Delta)\, P_b(\Delta) \tag{31}$$

where $P_b(\Delta) = \Pr\{B(\boldsymbol{x}) \cap B(\boldsymbol{x}+\Delta)\}$ and $P_o(\Delta) = \Pr\{O(\boldsymbol{x}, \boldsymbol{x}+\Delta)\}$ where the event $O(\boldsymbol{x}, \boldsymbol{x}+\Delta) =$ {parts of the *same* object are visible at $\boldsymbol{x}$ and $\boldsymbol{x}+\Delta$}. By comparing (27) with (31) we find that

$$P_b(\Delta) = e^{-\lambda\overline{A}[2-\overline{C}_n(\Delta)]} \tag{33}$$

$$P_o(\Delta) = \left\{\frac{\overline{C}_n(\Delta)}{2-\overline{C}_n(\Delta)}\right\}\left\{1-e^{-\lambda\overline{A}[2-\overline{C}_n(\Delta)]}\right\}. \tag{34}$$

4.3 Autocovariance for Non-Zero-Mean Image

For the general case of nonzero $\eta_{\mathbb{I}}$ and $\eta_b(\boldsymbol{x})$, each sample function $s(\boldsymbol{x},\zeta)$ of the random scene-image process $s(\boldsymbol{x})$ is the sum of $s'(\boldsymbol{x},\zeta)$ and $s''(\boldsymbol{x},\zeta)$ where $s''(\boldsymbol{x},\zeta) = \eta_{\mathbb{I}}$ if an object-image exists at $\boldsymbol{x}$ and $s''(\boldsymbol{x},\zeta) = \eta_b(\boldsymbol{x})$ otherwise. The autocorrelation function of $s(\boldsymbol{x})$ is

$$\begin{aligned} R_s(\boldsymbol{x}_1, \boldsymbol{x}_2) = {} & \left[K_{\mathbb{I}}(\boldsymbol{x}_2 - \boldsymbol{x}_1) + \eta_{\mathbb{I}}^2\right]\Pr\{O(\boldsymbol{x}_1), O(\boldsymbol{x}_2)\} \\ & + \eta_{\mathbb{I}}\eta_b(\boldsymbol{x}_2)\Pr\{O(\boldsymbol{x}_1), B(\boldsymbol{x}_2)\} \\ & + \eta_b(\boldsymbol{x}_1)\eta_{\mathbb{I}}\Pr\{B(\boldsymbol{x}_1), O(\boldsymbol{x}_2)\} \\ & + R_b(\boldsymbol{x}_1, \boldsymbol{x}_2)\Pr\{B(\boldsymbol{x}_1), B(\boldsymbol{x}_2)\} \end{aligned} \tag{35}$$

We also have:

$$R_s(\boldsymbol{x}, \boldsymbol{x}+\Delta) = K_s(\boldsymbol{x}, \boldsymbol{x}+\Delta) + \eta_s(\boldsymbol{x})\eta_s(\boldsymbol{x}+\Delta) \tag{36}$$

where $\eta_s(\boldsymbol{x})$ is given by (17). We evaluate the probabilities in (35) in Appendix C. When we combine these probabilities with (35) and equate the result to (36) we find that

$$\begin{aligned} K_s(\boldsymbol{x}, \boldsymbol{x}+\Delta) = {} & K_{\mathbb{I}}(\Delta)P_o(\Delta) \\ & + K_b(\boldsymbol{x}, \boldsymbol{x}+\Delta)P_b(\Delta) \\ & + [\eta_{\mathbb{I}} - \eta_b(\boldsymbol{x})][\eta_{\mathbb{I}} - \eta_b(\boldsymbol{x}+\Delta)][P_b(\Delta) - P_b(0)^2] \end{aligned} \tag{37}$$

We see from (33), (34) and (37) that $K_s(\boldsymbol{x}, \boldsymbol{x}+\Delta)$ is isotropic if and only if $K_{\mathbb{I}}(\Delta)$, $K_b(\boldsymbol{x}, \boldsymbol{x}+\Delta)$, and $\overline{C}(\Delta)$ are isotropic and $\eta_b(\boldsymbol{x})$ is a constant. The assumption of isotropic scene-image autocovariance therefore fits reasonably within the framework of our model. We also see from the same equations that separable $\overline{C}(\Delta)$ does not imply separable $K_s(\boldsymbol{x}, \boldsymbol{x}+\Delta)$ and vice versa. Therefore, the assumption of separable scene-image autocovariance seems less reasonable within the context of the model.

4.4 Example

For a simple example of the above development, assume that each object image is a disk having known radius r and uniform but random intensity $\mathbb{I}$. (The radius, r, is identical for each disk, but the intensity, $\mathbb{I}$, varies from disk to disk.) Assume further that $\lambda\overline{A} \gg 1$. One tenth of the scene image is shown below.

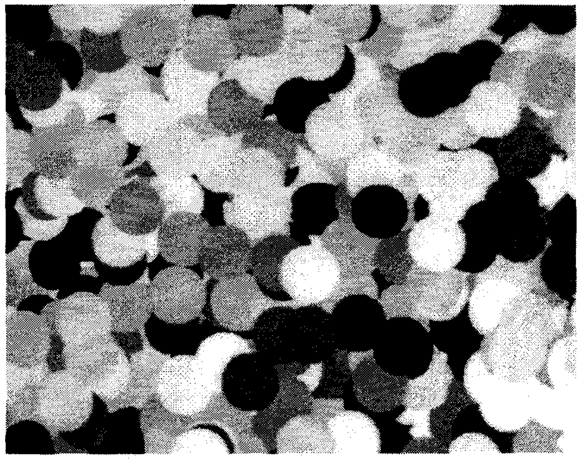

Figure 2. Occluding Disks

To obtain the theoretical autocovariance, we set $\boldsymbol{v}$ to the scalar $v = r$, so $\mathcal{DO}(v) = \{\boldsymbol{x} : |\boldsymbol{x}| \le v\}$ and

$$I(\boldsymbol{x}; v) = \begin{cases} 1 & \text{for } |\boldsymbol{x}| \le v \\ 0 & \text{otherwise,} \end{cases}$$

where $p_v(V) = \delta(V - r)$. Straightforward evaluation of (28)-(30) yields

$$\overline{C}_n(\Delta) = \left(\frac{2}{\pi}\cos^{-1}\frac{|\Delta|}{2r} - \frac{|\Delta|}{\pi r}\sqrt{1 - \frac{|\Delta|^2}{4r^2}}\right)I(\Delta; 2r)$$

and (37) reduces to

$$K_s(\boldsymbol{x}, \boldsymbol{x}+\Delta) = \sigma^2\frac{\overline{C}_n(\Delta)}{2-\overline{C}_n(\Delta)}$$

where σ^2 is the variance of $\mathbb{I}$. The sample (spatial average) autocovariance function of the image was computed using a discrete-space approximation to the optical technique described in [10]. The result is shown in Figure 3 along with the theoretical autocovariance function.

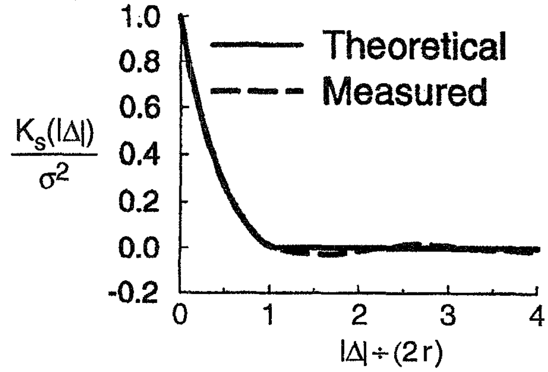

Figure 3. Normalized Autocovariance Functions

5. Concluding Remarks

The autocovariance expressions derived in this paper appear well suited for modeling autocovariance functions of certain ensembles of real-world scenes. An example is given by the ensemble of images of leaves from a tree randomly scattered on the earth, where shading, and perspective are not significant factors. The theoretical autocovariance expressions described here are much more general than the example suggests. Object image shapes can be different, and their intensities need not be uniform. Also, a scene image model can be used as a background model for generating a second scene image model, etc.

APPENDIX A

We show here the steps leading to (15) and (17). As a preliminary development, we evaluate

$$\xi \triangleq E\{I^c(\boldsymbol{x}-\boldsymbol{p};\,\boldsymbol{v})\} = 1 - E\{I(\boldsymbol{x}-\boldsymbol{p};\,\boldsymbol{v})\} \qquad \text{A1}$$

for $\boldsymbol{x} \in \mathcal{DI}(L)$. From (11) we find that

$$E\{I(\boldsymbol{x}-\boldsymbol{p}_i;\boldsymbol{v}_i)|\,\boldsymbol{v}_i = \boldsymbol{V}_i\} = \frac{1}{[2L+W(\boldsymbol{V})][2L+H(\boldsymbol{V})]}A(\boldsymbol{V})$$

for $\boldsymbol{x} \in \mathcal{DI}(L)$ and $\boldsymbol{V} \in \Psi(L)$ where $A(\boldsymbol{V})$ is defined by (19). Removal of the condition $\boldsymbol{v} = \boldsymbol{V}$ yields

$$\xi = 1 - \int_{-\infty}^{\infty} \frac{1}{[2L+W(\boldsymbol{V})][2L+H(\boldsymbol{V})]}A(\boldsymbol{V})f_{\boldsymbol{v}}^{(L)}(\boldsymbol{V})d\boldsymbol{V}$$

which, by algebraic manipulations, becomes

$$\xi = 1 - \frac{1}{(2L)^2}\overline{A}(L) +$$

$$\frac{1}{(2L)^2}\int_{-\infty}^{\infty} A(\boldsymbol{V})\frac{2L[W(\boldsymbol{V})+H(\boldsymbol{V})]+W(\boldsymbol{V})H(\boldsymbol{V})}{[2L+H(\boldsymbol{V})][2L+W(\boldsymbol{V})]}f_{\boldsymbol{v}}^{(L)}(\boldsymbol{V})d\boldsymbol{V}$$

A2

where

$$\overline{A}(L) = \int_{-\infty}^{\infty} A(\boldsymbol{V})f_{\boldsymbol{v}}^{(L)}(\boldsymbol{V})d\boldsymbol{V}$$

We now describe the steps leading to (15). When we remove the conditions from the LHS of (13) we obtain the LHS of (15) by definition. Consider the first term on the RHS of (13):

$$\eta_o(\boldsymbol{x}-\boldsymbol{P}_i \mid \boldsymbol{V}_i) = E\{o(\boldsymbol{x}-\boldsymbol{p}_i)|\boldsymbol{p}_i = \boldsymbol{P}_i,\, \boldsymbol{v}_i = \boldsymbol{V}_i\}$$

$$= E\{\mathbb{I}(\boldsymbol{x}-\boldsymbol{P}_i)I(\boldsymbol{x}-\boldsymbol{P}_i;\boldsymbol{V}_i)\} = \eta_{\mathbb{I}}\, I(\boldsymbol{x}-\boldsymbol{P}_i;\,\boldsymbol{V}_i)$$

Removal of the conditions $\boldsymbol{p}_i = \boldsymbol{P}_i$, $\boldsymbol{v}_i = \boldsymbol{V}_i$ yields

$$\eta_{\mathbb{I}} E\{I(\boldsymbol{x}-\boldsymbol{p}_i;\boldsymbol{v}_i)\} = \eta_{\mathbb{I}}(1-\xi)$$

which is the first term on the RHS of (15). Removal of the conditions $\boldsymbol{p}_i = \boldsymbol{P}_i$, $\boldsymbol{v}_i = \boldsymbol{V}_i$ for the second term on the RHS of (13) yields

$$\eta_{s_{i-1}}(\boldsymbol{x})E\{I^c(\boldsymbol{x}-\boldsymbol{p};\,\boldsymbol{v})\} = \eta_{s_{i-1}}(\boldsymbol{x})\xi$$

which is the second term on the RHS of (15).

To derive (17), note that zero is a lower bound for the integral on the RHS of A2. An upper bound for the same integral is obtained by dropping the $W(\boldsymbol{V})$ and the $H(\boldsymbol{V})$ in the denominator and applying the inequality $A(\boldsymbol{V}) \le W(\boldsymbol{V})H(\boldsymbol{V})$. This yields the upper bound:

$$\frac{1}{(2L)^3}\Big[m_{21}(L) + m_{12}(L)\Big] + \frac{1}{(2L)^4}m_{22}(L)$$

where

$$m_{21}(L) = \int_{-\infty}^{\infty} W^2(\boldsymbol{V})H(\boldsymbol{V})f_{\boldsymbol{v}}^{(L)}(\boldsymbol{V})d\boldsymbol{V}, \qquad \text{A3}$$

$$m_{12}(L) = \int_{-\infty}^{\infty} W(\boldsymbol{V})H^2(\boldsymbol{V})f_{\boldsymbol{v}}^{(L)}(\boldsymbol{V})d\boldsymbol{V}, \qquad \text{A4}$$

and

$$m_{22}(L) = \int_{-\infty}^{\infty} W^2(\boldsymbol{V})H^2(\boldsymbol{V})f_{\boldsymbol{v}}^{(L)}(\boldsymbol{V})d\boldsymbol{V} \qquad \text{A5}$$

It follows from A2 that

$$1-\frac{1}{(2L)^2}\overline{A}(L) < \xi < 1-\frac{1}{(2L)^2}\overline{A}(L)+$$

$$\frac{1}{(2L)^3}\Big[m_{21}(L)+m_{12}(L)\Big]+\frac{1}{(2L)^4}m_{22}(L)$$

$\overline{A}(L)$, $m_{21}(L)$, $m_{12}(L)$ and $m_{22}(L)$ are all nonnegative. They are also all bounded functions of L because we have assumed that $\mathcal{DO}(V)$ can be drawn physically. If we operate on the above inequality with $\lambda(2L)^2\log_e$ and use the inequality

$$\epsilon-\epsilon^2 < \ln(1+\epsilon) < \epsilon \qquad \text{A6}$$

for $|\epsilon| < 0.5$, we obtain, for sufficiently large L,

$$-\lambda\overline{A}(L)-\lambda\Big[\frac{\overline{A}(L)}{2L}\Big]^2 < \ln\xi^{\lambda(2L)^2} <$$

$$-\lambda\overline{A}(L)+\frac{\lambda}{2L}\Big[m_{21}(L)+m_{12}(L)\Big]+\frac{\lambda}{(2L)^2}m_{22}(L)$$

It follows that $\xi^{\lambda(2L)^2}\rightarrow e^{-\lambda\overline{A}}$ as $L\rightarrow\infty$ where

$$\overline{A} \overset{\Delta}{=} \lim_{L\rightarrow\infty}\overline{A}(L)=\int_{-\infty}^{\infty}A(\boldsymbol{V})p_v(\boldsymbol{V})d\boldsymbol{V}$$

Therefore, we have established (19).

APPENDIX B

We show here the steps leading to (27). As a preliminary step, we evaluate

$$\phi(\boldsymbol{\Delta}) \overset{\Delta}{=} E\{I^c(\boldsymbol{x}_1-\boldsymbol{p}_i;\boldsymbol{v}_i)I^c(\boldsymbol{x}_2-\boldsymbol{p}_i;\boldsymbol{v}_i)\} \qquad \text{B1}$$

and

$$\phi(\boldsymbol{\Delta})-\theta(\boldsymbol{\Delta}) \overset{\Delta}{=} E\{I(\boldsymbol{x}_1-\boldsymbol{p}_i;\boldsymbol{v}_i)I(\boldsymbol{x}_2-\boldsymbol{p}_i;\boldsymbol{v}_i)\} \qquad \text{B2}$$

for $\boldsymbol{x}_1,\boldsymbol{x}_2\in\mathcal{DI}(L)$. We substitute (2) into B1 and expand the result into four terms, each of which is an expectation with respect to $f_{\boldsymbol{p}|\boldsymbol{v}}^{(L)}(\boldsymbol{P}\mid\boldsymbol{V})f_{\boldsymbol{v}}^{(L)}(\boldsymbol{V})$. Evaluation of these expectations for $\boldsymbol{x}_1$, $\boldsymbol{x}_2\in\mathcal{DI}(L)$ yields:

$$\phi(\boldsymbol{\Delta})=1$$

$$-\int_{-\infty}^{\infty}\frac{2}{[2L+W(\boldsymbol{V})][2L+H(\boldsymbol{V})]}A(\boldsymbol{V})f_{\boldsymbol{v}}^{(L)}(\boldsymbol{V})d\boldsymbol{V}$$

$$+\int_{-\infty}^{\infty}\frac{1}{[2L+W(\boldsymbol{V})][2L+H(\boldsymbol{V})]}\mathcal{C}(\boldsymbol{\Delta};\boldsymbol{V})f_{\boldsymbol{v}}^{(L)}(\boldsymbol{V})d\boldsymbol{V}$$

where $A(\boldsymbol{V})$ and $\mathcal{C}(\boldsymbol{\Delta};\boldsymbol{V})$ are given by (19) and (30), respectively for $\boldsymbol{V}\in\Psi(L)$. Algebraic manipulations yield

$$\phi(\boldsymbol{\Delta})=1-\frac{2}{(2L)^2}\overline{A}(L)$$

$$+\frac{1}{(2L)^2}\overline{\mathcal{C}}(\boldsymbol{\Delta};L)+\frac{1}{(2L)^3}\overline{\gamma}(\boldsymbol{\Delta},L) \qquad \text{B3}$$

where

$$\overline{\mathcal{C}}(\boldsymbol{\Delta};L)=\int_{-\infty}^{\infty}\mathcal{C}(\boldsymbol{\Delta};\boldsymbol{V})f_{\boldsymbol{v}}^{(L)}(\boldsymbol{V})d\boldsymbol{V}$$

and

$$\overline{\gamma}(\boldsymbol{\Delta},L)=\int_{-\infty}^{\infty}\gamma(\boldsymbol{\Delta},L)f_{\boldsymbol{v}}^{(L)}(\boldsymbol{V})d\boldsymbol{V}$$

where

$$\gamma(\boldsymbol{\Delta},L)=2L\frac{2L[W(\boldsymbol{V})+H(\boldsymbol{V})]+W(\boldsymbol{V})H(\boldsymbol{V})}{[2L+H(\boldsymbol{V})][2L+W(\boldsymbol{V})]}[2A(\boldsymbol{V})-\mathcal{C}(\boldsymbol{\Delta};\boldsymbol{V})]$$

Similarly, the expectation in B2 is with respect to $f_{\boldsymbol{p}|\boldsymbol{v}}^{(L)}(\boldsymbol{P}\mid\boldsymbol{V})f_{\boldsymbol{v}}^{(L)}(\boldsymbol{V})$. Evaluation of this expectation yields

$$\phi(\boldsymbol{\Delta})-\theta(\boldsymbol{\Delta})=$$

$$\int_{-\infty}^{\infty}\frac{1}{[2L+W(\boldsymbol{V})][2L+H(\boldsymbol{V})]}\mathcal{C}(\boldsymbol{\Delta};\boldsymbol{V})f_{\boldsymbol{v}}^{(L)}(\boldsymbol{V})d\boldsymbol{V}$$

which, by algebra, becomes

$$\phi(\boldsymbol{\Delta})-\theta(\boldsymbol{\Delta})=$$

$$=\frac{1}{(2L)^2}\overline{\mathcal{C}}(\boldsymbol{\Delta};L)-\frac{1}{(2L)^3}\overline{\alpha}(\boldsymbol{\Delta},L) \qquad \text{B4}$$

where

$$\overline{\alpha}(\boldsymbol{\Delta},L)=$$

$$2L\int_{-\infty}^{\infty}\frac{2L(W(\boldsymbol{V})+H(\boldsymbol{V}))+W(\boldsymbol{V})H(\boldsymbol{V})}{(2L+W(\boldsymbol{V}))(2L+H(\boldsymbol{V}))}\mathcal{C}(\boldsymbol{\Delta};\boldsymbol{V})f_{\boldsymbol{v}}^{(L)}(\boldsymbol{V})d\boldsymbol{V}$$

We now describe the steps leading to (27). We have

$$0\le\mathcal{C}(\boldsymbol{\Delta};\boldsymbol{V})\le\mathcal{C}(0;\boldsymbol{V})=A(\boldsymbol{V})\le W(\boldsymbol{V})H(\boldsymbol{V})$$

so

$$0\le 2A(\boldsymbol{V})-\mathcal{C}(\boldsymbol{\Delta};\boldsymbol{V})\le 2W(\boldsymbol{V})H(\boldsymbol{V})$$

Therefore,

$$0\le\overline{\gamma}(\boldsymbol{\Delta},L)\le$$

$$\int_{-\infty}^{\infty}2L\frac{2L[W(\boldsymbol{V})+H(\boldsymbol{V})]+W(\boldsymbol{V})H(\boldsymbol{V})}{[2L+H(\boldsymbol{V})][2L+W(\boldsymbol{V})]}2W(\boldsymbol{V})H(\boldsymbol{V})f_{\boldsymbol{v}}^{(L)}(\boldsymbol{V})d\boldsymbol{V}$$

The RHS of the above can be upper bounded by dropping the $H(\boldsymbol{V})$ and $W(\boldsymbol{V})$ from the denominator. Application of A3-A5 then yields

$$0 \leq \overline{\gamma}(\Delta,L) \leq 2m_{21}(L) + 2m_{12}(L) + \frac{1}{L}m_{22}(L)$$

We use the above to bound $\phi(\Delta)$ of B3. If we operate on the resulting bound by $\lambda(2L)^2\log_e$ and apply A6, we obtain for sufficiently large L,

$$\lambda\big[-2\overline{A}(L) + \overline{C}(\Delta;L)\big] - \lambda\big[-\tfrac{1}{L}\overline{A}(L) + \tfrac{1}{2L}\overline{C}(\Delta;L)\big]^2$$
$$< \ln\phi^{\lambda(2L)^2}(\Delta) <$$
$$-\lambda 2\overline{A}(L) + \lambda\overline{C}(\Delta;L) +$$
$$\lambda\tfrac{1}{2L}\big(2m_{21}(L) + 2m_{12}(L) + \tfrac{1}{L}m_{22}(L)\big)$$

It follows that

$$\lim_{L\to\infty} \phi^{\lambda(2L)^2}(\Delta) = e^{-\lambda[2\overline{A}-\overline{C}(\Delta)]}$$

which is used in (27) where

$$\overline{C}(\Delta) \triangleq \lim_{L\to\infty} \overline{C}(\Delta;L) = \int_{-\infty}^{\infty} C(\Delta;V)p_v(V)dV$$

The substitution of B3 and B4 into the fraction in (25) yields

$$\frac{\phi(\Delta)-\theta(\Delta)}{1-\phi(\Delta)} = \frac{\overline{C}(\Delta;L) - \frac{1}{2L}\overline{\alpha}(\Delta,L)}{2\overline{A}(L) - \overline{C}(\Delta;L) - \frac{1}{2L}\overline{\gamma}(\Delta,L)}$$

We bound $\overline{\alpha}(\Delta,L)$ by techniques similar to those we used to bound $\overline{\gamma}(\Delta,L)$. The result is

$$0 \leq \overline{\alpha}(\Delta,L) \leq m_{21}(L) + m_{12}(L) + \frac{1}{2L}m_{22}(L)$$

from which,

$$\lim_{L\to\infty} \frac{\phi(\Delta)-\theta(\Delta)}{1-\phi(\Delta)} = \frac{\overline{C}(\Delta)}{2\overline{A} - \overline{C}(\Delta)}$$

which is used in (27).

APPENDIX C

To evaluate the probabilities in (35) we first note that

$$\Pr\{B(\boldsymbol{x}_2)|B(\boldsymbol{x}_1)\} = \frac{\Pr\{B(\boldsymbol{x}_1),B(\boldsymbol{x}_2)\}}{\Pr\{B(\boldsymbol{x}_1)\}}$$

$$= \frac{P_b(\Delta)}{P_b(0)} = \frac{e^{-\lambda\overline{A}[2-\overline{C}_n(\Delta)]}}{e^{-\lambda\overline{A}}} = e^{-\lambda\overline{A}[1-\overline{C}_n(\Delta)]}$$

From the above, (22), and probability theory we obtain:

$$\Pr\{O(\boldsymbol{x}_2)|B(\boldsymbol{x}_1)\} = 1 - e^{-\lambda\overline{A}[1-\overline{C}_n(\Delta)]}$$

$$\Pr\{B(\boldsymbol{x}_1)|O(\boldsymbol{x}_2)\} = \frac{1-e^{-\lambda\overline{A}[1-\overline{C}_n(\Delta)]}}{1-e^{-\lambda\overline{A}}}e^{-\lambda\overline{A}}$$

$$\Pr\{O(\boldsymbol{x}_1)|O(\boldsymbol{x}_2)\} = \frac{1-2e^{-\lambda\overline{A}} + e^{-\lambda\overline{A}[2-\overline{C}_n(\Delta)]}}{1-e^{-\lambda\overline{A}}}$$

$$\Pr\{O(\boldsymbol{x}_1),O(\boldsymbol{x}_2)\} = 1 - 2e^{-\lambda\overline{A}} + e^{-\lambda\overline{A}[2-\overline{C}_n(\Delta)]}$$

$$\Pr\{O(\boldsymbol{x}_2),B(\boldsymbol{x}_1)\} = \Big(1 - e^{-\lambda\overline{A}[1-\overline{C}_n(\Delta)]}\Big)e^{-\lambda\overline{A}}$$

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References

[1] W. K. Pratt (ed.), *Image Transmission Techniques*, Academic, 1979.
[2] W. K. Pratt, *Digital Image Processing*, Wiley, 1978.
[3] A. Rosenfield and A. C. Kak, *Digital Picture Processing*, Academic, second edition, 1982.
[4] A. N. Netravali and B. G. Haskell, *Digital Pictures, Representation and Compression*, Plenum, 1988.
[5] A. K. Jain, *Fundamentals of Digital Image Processing*, Prentice Hall, 1989.
[6] W. F. Schreiber, "*Fundamentals of Electronic Imaging Systems, Some Aspects of Image Processing*", Springer Series in Information Sciences, Springer-Verlag, third edition, 1993.
[7] A. Habibi, P.A. Wintz, "Image coding by Linear Transformation and Block Quantization", *IEEE Trans. Commun. Tech*, **19** (1971) 50-62.
[8]L. E. Franks, "A Model for the Random Video Process", *Bell System Technical Journal*, **45** (1966) 609-630.
[9] John A. Stuller, "A Stochastic One-Dimensional Image Model Based on Occluding Object Images", *Proceedings of the Twenty-eighth Asilomar Conference on Signals, Systems & Computers*, October 30-November 2, 1994. (IEEE).
[10] E. R. Kretzmer, "Statistics of Television Signals", *Bell System Technical Journal*, **31** (1952) 751-763.