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Generic Singularities of Robot Manipulators

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Abstract

The singularities of the differential kinematic map, i.e., of the manipulator Jacobian, are considered. We first examine the notion of a "generic" kinematic map, whose singularities form smooth manifolds of prescribed dimension in the joint space of the manipulator. For 3-joint robots, an equivalent condition for genericity using determinants is derived. The condition lends itself to symbolic computation and is sufficient for the study of decoupled manipulators, i.e., manipulators which can be separated into a 3-joint translating part and a 3-joint orienting part. The results are illustrated by analyzing the singularities of two classes of 3-joint positioning robots.

1 Introduction

The kinematics of robot manipulators is of importance in almost all areas of robotics, including dynamics, control and motion planning. Of particular interest is the differential kinematic map, commonly known as the manipulator Jacobian, which plays a central role in, among other things, trajectory planning, velocity and force control, and the numerical solution to the inverse kinematics problem.

Since the Jacobian is the best linear approximation to the kinematic map at a configuration, the manipulator's performance is profoundly affected by the value of the Jacobian. In particular, if the Jacobian is non-singular, the Implicit Function Theorem [1] of differential calculus guarantees a smooth right inverse to the kinematic map locally. However, if the Jacobian is singular, the kinematic map may not be smoothly invertible.

Further, the singular manipulator can not impose any velocities of the end-effector reference frame in certain directions. This causes local control methods such as Resolved Rate Control [2] and Operational Space Control [3] to fail at a singularity. The robot is also able to withstand, in principle, infinite forces along the same directions. The rank of the Jacobian is the number of degrees of freedom the end-effector of the manipulator has locally. Hence, the lower the rank of the Jacobian, the more constrained is the motion of the end-effector. Determining the sets of singular points of various ranks and the images of these singular points is thus of importance.

The problem of determining the singularities of robot manipulators has received some attention. However, the problem in its full generality is difficult. Borrel and Liegeois [4] have discussed the calculation of the set of singular points when a specific manipulator is given, i.e., when all link parameters are known and the Jacobian matrix can be computed numerically. They further show that these sets may be used for computing the workspace of a robot manipulator and for planning motions. While this is useful for analyzing specific manipulators, it yields little insight into the effect of various link parameter values on the singularities. Gorla [5] was able to get expressions for the set of singular points by assuming that link twists were multiples of $\frac{\pi}{2}$. Recently, Pai [6] examined the singularities of robot manipulators based on the genericity of the kinematic map, and classified the singularities of separable manipulators. Burdick [7] presented a detailed analysis of singularities using screw theory. Burdick also showed the significance of manipulator singularities in the design of robot manipulators.

The remainder of this paper is organized as follows. Section 2 provides the definitions of relevant terms used in the paper. Section 3 introduces the concept of a "generic" kinematic map, and examines the properties of the singularities of such maps. In Section 4 we derive an alternate criterion for genericity. This criterion is easy to apply and can be used for 3-joint manipulators with a 3-dimensional task space. The utility of the criterion is illustrated in Section 5, where it is used to analyze the singularities of PPR manipulators and SCARA type manipulators.

2 Preliminaries

In the following, a robot manipulator is taken to be any open linkage, i.e., a sequence of rigid bodies connected by joints, which are assumed to be either prismatic (sliding) or revolute (turning).

Definition 1. The **joint space** $\mathcal{J}$ of a manipulator is the space of all joint variables $(q_1, q_2, \ldots, q_n)$ of the manipulator. The variables are defined in the usual sense of Denavit and Hartenberg [8].

If the manipulator has r revolute joints and $n-r$ prismatic joints, the joint space is actually $\mathcal{T}^r \times \Re^{n-r}$, where $\mathcal{T}^r$ is an r-torus, $\mathcal{T}^r = \mathcal{S}^1 \times \cdots \times \mathcal{S}^1$. This is due to the fact that a rotation of $q_i + 2\pi$ is equivalent to a rotation of q_i. Since the distinction is not important for our purposes, we shall consider $\mathcal{J}$ to be $\Re^n$, the space of n-tuples of real numbers, with the understanding that $q_i + 2k\pi \equiv q_i$. We shall denote by $\mathbf{q} = (q_1\, q_2\, \dots\, q_n)^T$ a point in the joint space [1]. The joint space is a configuration space, i.e., by specifying $\mathbf{q}$ we completely specify the configuration of the robot. Hence, we will speak of $\mathbf{q}$ as a *configuration* of the robot.

A robot manipulator's motion is typically required in terms of the motion of a reference frame E attached to the manipulator. This is usually a coordinate frame attached to the end-effector of the manipulator [9].

Definition 2. The **task space** $\mathcal{K}$ of a manipulator is the space of all required rigid motions of E.

The task space is so called because it is the space in which the task is specified. For typical 3-dimensional tasks, the task space is the 6-dimensional space of rigid translations and rotations. This space is the manifold $\Re^3 \times \mathcal{SO}(3)$, where $\mathcal{SO}(3)$ is the manifold of the Special Orthogonal Group (the group of 3-dimensional rotations). This will be our default task space. However, the task space is actually defined by the application. For example, if one is only interested in the translation of the end-effector in the plane, the task space is $\Re^3$. An element of the task space is called a *generalized position*, or simply *position*. Note that this may have both a *translational* part belonging to $\Re^3$ and a *rotational* part belonging to $\mathcal{SO}(3)$. This is *not* standard terminology, since none exists at the present time. In the literature, positions have also been called locations, displacements, motions, configurations, transformations, etc., which unfortunately convey different meanings to different readers.

The kinematics of the manipulator defines a map from the joint space to the task space.

Definition 3. The **kinematic map** of a manipulator is the map $\kappa : \mathcal{J} \to \mathcal{K}$, which maps a configuration $\mathbf{q}$ of the robot to the position of the end-effector reference frame E.

The map can be considered to be the cartesian product of two maps,

$$\kappa = \begin{pmatrix} \kappa_t \\ \kappa_r \end{pmatrix}, \tag{1}$$

where

$$\kappa_t : \mathcal{J} \to \Re^3, \tag{2}$$

and

$$\kappa_r : \mathcal{J} \to \mathcal{SO}(3). \tag{3}$$

κ_t will be called the translation map and κ_r will be called the rotation, or orientation, map.

[1] The symbol $()^T$ denotes the transpose

The derivative $D_{\mathbf{q}}\kappa$ of the kinematic map at a configuration $\mathbf{q}$ is a linear map from the tangent space of $\mathcal{J}$ at $\mathbf{q}$ to the tangent space of $\mathcal{K}$ at $\kappa(\mathbf{q})$. When represented as a matrix in coordinates, it is commonly known as the *manipulator Jacobian*. The Jacobian matrix may be conveniently computed by the vector cross-product method [2] and other methods.

The matrix is written as

$$D_{\mathbf{q}}\kappa = \begin{pmatrix} D_{\mathbf{q}}\kappa_t \\ D_{\mathbf{q}}\kappa_r \end{pmatrix}, \tag{4}$$

where

$$\begin{aligned} D_{\mathbf{q}}\kappa_t &= (\sigma_0 \mathbf{z}_0 \times \mathbf{p}_0 + \bar{\sigma}_0 \mathbf{z}_0 \;\; \dots \\ &\qquad \sigma_{n-1}\mathbf{z}_{n-1} \times \mathbf{p}_{n-1} + \bar{\sigma}_{n-1}\mathbf{z}_{n-1}) \\ D_{\mathbf{q}}\kappa_r &= (\; \sigma_0 \mathbf{z}_0 \;\; \dots \;\; \sigma_{n-1}\mathbf{z}_{n-1} \;), \end{aligned} \tag{5}$$

$$\sigma_i = \begin{cases} 0 & \text{if joint } i+1 \text{ is prismatic} \\ 1 & \text{if joint } i+1 \text{ is revolute} \end{cases}$$

$$\bar{\sigma}_i = \begin{cases} 0 & \text{if } \sigma_i = 1 \\ 1 & \text{if } \sigma_i = 0 \end{cases}$$

Also, $\mathbf{z}_i$ is the unit vector along the axis of joint $i+1$ and $\mathbf{p}_i$ is the vector from a point on the axis of joint $i+1$ [2] to the origin of the end-effector coordinate frame E.

The kinematic map is easily derived for typical robot manipulators, for instance by using homogeneous transformations. The Jacobian can be obtained in many cases by simply differentiating the kinematic map. Paul [9] offers a method for computing $D_{\mathbf{q}}\kappa$ using homogeneous transformations.

Definition 4. A manipulator is said to be **singular** at a configuration $\mathbf{q}$ if $D_{\mathbf{q}}\kappa$ is singular, i.e., if it is not of maximal rank. The configuration $\mathbf{q}$ is then called a **singular point** and its image $\kappa(\mathbf{q})$ is called a **singular image**.

Some authors also call a singular point a *critical* point and a singular image a *critical value*, especially when dealing with real-valued maps. A point in $\mathcal{J}$ is called a *regular point* if it is not a critical point. A point in $\mathcal{K}$ is called a *regular value* if it is not the image of a singular (critical) point.

In this paper *we shall always assume that the dimension of $\mathcal{J}$ is at least as large as that of $\mathcal{K}$*, i.e., we shall deal with manipulators with at least as many degrees of freedom as required by the task. Hence, a configuration $\mathbf{q}$ is singular if and only if rank $(D_{\mathbf{q}}\kappa)$ is less than the dimension of $\mathcal{K}$.

3 Singularities of Generic Mappings

In this section we introduce the important concept of genericity of a smooth mapping and related results from

[2] Usually taken to be the origin of link i's coordinate frame.

the field of differential topology, and demonstrate their relevance to the singularities of kinematic maps. In general, the types of singular sets that can occur depend on the actual mapping. Of particular interest are mappings whose singular points form smooth manifolds in the domain. Smooth manifolds have several important properties, including the fact that they can be traced out by local methods.

Generic mappings constitute a large class of mappings whose singular points form smooth manifolds. In fact, almost all smooth maps are generic. The book by Golubitsky and Guillemin [10] provides more information for the reader. Elementary definitions of smooth manifolds, tangent spaces, etc., can be found in textbooks on differential topology such as [1].

Let $\mathcal{L}$ be the space of all linear maps from the tangent space to $\mathcal{J}$ at $\mathbf{q}$, denoted $T_q\mathcal{J}$, to the tangent space to $\mathcal{K}$ at $\kappa(\mathbf{q})$, denoted $T_{\kappa(q)}\mathcal{K}$. Let $\dim(\mathcal{J}) = j$ and $\dim(\mathcal{K}) = k$. With local coordinates on $\mathcal{J}$ and $\mathcal{K}$, $\mathcal{L}$ is isomorphic to the space of $k \times j$ matrices. Note that $\mathcal{L}$ is a vector space under scalar multiplication and addition of matrices, isomorphic to $\Re^{jk}$.

We denote by $\mathcal{L}_r$ the set of points in $\mathcal{L}$ of rank r. It is well known that each $\mathcal{L}_r$ is a manifold with $\text{codim}(\mathcal{L}_r) = (j-r)(k-r)$, where codim is the codimension of a submanifold in its containing manifold. Thus the $\mathcal{L}_r$ partition $\mathcal{L}$, i.e., with $\nu = \min\{j,k\}$,

$$\mathcal{L}_0 \cup \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_\nu = \mathcal{L}, \tag{6}$$

and

$$\mathcal{L}_r \cap \mathcal{L}_s = \emptyset \text{ for } r \neq s. \tag{7}$$

Further, the limit points of $\mathcal{L}_r$ not in it are in some $\mathcal{L}_s$, $s > r$. Such a set of manifolds is called a "manifold collection".

Definition 5. Let $\mathbf{f} : \mathcal{M} \to \mathcal{N}$ be a smooth map between manifolds $\mathcal{M}$ and $\mathcal{N}$. The map $\mathbf{f}$ is *transversal* to a submanifold $\mathcal{U}$ of $\mathcal{N}$ if and only if for each point $x \in \mathbf{f}^{-1}(\mathcal{U})$

$$\text{Image}(D_x\mathbf{f}) + T_{\mathbf{f}(x)}\mathcal{U} = T_{\mathbf{f}(x)}\mathcal{N}. \tag{8}$$

Also, $\mathbf{f}$ is transversal to a manifold collection $\{\mathcal{L}_i\}$ in $\mathcal{N}$ if and only if $\mathbf{f}$ is transversal to each $\mathcal{L}_i$.

We write $\mathbf{f} \pitchfork \mathcal{U}$ to indicate that $\mathbf{f}$ is transversal to $\mathcal{U}$. Transversality is one of the most important concepts in differential topology. We note some of the applications below.

Theorem 1 (Preimage Theorem) Let $\mathbf{f}$, $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{U}$ be as above. Then the preimage $\mathbf{f}^{-1}(\mathcal{U})$ is a submanifold of $\mathcal{M}$ and

$$\text{codim}(f^{-1}(\mathcal{U})) = \text{codim}(\mathcal{U}). \tag{9}$$

The Jacobian $D_{\mathbf{q}}\kappa$ is a linear map from $T_q\mathcal{J}$ to $T_{\kappa(q)}\mathcal{K}$. We can view the collection of $D_{\mathbf{q}}\kappa$, for all $\mathbf{q} \in \mathcal{J}$, as a map from $\mathcal{J}$ to the space of linear maps from $T_q\mathcal{J}$ to $T_{\kappa(q)}\mathcal{K}$, viz., $\mathcal{L}$. We will denote this by $D\kappa : \mathcal{J} \to \mathcal{L}$, with $D\kappa(\mathbf{q}) = D_{\mathbf{q}}\kappa$. The map $D\kappa$ is smooth.

Definition 6. A kinematic map κ of a manipulator is *generic* [3] if $D\kappa \pitchfork \{\mathcal{L}_i\}$. We shall call a manipulator generic if it has a generic kinematic map.

Proposition 1. Let $\mathcal{S}_r \subset \mathcal{J}$ be the set of all singular points of rank r and let $\kappa : \mathcal{J} \to \mathcal{K}$ be generic, with $\dim(\mathcal{J}) = j$ and $\dim(\mathcal{K}) = k$. Then, $\mathcal{S}_r$ is a smooth submanifold of $\mathcal{J}$. Further, if $\mathcal{S}_r$ is not empty

$$\text{codim}(\mathcal{S}_r) = (j-r)(k-r). \tag{10}$$

Proof. Since κ is generic, $D\kappa \pitchfork \mathcal{L}_r$, $\text{codim}(\mathcal{L}_r) = (j-r)(k-r)$. From the Preimage Theorem, $\mathcal{S}_r = D\kappa^{-1}(\mathcal{L}_r)$ is a smooth manifold of $\mathcal{J}$ and $\text{codim}(\mathcal{S}_r) = \text{codim}(\mathcal{L}_r) = (j-r)(k-r)$. □

Proposition 1 may be used to determine the dimension of sets of singular points of generic kinematic maps. Observe that one way for κ to be generic is to have no singular points at all. In this case, all $\mathcal{S}_r$ for $r < k$ will be empty. The above proposition describes the dimension of $\mathcal{S}_r$ when it is not empty. The proposition also allows us to preclude the existence of generic singularities of certain low ranks. If $(j-r)(k-r) > j$, then the codimension of $\mathcal{S}_r$ is greater than the dimension of the joint space. Hence a rank r singular point can not exist.

We examine three examples of singularities of generic kinematic maps below.

Example 1: A 3-joint generic manipulator used for translation only or orientation only. Here the dimension of the joint space j is 3 and the dimension of the task space k is also 3. Hence, if $\mathcal{S}_2$ is not empty, dim $(\mathcal{S}_2) = 2$, and both $\mathcal{S}_1$ and $\mathcal{S}_0$ have to be empty. Therefore, only the rank 2 singularity is possible.

Example 2: A 6-joint generic manipulator used for both translation and orientation. Here the dimension of the joint space is 6 and the dimension of the task space is also 6. Hence, if singularities of ranks 4 and 5 exist, dim $(\mathcal{S}_5) = 5$, dim $(\mathcal{S}_4) = 2$, and the smaller rank singular sets are empty.

Example 3: An 8-joint generic manipulator used for both translation and orientation. This robot is redundant and has two extra degrees of freedom. Here, if the manipulator can become singular, dim $(\mathcal{S}_5) = 5$, dim $(\mathcal{S}_4) = 0$ and smaller rank singularities can not occur.

4 A Condition for Genericity

We saw in Section 3 that generic mappings possess several desirable properties. However, it is difficult to determine if a map is generic using only the definition of genericity. In this section, we derive an algebraic criterion for determining if a 3-joint robot ($j = 3$) in a 3-dimensional task space ($k = 3$) is generic. This criterion lends itself well

[3] Also called *one*-generic, to distinguish it from genericity with respect to higher derivatives.

to symbolic computation, and has been implemented in MACSYMA.

The restriction to 3-joint manipulators does not limit us in analyzing the singularities of many general spatial manipulators. Almost all current manipulators can be decoupled into a 3-joint translational part and a 3-joint orienting part (see Appendix B). The translational part corresponds to the large links towards the base of the manipulator, while the orienting part corresponds to the small terminal links that constitute a wrist. Our result will allow us to analyze each part separately.

Lemma 1. Let $A = (a_{lm})$ be a $n \times n$ matrix, and A^* be the matrix of cofactors of A. A and $A^* \in \mathcal{L}$, where $\mathcal{L}$ is the space of $n \times n$ matrices. Let $\det : \mathcal{L} \rightarrow \Re$ be the determinant function. Then

$$D_A \det = \mathrm{A}^*. \tag{11}$$

Proof. Let a_{lm} be the l, m element of A. Therefore, the determinant of A can be written as

$$\det(A) = a_{lm}\mathrm{cofactor}(a_{lm}) + \text{terms not involving } a_{lm}. \tag{12}$$

Therefore,

$$\tfrac{\partial}{\partial a_{lm}}\det(A) = \mathrm{cofactor}(a_{lm}). \tag{13}$$

Hence

$$D_A \det = (\mathrm{cofactor}(a_{lm})) = \mathrm{A}^*. \tag{14}$$

□

When $j = k = n$, $\mathcal{L}$ is a manifold isomorphic to $\Re^{n^2}$. We have seen that $\mathcal{L}_{n-1}$, the set of all matrices of rank $n-1$, is a submanifold of $\mathcal{L}$ with codimension $(n-(n-1))(n-(n-1)) = 1$. Hence, there is a one-dimensional vector space normal to the tangent space of $\mathcal{L}_{n-1}$. With the identification of $\mathcal{L}$ with $\Re^{n^2}$, a vector in $\mathcal{L}$ is just another $n \times n$ matrix.

Lemma 2. The normal subspace to $\mathcal{L}_{n-1}$ at $A \in \mathcal{L}_{n-1}$ is spanned by $D_A \det = \mathrm{A}^*$.

Proof. Since $\mathcal{L}_{n-1}$ is a singular set, $\det(A) = 0$. Further, since A is of rank $n-1$, at least one cofactor of $A \neq 0$. Therefore, $D_A \det = A^* \neq 0$. So 0 is a regular value of the function det. Hence, locally $\mathcal{L}_{n-1}$ is an $n^2 - 1$ surface and $D_A \det \neq 0$ is normal to it. □

In the proof of the theorem below, we use the following identity which relates derivatives of determinants in $\mathcal{L}$ to derivatives in $\mathcal{J}$.

Lemma 3.

$$\left(\tfrac{\partial}{\partial q_i} D\kappa\right)\Big|_{\mathbf{q}} \cdot D_{\mathbf{q}}\kappa^* = \left(\tfrac{\partial}{\partial q_i}\det D\kappa\right)\Big|_{\mathbf{q}}, \tag{15}$$

where $\cdot$ is the usual inner product in $\Re^{n^2}$.

Proof. Let d_{lm} be an element of $D\kappa$ and d_m be a column.

$$\tfrac{\partial}{\partial q_i}\det[d_1 \ldots d_n]\Big|_{\mathbf{q}} \tag{16}$$

$$= \sum_m \det\left[d_1 \ldots \tfrac{\partial}{\partial q_i} d_m \ldots d_n\right]\Big|_{\mathbf{q}}$$

$$= \sum_l \sum_m \left(\tfrac{\partial}{\partial q_i} d_{lm}\right) \mathrm{cofactor}(d_{lm})\Big|_{\mathbf{q}}$$

$$= \left(\tfrac{\partial}{\partial q_i} D\kappa\right)\Big|_{\mathbf{q}} \cdot D_{\mathbf{q}}\kappa^*. \tag{17}$$

□

We now use these facts to show the main result of this section.

Theorem 2. For a 3-joint robot, with a 3-dimensional task space,

κ is generic if and only if $\forall\, \mathbf{q}$ such that

$$\det(D_{\mathbf{q}}\kappa) = 0,$$
$$D_{\mathbf{q}}\det(D\kappa) \neq 0\ .$$

Proof. By definition, κ is generic if and only if $D\kappa \pitchfork \{\mathcal{L}_i\}, i = 0,1,2,3$. $D\kappa$ is obviously transversal to $\mathcal{L}_3$. Now, $\mathrm{codim}(\mathcal{L}_0) = 9$ and $\mathrm{codim}(\mathcal{L}_1) = 4$, while $\dim(\mathrm{T}_q\mathcal{J}) = 3$. Therefore the only way for $D\kappa$ to be transversal to $\mathcal{L}_0$ and $\mathcal{L}_1$ is to avoid them altogether. Hence

genericity $\Leftrightarrow$ (a) Only rank 2 singularities can occur
AND
(b) $D\kappa \pitchfork \mathcal{L}_2$.

First look at (b). Let $\mathbf{q} \in D\kappa^{-1}(\mathcal{L}_2)$. $D\kappa \pitchfork \mathcal{L}_2 \Leftrightarrow \exists \underset{\sim}{v} \in \mathrm{T}_q\mathcal{J}$ such that $(D_{\mathbf{q}}D\kappa)(\underset{\sim}{v}) \cdot D_{\mathbf{q}}\kappa^* \neq 0$. Note that $D_{\mathbf{q}}\kappa^*$ is the normal to $\mathcal{L}_2$ at $D_{\mathbf{q}}\kappa$, from Lemma 2. This is equivalent to

$$\left(\sum_{i=1}^{3}\left(\tfrac{\partial}{\partial q_i}D\kappa\right)\Big|_{\mathbf{q}} v_i\right) \cdot D_{\mathbf{q}}\kappa^* \neq 0,$$

i.e.,

$$\sum_{i=1}^{3}\left(\left(\tfrac{\partial}{\partial q_i}D\kappa\right)\Big|_{\mathbf{q}} \cdot D_{\mathbf{q}}\kappa^*\right) v_i \neq 0,$$

i.e.,

$$\text{for some } i,\ \left(\tfrac{\partial}{\partial q_i}D\kappa\right)\Big|_{\mathbf{q}} \cdot D_{\mathbf{q}}\kappa^* \neq 0,$$

from Lemma 3,

$$\text{for some } i,\ \tfrac{\partial}{\partial q_i}\det(D\kappa) \neq 0,$$

i.e.,

$$D_{\mathbf{q}}\det(D\kappa) \neq 0.$$

Now (a) is equivalent to the condition that $D_{\mathbf{q}}\kappa^* \neq 0$ for all $\mathbf{q} \in \mathcal{J}$. This is because each 2×2 submatrix of $D_{\mathbf{q}}\kappa$ is an element of $D_{\mathbf{q}}\kappa^*$; if they are all zero, $D_{\mathbf{q}}\kappa$ has rank less than 2. Clearly, $D_{\mathbf{q}}\kappa^* \neq 0$ when the robot is non-singular. When the robot is singular, i.e., when $\det(D_{\mathbf{q}}\kappa) = 0$, if $D_{\mathbf{q}}\det(D\kappa) \neq 0$, there exists an i such that $\left(\tfrac{\partial}{\partial q_i}D\kappa\right)\Big|_{\mathbf{q}} \cdot D_{\mathbf{q}}\kappa^* \neq 0$. Hence $D_{\mathbf{q}}\kappa^* \neq 0$. □

Corollary 1. For 3-joint robots, genericity implies that the set of singular points is either empty or a regular level surface of dimension 2.

5 Examples

The genericity condition of Theorem 2 has many applications, including the classification of singularities for separable manipulators (see Pai [6]). In this section, we present only two examples to illustrate the utility of the condition. First, we show that the singularities of all non-trivial PPR positioning manipulators are generic. Second, we derive a necessary and sufficient condition for genericity for a SCARA-type positioning manipulator (i.e., an RRP manipulator with the first two revolute joints parallel). Appendix A describes the notation used in this section for the kinematic parameters, which is similar to that of Paul [9].

5.1 PPR manipulator singularities

This manipulator has two prismatic joints followed by a terminal revolute joint. The joint variables are d_1, d_2, and θ_3. The expression for the determinant of the Jacobian simplifies to:

$$\det D_{\mathbf{q}}\kappa_t = a_3 \sin\alpha_1 \left(\cos\theta_2 \sin\theta_3 + \cos\alpha_2 \sin\theta_2 \cos\theta_3\right). \quad (18)$$

The factor a_3 indicates that if $a_3 = 0$, the end-effector point lies on the axis of revolution of joint 3. Joint 3 can not contribute any translational velocity to the end-effector in this case. Also, $\sin\alpha_1 = 0$ makes the two prismatic joints parallel. Therefore, unless the manipulator is always singular, $a_3 \sin\alpha_1 \neq 0$. The only interesting factor is $\cos\theta_2 \sin\theta_3 + \cos\alpha_2 \sin\theta_2 \cos\theta_3$.

Unless it is identically zero, the zero sets of Equation 18 are singular planes in the d_1–d_2–θ_3 space, at $\theta_3 = \tan^{-1}(-\cos\alpha_2 \tan\theta_2)$, separated in θ_3 by π. These are generic since

$$\begin{aligned} & D_{\mathbf{q}} \det D\kappa_t \\ = \;& a_3 \sin\alpha_1 \begin{pmatrix} 0 \\ 0 \\ \cos\theta_2\cos\theta_3 - \cos\alpha_2 \sin\theta_2 \sin\theta_3 \end{pmatrix} \\ \neq \;& 0, \end{aligned} \quad (19)$$

when Equation 18 is 0. Hence all non-trivial singularities are generic.

5.2 SCARA-type manipulator singularities

This is a manipulator with two revolute joints followed by a terminal prismatic joint. In addition, the first two revolute joints are parallel. The joint variables are θ_1, θ_2, and d_3. The determinant of the Jacobian simplifies to

$$\det(D_{\mathbf{q}}\kappa_t) = a_1 \cos\alpha_2 (a_2 \sin\theta_2 - \sin\alpha_2 d_3 \cos\theta_2). \quad (20)$$

The case of $a_1 \cos\alpha_2 = 0$ is trivial, since the manipulator is always singular. Hence we only need to analyze the factor $(a_2 \sin\theta_2 - \sin\alpha_2 d_3 \cos\theta_2)$.

From Theorem 2, the kinematic map is non-generic if and only if there is a simultaneous solution to the following equations:

$$a_2 \sin\theta_2 - \sin\alpha_2 d_3 \cos\theta_2 = 0, \quad (21)$$

$$a_2 \cos\theta_2 + \sin\alpha_2 d_3 \sin\theta_2 = 0, \quad (22)$$

$$\sin\alpha_2 \cos\theta_2 = 0. \quad (23)$$

Hence the manipulator is non-generic if and only if $a_2 = 0$, i.e., if and only if the axes of joints 2 and 3 intersect.

6 Conclusions

Some results from differential topology were applied to the manipulator singularity problem. We showed that for the class of generic robots, important information could be obtained about the ranks of the possible singularities and the differential topology of the set of singular points. We saw that for generic robots, the set of singular points of rank r are smooth manifolds in joint space of codimension $(j-r)(k-r)$, where j is the dimension of joint space and k is the dimension of task space. This result also allows us to automatically exclude certain low-rank singularities from occurring in generic robots. Hence by designing a robot to be generic, we can eliminate singularities of low rank.

Since generic singularities are so well behaved, the question naturally arises: what types of robots are generic? Genericity was originally defined in terms of transversality of the map $D\kappa$ to a manifold collection in the space $\mathcal{L}$ of all $k \times j$ matrices. This definition is not well suited for determining the values of the kinematic parameters which cause the manipulator to be generic. An equivalent condition for genericity of 3-joint manipulators in a 3-dimensional task space was derived (Theorem 2). The condition uses the determinant of the Jacobian and its derivatives, and is amenable to symbolic computation. It is directly applicable to the common class of robots which can be decoupled into a translating part and an orienting part. All 6-joint manipulators with a so-called "spherical wrist" are of this class. The condition for genericity can be used to analyze the singularities of such robot manipulators, as demonstrated in Section 5.

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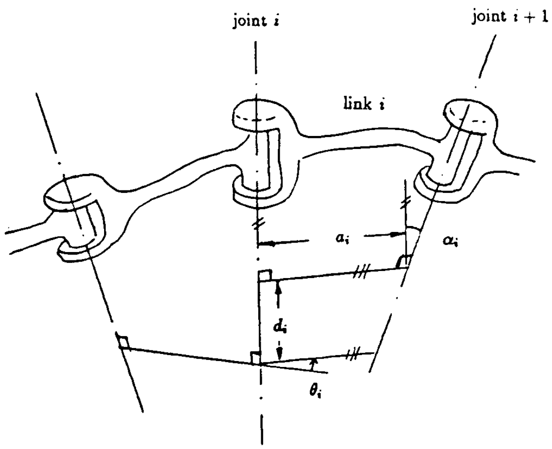

Figure 1: Kinematic parameters of a link

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Appendices

A Kinematic Parameter Convention

The convention used for the kinematic parameters of robot manipulators is that of Paul [9] and is based on the work of Denavit and Hartenberg [8]. Figure 1 summarizes the convention.

Each joint is assigned an axis. For a revolute joint, the axis is uniquely defined as the axis of revolution of the joint. In the case of prismatic joints, only the direction of the axis is uniquely defined and the axis is taken to pass through any convenient point. The links are numbered starting from 0 for the base (fixed) link. The joints are numbered starting from 1 for the first joint.

- a_i: The *length* of link i, defined as the shortest distance between the axis of joint i and the axis of joint $i+1$.
- α_i: The *twist* of link i, defined as the angle between the axis of joint i and the axis of joint $i+1$.
- d_i: The *offset* of link i, defined as the distance along the axis of joint i, between the foot of the common normal to joint $i-1$ and the foot of the common normal to joint $i+1$.
- θ_i: The *angle* of joint i, defined as the angle between the common normal to joint $i-1$ and the common normal to joint $i+1$.

It is also customary to associate a coordinate frame with each link. Reference [9] describes the specification of the link coordinate frames. For our purposes, the most important feature of the coordinate frame of link i is that the unit vector $\mathbf{z}_i$ is aligned with the axis of joint $i+1$. If joint $i+1$ is revolute, a right hand rotation about $\mathbf{z}_i$ corresponds to a positive rotation of θ_{i+1}; if the joint is prismatic, a displacement along $\mathbf{z}_i$ corresponds to increasing d_{i+1}.

B Decoupling

We have seen that an n-joint robot operating in the task space $\Re^3 \times SO(3)$ is singular if and only if the $6 \times n$ Jacobian matrix $D_{\mathbf{q}}\kappa$ is singular. It is clear that the singular points of the two $3 \times n$ matrices $D_{\mathbf{q}}\kappa_t$ and $D_{\mathbf{q}}\kappa_r$ are subsets of the singular points of $D_{\mathbf{q}}\kappa$. A question that naturally arises is: can the translation and rotation singularities be decoupled? The answer is generally "no". However, for certain extremely common manipulator designs, the decoupling is possible [11].

Consider a 6-joint robot in which the last 3 joints are revolute and intersect at a point W. This design is widespread since it makes the inverse kinematics solution in closed form tractable [4] [12]. For such a manipulator, we can take $\mathbf{p}_3 = \mathbf{p}_4 = \mathbf{p}_5 =$ vector from W to the end-effector origin. Now,

$$\begin{aligned} D_{\mathbf{q}}\kappa_t &= (\sigma_0 \mathbf{z}_0 \times \mathbf{p}_0 + \bar{\sigma}_0 \mathbf{z}_0 \ \ldots \\ &\qquad \mathbf{z}_3 \times \mathbf{p}_5 \ \ldots \ \mathbf{z}_5 \times \mathbf{p}_5). \end{aligned} \tag{24}$$

Hence, $D_{\mathbf{q}}\kappa$ can be written as

$$\begin{aligned} D_{\mathbf{q}}\kappa &= \begin{pmatrix} I & [\mathbf{p}_5] \\ 0 & I \end{pmatrix} \times \\ &\begin{pmatrix} \sigma_0 \mathbf{z}_0 \times \mathbf{p}_0' + \bar{\sigma}_0 \mathbf{z}_0 & \ldots & \sigma_2 \mathbf{z}_2 \times \mathbf{p}_2' + \bar{\sigma}_2 \mathbf{z}_2 & 0 & \ldots & 0 \\ \mathbf{z}_0 & \ldots & \mathbf{z}_2 & \mathbf{z}_3 & \mathbf{z}_4 & \mathbf{z}_5 \end{pmatrix}, \end{aligned} \tag{25}$$

where

$$[\mathbf{p}_5] = \begin{pmatrix} 0 & -\mathbf{p}_{5_z} & \mathbf{p}_{5_y} \\ \mathbf{p}_{5_z} & 0 & -\mathbf{p}_{5_x} \\ -\mathbf{p}_{5_y} & \mathbf{p}_{5_x} & 0 \end{pmatrix}, \tag{26}$$

I is the 3×3 identity matrix and the $\mathbf{p}_i'$ are vectors from a point on the axes of joint $i+1$ to W. Therefore, $D_{\mathbf{q}}\kappa$ is singular if and only if

$$\begin{aligned} &\left(\sigma_0 \mathbf{z}_0 \times \mathbf{p}_0' + \bar{\sigma}_0 \mathbf{z}_0 \ \ \ldots \ \ \sigma_2 \mathbf{z}_2 \times \mathbf{p}_2' + \bar{\sigma}_2 \mathbf{z}_2 \right) \\ &\text{or} \\ &\left(\mathbf{z}_3 \ \ \mathbf{z}_4 \ \ \mathbf{z}_5 \right) \end{aligned} \tag{27}$$

is singular.

Therefore, such a manipulator can be treated as two separate manipulators. The first 3 joints constitute a translating robot, used to locate the wrist point W, and the last 3 joints serve as an orienting robot. The singularities of these "sub-manipulators" can be studied separately.

[4] A fact linked to the decoupling of the translation and rotation functions of the manipulator.

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