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ABSTRACT

Machining process control technologies are currently not well integrated into machine tool controllers and, thus, servomechanism dynamics are often ignored when designing and implementing process controllers. In this paper, a hierarchical controller is developed that simultaneously regulates the servomechanism positions and cutting forces in a lathing operation. The force process and servomechanism system are separated into high and low levels, respectively, in the hierarchy. The high level goal is to maintain a constant cutting force to maximize productivity while not violating a spindle power constraint. This goal is systematically propagated to the lower level and combined with the low level goal to track the reference position. Since there are only control signals at the lower level, in this case motor voltages, a single controller is designed at the bottom level that will meet both the high level and low level goals. Simulations are conducted to validate the developed methodology.

INTRODUCTION

Process control technologies (e.g., force control, chatter suppression) have a tremendous potential to impact machining operations by improving productivity and part quality. However, machining process control technologies are currently not well integrated into machine tool controllers and, thus, servomechanism dynamics are often ignored when designing and implementing process controllers. In this paper a novel approach to the design and implementation of process control technologies is developed based upon the concepts of hierarchical control. The approach is applied to the simultaneous regulation of cutting forces and positional errors in a lathing operation. A schematic of a lathing operation is shown in Fig. 1. A servomechanism drives the cutting tool, thus, creating the feed (i.e., chip thickness) that determines the magnitude of the cutting forces. The objectives are to maintain a constant cutting force corresponding to maximum productivity while tracking a tool reference trajectory.

The cutting force depends on the cutting speed, feed, and the depth–of–cut of the cutting tool and is related to these parameters by the following nonlinear relation (Landers and Ulsoy, 2000)

$$F = Kf^{\alpha} d^{\beta} V^{\gamma} \qquad (1)$$

where F is the cutting force in *kN*, f is the feed in *mm*, d is the depth–of–cut in *mm*, V is the cutting speed in *km/min*, and K, α, β, and γ are empirically determined constants.

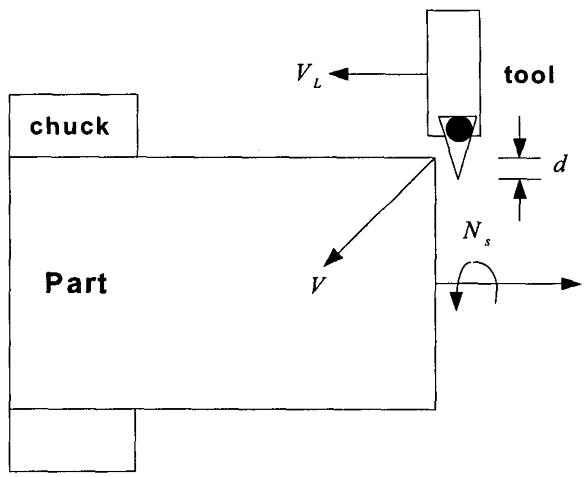

Fig. 1: Schematic of a Lathing Operation.

The subject of force control has been studied extensively in the literature using many types of control methodologies. Some examples of adaptive machining force control include Ulsoy and Koren (1989) and Elbestawi *et al.* (1990). In these studies, model parameters were estimated on–line and control gains were adjusted to maintain stability over a wide range of parameter variations. As an example of direct model based control, Landers and Ulsoy (2000) developed a nonlinear controller that directly incorporated the force–feed nonlinearity, and the stability properties of such a controller were studied. Punyko and Bailey (1994) and Rober *et al.* (1997) designed Quantitative Feedback Theory (QFT) machining force controllers in the discrete domain utilizing the delta transform. Their designs were based on a linear plant with uncertainty in pole and zero locations as well as the magnitude of a gain factor that indirectly accounts for variations in the depth–of–cut and nonlinear process parameters. While some studies (e.g., Furness *et al.*, 1999) have directly incorporated the servomechanism dynamics into the force controller design, machining force controllers typically have been treated separately from the servomechanisms.

In this paper, simultaneous force control and position control in a lathing operation is performed using the optimal hierarchical architecture proposed in Dasgupta *et al.* (2002). The force process and servomechanism system are separated into high and low levels, respectively, in the hierarchy. The high level goal is to maintain a constant cutting force to maximize operation productivity while not violating a spindle power constraint. This goal is propagated to the lower level and combined with the low level goal to track the reference position. This propagation requires an aggregation between the high and low levels; in this case, between the servomechanism and the cutting force process. Since there are only control signals at the lower level, in this case motor voltages, a single controller is designed at the bottom level that will meet both the high level and low level goals.

HIERARCHICAL SYSTEMS

Complicated systems are typical of the real world. Examples of such complicated systems are aeroplanes, highway systems, large power plants, manufacturing systems, etc. To formulate an optimal control problem for large–scale systems, the cost function/objective function in terms of all the state and control variables must be formulated. This further involves an individual goal setting for each of the variables. Typically, reference goal values for each of the variables may not be available. Also, if the number of variables is very large, it is extremely tedious and complicated to set the goals for each of the variables separately. The description of goals for such systems is mostly in terms of certain abstract variables that are not available in the original system. These factors make a large–scale system extremely complex to analyze and solve.

Some examples of highly complex systems are Air Traffic Control (ATC) system (Pappas *et al.*, 1997) and Intelligent Vehicle / Highway Systems (IVHS) (Varaiya, 1993). In Pappas *et al.*, (1997), a hierarchical Air Traffic Management (ATM) structure that is a trade–off between the completely centralized and the completely decentralized decision–making is proposed. Such a hierarchical system, which consists of four layers (strategic planner, tactical planner, trajectory planner, and regulation), reduces the complexity and assists the pilots to perform their tasks better as compared to a centralized ATC. In Varaiya (1993), a method for reducing highway congestion is formulated. The resulting method is an IVHS whose control module is a four–layered hierarchical architecture, namely, planning, network, link and regulation, which take care of the pre–trip, in–trip, and the post–trip decisions of the vehicle, respectively.

In a hierarchical control system, the complicated system is at the bottom level in the hierarchy. The upper levels in the system hierarchy are obtained by aggregating the lower levels (Fig. 2). The state variables at the upper level are chosen such that the goal descriptions for these variables are available. A cost function is then formulated at the upper level using the available values. The problem is then solved at the upper level at a reduced complexity as a result of the aggregation. The results of the upper level are then propagated to the lower levels in the hierarchy and these act as goal setting values for the lower levels due to the aggregation process. The lower level has to simply track these goals. The cost function formulation at the lower level is in terms of the original state and control variables and is formulated in such a way that the tracking of the goals passed by the upper level is possible. Since, at the lower level the task is to simply track the goals that are already evaluated at the upper levels, the complexity is reduced.

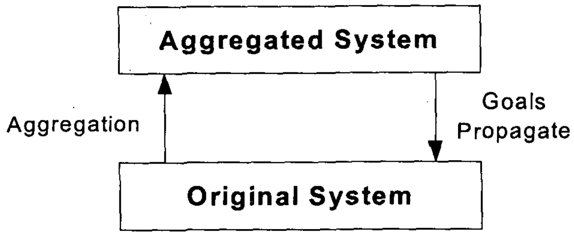

Fig. 2: Hierarchical System.

A hierarchical system provides other advantages. In a large–scale system, due to the increased complexity, the evaluation of the system properties like controllability, observability, and stability become even more difficult. In hierarchical systems, the upper level dynamics are obtained as a result of the aggregation of the lower level (Dasgupta *et al.*, 2002). The aggregated system captures the complete system behavior (Pappas and Sastry, 1999). Thus, the system properties can be evaluated from the aggregated system dynamics. Another reason for aggregation is that not every state and control variable of the original system may be accessible. During aggregation, the states and control variables of the original system are combined in a way so that the states and control variables of the aggregated system are now accessible. A third advantage of using a hierarchical system arises from the fact that the overall system goals may be expressed in terms of variables that are not present in the original system representation. Using aggregation, certain variables are obtained from the original variables such that the overall system goals are now defined in terms of these new variables. Thus, an optimal control formulation is possible at the upper level with the aggregated state variables and the lower levels, which have more state and control variables, have to simply track the goals propagated from the top. Therefore, extensive computations at the lower levels are minimized.

MULTIRESOLUTIONAL ASPECTS

In a hierarchical system, each level in the hierarchy has a different resolution. The upper levels have a lower resolution than those further down in the hierarchy. The basic problem (overall goal of the system) remains the same at all the levels in the hierarchy, but the problem description

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at each level is different. The problem is expressed in a coarser sense as we go higher in the hierarchy while the description becomes more and more detailed as we go lower down the hierarchy (Meystel and Maximov, 1993). The problem description is least detailed (coarsest) at the topmost level while it is most detailed (finest) at the lowest level in the hierarchy. Thus, we have a multiresolutional representation. In the force control problem, the servomechanism is placed at the lower level. The machining process with the cutting forces represented by (1) is at the top of the hierarchy. The dynamics at the top level are in terms of the control voltage supplied to the electric motor and the cutting force. In other words, there is a model where the control voltage directly affects the cutting force. We are interested to determine the voltage trajectory that ensures the cutting force follows a certain trajectory. At the lower level, the model is in terms of the feed and the voltage supplied to the electric motor. Thus, this description of the problem is more detailed than the description at the top where the voltage directly affects the cutting force. We are interested to determine what the voltage should be so that the feed trajectory is such that the force trajectory evaluated at the upper level is implemented. Thus, there are two different resolution levels. At the upper level, the resolution is lower as we speak in terms of the voltage affecting the cutting forces directly. At the lower level, the results of the upper level are refined as we speak in terms of the voltage affecting the feed. The lower level is at a higher resolution. Fig. 3 shows the multiresolutional aspects of the hierarchical system and Fig. 4 shows the block diagram. The upward arrow indicates 'aggregation' while the downward arrow indicates 'goal propagation.'

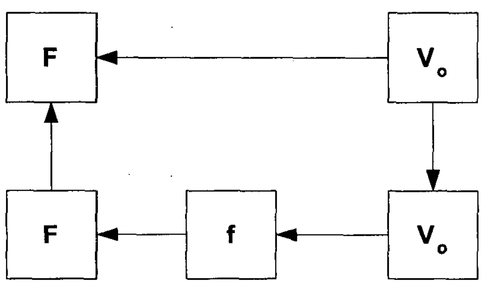

Fig. 3: Multiresolutional Aspects.

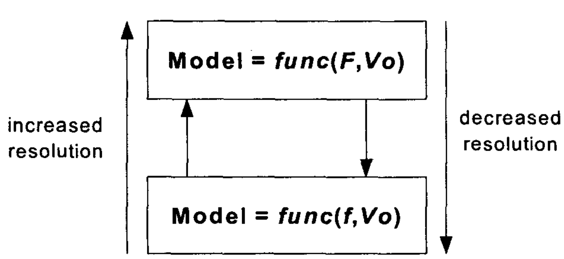

Fig. 4: Multiresolutional Diagram.

SERVOMECHANISM MODELING

This section provides a model of the servomechanism system that will be utilized in the subsequent controller design. The system consists of an interpolator that determines the reference positions along the paths specified in the part program, the controller that determines the motor voltage, and the physical servomechanism system that consists of motors, leadscrews, gears, tables, etc.

The interpolator calculates the linear axis reference positions at each sample period based on the paths and velocities specified by the part program. The transfer function of the linear motion interpolator is

$$G_I(s) = \frac{P_{ref}}{V_{L_{ref}}} = \frac{1}{s} \tag{2}$$

where $V_{L_{ref}}$ is the reference liner velocity in *mm/s* and P_{ref} is the reference position in *mm*. The reference linear velocity is related to the reference feed (f_{ref} in *mm*) and reference spindle speed (N_{sref} in *rpm*) by

$$V_{L_{ref}} = \frac{N_{sref}}{60} f_{ref} \tag{3}$$

The reference feed is calculated from (1) corresponding to the spindle speed, depth–of–cut, and maximum force. The controller (described below) outputs the motor voltage (V_0 in *V*) that drives the servomechanism. The servomechanism states are the actual cutting tool velocity (V_L in *mm/s*) and the actual cutting tool position (P_{act} in *mm*). The servomechanism transfer function, neglecting disturbance torques and electrical dynamics, is

$$G_{SM}(s) = \frac{V_L}{V_0} = \frac{K_{SM}}{\tau s + 1} \tag{4}$$

where τ is the servomechanism time constant in *s* and K_{SM} is the servomechanism gain in *(mm/s)/V*. The relationship between the servomechanism actual position and the servomechanism actual velocity is

$$\frac{P_{act}}{V_L} = \frac{1}{s} \tag{5}$$

The actual feed is

$$f = \frac{60 V_L}{N_{sref}} \tag{6}$$

The system block diagram is shown in Figure 5.

The state space realization of the servomechanism is

$$\begin{bmatrix} \dot{P}_{act} \\ \dot{f} \end{bmatrix} = \begin{bmatrix} 0 & \frac{N_{sref}}{60} \\ 0 & -\frac{1}{\tau} \end{bmatrix} \begin{bmatrix} P_{act} \\ f \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{60 K_{SM}}{\tau N_{sref}} \end{bmatrix} V_0 \tag{7}$$

Since the aggregation relation will be described in terms of the perturbed variables, the state variables chosen are the perturbed actual position $\Delta P_{act} = P_{ref} - P_{act}$ and the perturbed feed $\Delta f = f_{ref} - f$. Hence, the state space representation in terms of the perturbed state variables is

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$$\begin{bmatrix} \Delta\dot{P}_{act} \\ \Delta\dot{f} \end{bmatrix} = \begin{bmatrix} 0 & \frac{N_{sref}}{60} \\ 0 & -\frac{1}{\tau} \end{bmatrix} \begin{bmatrix} \Delta P_{act} \\ \Delta f \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{60K_{SM}}{\tau N_{sref}} \end{bmatrix} \Delta V_0 \tag{8}$$

where $\Delta V_0 = V_{0_{ref}} - V_0$ and $V_{0_{ref}} = \frac{V_{L_{ref}}}{K_{SM}}$ is the equilibrium voltage. Equation (8) may be written compactly as

$$\Delta\dot{X} = A_{bot}\Delta X + B_{bot}\Delta V_0 \tag{9}$$

where

$$\Delta X = \begin{bmatrix} \Delta P_{act} \\ \Delta f \end{bmatrix} \tag{10}$$

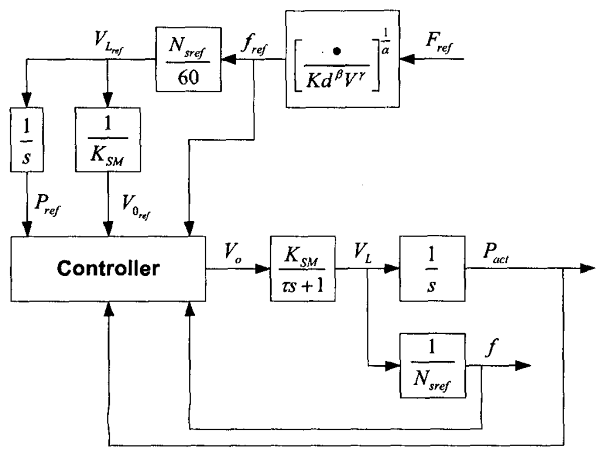

Fig. 5: System Block Diagram.

HIERARCHICAL CONTROLLER DESIGN

As stated earlier, it is assumed that the depth–of–cut and the cutting speed are constant. The equilibrium feed can be calculated from (1) corresponding to the maximum allowable force (F_{max}). If P_{max} is the maximum power that can be supplied by the spindle, the reference force and feed, respectively, are

$$F_{ref} = F_{max} = \frac{P_{max}}{V} \tag{11}$$

$$f_{ref} = \left[\frac{F_{ref}}{KV^{\gamma}d^{\beta}}\right]^{\frac{1}{\alpha}} \tag{12}$$

Linearizing (1) about the operating (i.e., equilibrium) conditions yields

$$\Delta F = \left[KV^{\gamma}d^{\beta}\alpha f_{ref}^{\alpha-1}\right]\Delta f = \Theta\Delta f \tag{13}$$

where $\Delta F = F_{max} - F$. The linearized force–feed relation (13) is used to aggregate the perturbed cutting force with the perturbed feed. The aggregation matrix is

$$C = [0 \quad \Theta] \tag{14}$$

where C is the aggregation matrix that maps the state variables of the lower level (i.e., Δf and ΔP_{act}) to the state variables of the upper level (i.e., ΔF). Thus

$$\Delta F = [0 \quad \Theta]\begin{bmatrix} \Delta P_{act} \\ \Delta f \end{bmatrix} = C\Delta X \tag{15}$$

Differentiating (15) with respect to time

$$\Delta\dot{F} = C\Delta\dot{X} \tag{16}$$

Substituting the right hand side from (9) into (16)

$$\Delta\dot{F} = CA_{bot}\Delta X + CB_{bot}\Delta V_o \tag{17}$$

Inserting (15) in (17)

$$\Delta\dot{F} = \left[CA_{bot}C^{\#}\right]\Delta F + \left[CB_{bot}\right]\Delta V_o = A_{top}\Delta F + B_{top}\Delta V_o \tag{18}$$

where $C^{\#}$ is the pseudo inverse of C. Equation (18) represents the dynamic model from the servomechanism voltage input to the cutting force. The nonlinear dynamics of the cutting force can be expressed as

$$\dot{F} = Func(A_{top}, F) + Func(B_{top}, V_0) \tag{19}$$

Theorem 1: Linearizing (19) about the equilibrium force is equivalent to linearizing the aggregation relation (1) about the reference feed in the sense of (15) and forming the linearized dynamics in the sense of (18). The proof is provided in Appendix A.

The goal at the top level (i.e., $\Delta F = 0$) is broad based and represents the overall system goal. The next step is to propagate this goal to the lower level; thus, the feed trajectory evaluated at the lower level should produce the cutting force trajectory $\Delta F = 0$. Due to the aggregation given by (13), there is a constraint on the lower level to track the trajectories of the upper level. This constraint is the goal propagation from the upper level to the lower level. The optimal control problem at the lower level can now be formulated as, minimize the following cost function

$$J_{bot} = \frac{1}{2}\left[\Delta F(t_f)^T S\Delta F(t_f)\right] + \tag{20}$$
$$\frac{1}{2}\int_0^{t_f}\left\{[\Delta F - C\Delta X]^T Q_{bot}[\Delta F - C\Delta X] + \Delta V_0^T R_{bot}\Delta V_0 + \Delta X^T\bar{Q}\Delta X\right\}dt$$

subject to the dynamics given by (9). The first term under the integral is the cost associated with satisfying the aggregation relation (13). In other words, the trajectory of ΔX should be such that the aggregation relation (13) is satisfied. The second term under the integral is the control effort cost. The third term under the integral is the cost associated with maintaining the local objectives at the bottom level.

Using optimal control principles the Hamiltonian is assumed of the form

$$H_{bot} = J_{bot} + \lambda_{bot}\left(A_{bot}\Delta X + B_{bot}\Delta V_0\right) \tag{21}$$

where

$$\lambda_{bot} = p_{bot}\Delta X + h_{bot} \tag{22}$$

is the Lagrange multiplier. The values of p_{bot} and h_{bot} are determined by integrating the following differential equations backwards in time

$$\begin{aligned} \dot{p}_{bot} &= -p_{bot}A_{bot} - A_{bot}^T p_{bot} \\ &+ p_{bot}B_{bot}R_{bot}^{-1}B_{bot}^T p_{bot} - C^T Q_{bot}C - \bar{Q} \end{aligned} \tag{23}$$

$$\dot{h}_{bot} = -A_{bot}^T h_{bot} + p_{bot}B_{bot}R_{bot}^{-1}B_{bot}^T h_{bot} + C^T Q_{bot}\Delta F(t_f) \tag{24}$$

The terminal boundary conditions (Lewis and Syrmos, 1995) are

$$p_{bot}\left(t_f\right)=\frac{\partial H_{bot}}{\partial \Delta X_{bot}}=C^T SC \tag{25}$$

$$h_{bot}\left(t_f\right)=-CS\Delta F\left(t_f\right) \tag{26}$$

To find the optimal control signal at the bottom level, (20) is partially differentiated with respect to V_0 and equated to zero. Thus, the optimal control signal is

$$\Delta V_0=-R_{bot}^{-1}B_{bot}^{T}\lambda_{bot}=-R_{bot}^{-1}B_{bot}^{T}\left[p_{bot}\Delta X+h_{bot}\right] \tag{27}$$

SIMULATION RESULTS AND DISCUSSION

In this section, simulation results of the hierarchical controller are provided. Based on the experiments of a steel part using a coated carbide insert conducted in Sandoval *et al.* (2001), the following parameters for the force process are α = 0.891, β = 0.877, γ = –0.273, K = 1.17. The maximum power is 10 *hp* (7.46 *kW*) and the operation parameters are N_{sref} = 6000 *rpm*, d = 1 *mm*, and V = 0.942 *km/min*. The servomechanism time constant and gain are τ = 0.055 *s* and K_{SM} = 20 (*mm/s*)/*V*, respectively. The value of the maximum force is F_{max} = 0.4772 *kN*, the equilibrium feed is f_{ref} = 0.3571 *mm*, the aggregation matrix is $C=\begin{bmatrix}0 & 1.186\end{bmatrix}$, and the matrices A_{bot} and B_{bot}, respectively, are

$$A_{bot}=\begin{bmatrix}0 & 100\\ 0 & -18.2\end{bmatrix} \tag{28}$$

$$B_{bot}=\begin{bmatrix}0\\ 3.64\end{bmatrix} \tag{29}$$

The weighing matrices are Q_{bot} = 2, S = 0.08, R_{bot} = 20, and $\bar{Q}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. The system is simulated for a final time of 2.5 *s* and the results are shown in Fig. 6 and Fig. 7. In Fig. 6 it is seen that the feed tracks the reference feed and, thus, the force tracks the reference force. Also, the position is tracks the reference position. Thus, the hierarchical controller is able to simultaneously meet the upper level and lower level objectives. The time histories of the elements of the P matrix and the h vector are shown in Fig. 7. It is interesting to note that the elements of the P matrix are constant as $t \rightarrow 0$ and the elements of the h vector are always zero. Since the terminal conditions for h given by (26) and the third term on the right hand side of (24) are zero, the vector h will be zero for all time. This fact makes sense since the problem was converted from a tracking problem to a regulation problem. Since the elements of the matrix P were constant as $t \rightarrow 0$, these values were used in (27) and the results are shown in Fig. 8. A comparison of Fig. 6 with Fig. 8 shows that there is no difference when using the constant values. Thus, implementation of this controller will be greatly aided.

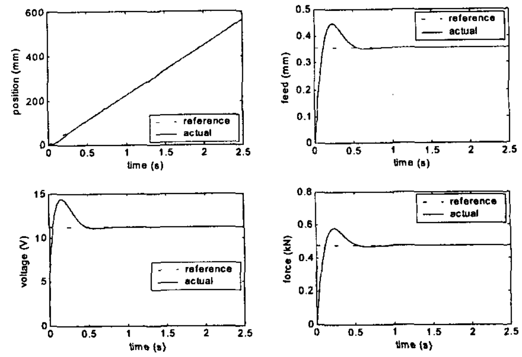

Fig. 6: Simulation Results for Time Varying *P*.

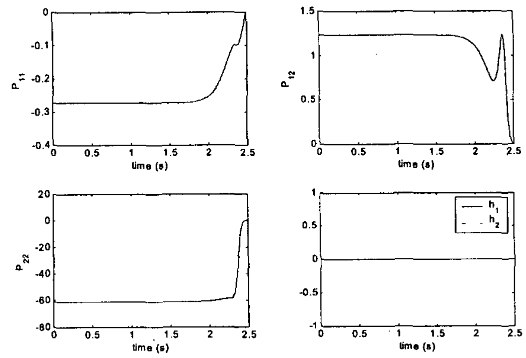

Fig. 7: Time History of *P* and *h*.

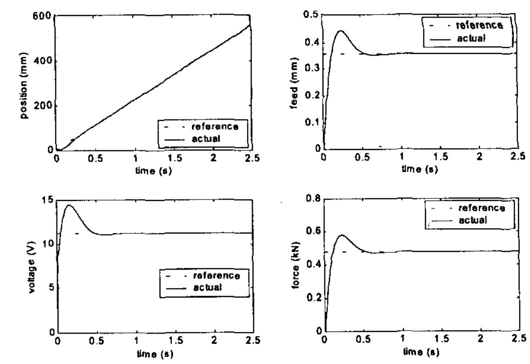

Fig. 8: Simulation Results for Constant *P*.

SUMMARY, CONCLUSIONS, AND FUTURE WORK

In this paper a hierarchical multiresolutional controller was developed to simultaneously regulate the machining forces and servomechanism position in a turning operation. Using the force–feed relation, the cutting force was aggregated from the cutting tool feed and position. An optimal control problem was solved to form a control law for the voltage trajectory that guarantees the desired force and position trajectories. Simulations were conducted to verify the developed controller. The results showed that the steady–state solution for the matrix P may be used, which will greatly aid implementation. Future work will include the investigation of robustness to parameter uncertainty and known changes in process variables, and the incorporation of contour control into the hierarchical control scheme. Experiments will also be conducted to verify the hierarchical controllers.

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APPENDIX A: THEOREM 1 PROOF

The aggregation relation (13) can be expressed as

$$\Delta\dot{F} = \begin{bmatrix} 0 & \alpha w f_{ref}^{\alpha-1} \end{bmatrix} \begin{bmatrix} \Delta\dot{P}_{act} \\ \Delta\dot{f} \end{bmatrix} \quad \text{(A1)}$$

where $w = Kd^{\beta}V^{\gamma}$. Using (9), (A1) can be rewritten as

$$\Delta\dot{F} = \begin{bmatrix} 0 & \alpha w f_{ref}^{\alpha-1} \end{bmatrix} \left[A_{bot}\Delta X + B_{bot}\Delta V_0 \right] \quad \text{(A2)}$$

Using (8) in (A2)

$$\Delta\dot{F} = \frac{1}{\tau}\Delta F + k\alpha w f_{ref}^{\alpha-1}\Delta V_0 \quad \text{(A3)}$$

where $k = \frac{60K_{SM}}{\tau N_{sref}}$. Equation (1) can be rewritten as

$$F = \left[Kd^{\beta}V^{\gamma}f^{\alpha-1} \right] f \quad \text{(A4)}$$

The time derivative of (A4) may be written as

$$\dot{F} = \begin{bmatrix} 0 & \alpha w f^{\alpha-1} \end{bmatrix} \begin{bmatrix} \dot{P}_{act} \\ \dot{f} \end{bmatrix} \quad \text{(A5)}$$

Using (7) in (A5)

$$\dot{F} = \begin{bmatrix} 0 & \alpha w f^{\alpha-1} \end{bmatrix} \left(A_{bot}X + B_{bot}V_0 \right) \quad \text{(A6)}$$

Substituting the values for A_{bot} from (7)

$$\dot{F} = -\left[\frac{\alpha w f^{\alpha-1}}{\tau} \right] f + \left[k\alpha w f^{\alpha-1} \right] V_0 \quad \text{(A7)}$$

Using (A4) in (A7)

$$\dot{F} = -\frac{\alpha}{\tau}F + \left[k\alpha w f^{\alpha-1} \right] V_0 \quad \text{(A8)}$$

Using (1) in (A8)

$$\dot{F} = -\frac{\alpha}{\tau}F + \left[k\alpha w^{\frac{1}{\alpha}} F^{\frac{\alpha-1}{\alpha}} \right] V_0 \quad \text{(A9)}$$

Linearizing (A9) about the equilibrium force, feed, and voltage

$$\Delta\dot{F} = \left[-\frac{\alpha}{\tau} + k\alpha w^{\frac{1}{\alpha}} \frac{\alpha-1}{\alpha} F_{\max}^{-\frac{1}{\alpha}} V_{0_{ref}} \right] \Delta F + \left[k\alpha w^{\frac{1}{\alpha}} F_{\max}^{\frac{\alpha-1}{\alpha}} \right] \Delta V_0 \quad \text{(A10)}$$

Using (1) in (A10)

$$\Delta\dot{F} = -\frac{1}{\tau}\Delta F + K_{SM}\alpha w f_{ref}^{\alpha-1}\Delta V_0 \quad \text{(A11)}$$

Q.E.D