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Decision-based design under uncertainty with intervals

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DECISION-BASED DESIGN UNDER UNCERTAINTY WITH INTERVALS

by

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ABSTRACT

Design requires a series of decisions, and one of the major challenges engineering designers face is making decisions under uncertainty. The engineering designer must choose the best design among various alternatives.

This work evaluates the use of utility functions to make decisions under uncertainty with both random and interval variables. Uncertainty cannot be eliminated completely, but it can be reduced. Most current methods treat only aleatory uncertainty. Here, however, both aleatory and epistemic uncertainties are addressed in the context of engineering design. Methods such as moment matching method and worst case analysis are used to incorporate uncertainty into design problems.

Multi-objective optimization problems usually involve conflicting objectives, and tradeoffs are necessary. This work assigns to each objective a utility function based on the preferences and judgments of the decision maker, and these functions are then combined into a single function. The objective of the problem is now to maximize the expected utility. The design alternative with the highest utility value will produce the optimal design. Due to the presence of epistemic uncertainty, a penalty is applied to the utility value in this work.

This method is tested by application to a two-bar pin-jointed truss problem and a flag pole design.

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1. INTRODUCTION

Deterministic problems present little difficulty, but design decisions that must be made under uncertainty present a challenge for engineers. Utility theory plays a significant role in making decisions under uncertainty [1, 2, 3]. Utility theory models a decision situation by specifying the performance attributes important to the design. The designer's preferences among the attributes are determined, thus making tradeoffs among the attributes. The foundation of utility theory has its roots in Daniel Bernoulli's St. Petersburg Paradox [4]. Utility analysis was developed by Von Neumann and Morgenstern, Savage, and Keeney and Raiffa [2, 3, 5]. Utility is defined as the measure of usefulness, value in use or the degree of satisfaction achieved [6, 7].

1.1. UNCERTAINTY

Uncertainty is ubiquitous in engineering design. It is a state in which knowledge is limited and the future cannot be predicted exactly. Uncertainty is measured by assigning probabilities to each possible outcome or by applying a probability density function to continuous random variables. Where it is not possible to represent uncertainty by a probability density function, it may be represented by intervals.

Uncertainty can be either aleatory or epistemic uncertainty [8, 9, 10, 11].

1.1.1. Aleatory Uncertainty. Aleatory uncertainty, also called objective uncertainty, stochastic uncertainty, irreducible uncertainty, or statistical uncertainty, describes the inherent variation associated with a physical system or environment [8, 9, 10, 11, 12]. An example of this kind of uncertainty is the roll of a die. The outcome of a particular roll cannot be known, but its probability can be calculated [13]. Sources of aleatory uncertainty include manufacturing imprecision, usage conditions, and material properties such as flexural strength or yield strength. Aleatory uncertainty is inherent in a system and cannot be reduced.

1.1.2. Epistemic Uncertainty. Epistemic uncertainty, or reducible uncertainty, arises due to a lack of knowledge about a model, system or environment. In theory it can be reduced by gathering more information [8, 9, 10, 11, 12]. Under some circumstances, however, it may be impossible to gather more information. In such cases, epistemic uncertainty may be represented as an interval or range.

1.2. ENGINEERING DESIGN

Engineering design is defined as the systematic and intelligent creation of things or systems that perform the stated objectives and satisfy specified constraints [14]. The discipline seeks solutions to new engineering problems or offers new ways to address long-standing problems [8, 15]. Engineering design requires a series of decisions that must be made with limited information [1, 16, 17].

1.3. DECISION MAKING UNDER UNCERTAINTY

Decision making is relatively easy when conditions are known with certainty. Since randomness is intrinsic to most systems and environments, decision making demands rational thinking and sound judgment. For example, in selecting the material and dimensions for a beam, one must consider randomness inherent in beam strength, loads, and other properties. In such cases, the engineer must make decisions under uncertainty.

1.4. OBJECTIVES IN DECISION MAKING

Various and often competing objectives determine the design of a product. The manufacturer may have one set of objectives, and the customer another as shown in Figures 1.1, 1.2, and 1.3. These require compromise. For example, the desire for high product quality conflicts with the need to minimize costs. And high stiffness may demand high volume in the design.

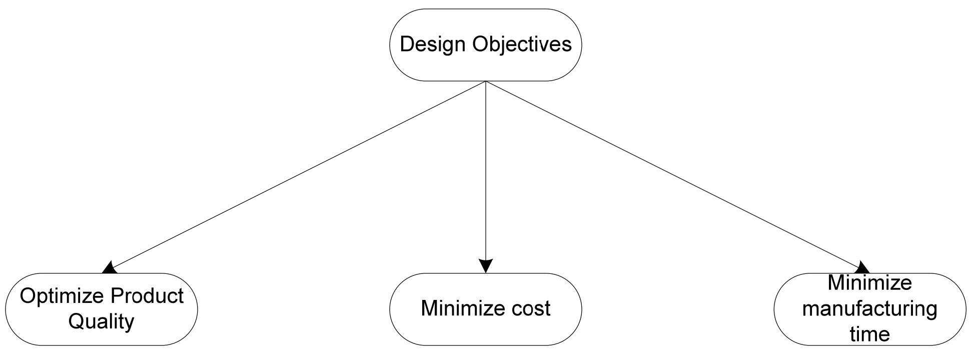

Figure 1.1. Design Objectives

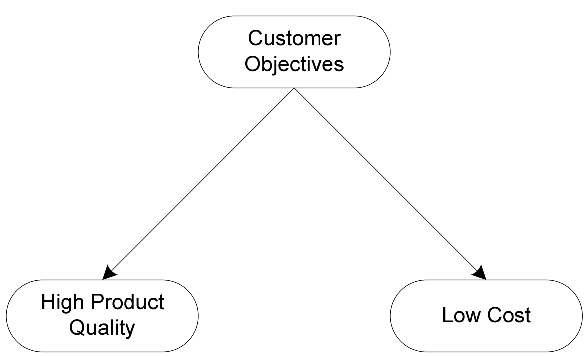

Figure 1.2. Customer Objectives

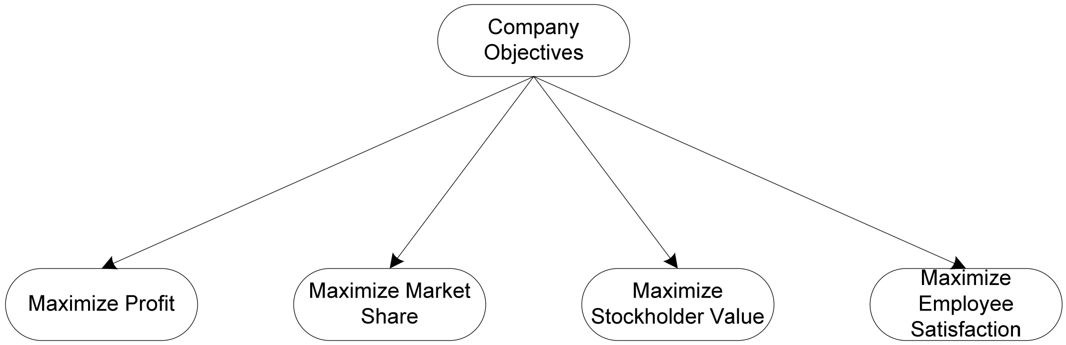

Figure 1.3. Company Objectives

1.5. RESEARCH NEEDS

This research seeks to use utility functions to solve multi-objective optimization problems. When uncertainty is due to intervals, no single correct solution exists. Previous applications in this area offer a solution set [18, 19]. The engineer then faces the challenge of determining which solution should be chosen from this set. A need for making a point decision and finding a single solution rather than a solution set is realized. This work uses methods such as moment matching and worst case analysis to address uncertainty in design. Multi-attribute utility analysis compares attributes of different metrics by normalizing the value of the attributes between 0 and 1. This work also facilitates decision making by maximizing the expected utility of the alternative selected from the solution set. The design point of maximum utility represents the preferred solution, thus simplifying the engineer’s task.

1.6. ORGANIZATION OF THESIS

Section 2 presents a survey of the literature on decision making under uncertainty in engineering design. It addresses methods that rely on utility functions and those that do not. Section 3 explains the basics of optimization and uncertainty modeling methods. Section 4 discusses utility theory and utility functions. Section 5 describes the proposed methodology. Section 6 validates the proposed methodology using two engineering design problems of a two-bar pinned truss system and a flag pole.

Finally, Section 7 presents the conclusions and describes some future work to be done in the area.

2. LITERATURE REVIEW

The St. Petersburg paradox is a game of probability [4, 21, 22]. A player pays a specified amount of money to enter the game. A fair coin is tossed repeatedly until tail appears, at which point the game ends. The payoff is $2 initially; and it doubles for each head that appears. If a tail appears on the first toss, the payoff is $2. If a tail appears on the second toss, the payoff is $4; on the third toss the payoff is $8, and so on. Thus if a tail appears on the n-th toss, the payoff is $\$2^n$.

The expected value of the game can be expressed as:

$$E = \frac{1}{2}(2) + \frac{1}{4}(4) + \frac{1}{8}(8) + \frac{1}{16}(16) + ... = \infty \qquad (1)$$

Since the expected value is so high, a rational person should pay any finite amount of money to enter this game, but in fact many people would not. Addressing this paradox, Daniel Bernoulli says that "the determination of the value of an item must not be based on the price, but rather on the utility it yields" [4]. In other words, the solution to this paradox involves a utility function. For this particular game, Bernoulli offers a logarithmic utility function, given as:

$$U(x) = \ln(x) \qquad (2)$$

where x is the wealth or money.

Bernoulli chose the logarithmic utility function because of the diminishing increase in the utility of the money. A $1000 gain has a higher utility for a pauper than for a millionaire.

2.1. UTILITY ANALYSIS

Utility analysis was developed by Von Neumann and Morgenstern [3], Savage [5], and Keeney and Raiffa [2]. The present utility theory is based on the axioms developed by Von Neumann and Morgenstern. Utility is a unitless measurement that allows comparison of various attributes. In utility analysis, human preferences are taken into account, along with risk attitudes. The goal in utility analysis is to maximize the expected utility of the design or system. Utility analysis helps humans to make better decisions. Decision making without the help of tools exhibits inconsistencies, irrationality, and suboptimal choices, especially when complex trade-offs must be made in conditions of uncertainty [23]. To ensure that decisions are rational, unbiased, and consistent, decision making with utility analysis should follow axioms developed by [2, 3]. These axioms establish the ground rules for proper decision making, and they structure the problem so that a decision maker's utility function can be assessed mathematically.

Utility analysis helps a decision maker construct a real valued function, or a utility function, which models the decision maker's preferences. This utility function is then used to determine which course of action will have the best outcome. Utility function is used in engineering design primarily for multiattribute optimization under uncertainty. The utility function captures the decision maker's attitude towards risk.

Solving design problems without uncertainty in the variables with the help of utility analysis and utility functions has been demonstrated in [7, 24, 25, 26, 27, 28].

2.2. SIMPLE METHODS FOR DECISION MAKING

Making decisions with the help of weights, ranking, group preferences, and dropping or adding alternatives is discussed in [29]. Other methods, such as pairwise comparison, ranking of alternatives, normalization rating, strength of preferences, and weighted sums, are presented in [30].

2.2.1. Pairwise Comparison. In pairwise comparison, each alternative is compared attribute by attribute, and the alternative that has the greatest net number of advantageous attribute wins. The winning alternative is then compared to the next alternative, and so on. However, this method does not take into account the strength of the decision maker's preference. Therefore, an alternative which is a little better on most attributes and a lot worse on the remaining attributes could still be chosen as the best design. This procedure also does not consider the relative importance of attributes.

2.2.2. Decision Matrix Method. The decision matrix method, also called the Pugh selection method, selects one alternative as a reference. The attributes of this reference and the other alternatives are then compared [31]. For each attribute, if the alternative is better than the reference, the alternative scores "+1". If it is worse it scores "-1", and it scores "0" if both are same. The scores are totaled for each alternative, and the alternative with the highest score wins. With this method, however, the best alternative may vary depending on which alternative is chosen as the reference.

2.2.3. Ranking of Alternatives. In the ranking of alternatives, attributes are assigned points or ranks. Each alternative receives a separate rank for each attribute. For example, given four alternatives, the one that is best in terms of a specific attribute will receive four points; the next best will receive three points, and so on. The alternatives are ranked for each attribute, and the scores are totaled. The alternative with the highest total score wins.

2.2.4. Normalization Rating. In normalization rating, the lowest value of an attribute among the alternatives is assigned 0 points and the highest value is assigned 100 points. All other values in between can be assigned points based on linear interpolation. Points are totaled for each alternative, and the one with the highest score wins.

2.2.5. Strength of Preferences and Weighted Sums. The strength of preferences and weighted sums method uses a non-linear point value that might more accurately reflect a decision maker's preference. The lowest value of an attribute among the alternatives is assigned 0 points, the highest value is assigned 100 points, and those in between are assigned points based on the preferences of the decision maker. For example, an increase in an individual's assets from $0 to $10,000 may be more important than an increase from $10,000 to $20,000, even though the increase is in the same amount. The individual can give 0 points for $0, 100 points for $20,000 and 65 points for $10,000. After all attributes are scored, the scores are totaled for each alternative, and the alternative with the highest total score wins. However, one attribute may be significantly more important than the other attribute. Therefore, the attributes are weighted. The weight of each attribute is then multiplied by its assigned point value to arrive at a final score. The scores for all attributes of an alternative are then totaled, and the alternative with the highest score wins.

2.3. COMPLEX METHODS FOR DECISION MAKING

To select one aircraft among four alternatives, See, Gurnani and Lewis [30] use a hypothetical equivalents approach. They determined that the attributes on which they would base their selection were speed, maximum cruise range, and passenger's capacity. They developed four hypothetical alternatives based on combinations of theses attributes. Normalized scores were then assigned to each of the attributes, with 0 for the least important, 1 for the most important, and interpolated values for the others. The total value of an alternative was the weight of the attribute multiplied by its normalized score. An optimization problem was then set up as:

$$\text{Min } F = \left[1 - \sum_{i=1}^{n_a} w_i\right]^2 \tag{3}$$

where n_a is the total number of attributes and w_i is the weight of the i-th attribute.

This optimization problem is subject to constraints obtained by determining the difference in the totals of the alternative values [30]. The weight of each attribute is then multiplied with its preference score, which is obtained using the strength of preferences method as described above, and the alternative with the highest total score wins.

Hazelrigg compares decision making methods, including weighted sum of attributes, analytical hierarchy process, physical programming, Pugh matrix, quality function deployment, Taguchi loss function, Suh's axiomatic design, and six sigma method [32].

Targets provide an alternative to utility functions [33]. The alternative with the highest probability of reaching a specified target is selected as the best design.

Information-gap decision theory approach is applied to problems involving uncertainty, but this approach has numerous limitations [34, 35].

Gurnani and Lewis present the concept of overlap measure [36]. For an attribute i of an alternative x, overlap is calculated as:

$$\text{Overlap measure} = \int_{-\infty}^{+\infty} f_i(x) U_i(x) dx \qquad (4)$$

where $f_i(x)$ is the probability density function of the i-th attribute of the alternative x, and $U_i(x)$ is the utility function of the i-th attribute. Figure 2.1 shows a graphical representation of the overlap measure for an attribute of an alternative.

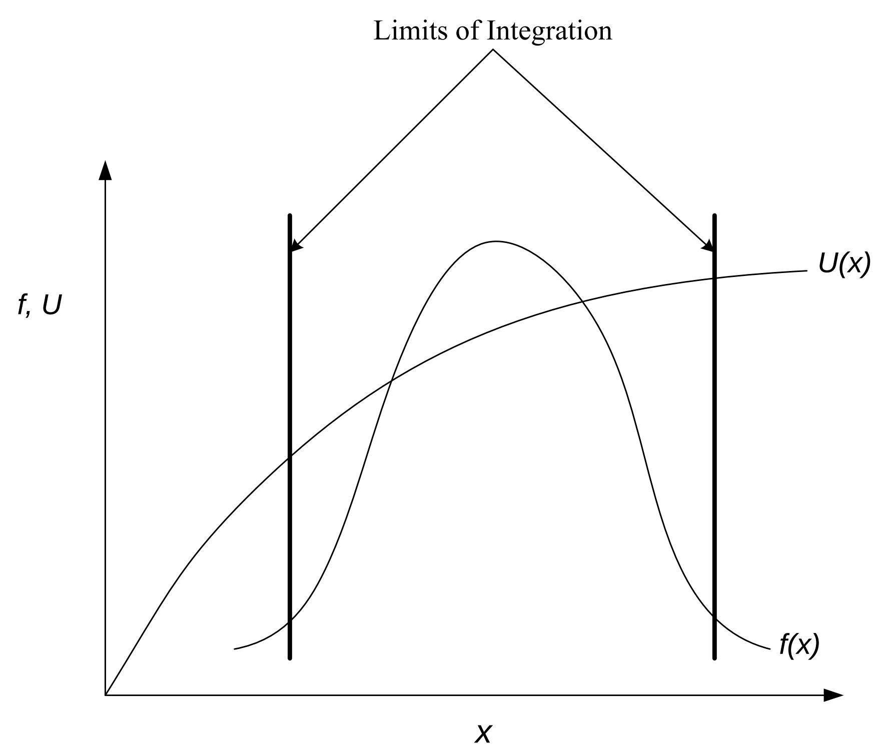

Figure 2.1. Overlap Measure for an Attribute of an Alternative. Source: [36]

Depending on the weights of the attributes, the alternative with the highest overlap measure wins. The overlap measure is determined for every attribute of each alternative. This measure is then multiplied by the attribute weight to arrive at a final score. The final scores of all attributes are totaled to reach the final score of the alternative. Gurnani and Lewis convert a problem of uncertainty with random or interval distribution into a single meaningful score [36]. This method also considers the decision maker’s preference using the utility function. The weights are obtained by the hypothetical approach [30]. The alternative with the highest score wins.

With more information, uncertainty is reduced and decisions can be made with less risk. This concept has been applied in [37], which uses the principles of utility theory and probability bounds analysis to obtain additional information and thus reduce epistemic uncertainty. By this means, some design points may be eliminated, reducing the number of alternatives and narrowing the design space. The decision maker thus chooses from fewer designs and is subject to less risk.

When uncertainty is involved, the decision maker must choose a design from a set of alternatives. This choice involves considerable risk. As demonstrated by [18, 19], by considering shared epistemic uncertainty, the set of alternatives can be reduced. Under interval-based uncertainty, some design alternatives can be eliminated allowing the decision maker to choose from a smaller set, thus reducing the risk. Even with this method, however, no single design can be considered optimal. Uncertainty implies a maximum utility and a minimum utility. By plotting the maximum and minimum utility curves, the decision maker can eliminate all designs having a maximum utility is lower than the minimum utility of any other design [18, 19]. Figure 2.2 illustrates this method.

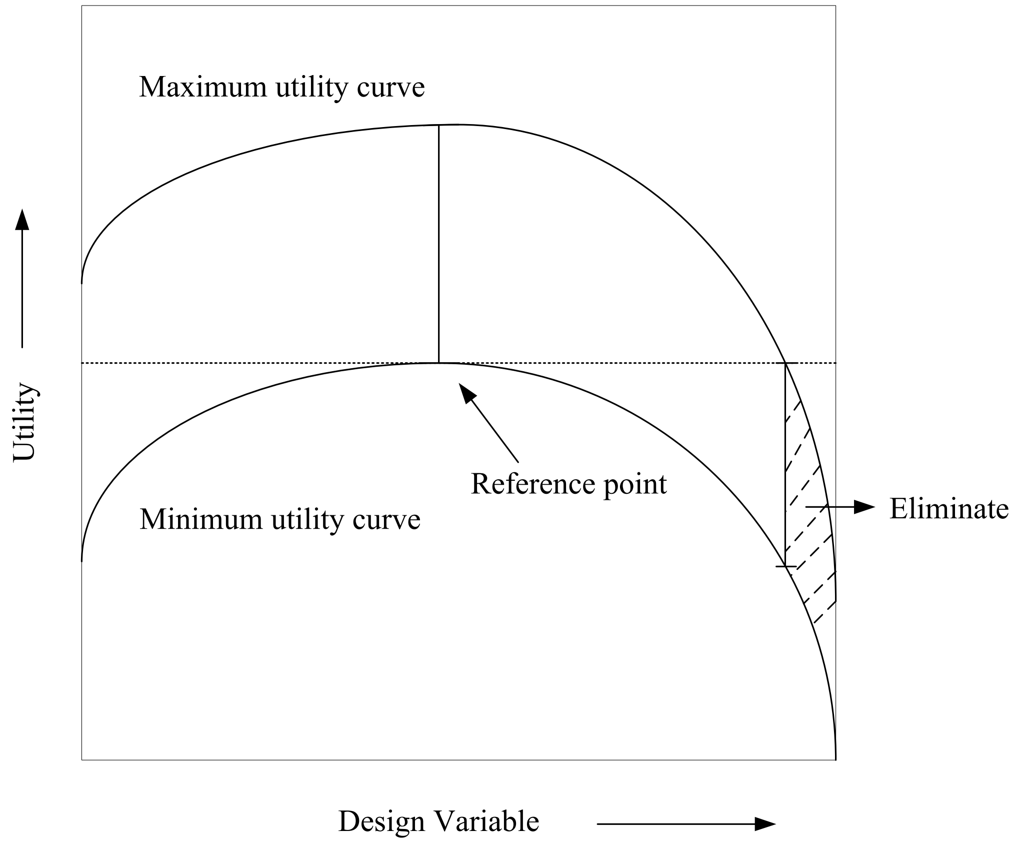

Figure 2.2. Elimination Using the Alternative with Highest Lower Bound. Source: [18]

2.4. OBJECTIVES OF THIS RESEARCH

Engineering design is influenced by decision making, and proper decisions must be made if a product or system is to succeed. Decision making with no tools or aid is irrational and generally inconsistent. This research uses utility analysis to make reliable, rational, and consistent decisions. The methods currently available for decision making give little attention to uncertainty. Of those methods that focus on uncertainty, most focus only on aleatory uncertainty.

When uncertainty is due to intervals, no single correct solution exists. Existing applications in this area offer a solution set [18, 19]. The engineer faces the challenge of choosing a solution from this set. This work addresses the critical issue of finding a single solution rather than a solution set.

This research treats both aleatory and epistemic uncertainty and introduces a simple yet effective penalty approach. Epistemic uncertainty is mainly due to a lack of knowledge, and the penalty approach penalizes the variations in the utility due to epistemic uncertainty. Utility analysis with the help of this penalty approach offers a single solution rather than a set of solutions. This helps the designer make consistent decisions.

3. BASICS OF OPTIMIZATION AND UNCERTAINTY MODELING

3.1. OPTIMIZATION

Engineers generally find it challenging to design effective and efficient systems without compromising on the basic essentials of the system. Competition from others has forced the engineers to design economical and better designs. Optimization can be defined as finding the solution that is a best possible fit for the available resources. In mathematical terms, optimization can be defined as the minimization or maximization of a real valued function. Decision making involves the allocation of specific resources to specific problems. Optimization can be used in decision making, and as an example, the solution that minimizes cost for a particular task is the best chosen solution. Thus, optimization is a very useful tool in decision making.

An optimization problem may be formulated from a problem statement in which the objective function, the design variables, and the constraints are identified [38]. Such a problem can be expressed mathematically as follows:

$$\min_{\mathbf{X}} \; f(\mathbf{X})$$

subject to

$$\begin{aligned} & g_i(\mathbf{X}) \le 0, \; i = 1, 2, \ldots, n_g \\ & h_j(\mathbf{X}) = 0, \; j = 1, 2, \ldots, n_h \\ & \mathbf{X}_{\min} \le \mathbf{X} \le \mathbf{X}_{\max} \end{aligned} \tag{5}$$

where $\mathbf{X}$ is the vector of design variables,

$f(\mathbf{X})$ is the objective function,

$g_i(\mathbf{X})$ are the inequality constraints,

n_g is the number of inequality constraints,

$h_j(\mathbf{X})$ are the equality constraints,

n_h is the number of equality constraints, and

$\mathbf{X}_{\min}$ and $\mathbf{X}_{\max}$ are the lower and upper bounds of the vector of design variables respectively.

The objective function is of primary concern here and must be optimized, either by minimization or maximization. Generally, the function is expressed as minimization, and the maximization is expressed as negative minimization. The constraint functions are the conditions that must be satisfied if the design is to be feasible. The designer seeks to offer the most feasible, or optimum design.

In Matlab, fmincon is a function generally used to solve complex nonlinear multivariable optimization problems [39]. In this work, the optimization is formulated using fmincon as:

$$\min_{\mathbf{X}} \; f(\mathbf{X})$$

subject to

$$\begin{aligned} & c(\mathbf{X}) \le 0 \\ & ceq(\mathbf{X}) = 0 \\ & A\mathbf{X} \le \mathbf{b} \\ & Aeq\mathbf{X} = \mathbf{beq} \\ & \mathbf{X}_{\min} \le \mathbf{X} \le \mathbf{X}_{\max} \end{aligned} \tag{6}$$

where $f(\mathbf{X})$ is the objective function that returns a scalar value,

b and **beq** are vectors,

A and Aeq are matrices,

$c(\mathbf{X})$ and $ceq(\mathbf{X})$ are functions that return vectors, and

$f(\mathbf{X})$, $c(\mathbf{X})$, and $ceq(\mathbf{X})$ can be nonlinear functions.

3.1.1. Multi-Attribute Optimization. Engineering designers must often consider multiple conflicting objectives. One of the simpler approaches considers the most important of the objective functions and sets the remaining functions as constraints restricted to within acceptable limits [40, 41].

Another common approach weights all the objectives and combines them into a single objective function [40] as follows:

$$\text{Optimize } f = w_1 f_1 + w_2 f_2 + \ldots + w_{n_a} f_{n_a} \tag{7}$$

where $f_1, f_2, \ldots, f_n$ are the individual attributes, $w_1, w_2, \ldots, w_n$ are the weights given to the attributes, and n_a is the total number of attributes. The sum of the weights equals 1. This is shown in equation 8.

$$\sum_{i=1}^{n_a} w_{n_a} = 1 \tag{8}$$

3.1.2. Tradeoffs in Decision Making. Complex engineering design problems often involve many conflicting objectives. Generally, no single design simultaneously maximizes all the objectives. Lesser mass precludes optimal deflection; high reliability cannot be achieved at low cost; smaller mass cannot carry higher loads.

When objectives conflict, the decision maker must make a tradeoff, sacrificing the value of one objective to increase the value of another. Ultimately, the choice depends on the value assigned by the decision maker to each objective, and different decision makers will have different preferences. A Pareto optimal solution is can achieve effective compromise. The Pareto optimal solution is not unique, and each solution in a Pareto optimal set trades improvement in one attribute for deterioration in one or more attributes.

The Pareto frontier is the set of design points not dominated by any other point in the design space. A Pareto frontier for a problem with two objectives, less mass and less deflection, will look something like that in Figure 3.1. The optimal point lies on the Pareto frontier, and the selection of a single Pareto point is at the discretion of the decision maker. The decision maker must base his choice on a compromise among the objectives.

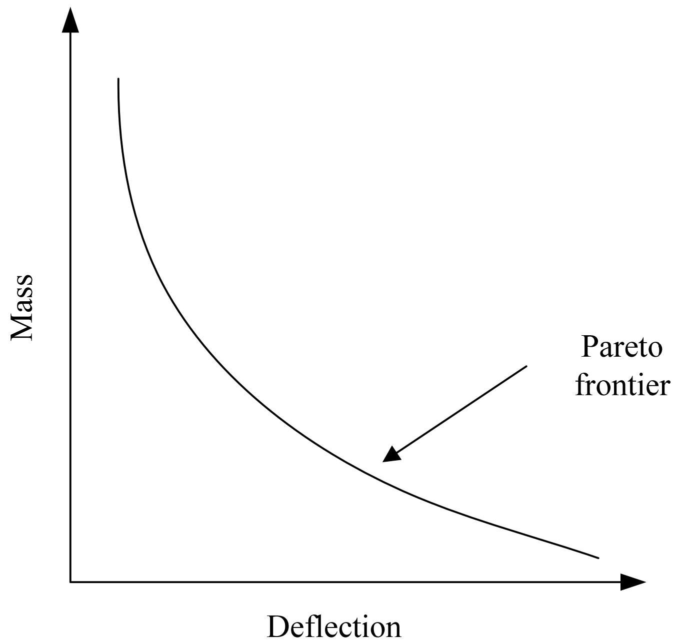

Figure 3.1. Pareto Frontier

3.2. UNCERTAINTY MODELING

Uncertainty occurs in a system in many different ways, and better methods are required to represent it. The uncertainties in a system or a design problem can be represented using probabilistic or interval methods. Uncertainty is represented probabilistically as a random variable that follows a specific probability distribution. The interval method represents uncertainty in a range as an interval variable. As mentioned above, if the uncertainty is aleatory, it is modeled by the probabilistic approach, and if it is epistemic, it is modeled by the interval approach.

3.2.1. Random Variable. A random variable reflects the numerical values of the outcome of any random event. The diameter of a shaft, the length of a beam, yield strength, or the roll of a die may be examples of a random variable. Aleatory uncertainty is modeled by random variables to which probability distributions are assigned. A random variable is defined by its mean, the standard deviation; and the types of distribution it follows.

3.2.2. Interval Variable. In some situations, data on the distribution or occurrence of variables may be insufficient, and the only information available may be the range in which the variable falls. These variables are treated as interval variables; they have an upper bound and a lower bound. A few examples of interval variables are given in [21, 22].

The interval variable Y is represented as

$$Y = [Y_{\min}, Y_{\max}] \tag{9}$$

where $Y_{\min}$ and $Y_{\max}$ are the lower and upper bounds of the interval variable respectively. The range of the interval variable Y is given as

$$\delta Y = Y_{\max} - Y_{\min} \tag{10}$$

The average value of the interval variable Y is given as

$$\overline{Y} = \frac{1}{2}(Y_{\max} + Y_{\min}) \tag{11}$$

3.3. UNCERTAINTY ANALYSIS METHODS

Two commonly used techniques for uncertainty analysis are the worst case analysis (or the extreme condition approach) and the statistical approach [42]. The statistical approach relies heavily on the use of data sampling to generate a cumulative distribution function (CDF) of system outputs [42].

3.3.1. Moment Matching Method. Knowing only the first two moments (the mean and standard deviation) of a random variable, moment matching method may be used to estimate the mean and standard deviation of a function [43, 44].

Let $\mathbf{X} = \left(X_1, X_2, X_3 ......X_m\right)$ be m independent random variables that have means $\boldsymbol{\mu} = \left(\mu_1, \mu_2, \mu_3 ......\mu_m\right)$ and standard deviations $\boldsymbol{\sigma} = \left(\sigma_1, \sigma_2, \sigma_3 ......\sigma_m\right)$

The performance function is linearized by the first order Taylor expansion at the means of the random variables as follows:

$$g(\mathbf{X}) \approx L(\mathbf{X}) = g(\boldsymbol{\mu}) + \sum_{i=1}^{m} \left[\frac{\partial g(\mathbf{X})}{\partial X_i} \right]_{\mu} (X_i - \mu_i) \tag{12}$$

The mean of $g(\mathbf{X})$ is approximated by the mean of the linearized function $L(\mathrm{X})$ and is given by

$$\mu_g \approx g(\boldsymbol{\mu}) \tag{13}$$

The standard deviation of $g(\mathbf{X})$ is given by

$$\sigma_g = \sqrt{\sum_{i=1}^{m} \left[\left[\frac{\partial g(\mathbf{X})}{\partial X_i} \right]_{\boldsymbol{\mu}} \right]^2 \sigma_i^2} \tag{14}$$

Using the moment matching method, the constraint can be

$$g = \mu_g + k_\sigma \sigma_g \tag{15}$$

where k_σ is the number of standard deviations.

The probabilities of constraint satisfaction for different values of k_σ are as shown in Table 3.1.

Table 3.1. Probability of Constraint Satisfaction

Number of Standard Deviations (k_σ)	Percentage of feasible designs
1	84.13
2	97.725
3	99.865
4	99.9968

3.3.2. Worst Case Analysis. For situations in which uncertainty is the result of interval variables, worst case analysis is used to find the interval of a performance function [43]. Worst case analysis assumes that all fluctuations may occur simultaneously in the worst possible combination [42]. Since this situation is unlikely, worst case analysis is almost always too conservative [44, 45].

Let the range of the interval variable Y_i be $[Y_{i\min}, Y_{i\max}]$

The performance function is linearized by the first order Taylor expansion at the average of the input random variables as follows:

$$g(Y) \approx g(\overline{\mathbf{Y}}) + \Delta g(Y) \tag{16}$$

where $\overline{\mathbf{Y}}$ is the average of the interval variable given by equation 11, and $\Delta g(Y)$ is the effect of variations of the interval variables on the performance function. It is estimated using a first-order Taylor's series as follows:

$$\Delta g(Y) = \sum_{i=1}^{n} \left[\frac{\partial g(\mathbf{Y})}{\partial Y_i} \right]_{\overline{\mathbf{Y}}} (Y_i - \overline{Y}_i) \tag{17}$$

Therefore, worst case analysis gives the constraint function as:

$$g(Y) \approx g(\overline{\mathbf{Y}}) + \sum_{i=1}^{n} \left[\frac{\partial g(\mathbf{Y})}{\partial Y_i} \right]_{\overline{\mathbf{Y}}} (Y_i - \overline{Y}_i) \tag{18}$$

Worst case analysis is only an approximation, and an error will occur in the calculation of the effect of variations, $\Delta g(Y)$. This error comes from using the first order Taylor expansion and taking the absolute values of the derivates. Nonetheless, this method is used because of its simplicity.

4. UTILITY THEORY AND UTILITY FUNCTIONS

4.1. UTILITY THEORY

Utility theory models a decision situation by specifying the performance attributes important to the design. The designer's preferences among the attributes are determined, thus making tradeoffs among the attributes. Finally a utility function of the attributes is constructed to represent the overall value of the design problem.

Utility theory has gained tremendous attention in decision making. First, an appropriate utility is assigned to each possible consequence. The expected utility of each alternative is then calculated. The best course of action is the alternative with the highest expected utility [2].

Multiattribute utility analysis also helps design engineers decide among numerous alternatives. Its strength lies in its ability to accommodate different metrics and to take into account human preferences and risk attitudes.

Utility theory relies heavily on the concept of lottery. A lottery is a set of possible outcomes, each of which has a specific probability of occurrence. A simple lottery is represented in Figure 4.1 in which $\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3, ..., \mathrm{p}_\mathrm{n}$ are the probabilities of the outcomes $A_1, A_2, A_3, ..., A_n$.

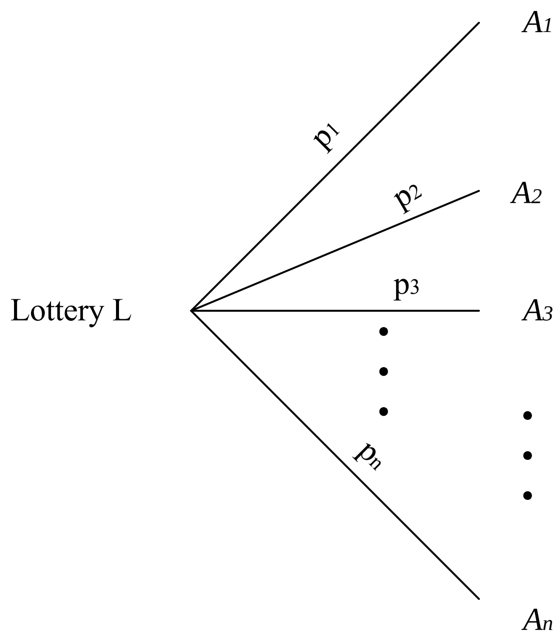

Figure 4.1. A Simple Lottery

A set of axioms must be followed to ensure that utility theory is an appropriate guide for decision making [2, 3, 46]. These axioms are

Orderability: A preference on part of the decision maker exists. Given two outcomes A and B, he either prefers A over B, or B over A, or he is indifferent to them. This axiom is shown as:

$$\begin{aligned} &A \succ B \quad \text{or} \\ &A \prec B \quad \text{or} \\ &A \sim B \end{aligned} \tag{19}$$

where $\succ$ means "is preferred to" and $\sim$ means "is equally preferred to".

Transitivity: The decision maker's rank ordering of preferences should be transitive. Given three outcomes *A*, *B*, and *C*,

$$\begin{array}{ll} \text{If} & A \succ B \\ \text{and} & B \succ C \\ \text{then} & A \succ C \end{array} \tag{20}$$

Monotonicity: The decision maker's preferences regarding an attribute must either increase or decrease monotonically. For example, more money is always preferred to less money, and lower cost is always preferred to more cost.

Continuity: If $A \succ B \succ C$, then there exists a probability p such that $pA + (1-p)C \sim B$. Thus the decision maker is indifferent in a choice between a certain outcome B and a lottery involving outcome *A* with probability p and outcome C with probability $1-p$. This axiom is illustrated in Figure 4.2.

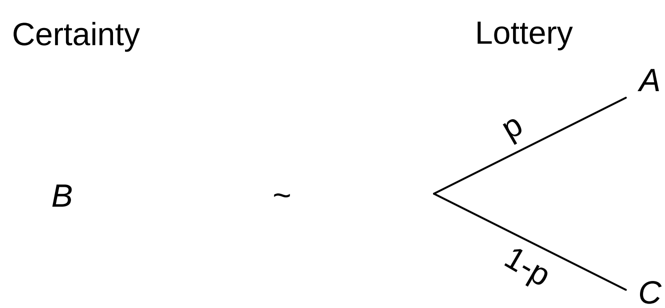

Figure 4.2. Certainty Equivalent Vs. a Lottery

Substitutability or Independence: If $A \sim B$, and C is any outcome, then the decision maker should have no preference between the two lotteries, as shown in Figure 4.3. If $A \succ B$, then lottery 1 should be preferred to lottery 2.

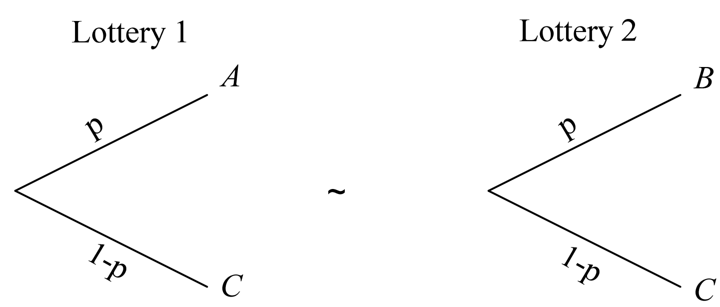

Figure 4.3. Independence Axiom Lottery

Decomposability: A compound lottery such as lottery 1 shown in Figure 4.4, broken down into a simple lottery that has all the outcomes of the compound lottery with the associated probabilities among its outcomes. A compound lottery is a lottery in which at least one outcome is another lottery.

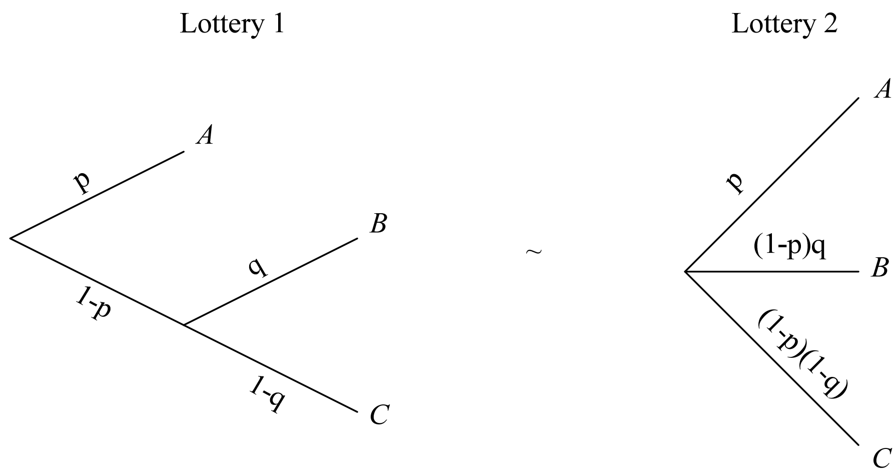

Figure 4.4. A Compound Lottery Broken Down Into a Simple Lottery

4.2. UTILITY FUNCTIONS

Utility functions provide a means to measure a designer's preferences. A utility is unitless; therefore a utility function can be compared to other utility functions.

One common and popular approach to multi-attribute optimization problems is defining a formulation that transforms an n-dimensional vector objective to a scalar performance measurement. This is called a multi-attribute utility function. It is composed of individual utility functions for each attribute, and it weighs attributes to show the significance for the design as a whole. The linear additive utility model is given as:

$$U(x) = \sum_{i=1}^{n_a} w_i U_i(x_i) \tag{21}$$

where $U(x)$ is the combined utility,

n_a is the total number of attributes,

x_i is the i-th attribute,

w_i is the weight of the i-th attribute, and

$U_i(x_i)$ is the utility of the i-th attribute. The sum of the weights is equal to 1.

The individual utility function of a particular attribute indicates the attribute's utility for the decision maker and reflects his attitude towards risk. These functions are usually evaluated by the certainty equivalent method presented in [2]. The utility values are normalized between a 0 and 1. A utility of 1 is assigned for the best possible outcome, and a utility of 0 is assigned for the worst possible outcome.

4.2.1. Types of Utility Functions. A number of utility functions can be used in decision making. The history and types of utility functions are listed in [35]. A few basic types and frequently used utility functions are described below.

Linear utility function: A utility function is said to be linear if it is of the form

$$U = ax + b \tag{21}$$

where x is the attribute value, and a and b are constants that are determined based on the preferences and risk attitude of the decision maker.

Logarithmic Utility function: A logarithmic utility function is expressed as:

$$U = \ln(x) \tag{22}$$

This is a monotonically increasing function.

Exponential utility function: A exponential utility function is expressed as:

$$U = a + b\exp(cx) \tag{23}$$

where $a, b,$ and c are constants that depend on the preferences and risk attitude of the decision maker. The exponential utility function has been favored in the past because it has risk aversion properties [40].

Power utility function: As risk aversion has become increasingly irrelevant in today's practical problems, the power utility function has gained popularity. The power utility function is given as

$$U = a + bx^c \tag{24}$$

Quadratic utility function: A quadratic utility function has the form

$$U = a + bx + cx^2 \tag{25}$$

4.2.2. Risk Attitude. The decision maker might be risk averse, risk seeking, or risk neutral. A risk neutral individual has no preference for either the expected value of a lottery or its certainty equivalent. For example, given a choice between the certainty of winning \$50 and a 0.5 probability of winning \$100, such an individual is indifferent. The risk neutral utility function is linear as shown in Figure 4.5.

A risk averse individual is one who prefers less risk for the same expected return. He sacrifices high returns to ensure low risk. Such a person would rather accept a sure return of \$50 than a 0.5 probability of a \$100 return. A utility function that is concave downwards is risk averse as shown in Figure 4.5.

A risk seeking individual is one who will accept the risk of a bad return for the chance of a very good return. The risk seeking person will choose to gamble on a 0.5

probability of winning $100 rather than accept a sure $50. A utility function that is concave upwards is risk seeking as shown in Figure 4.5.

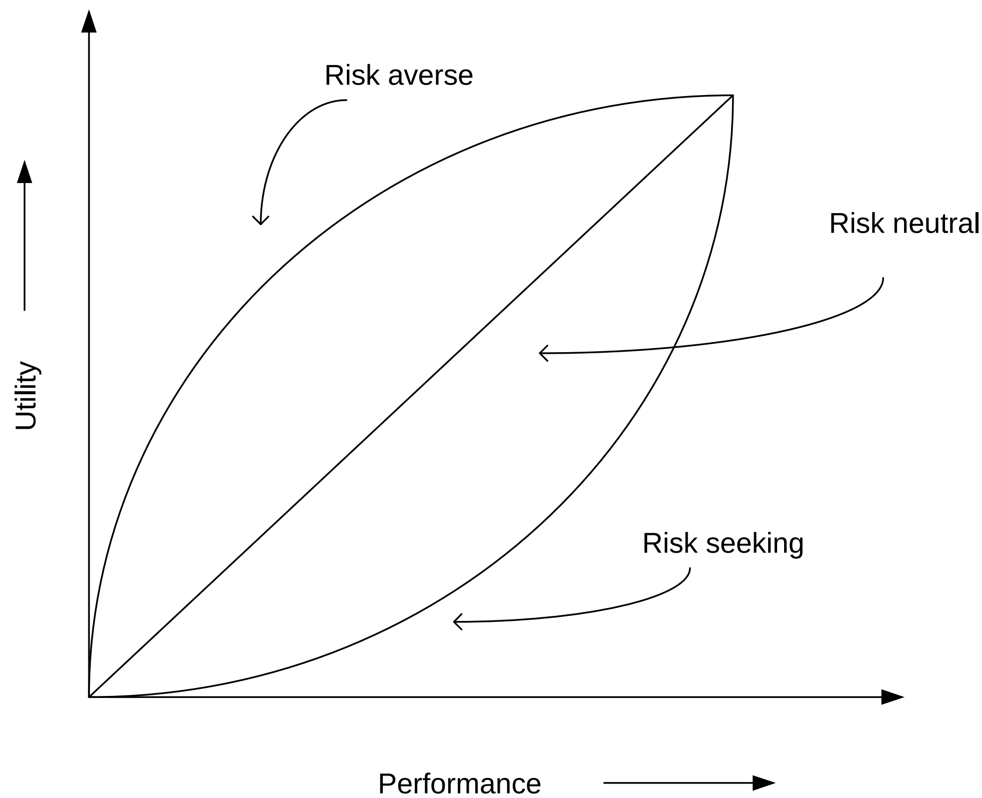

Figure 4.5. Utility Functions Based on Risk Attitude

4.3. CONSTRUCTION OF A QUADRATIC UTILITY FUNCTION

This work uses utility functions of the second order polynomial, (i.e., quadratic utility function). The quadratic utility function has the form

$$U(x) = a + bx + cx^2 \tag{26}$$

where $a, b,$ and c are the constants to be determined to complete the utility function, and x is the value of the attribute.

Suppose the best or the most desirable value of an attribute x is x_{best}, and the worst or the least desirable value of attribute x is x_{worst}. By assigning a utility of 1 to the best value of the attribute and a utility of 0 to the worst, two equations are generated as:

$$U(x_{best}) = 1 = a + bx_{best} + cx_{best}{}^2 \tag{27}$$

$$U(x_{worst}) = 0 = a + bx_{worst} + cx_{worst}{}^2 \tag{28}$$

A third equation is necessary to solve for all three constants $a, b,$ and c.

Since the decision maker's preferences and attitude towards risk are reflected in the utility function, he is asked to assign a value he believes will give him a utility of 0.5. Suppose that $x_{0.5}$ is the value at which the utility is 0.5, the third equation would be

$$U(x_{0.5}) = 0.5 = a + bx_{0.5} + cx_{0.5}{}^2 \tag{29}$$

This process of finding out utilities at other values of the attribute is called the certainty equivalent method and can also be shown as a lottery in Figure 4.6 [2]. The certainty equivalent is the value at which the decision maker is indifferent between receiving this value for certain or playing a lottery where the expected payoffs are the most desired value and the least desired value with probabilities of 0.5 each.

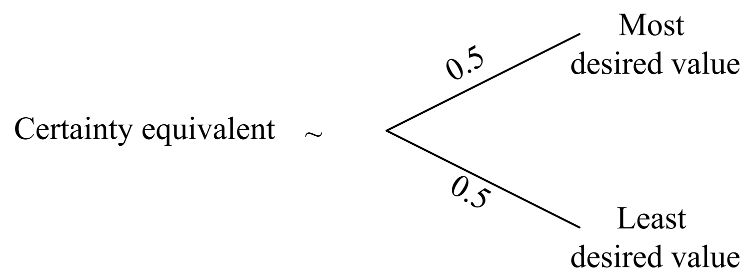

Figure 4.6. Certainty Equivalent

These three equations 27, 28, and 29 are solved for the constants $a, b,$ and c, and the utility function for a particular attribute x is given by $U(x) = a + bx + cx^2$.

This process is used to generate a utility function for each attribute. The attributes are then weighted, and the utility functions are added up assuming a linear additive utility model as:

$$U = \sum_{i=1}^{n_a} w_i U_i \tag{30}$$

where n_a is the number of attributes, w_i is the weight of the i-th attribute, and U_i is the utility function for the i-th attribute

4.4. OPTIMIZATION WITH UTILITY FUNCTIONS

The first step in solving an optimization problem with utility functions is to identify the attributes and constraints. Individual utility functions are then constructed for each attribute. The utility function is chosen based on the needs of the decision maker. It affects the design, and different utility functions may give different optimum designs. Each decision maker will use his own utility function to solve an optimization problem; therefore, unique designs might be produced.

Next, a single utility function is constructed which is a weighted sum of the utility functions of the attributes. When multiple attributes are combined into a single utility function, care must be taken to assign weights to the attributes in such a way that no attribute dominates the function. The expected utility from this function is found, and the design point at which the maximum utility occurs is chosen as the optimum design. Figure 4.7 shows a simple flowchart to solve an optimization problem using utility functions.

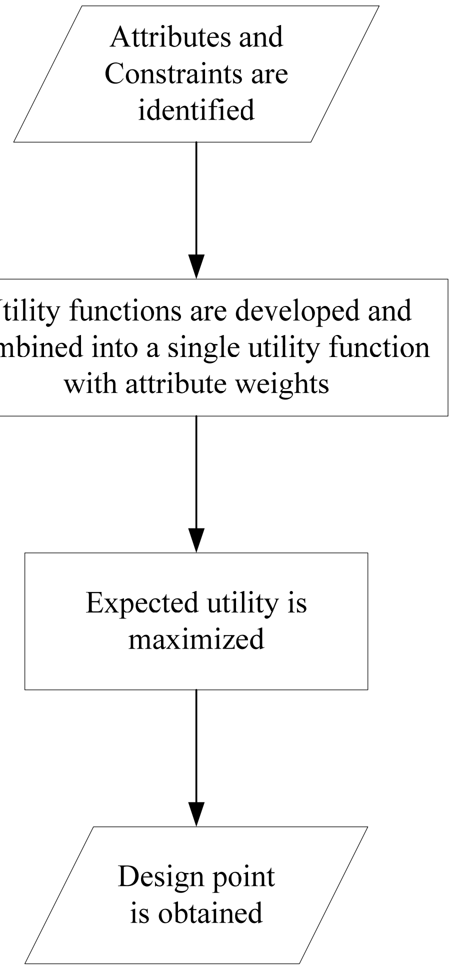

Figure 4.7. Solution to an Optimization Problem Using Utility Functions

5. PROPOSED METHODOLOGY

Most current methods for uncertainty focus only on aleatory uncertainty. When uncertainty is due to intervals, no single correct solution exists. Previous applications in this area offer a solution set [18, 19]. The engineer faces the challenge of choosing a solution from this set. The methodology presented here addresses the critical issue of finding a single solution rather than a solution set. Most previous methods are complicated, confusing the user and requiring much thought.

Utility analysis permits comparison of the attributes of various metrics by normalizing the value of the attributes between 0 and 1. It helps designers make reliable, rational, and consistent decisions.

The method proposed here facilitates decision making by maximizing the expected utility. The design point of maximum utility represents the preferred solution, thus simplifying the engineer’s task.

The proposed method treats both aleatory and epistemic uncertainties introducing a simple yet effective penalty approach. Epistemic uncertainty is mainly due to a lack of knowledge, and the penalty approach penalizes the variations in the utility due to epistemic uncertainty. With the help of this penalty approach, utility analysis offers a single solution rather than a set of solutions, thus helping the designer to make consistent decisions.

5.1. PROPOSED METHOD

Due to the existence of aleatory and epistemic uncertainties, the utility function depends on both random and interval variables. For given values of the random variables, the utility function is still a function of interval variables. Therefore, over a range of values for interval variables, there exists a maximum utility value and a minimum utility value. Since the utility value is in a range, it is difficult to choose a design that provides the highest utility. The highest utility may not be presented by an average of the maximum and minimum values. Such an average might, for example, produce a situation like the following.

$$\begin{aligned}&\text{Design 1:} U_{\max} = 0.9, U_{\min} = 0.1\\&\text{Design 2:} U_{\max} = 0.6, U_{\min} = 0.4\end{aligned} \tag{31}$$

where $U_{\max}$ and $U_{\min}$ represent the maximum and minimum utilities

For both these designs, the average utility is 0.5. In the first design, however, the difference between the maximum and minimum utilities is much larger than in the second design. The closer the values of maximum and minimum utilities, the smaller the effect of interval variables on the design. Therefore, a penalty is introduced into the average utility formula that depends on this difference. This penalty is due to the lack of knowledge on the interval variables. The designer determines the penalty factor, which is denoted by k as shown in equation 32. The final utility is now given as

$$U = 0.5(U_{\max} + U_{\min}) - k(U_{\max} - U_{\min}) \tag{32}$$

Using the penalty method, and a penalty factor, k of 0.2, the final utility values for the above two designs are

$$\begin{aligned} &\text{Design 1: } U = 0.34 \\ &\text{Design 2: } U = 0.46 \end{aligned} \tag{33}$$

Although the maximum utility of design 1 is higher than that of design 2, these utility values indicate that design 2 is the better choice.

The above procedure applies only to a single sample of the random variables. It must be modified so that the uncertainty due to random variables is also included in the design.

For n samples of the random variables, the maximum and minimum utility can now be determined based on the values of the interval variables. Over the range of the interval variables, using an optimization loop, the maximum utility and the minimum utility are found. For each sample of the random variable, there exists a maximum utility, $U_{\max}$ and a minimum utility, $U_{\min}$; the average of these n values is expressed as $\overline{U}_{\max}$ and $\overline{U}_{\min}$ respectively.

$$\begin{aligned} \overline{U}_{\max} &= \sum_{i=1}^{n} \frac{U^i_{\max}}{n} \\ \overline{U}_{\min} &= \sum_{i=1}^{n} \frac{U^i_{\max}}{n} \end{aligned} \tag{34}$$

The net or expected utility is therefore given as:

$$U_{net} = 0.5(\overline{U}_{\max} + \overline{U}_{\min}) - k(\overline{U}_{\max} - \overline{U}_{\min}) \tag{35}$$

For a single starting design point, there is an expected utility, U_{net}. The goal now is to identify a design for which the expected utility is maximized. In addition, if the design is to be feasible, the constraints must be satisfied. The overall problem can now be formulated as:

$$\begin{aligned} &\underset{\mathbf{DV}}{Max}\ U_{net} = 0.5(\overline{U}_{\max} + \overline{U}_{\min}) - k(\overline{U}_{\max} - \overline{U}_{\min}) \\ &s.t.\ \text{constraints} \end{aligned} \tag{36}$$

where $\mathbf{DV}$ are design variables. In many engineering design problems, the means of the random variables $\mathbf{X}$ and the averages of interval variables $\mathbf{Y}$ are to be determined during a design process. In this work, therefore, we use $\boldsymbol{\mu}_X$ and $\overline{\mathbf{Y}}$ as design variables. $\boldsymbol{\mu}_X$ and $\overline{\mathbf{Y}}$ are the means of the random variables and averages of the interval variables respectively.

$$\mathbf{DV} = \left(\boldsymbol{\mu}_X,\ \overline{\mathbf{Y}}\right)$$

The constraints to be satisfied might involve uncertainty, which might be due to random and interval variables. To address the uncertainty due to intervals, worst case analysis is applied to the constraints. The constraint is then given by equation 18 as shown above in Section 3.

$$g(Y) \approx g(\overline{\mathbf{Y}}) + \sum_{i=1}^{n} \left[\frac{\partial g(\mathbf{Y})}{\partial Y_i} \right]_{\overline{\mathbf{Y}}} (Y_i - \overline{Y}_i)$$

To address uncertainty due to random variables, moment matching method is applied to the constraints. A Monte Carlo simulation for *n* number of samples of the random variables is performed to get *n* values of each of the constraints. The means and standard deviations of each constraint are then found. Using the moment matching method, the constraints are then given in equation 12, 14, and15 as shown above.

$$g(\mathbf{X}) \approx g(\boldsymbol{\mu}) + \sum_{i=1}^{m} \left[\frac{\partial g(\mathbf{X})}{\partial X_i} \right]_{\mu} (X_i - \mu_i)$$

$$\sigma_g = \sqrt{\sum_{i=1}^{m} \left[\left[\frac{\partial g(\mathbf{X})}{\partial X_i} \right]_{\boldsymbol{\mu}} \right]^2 \sigma_i^2}$$

$$g^{\max} = \mu_g + k_\sigma \sigma_g$$

Depending on the probability, *R* of satisfying the constraints, the value of k_σ is taken from Table 3.1.

The overall problem can now be shown as:

$$\begin{aligned} &\underset{\boldsymbol{\mu}_X, \overline{\mathbf{Y}}}{Max}\ U_{net} = 0.5(\overline{U}_{\max} + \overline{U}_{\min}) - k(\overline{U}_{\max} - \overline{U}_{\min}) \\ &s.t.\ \ P\left\{ g^{\max}(\mathbf{X}, \mathbf{Y}) \le 0 \right\} \ge R \end{aligned} \tag{37}$$

The probability of the maximum constraint satisfaction $g^{\max}(\mathbf{X}, \mathbf{Y}) \le 0$ should be greater than a required reliability level *R* desired by the designer for the application type. Figure 5.1 summarizes this procedure in a flowchart.

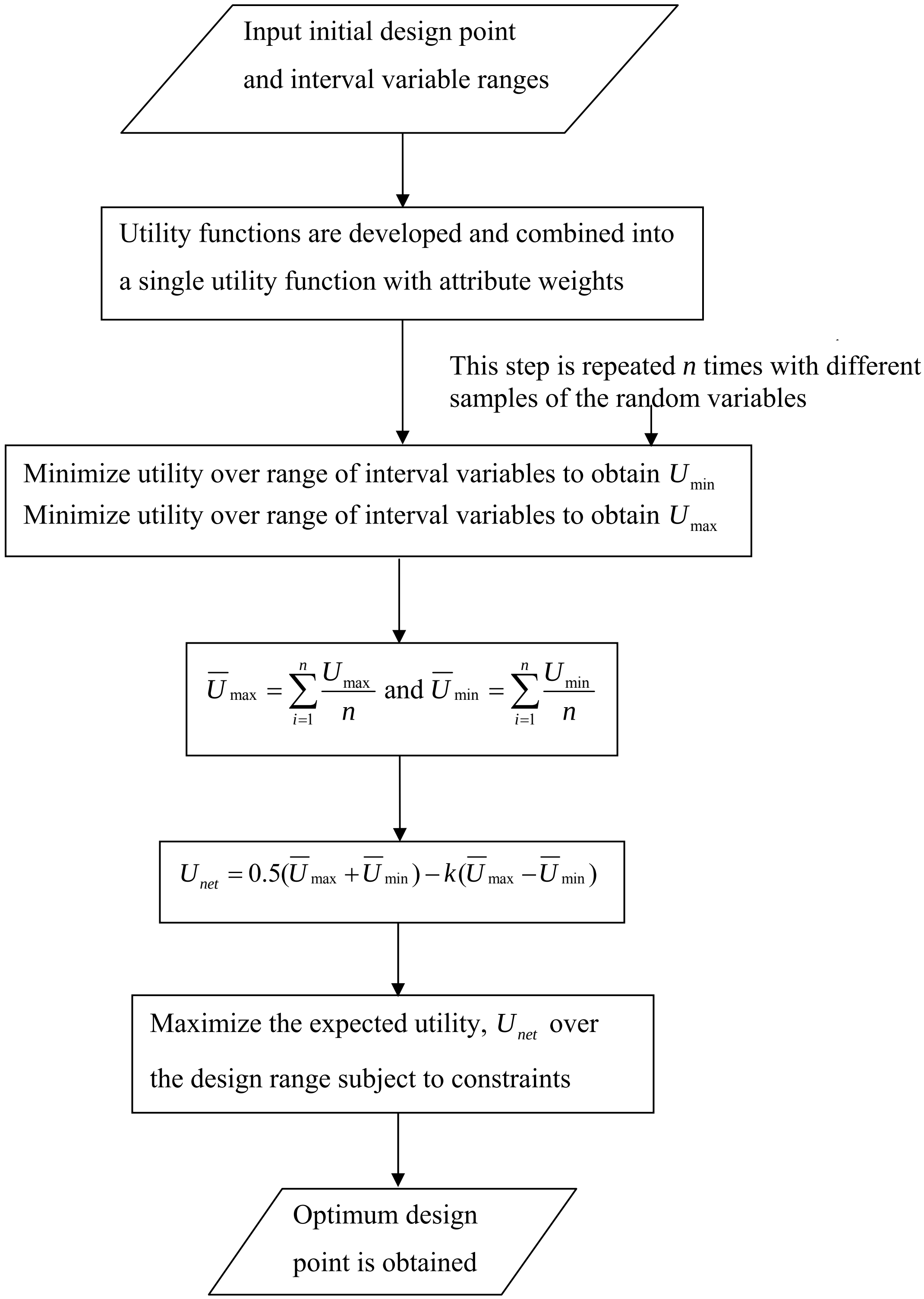

Figure 5.1. Flowchart of the Proposed Method

5.2. SUMMARY OF THE PROPOSED METHOD

The first step in the procedure is to identify the objectives, constraints, design variables, random variables, and interval variables. Next, individual utility functions are constructed for all the objectives involved in the problem. Giving a weight to each of the objective or attribute, a single utility function is determined as the weighted sum of the individual utility functions.

For a given start point of the design variables, and for a sample of the random variables, the minimum and maximum utilities over the range of the interval variables are found. The net utility is then calculated as given by the formula above, which reflects the lack of information due to epistemic uncertainty by using the penalty approach. For the design to be feasible, the constraints also need to be satisfied as discussed above. This is performed *n* number of times and the average minimum utility and the average maximum utility are determined for a particular value of the design variables. The optimization of the net utility yields the optimum value of the design variables. Using the moment matching method and worst case analysis, the constraints are simplified to address uncertainty.

To better understand the proposed method, two engineering problems are taken as examples and are presented in the next section.

6. EXAMPLES

6.1. EXAMPLE 1: A TWO-BAR PIN-JOINTED TRUSS PROBLEM

A two-bar pin-joined truss problem is taken from [12], and modified to test the optimization method described here. In Figure 6.1, *A* and *B* are two stationary pinned joints connected to one of two bars in the truss. The bars join at joint *C* where a random force of R_2 acts downwards on it. The design variables are the cross-sectional areas of bars *AC* and *BC*, represented by x_1 and x_2 respectively. The vertical distance between the line joining *A* and *B* to the point *C* is denoted by y .

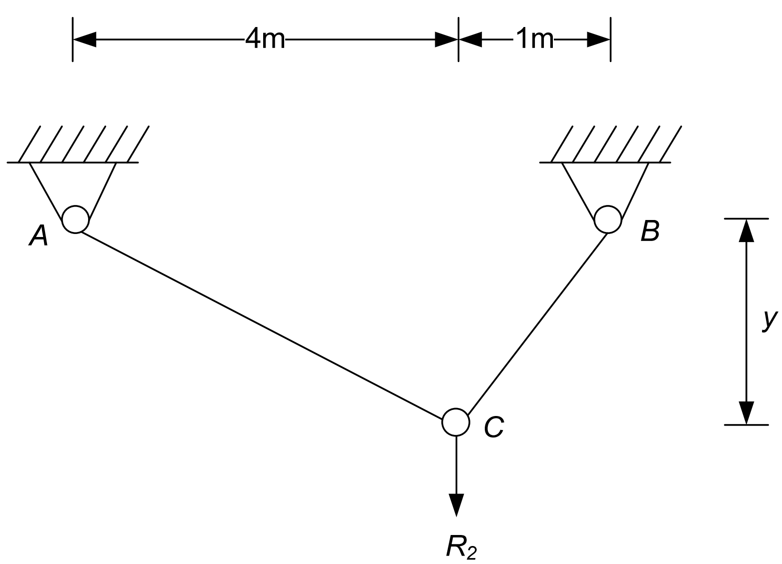

Figure 6.1. A Two-Bar Pin Jointed Truss Problem

The stresses in bars *AC* and *BC* are limited to R_2 (yield strength of the material), and the total volume of the two bars should not exceed $0.1\ \text{m}^3$. The objective is to minimize the volume of the material, while also minimizing the stress in bar *AC*.

The properties of the randomness of R_1 and R_2 are given in Table 6.1.

Table 6.1. Properties of the Random Variables

Symbol	**Mean**	**Standard Deviation**	**Distribution**
Yield Strength, R_1	100,000 kPa	10,000 kPa	Normal
Force applied, R_2	100 kN	10 kN	Normal

The vertical distance y between the line joining *A* and *B* to the point *C* is an interval variable and may range over 1 m and 3 m; namely, $y \in [1,3]\ \text{m}$.

The volume in the truss system is given by

$$V = x_1\sqrt{16+y^2} + x_2\sqrt{1+y^2} \tag{38}$$

The stress in *AC* is given by

$$\sigma_{AC} = \frac{R_2\sqrt{16+y^2}}{5x_1 y} \tag{39}$$

The stress in *BC* is given by

$$\sigma_{BC} = \frac{4R_2\sqrt{1+y^2}}{5x_2 y} \tag{40}$$

This problem has two objectives, to minimize the volume of material used and to minimize the stress in bar *AC*.

The problem can be formulated as

$$\begin{aligned}
&Minimize\ V = x_1\sqrt{16+y^2} + x_2\sqrt{1+y^2} \\
&Minimize\ \sigma_{AC} = \frac{R_2\sqrt{16+y^2}}{5x_1 y} \\
&Subject\ to:\ V \le 0.1\ \text{m}^3 \\
&\qquad\qquad\quad \sigma_{AC} \le R_1 \\
&\qquad\qquad\quad \sigma_{BC} \le R_1 \\
&\qquad\qquad\quad x_1 > 0,\ x_2 > 0
\end{aligned} \tag{41}$$

As the volume of the material decreases, the stress in *AC* increases; therefore, to reduce the stress in *AC*, the volume of the material must increase. However, since the goal is to reduce both, a trade-off must be made between these two objectives. This compromise can be achieved by assigning a utility function involving both material volume and stress in *AC*. The objective of the optimization problem then becomes to select the design that offers the highest utility while successfully satisfying all constraints.

6.1.1. Construction of the Utility Function. A less volume is desirable; therefore, a utility of 1 is assigned to a volume of 0 m^3. The maximum possible volume in this problem is 0.1 m^3. Therefore, a utility of 0 is assigned to a volume of 0.1 m^3. This is shown as

$$U_V(0) = 1$$
$$U_V(0.1) = 0$$

The volume at which the utility is 0.5 must be determined, and this lies between 0 m^3 and 0.1 m^3. In this case, a volume of 0.04 m^3 is given a utility of 0.5, this is represented as $U_V(0.04) = 0.5$

Using the quadratic function to express the utility of the volume, three equations are obtained as shown below. These are then solved for the constants *a, b,* and *c* of the quadratic equation $U(x) = a + bx + cx^2$.

$$\begin{aligned} U_V(0) &= 1 = a + 0b + 0^2c \\ U_V(0.1) &= 0 = a + 0.1b + 0.1^2c \\ U_V(0.04) &= 0.5 = a + 0.04b + 0.04^2c \end{aligned} \tag{42}$$

The constants obtained are

$$a = 1$$
$$b = -14.266$$
$$c = 42.379$$

Therefore, the quadratic utility function for the volume is

$$U_V = 1 - 14.266V + 42.379V^2 \tag{43}$$

This utility curve is plotted in Figure 6.2 for all feasible values of the volume.

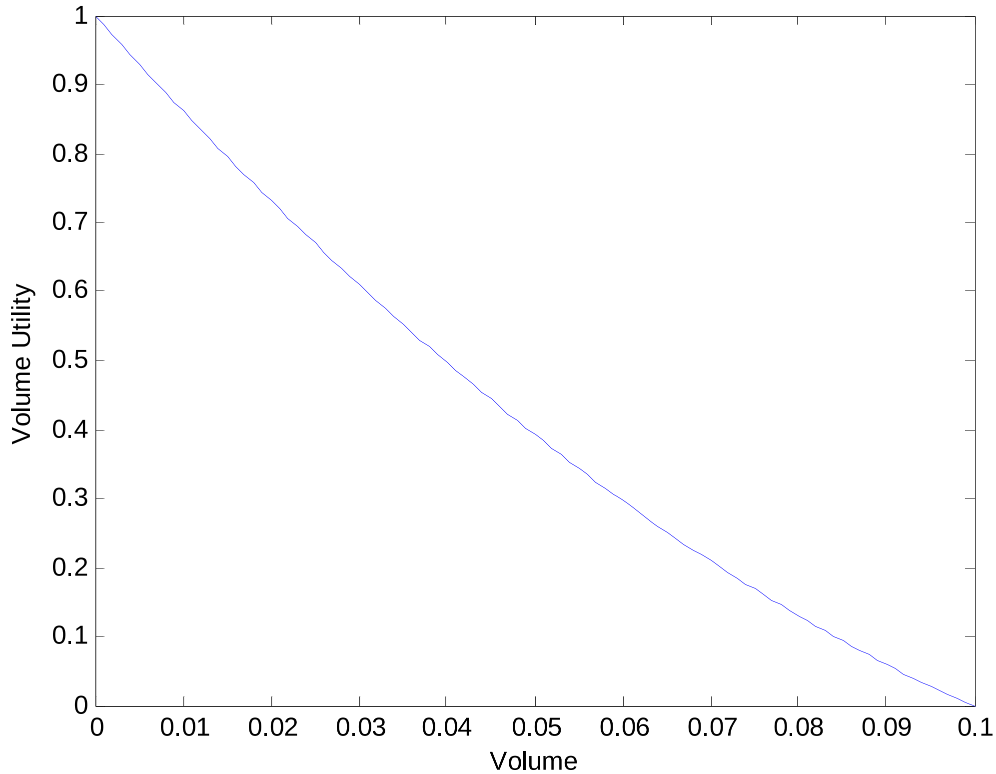

Figure 6.2. Utility Curve of Volume

A less stress in *AC* is desirable; therefore, a utility of 1 is assigned to a stress of 0 kPa. The stress in *AC* can be 100,000 kPa at maximum. Therefore, a utility of 0 is assigned to a stress of 100,000 kPa. This is shown as

$$U_{\sigma_{AC}}(0) = 1$$

$$U_{\sigma_{AC}}(100,000) = 0$$

The stress in *AC* at which the utility is 0.5 is to be determined, and this lies between 0 kPa and 100,000 kPa. In this case, a stress of 35,000 kPa is given a utility of 0.5. This is represented as $U_{\sigma_{AC}}(35,000) = 0.5$

Using the quadratic function to express the utility of the stress in *AC*, three equations are obtained as shown below, which are then solved for the constants *a, b,* and *c* of the quadratic equation $U(x) = a + bx + cx^2$.

$$\begin{aligned} U_{\sigma_{AC}}(0) &= 1 = a + 0b + 0^2 c \\ U_{\sigma_{AC}}(100,000) &= 0 = a + 100000b + 100000^2 c \\ U_{\sigma_{AC}}(35,000) &= 0.5 = a + 35000b + 35000^2 c \end{aligned} \tag{44}$$

The constants obtained are

$$\begin{aligned} a &= 1 \\ b &= -0.1659E-4 \\ c &= 0.659E-10 \end{aligned}$$

Therefore, the quadratic utility function for the stress in *AC* is

$$U_{\sigma_{AC}} = 1 - (0.1659E-4)\sigma_{AC} + (0.659E-10)\sigma_{AC}{}^2 \tag{45}$$

This utility curve is plotted in Figure 6.3 for all feasible values of the stress.

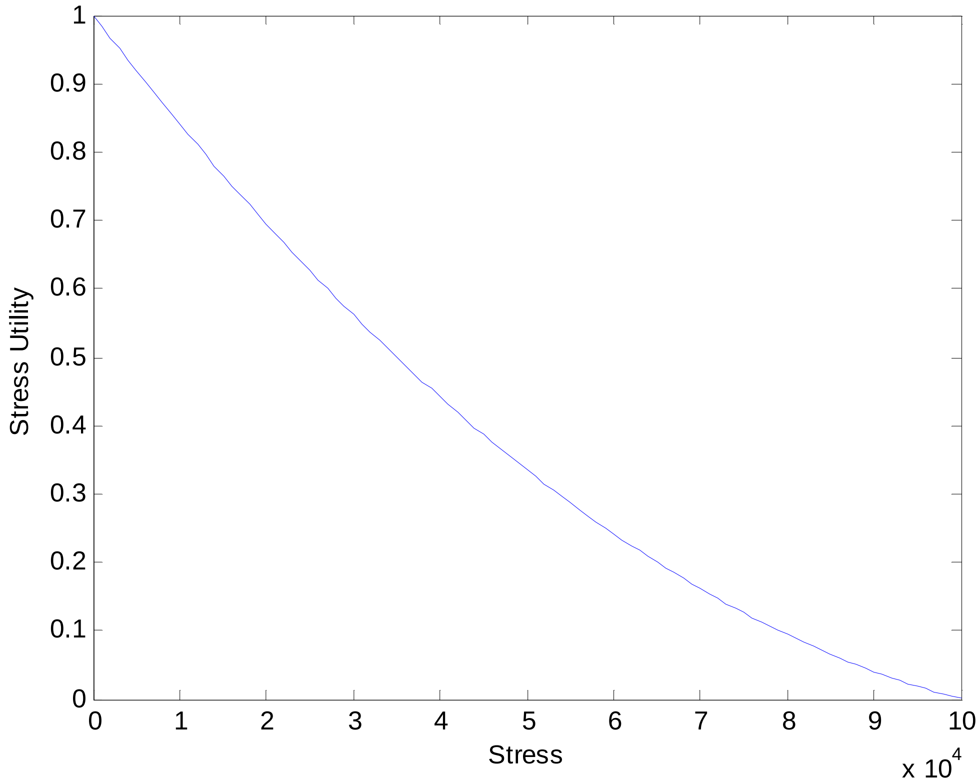

Figure 6.3. Utility Curve of Stress in *AC*

Assuming the linear additive utility model with a weight of 0.5 for both attributes, the overall utility function for this problem is

$$U = 0.5U_V + 0.5U_{\sigma_{AC}} \tag{46}$$

Substituting the values of the individual utilities from equations 43 and 45, the overall utility function becomes

$$U = 0.5\left(1 - 14.266V + 42.379V^2\right) + 0.5\left(1 - (0.1659E-4)\sigma_{AC} + (0.659E-10)\sigma_{AC}{}^2\right) \tag{47}$$

where the values of V and σ_{AC} are given by equations 38 and 39.

Since some parameters in the calculation of volume and stress are random variables and intervals, the exact value of the utility cannot be calculated. Before the problem is solved under uncertainty, it is first solved deterministically.

6.1.2. Deterministic Solution. The deterministic case takes into account only the mean and average values of the random and interval variables. The new values are R_1 = 100,000 kPa, R_2 = 100 kN, and y = 2.

The problem can now be formulated as

$$\textit{Maximize } U = 0.5U_V + 0.5U_{\sigma_{AC}}$$

$$\textit{or Minimize } -U = -\left(0.5U_V + 0.5U_{\sigma_{AC}}\right)$$

where

$$U_V = 1 - 14.266V + 42.379V^2$$

$$U_{\sigma_{AC}} = 1 - (0.1659E-4)\sigma_{AC} + (0.659E-10)\sigma_{AC}^{\;2}$$

$$V = x_1\sqrt{16+y^2} + x_2\sqrt{1+y^2}$$

$$\sigma_{AC} = \frac{R_2\sqrt{16+y^2}}{5x_1 y}$$

$$\sigma_{BC} = \frac{4R_2\sqrt{1+y^2}}{5x_2 y}$$

$$\begin{aligned} \textit{Subject to}\quad & V \le 0.1\ \mathrm{m}^3 \\ & \sigma_{AC} \le R_1 \\ & \sigma_{BC} \le R_1 \\ & x_1 > 0,\ x_2 > 0 \end{aligned} \tag{48}$$

Using the optimization tool of Matlab, the problem is solved; and the best utility value is found to be 0.4718. The design values at which the maximum utility occurs are

$x_1 = 0.000447$ m²
$x_2 = 0.0009$ m²

6.1.3. Solution under Uncertainty. Since the constraints also contain interval variables and random variables, worst case analysis and the moment matching method are applied to incorporate uncertainty into the constraints and simplify them. The constraints are

$$C_1 = x_1\sqrt{16+y^2} + x_2\sqrt{1+y^2} - 0.1 \text{ m}^3 \le 0$$

$$C_2 = \frac{R_2\sqrt{16+y^2}}{5x_1 y} - R_1 \le 0 \tag{49}$$

$$C_3 = \frac{4R_2\sqrt{1+y^2}}{5x_2 y} - R_1 \le 0$$

Constraint C_1 contains only design and interval variables, whereas constraints C_2 and C_3 contain design, random, and interval variables. Worst case analysis is used to treat uncertainty due to interval variables. From equations 16, 17 and 18, the constraint C_1 is written as

$$C_1 \approx C_1(\bar{y}) + \Delta C_1(y) \text{ or } C_1 \approx C_1(\bar{y}) + \left(\frac{\partial C_1(\mathrm{y})}{\partial y_i}\right)_{\bar{y}} (y-\bar{y}) \tag{50}$$

where $\bar{y} = 2$. Substituting this value into equations 49 and 50,

$$C_1(\bar{y}) = \sqrt{20}x_1 + \sqrt{5}x_2 - 0.1$$

$$\frac{\partial C_1(y)}{\partial y} = \frac{x_1(2y)}{2\sqrt{16+y^2}} + \frac{x_2(2y)}{2\sqrt{1+y^2}}$$

$$\left(\frac{\partial C_1(y)}{\partial y}\right)_{\bar{y}} = \frac{2x_1}{\sqrt{20}} + \frac{2x_2}{\sqrt{5}}$$

$$\Delta C_1(y) = \left(\frac{2x_1}{\sqrt{20}} + \frac{2x_2}{\sqrt{5}}\right)(3-2) = \left(\frac{2x_1}{\sqrt{20}} + \frac{2x_2}{\sqrt{5}}\right)$$

$$C_1 \approx \sqrt{20}x_1 + \sqrt{5}x_2 - 0.1 + \left(\frac{2x_1}{\sqrt{20}} + \frac{2x_2}{\sqrt{5}}\right) = 4.91x_1 + 3.13x_2 - 0.1$$

$$C_1 \approx 4.91x_1 + 3.13x_2 - 0.1 \tag{51}$$

Similarly, worst case analysis is applied on constraints C_2 and C_3 to yield

$$C_2 \approx 0.2683\frac{R_2}{x_1} - R_1 \tag{52}$$

$$C_3 \approx 0.805\frac{R_2}{x_1} - R_1 \tag{53}$$

The constraints C_2 and C_3 still contain random variables R_1 and R_2, and moment matching method is applied on them to address uncertainty. A Monte Carlo simulation is performed for 50 samples of R_1 and R_2 to get 50 values of the constraints C_2 and C_3. The means and standard deviations of the constraints on these 50 values are found. For a 99.865% probability of constraint satisfaction, a *k* value of 3 is taken from Table 3.1. The new constraints are then given as:

$$\begin{aligned} C_2 &\approx \text{mean}(C_2)\text{+}3\text{std}(C_2) \\ C_3 &\approx \text{mean}(C_3)\text{+}3\text{std}(C_3) \end{aligned} \tag{54}$$

where $\text{mean}(C_2)$ and $\text{mean}(C_3)$ are the means of the constraints C_2 and C_3 respectively, and $\text{std}(C_2)$ and $\text{std}(C_3)$ are the standard deviations of the constraints C_2 and C_3 respectively.

The solution generated by Matlab indicates that the highest expected utility is 0.4923, and the design variables corresponding to this utility are

x_1 = 0.0195 m²
x_2 = 0.0013 m²

Also, this optimum design was found to have a maximum utility of 0.5226 and a minimum utility of 0.4793.

In the deterministic case, the design variables obtained were

x_1 = 0.000447 m²
x_2 = 0.0009 m²

A Comparison of the solutions demonstrates that optimization under uncertainty suggests a more conservative design than that selected by the deterministic case.

6.2. EXAMPLE 2: DESIGN OF A FLAG POLE

An example of a flag pole design demonstrates the methodology presented in the previous section. This example is modified from a design presented in [38]. The goal here is to design a flag pole of height H with minimum mass. The pole is made of uniform circular tubing with d_0 and d_i as the outer and inner diameters respectively. This pole must withstand very high winds, and it should be between 9.5 m and 10 m high.

This work assumes that the pole is a cantilever subject to a uniform lateral wind load w. At its top, the pole carries a concentrated load P. The pole should not fail in bending or shear, and deflection at its top should not exceed 0.1 m. The ratio of mean diameter to thickness must not exceed 60. The design variables are the height H, the outer diameter d_o, and the inner diameter d_i. Figure 6.4 illustrates the design of the flag pole.

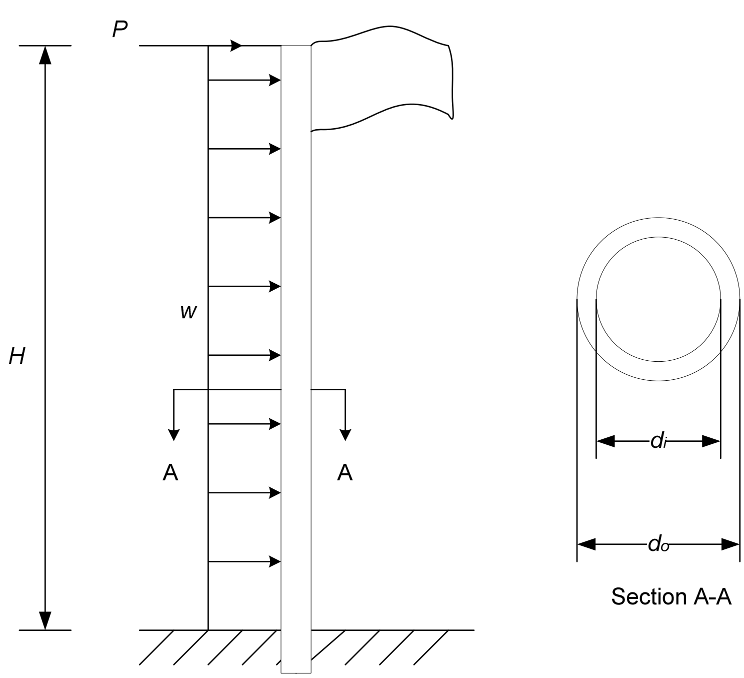

Figure 6.4. Flag Pole Design

The modulus of elasticity, allowable bending stress, and the allowable shear stress are random variables; they are shown in Table 6.2.

Table 6.2. Random Variables

Random Variables	**Mean**	**Standard Deviation**	**Distribution**
Modulus of elasticity, E	210 GPa	10.5 GPa	Normal
Allowable bending stress, σ_b	165 MPa	8.2 MPa	Normal
Allowable shear stress, τ_s	50 MPa	2.5 MPa	Normal

The mass density, wind load, and the load on the top of the pole are interval variables and are shown in Table 6.3

Table 6.3. Interval Variables

Interval Variables	**Lower Bound**	**Upper Bound**
Mass density, ρ	7300 Kg/m^3	8300 Kg/m^3
Wind load, w	1800 N/m	2200 N/m
Load on top of pole, P	3700 N	4300 N

Other equations related to this problem are

$$
\begin{aligned}
&\text{Cross-sectional area, } A = \frac{\pi}{4}(d_0^2 - d_i^2),\ \text{m}^2 \\
&\text{Moment of inertia, } I = \frac{\pi}{64}(d_0^4 - d_i^4),\ \text{m}^4 \\
&\text{Moment at the base, } M = (PH + 0.5wH^2),\ \text{kNm} \\
&\text{Bending stress, } \sigma = \frac{M}{2I} d_0,\ \text{kPa} \\
&\text{Shear at the base, } S = (P + wH),\ \text{kN} \\
&\text{Shear stress, } \tau = \frac{S}{12I}(d_0^2 + d_0 d_i + d_i^2),\ \text{kPa} \\
&\text{Deflection at the top, } \delta = \frac{PH^3}{3EI} + \frac{wH^4}{8EI},\ \text{m}
\end{aligned}
\tag{55}
$$

The constraints in this design problem are as follows:

1. Bending stress should be less than the allowable bending stress.

$$
\begin{gathered}
\sigma - \sigma_b \le 0 \\
C_1 = \frac{(PH + 0.5wH^2)}{2I} d_0 - \sigma_b \le 0
\end{gathered}
\tag{55}
$$

2. The shear stress should be less than the allowable shear stress.

$$
\begin{gathered}
\tau - \tau_s \le 0 \\
C_2 = \frac{(P + wH)}{12I}(d_0^2 + d_0 d_i + d_i^2) - \tau_s \le 0
\end{gathered}
\tag{56}
$$

3. The deflection should not exceed 0.1 m.

$$\delta - 0.1 \le 0$$
$$C_3 = \frac{PH^3}{3EI} + \frac{wH^4}{8EI} - 0.1 \le 0 \tag{57}$$

4. The thickness of the pole should be between 0.005 m and 0.02 m.

$$C_{4a} = d_i - d_o + 0.005 \le 0 \text{ and } C_{4b} = d_o - d_i - 0.02 \le 0 \tag{58}$$

5. The ratio of mean diameter to thickness must not exceed 60.

$$C_5 = \frac{0.5(d_o + d_i)}{d_o - d_i} - 60 \le 0 \tag{59}$$

6. The mass of the pole should not exceed 5000 kgs.

$$\rho AH - 5000 \le 0$$
$$C_6 = \rho \frac{\pi}{4}(d_0^2 - d_i^2)H - 5000 \le 0 \tag{60}$$

6.2.1. Construction of the Utility Function. Given these values, the minimum possible deflection is 0.0026 m, and the maximum allowable deflection is 0.1 m. Therefore, a utility of 1 is assigned for the minimum deflection, and a utility of 0 is assigned for the maximum. This is shown as

$$U_\delta(0.0026) = 1$$
$$U_\delta(0.1) = 0$$

The decision maker determines the deflection at which the utility is 0.5, and this lies between 0.0026 m and 0.1 m. In this case, a deflection of 0.03 m is given a utility of 0.5. This is represented as $U_{\delta}(0.03) = 0.5$

The quadratic function expresses the utility of the deflection. These three equations are then solved for the constants *a, b,* and *c* of the quadratic equation $U(x) = a + bx + cx^2$.

$$\begin{aligned} U_{\delta}(0.0026) &= 1 = a + 0.0026b + 0.0026^2 c \\ U_{\delta}(0.1) &= 0 = a + 0.1b + 0.1^2 c \\ U_{\delta}(0.03) &= 0.5 = a + 0.03b + 0.03^2 c \end{aligned} \tag{61}$$

The constants obtained are

$$\begin{aligned} a &= 1.056 \\ b &= -21.965 \\ c &= 114.017 \end{aligned}$$

Therefore, the quadratic utility function for the deflection is

$$U_{\delta} = 1.056 - 21.965\delta + 114.017\delta^2 \tag{62}$$

This utility curve is plotted in Figure 6.5. for all feasible values of the deflection.

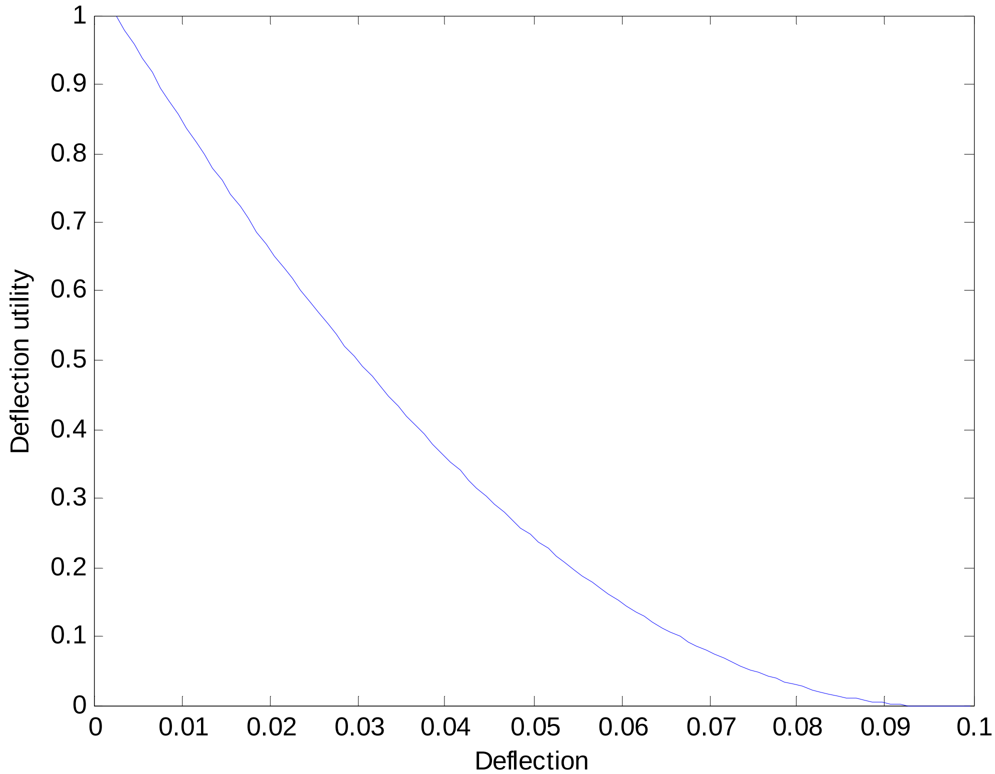

Figure 6.5. Utility Curve of Deflection

Given the constraint values listed above, the minimum possible mass is 322 kgs, and a mass above 5000 kgs is undesirable. Therefore, a utility of 1 is assigned for the minimum mass, and a utility of 0 is assigned for the maximum. This is shown as

$$U_m(322) = 1$$

$$U_m(5000) = 0$$

The decision maker determines the mass at which the utility is 0.5, and this lies between 322 kgs and 5000 kgs. In this case, a mass of 2000 kgs is given a utility of 0.5. This is represented as $U_m(2000) = 0.5$

Using the quadratic function to expresses the utility of the mass, three equations are obtained as shown below, which are then solved for the constants $a, b,$ and c of the quadratic equation $U(x) = a + bx + cx^2$.

$$\begin{aligned} U_m(322) &= 1 = a + 322b + 322^2 c \\ U_m(5000) &= 0 = a + 5000b + 5000^2 c \\ U_m(2000) &= 0.5 = a + 2000b + 2000^2 c \end{aligned} \tag{63}$$

The constants obtained are

$a = 1.114$
$b = -0.363\text{E-3}$
$c = 0.28\text{E-7}$

Therefore the quadratic utility function for the mass is

$$U_m = 1.114 - (0.363\text{E-3})m + (0.28\text{E-7})m^2 \tag{64}$$

This utility equation is plotted in Figure 6.6 for all the values of mass that are feasible in this design.

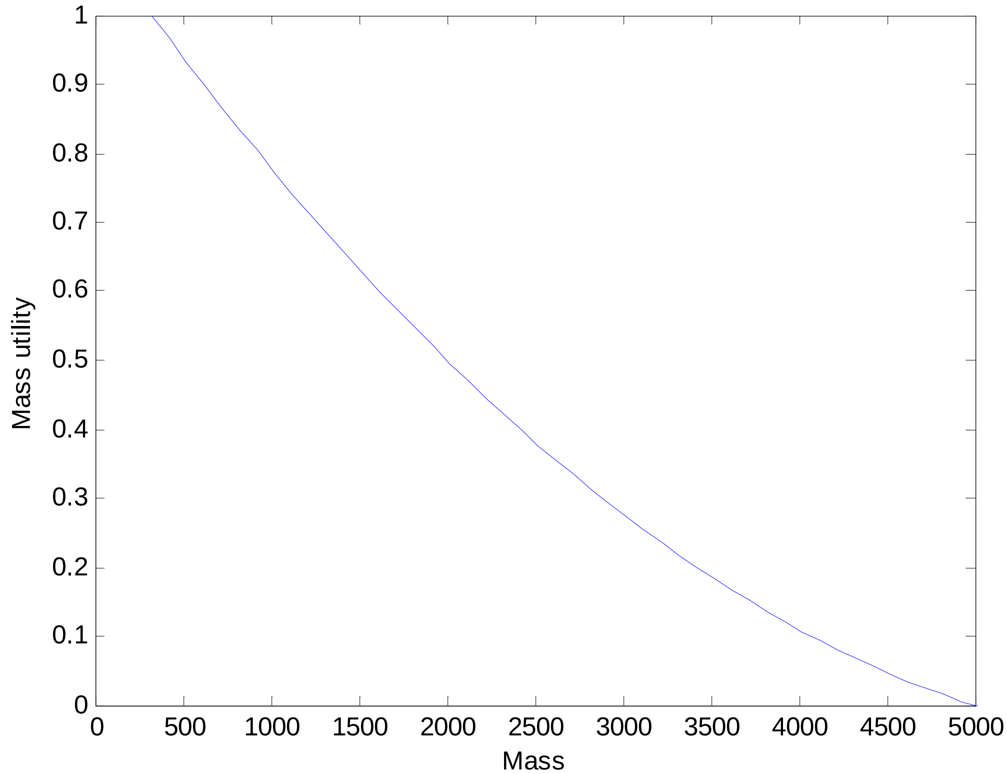

Figure 6.6. Utility Curve of Mass

Here, the deflection attribute is assigned a weight of 0.2, and the mass attribute is assigned a weight of 0.8. These are combined by the linear additive utility model into a single utility function.

$$U = 0.2U_{\delta} + 0.8U_{mass} \tag{65}$$

6.2.2. Deterministic Solution. The problem is solved deterministically, taking the mean of the random variables and the average value of the interval variables. Applying the procedure described in Section 5, the design variables are

$d_o = 0.4597m$ or $45.97cm$

$d_i = 0.45m$ or $45cm$

$H = 9.5m$

The expected utility at these design variables is $U = 0.7511$.

6.2.3. Solution under Uncertainty. Constraints C_1, C_2, and C_3 are a combination of design variables, random variables, and interval variables. Constraints C_4 and C_5 are a combination of only design variables. Constraint C_6 is a combination of design variables and interval variables only. Uncertainty must be incorporated into the constraints, and worst case analysis is used for all constraints that depend on interval variables. Moment matching method incorporates uncertainty into the constraints that depend on random variables.

Worst case analysis as described above is applied to constraints C_1, C_2, C_3, and C_6. The constraints now are

$$C_1 = \frac{(4300H + 1100H^2)}{2I} d_0 - \sigma_b \le 0 \tag{66}$$

$$C_2 = \frac{(4300 + 220H)}{12I}(d_0^2 + d_0 d_i + d_i^2) - \tau_s \le 0 \tag{67}$$

$$C_3 = \frac{4300H^3}{3EI} + \frac{2200H^4}{8EI} - 0.1 \le 0 \tag{68}$$

$$C_{4a} = d_i - d_o + 0.005 \le 0 \text{ and } C_{4b} = d_o - d_i - 0.02 \le 0 \tag{69}$$

$$C_5 = \frac{0.5(d_o + d_i)}{d_o - d_i} - 60 \le 0 \tag{70}$$

$$C_6 = 8300\frac{\pi}{4}(d_0^2 - d_i^2)H - 5000 \le 0 \tag{71}$$

Only constraints C_1, C_2, and C_3 depend on random variables, and moment matching method is used here.

The problem is then solved based on the methodology presented in Section 5. The design variables obtained are

$d_o = 0.4625m$ or $46.25cm$

$d_i = 0.45m$ or $45cm$

$H = 9.5m$

The expected utility at these design variables is $U = 0.7237$. The maximum and minimum utility dependent on the interval variables are 0.7508 and 0.7120 respectively.

In the deterministic design, the design variables obtained were

$d_o = 0.4597m$ or $45.97cm$

$d_i = 0.45m$ or $45cm$

$H = 9.5m$

A comparison of the solutions demonstrates that optimization under uncertainty suggests a more conservative design than that selected by the deterministic case.

7. CONCLUSIONS AND FUTURE WORK

7.1. CONCLUSIONS

The research has shown that utility analysis is a powerful tool for decision making. Utility functions express the designer's preferences and risk attitudes as a scalar value, and optimization of this utility value determines the best possible solution. This research also treats uncertainty using the moment matching method for aleatory uncertainty and worst case analysis for epistemic uncertainty. Thus the constraints incorporate both forms of uncertainty. Variation due to a lack of knowledge of the interval variables is penalized using a penalty on the lack of information. If more information is obtained, the penalty value is reduced and a more robust solution can be chosen.

Using the method presented here, the designer can effectively obtain point solutions to any problem instead of a solution set. This aids the designer in making decisions in his applications. In the two examples shown, the design under uncertainty is more conservative than the deterministic design because it uses the worst case of the interval variables, and all the worst possible fluctuations are assumed to occur simultaneously.

7.2. FUTURE WORK

Proper formulation and assessment of a utility function requires too much time and effort. Monte Carlo simulation is computationally expensive, and a more efficient algorithm is needed to reduce the computational burden. The feasibility of a particular type of utility function must be assessed. Depending on the application type, utility functions must be used. Also, the final designs obtained are extremely conservative; hence the cost to produce such a design is high. New methods must be formulated to produce efficient designs. Further, effective methods are required to gather more information on the interval variables and thus to reduce epistemic uncertainty.

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