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Monotonic Convergence of Iterative Learning Control for Uncertain Systems Using a Time-Varying Filter

Douglas A. Bristow and Andrew G. Alleyne

***Abstract*—Iterative learning control (ILC) is a learning technique used to improve the performance of systems that execute the same task multiple times. Learning transient behavior has emerged as an important topic in the design and analysis of ILC systems. In practice, the learning control is often low-pass filtered with a "*Q*-filter" to prevent transient growth, at the cost of performance. In this note, we consider linear time-invariant, discrete-time, single-input single-output systems, and convert frequency-domain uncertainty models to a time-domain representation for analysis. We then develop robust monotonic convergence conditions, which depend directly on the choice of the *Q*-filter and are independent of the nominal plant dynamics. This general result is then applied to a class of linear time-varying *Q*-filters that is particularly suited for precision motion control.**

***Index Terms*—Iterative learning control (ILC), monotonic convergence, motion control, robust, time-varying filters.**

I. Introduction

Iterative learning control (ILC) [1]–[3] uses the tracking errors from previous iterations of repeated motion to generate a feedforward control signal for use on subsequent iterations. Convergence of the learning process results in a feedforward control signal that is customized for the repeated motion, yielding very low tracking error.

Early work on ILC focused on first-order learning algorithms with a single filter

$$u_{j+1}(k) = u_j(k) + L(q)\, e_j(k) \tag{1}$$

where u is the control, e is the error, k is the discrete-time index, j is the iteration index, q is the forward time-shift operator $qx(k) \stackrel{\Delta}{=} x(k+1)$, and $L(q)$ is a linear time-invariant (LTI) filter called the learning function. Frequency-domain analysis reveals that the robustness of (1) to model uncertainty is limited [4], which can result in large transient growth or instability when high frequencies propagate through the learning. To improve robustness, a second LTI filter is often used [1]–[4], resulting in

$$u_{j+1}(k) = Q(q)\,(u_j(k) + L(q)e_j(k)) \tag{2}$$

where $Q(q)$ is called the Q-filter. The Q-filter is a low-pass filter that is added in a second design step, after designing $L(q)$ [4].

Recently, linear time-varying (LTV) Q-filters have demonstrated improved performance versus the LTI Q-filters while maintaining good transient learning behavior [5]. In this note, we examine the robustness of the LTV Q-filters in terms of monotonic convergence, a condition used to guarantee good transient learning behavior. Unlike many ILC papers that treat the performance problem, in this note, we focus solely on convergence transients. First, frequency-domain uncertainty models of the plant are converted into time-domain uncertainty models for time-domain analysis of the ILC system. Robust monotonic convergence conditions are developed in the time domain for the general LTV Q-filters. Second, a class of LTV Q-filters that is particularly suited for precision motion control is presented. The structure of this class of filters is exploited to obtain robust monotonic convergence conditions that provide insight into tradeoffs associated with the LTV Q-filter bandwidth design. This note differs from other recent studies on time-varying filters for monotonic ILC convergence, such as [6]–[8], in that we consider the robustness of the monotonicity with respect to a frequency-domain uncertainty model, whereas in [6] and [7], time-domain uncertainty models are considered, and in [8], uncertainty is not considered.

The rest of this note is organized as follows. The class of uncertain systems is described and the ILC problem is defined in Section II. In Section III, sufficient conditions for robust monotonic convergence with a general Q-filter are developed. In Section IV, a class of LTV Q-filters is presented, and robust monotonicity of this class is investigated. Finally, a discussion of the results is given in Section VI.

II. Problem Setup

Here, we consider a discrete-time framework because ILC requires the storage of signals from iteration to iteration, which is done digitally. Additionally, a single-input single-output (SISO) LTI system is considered because of its importance to precision motion control applications. Consider the discrete-time LTI SISO system written in convolution form as

$$y(k+m,j) = \sum_{r=0}^{k} p_r u(k-r,j) + d(k+m) \tag{3}$$

where $k = 1,\ldots,N$ is the time index, $j \in 0,1,\ldots$ is the iteration index, $y(k+m,j) \in R$, $u(k,j) \in R$, $d(k+m) \in R$, and m is the discrete delay. We assume the following.

A1) *Model uncertainty assumptions*: The Markov parameters of the system p_r are unknown, but $P(z) = \sum_{r=0}^{\infty} p_r z^{-r} = \hat{P}(z)(1+W(z)\Delta(z))$, where $\hat{P}(z) = \sum_{r=0}^{\infty} \hat{p}_r z^{-r}$, $W(z) = \sum_{r=0}^{\infty} w_r z^{-r}$, and $\Delta(z) = \sum_{r=0}^{\infty} \delta_r z^{-r}$. $P(z)$, $\hat{P}(z)$, $W(z)$, and $\Delta(z)$ are stable. $\hat{p}_r$ and w_r are known, while δ_r's are unknown, and $\|\Delta(z)\|_\infty \leq 1$.

A2) *Relative degree assumptions*: $m \geq 0$ and $\hat{P}(z)$ has zero relative degree, or $\hat{p}_0 \neq 0$.

A3) *Repeatability assumptions*: The signal $d(k)$, which includes external disturbances [9], nonzero initial conditions [4], and internal feedback control [9], is iteration-invariant. The desired trajectory $y_d(k+m) \in R$ is iteration-invariant.

A4) *First-order ILC assumption*: The ILC is the time-varying generalization of (2)

$$u(k,j+1) = \sum_{i_2=1}^{N} \lambda_{k,i_2} \left(u(i_2,j) + \sum_{i_1=1}^{N} l_{k,i_1} e(i_1+m,j) \right) \tag{4}$$

where $e(k+m,j) \in R$ is the error, given by

$$e(k+m,j) = y_d(k+m) - y(k+m,j) \tag{5}$$

and λ_{k,i_1} and l_{k,i_2} are the Q-filter and learning function parameters, respectively.

Remark 1: Assumption A1) is a standard uncertainty model for neglected and unmodeled dynamics [10]. The first part of A2) ensures that (3) is causal, while the second part ensures that $\hat{p}_r$ and m are uniquely defined. Otherwise, if $\hat{p}_s$, $s \neq 0$, is the first nonzero $\{\hat{p}_r\}$, then m and $\hat{p}_r$ can be redefined as $m_{\rm new} = m + s$ and $\hat{p}_{r,\rm new} = \hat{p}_{r+s}$, $r = 0,1,\ldots$.

Next, convert the class of systems (3) and learning algorithm (4) into a lifted system representation. Let $a_r \in R$, $b_{r,s} \in R$, $r,s = 1,\ldots,N$.

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Define the lower triangular Toeplitz operator $T(\cdot)$ as $T([a_r]) = [b_{r,s}]$, where $b_{r,s} = a_{r-s}$ for $r \geq s$ and $b_{r,s} = 0$ for $r < s$. Equation (3) becomes

$$\mathbf{y}_j = \mathbf{P}\mathbf{u}_j + \mathbf{d} \tag{6}$$

where

$$\begin{aligned}\mathbf{P} &= T([p_0 \cdots p_{N-1}]^T)\\ \mathbf{y}_j &= [y(m+1,j)\cdots y(m+N,j)]^T\\ \mathbf{u}_j &= [u(1,j)\cdots u(N,j)]^T\\ \mathbf{d} &= [d(m+1,j)\cdots d(m+N,j)]^T.\end{aligned}$$

Equation (4) can be written as

$$\mathbf{u}_{j+1} = \mathbf{Q}(\mathbf{u}_j + \mathbf{L}\mathbf{e}_j), \qquad \mathbf{e}_j = \mathbf{y_d} - \mathbf{y}_j \tag{7}$$

where $\mathbf{y_d} = [y_d(m+1)\cdots y_d(m+N)]^T$, $\mathbf{Q} = [\lambda_{k,i}]$, and $\mathbf{L} = [l_{k,i}]$, $k, i = 1, 2, \ldots, N$.

It is easy to verify that

$$\mathbf{P} = \hat{\mathbf{P}}(\mathbf{I} + \mathbf{W}\boldsymbol{\Delta}) \tag{8}$$

where

$$\begin{aligned}\hat{\mathbf{P}} &= T([\hat{p}_0 \cdots \hat{p}_{N-1}]^T)\\ \mathbf{W} &= T([w_0 \cdots w_{N-1}]^T)\\ \boldsymbol{\Delta} &= T([\delta_0 \cdots \delta_{N-1}]^T).\end{aligned}$$

As in [1] and [11], we select $\mathbf{L}$ as the inverted plant model

$$\mathbf{L} = \hat{\mathbf{P}}^{-1} \tag{9}$$

since the nominal plant model is available for the class of systems (3). Note that $\hat{\mathbf{P}}$ is nonsingular since $\hat{p}_0 \neq 0$. If $\mathbf{u}_j$ converges, we define its fixed point as $\mathbf{u}_\infty \triangleq \lim_{j\to\infty} \mathbf{u}_j$.

Definition 1: Equation (4) is monotonically convergent if $\|\delta\mathbf{u}_{j+1}\|_2 < \|\delta\mathbf{u}_j\|_2$, where $\delta\mathbf{u}_j \triangleq \mathbf{u}_\infty - \mathbf{u}_j$.

Remark 2: Monotonic convergence of $\|\delta\mathbf{u}_j\|_2$ does not necessarily give monotonic convergence of the system error, although $\|\delta\mathbf{u}_j\|_2$ bounds the error growth. For example, $\|\delta\mathbf{e}_j\|_\infty \leq \|\delta\mathbf{e}_j\|_2 \leq \|\mathbf{P}\|_2\|\delta\mathbf{u}_j\|_2$, where $\delta\mathbf{e}_j \triangleq \mathbf{e}_\infty - \mathbf{e}_j$ and $\mathbf{e}_\infty \triangleq \mathbf{y_d} - \mathbf{P}\mathbf{u}_\infty - \mathbf{d}$.

A. Problem Statement

Let (3) satisfy assumptions A1)–A3) and let the learning algorithm be given by (4) and (9). Determine $\mathbf{Q}$ that will result in monotonic convergence. Then, structure $\mathbf{Q}$ as in Section IV and reexamine monotonic convergence.

III. Robust Monotonic Learning Convergence

The iteration-domain dynamics of $\mathbf{u}_j$ can be found using (6)–(9)

$$\mathbf{u}_{j+1} = -\mathbf{Q}\mathbf{W}\boldsymbol{\Delta}\mathbf{u}_j + \mathbf{Q}\hat{\mathbf{P}}^{-1}(\mathbf{y_d} - \mathbf{d}). \tag{10}$$

The ILC converges if the eigenvalues of $\mathbf{Q}\mathbf{W}\boldsymbol{\Delta}$ are inside the unit circle. If the ILC converges, the fixed point can be found as

$$\mathbf{u}_\infty = [\mathbf{I} + \mathbf{Q}\mathbf{W}\boldsymbol{\Delta}]^{-1}\mathbf{Q}\hat{\mathbf{P}}^{-1}(\mathbf{y_d} - \mathbf{d}). \tag{11}$$

The following theorem offers some relationships between the time- and frequency-domain representations that will be useful for characterizing Δ.

Theorem 1 ([9], [12]): Let $F(z) = f_0 + f_1 z^{-1} + f_2 z^{-2} + \cdots$ be stable and SISO. Then

$$\bar{\sigma}(\mathbf{F}_N) \leq \bar{\sigma}(\mathbf{F}_{N+1}) \leq \sup_{\theta\in[-\pi,\pi]} |F(e^{j\theta})|$$

where $\mathbf{F}_N = T([f_0\ f_1 \cdots f_{N-1}]^T)$.

Using Theorem 1, we can now summarize three properties of the unknown matrix $\boldsymbol{\Delta}$ as follows.

P1) $\boldsymbol{\Delta}$ is lower triangular.
P2) $\boldsymbol{\Delta}$ is Toeplitz.
P3) $\bar{\sigma}(\boldsymbol{\Delta}) \leq 1$.

Theorem 2: The ILC is monotonically convergent for $\boldsymbol{\Delta}$ with properties P1)–P3) if

$$\bar{\sigma}(\mathbf{Q}\mathbf{W}) < 1. \tag{12}$$

Proof: We first assume that $\mathbf{Q}\mathbf{W}\boldsymbol{\Delta}$ has eigenvalues inside the unit circle. Then, $\mathbf{u}_\infty$ exists and the $\delta\mathbf{u}_j$ dynamics are given by

$$\delta\mathbf{u}_{j+1} = -\mathbf{Q}\mathbf{W}\boldsymbol{\Delta}\delta\mathbf{u}_j \tag{13}$$

which can be verified by substituting for $\delta\mathbf{u}_j$ and rearranging to agree with (10). Since $\bar{\sigma}(\boldsymbol{\Delta}) \leq 1$, (12) implies $\bar{\sigma}(\mathbf{Q}\mathbf{W}\boldsymbol{\Delta}) < 1$, which gives monotonic convergence and also that $\mathbf{Q}\mathbf{W}\boldsymbol{\Delta}$ has eigenvalues inside the unit circle. ∎

It is well known that the higher the bandwidth of the Q-filter, the lower the asymptotic tracking error will tend to be [4]. Therefore, we suggest as a heuristic that the Q-filter with the highest bandwidth satisfying (12) should be used. Equation (12) implies that robust monotonicity is dependent on the size and structure of the uncertainty, the choice of the Q-filter, and the iteration length. The following example demonstrates how Theorem 2 can be used to maximize the Q-filter bandwidth while achieving robust monotonicity.

Example 1: Consider the uncertainty bound $W(z) = 1 - z^{-1}$ and the noncausal low-pass Q-filter $Q(z) = cz^{-1} + (1-2c) + cz$, where $c \in [0, 0.5]$ determines the bandwidth. For $N = 3$

$$\mathbf{W} = \begin{bmatrix} 1 & 0 & 0\\ -1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{bmatrix} 1-2c & c & 0\\ c & 1-2c & c\\ 0 & c & 1-2c\end{bmatrix}.$$

We use $N = 3$ for illustrative purposes only; N is typically larger in practice. By searching over c, we find that the highest bandwidth that satisfies (12) occurs when $c = 0.14$. If we consider a longer iteration length, such as $N = 4$, and calculate $\bar{\sigma}(\mathbf{Q}\mathbf{W})$, we find that $c = 0.14$ no longer satisfies (12) and a lower bandwidth must be selected. This example shows that the iteration length can affect the performance when robust monotonic convergence is enforced as a design requirement.

IV. Monotonic Convergence for a Specific Time-Varying Q-Filter

Here, we consider a structure of the Q-filter [13], with the following assumptions that are particularly suited to precision motion control applications.

A5) *Structure assumption*: The Q-filter consists of long segments with low bandwidth and short segments with high bandwidth. Low-bandwidth segments are referred to as β-segments and high-bandwidth segments as α-segments. We assume n_α α-segments and $n_\alpha + 1$ β-segments are arranged so that segments alternate as $\beta_1, \alpha_1, \beta_2, \ldots, \alpha_{n_\alpha}, \beta_{n_\alpha+1}$. The hth α-segment begins at time $k = T_h$ and has length $N_{a,h}$. The hth β-segment has length $N_{\beta,h}$.

A6) *FIR assumption*: The Q-filter is finite impulse response (FIR) with causal support $N_{Qa} \ll N$ and noncausal support $N_{Qb} \ll N$. That is, $\lambda_{k,i} = 0$ for $k - i > N_{Qa}$ and $k - i < -N_{Qb}$.

A7) *β-segment assumption*: The Q-filter has the bandwidth Ω_0 during β-segments. We define $\hat{\lambda}_r$, $r = -N_{Qb}, \ldots, N_{Qa}$ such that $Q_\beta(z) = \sum_{r=-N_{Qb}}^{N_{Qa}} \hat{\lambda}_r z^{-r}$ has a bandwidth Ω_0 and assume $\lambda_{k,i} = \hat{\lambda}_{k-i}$ for rows k of $\mathbf{Q}$ that are β-segments.

Remark 3: The Q-filter structure given in A5) is crucial for precision motion control profiles such as stepping motion profiles [14]–[16], where rapid motion is combined with low-tracking-error requirements. The α-segments provide high-bandwidth learning, while robustness to high-frequency learning transient growth is provided by the low-bandwidth β-segments.

In the lifted system representation, this class of Q-filters is given by

$$\mathbf{Q}_\beta \triangleq \mathrm{T}\left(\left[\frac{1}{2}\hat{\lambda}_0 \quad \hat{\lambda}_1 \cdots \hat{\lambda}_{N_{Qa}} \quad \mathbf{0}_{1xN-N_{Qa}-1}\right]^T\right) + \mathrm{T}^T\left(\left[\frac{1}{2}\hat{\lambda}_0 \quad \hat{\lambda}_{-1} \cdots \hat{\lambda}_{-N_{Qb}} \quad \mathbf{0}_{1xN-N_{Qb}-1}\right]^T\right) \quad (14)$$

and $\mathbf{Q}_\alpha \triangleq \mathbf{Q} - \mathbf{Q}_\beta$.

The robust monotonic convergence condition (12) can be written as

$$\bar{\sigma}(\mathbf{QW}) \le \bar{\sigma}(\mathbf{Q}_\beta \mathbf{W}) + \bar{\sigma}(\mathbf{Q}_\alpha \mathbf{W}) < 1 \quad (15)$$

which allows for a design tradeoff between the bandwidth of the β-segments in $\bar{\sigma}(\mathbf{Q}_\beta\mathbf{W})$ and α-segments in $\bar{\sigma}(\mathbf{Q}_\alpha\mathbf{W})$.

Let $\mathbf{Q}_{\alpha,h}$, $h = 1, 2, \ldots, n_a$ be a submatrix of $\mathbf{Q}_\alpha$ containing the rows T_h to $T_h + N_{\alpha,h} - 1$ and columns $T_h - N_{Qa}$ to $T_h + N_{Qb} + N_{\alpha,h} - 1$ as

$$\mathbf{Q}_{\alpha,h} = \mathbf{Q}_\alpha\left(T_h : T_h + N_{\alpha,h} - 1, T_h - N_{Qa} : T_h + N_{Qb} + N_{\alpha,h} - 1\right). \quad (16)$$

All nonzero elements of $\mathbf{Q}_\alpha$ are contained in $\mathbf{Q}_{\alpha,h}$, $h = 1, 2, \ldots, n_a$.

We note that because $W(z)$ is stable, and thus, w_r decays exponentially, $\mathbf{Q}_\alpha\mathbf{W}$ is approximately block diagonal. To separate the nonblock-diagonal terms, we approximate the IIR $W(z)$ dynamics as FIR dynamics. There exists a convergent geometric sequence that upper bounds the impulse response of $W(z)$ as

$$|w_r| \le \kappa_W (\gamma_W)^r, \qquad \text{for } r = N_W, \ldots, N \quad (17)$$

where $\kappa_W > 0$, $\gamma_W \in [0, 1)$, and N_W is the length of the FIR approximation. Let

$$\mathbf{W}_{\mathrm{FIR}} \triangleq \mathrm{T}([w_0 \cdots w_{N_W-1} \quad \mathbf{0}_{1xN-N_W}]^T) \quad (18)$$

and

$$\mathbf{W}_\varepsilon \triangleq T([\mathbf{0}_{1xN_W} \quad w_{N_W} \cdots w_{N-1}]^T) \quad (19)$$

so $\mathbf{W} = \mathbf{W}_{\mathrm{FIR}} + \mathbf{W}_\varepsilon$.

The product $\mathbf{Q}_\alpha\mathbf{W}_{\mathrm{FIR}}$ is composed of blocks $\mathbf{Q}_{\alpha,h}\mathbf{W}_{\mathbf{FIR},h}$, $h = 1, 2, \ldots, n_a$, and zeros, where

$$\mathbf{W}_{\mathbf{FIR},h} \triangleq \mathrm{T}^T\left([w_{N_W-1} \cdots w_0 \quad \mathbf{0}_{1xN_{Qa}+N_{Qb}+N_{\alpha,h}}]^T\right). \quad (20)$$

Theorem 3: If

1) the spacing requirement

$$\min_{h=1,\ldots,n_\alpha+1} \{N_{\beta,h}\} \ge N_{Qa} + N_{Qb} + N_W - 1 \quad (21)$$

2) the starting requirement

$$T_1 \ge N_{Qa} + N_W \quad (22)$$

3) the ending requirement

$$T_{n_\alpha} \le N - N_{\alpha,n_\alpha} - N_{Qb} + 1 \quad (23)$$

are satisfied, and

$$\underbrace{\bar{\sigma}(\mathbf{Q}_\beta\mathbf{W})}_{\beta-\text{segments}} + \underbrace{\max_{h\in[1,n_\alpha]}\{\bar{\sigma}(\mathbf{Q}_{\alpha,h}\mathbf{W}_{\mathbf{FIR},h})\}}_{\alpha-\text{segments}} + \underbrace{\max_{h\in[1,n_\alpha]}\left\{\|\mathbf{Q}_{\alpha,h}\|_\infty\right\}\frac{\kappa_W(\gamma_W)^{N_W}\sqrt{\sum_{r=1}^{m} N_{\alpha,r}}}{1-\gamma_W}}_{W(z)\text{ FIR approximation error}} \le 1 \quad (24)$$

then the ILC is monotonically convergent.

Proof: From Theorem 2, robust monotonic convergence is assured if $\bar{\sigma}(\mathbf{QW}) \le 1$. Now,

$$\bar{\sigma}(\mathbf{QW}) \le \bar{\sigma}(\mathbf{Q}_\beta\mathbf{W}) + \bar{\sigma}(\mathbf{Q}_\alpha\mathbf{W}_{\mathbf{FIR}}) + \bar{\sigma}(\mathbf{Q}_\alpha\mathbf{W}_\varepsilon).$$

It can be shown that if $\mathbf{A}$ is an $N \times N$ matrix with M nonzero rows, then $\bar{\sigma}(\mathbf{A}) \le \sqrt{M}\,\|\mathbf{A}\|_\infty$. Therefore,

$$\begin{aligned}\bar{\sigma}(\mathbf{Q}_\alpha\mathbf{W}_\varepsilon) &\le \sqrt{\sum\nolimits_{h=1}^{n_\alpha} N_{\alpha,h}}\,\|\mathbf{Q}_\alpha\mathbf{W}_\varepsilon\|_\infty \\ &\le \sqrt{\sum\nolimits_{h=1}^{n_\alpha} N_{\alpha,h}}\,\|\mathbf{Q}_\alpha\|_\infty\|\mathbf{W}_\varepsilon\|_\infty \\ &\le \sqrt{\sum\nolimits_{h=1}^{n_\alpha} N_{\alpha,h}}\,\|\mathbf{Q}_\alpha\|_\infty \sum_{k=N_W}^{N}\kappa_W(\gamma_W)^k \\ &\le \sqrt{\sum\nolimits_{r=1}^{n_\alpha} N_{\alpha,h}}\,\max_{i\in[1,n_\alpha]}\{\|\mathbf{Q}_{\alpha,i}\|_\infty\}\frac{\kappa_W(\gamma_W)^{N_W}}{1-\gamma_W}.\end{aligned}$$

To bound the $\bar{\sigma}(\mathbf{Q}_\alpha\mathbf{W}_{\mathbf{FIR}})$ term, we first note that the nonzero elements of $\mathbf{Q}_\alpha$ are grouped in n_α blocks. The indexes of the elements of $\mathbf{Q}_\alpha$ corresponding to the n_α blocks are given by

$$S_{\mathbf{Q},h} = \{(r,s) : T_h \le r \le T_h + N_{\alpha,h}, \max\{1, T_h - N_{Q_a}\} \le s \le \min\{N, T_h + N_{Q_b} + N_{\alpha,h} - 1\}\}, \qquad h = 1, \ldots, n_\alpha$$

where (r, s) is the rth row and sth column. $\mathbf{W}_{\mathbf{FIR}}$ has its nonzero elements at the indexes

$$S_{\mathbf{W}} = \{(r,s) : 0 \le r - s \le N_W - 1\}$$

so the nonzero elements of $\mathbf{Q}_\alpha\mathbf{W}_{\mathbf{FIR}}$ are grouped in n_α blocks with indexes given by

$$S_{\mathbf{QW},h} = \{(r,s) : T_h \le r \le T_h + N_{\alpha,h}, \max\{1, T_h - N_{Q_a} - N_W + 1\} \le s \le \min\{N, T_h + N_{Q_b} + N_{\alpha,h} - 1\}\}, \qquad h = 1, \ldots, n_\alpha.$$

One can verify that the hth block of $\mathbf{Q}_\alpha\mathbf{W}_{\mathbf{FIR}}$ (indexes in $S_{\mathbf{QW},h}$) is $\mathbf{Q}_{\alpha,h}\mathbf{W}_{\mathbf{FIR},h}$, for $h = 2, \ldots, n_\alpha - 1$. Equations (22) and (23) imply that the first and last blocks of $\mathbf{Q}_\alpha\mathbf{W}_{\mathbf{FIR}}$ are given by $\mathbf{Q}_{\alpha,1}\mathbf{W}_{\mathbf{FIR},1}$ and $\mathbf{Q}_{\alpha,N_\alpha}\mathbf{W}_{\mathbf{FIR},N_\alpha}$, respectively. Thus, $\mathbf{Q}_\alpha\mathbf{W}_{\mathbf{FIR}}$

contains the blocks $\mathbf{Q}_{\alpha,h}\mathbf{W}_{\mathbf{FIR},h}$, $h = 1, \ldots, n_\alpha$. $\mathbf{Q}_\alpha \mathbf{W}_{\mathbf{FIR}}$ is block diagonal when $S_{\mathbf{QW},i}$ and $S_{\mathbf{QW},j}$, $i \neq j$, do not share any rows or columns. Row independence is assured by alternating α-/β-segments. The rightmost column of $S_{\mathbf{QW},j}$ is $T_j + N_{Q_b} + N_{\alpha,j} - 1$ and the leftmost column of $S_{\mathbf{QW},i}$ is $T_i - N_{Q_a} - N_w + 1$. From (21)

$$T_i \geq T_j + N_{Q_a} + N_{Q_b} + N_{\alpha,j} + N_w - 1$$

for $i > j$. Therefore, (21) is sufficient for column independence. Thus, $\mathbf{Q}_\alpha \mathbf{W}_{\mathbf{FIR}}$ is block diagonal and $\bar{\sigma}(\mathbf{Q}_\alpha \mathbf{W}_{\mathbf{FIR}}) = \max_{h \in [1, n_\alpha]} \bar{\sigma}(\mathbf{Q}_{\alpha,h}\mathbf{W}_{\mathbf{FIR},h})$. ■

Remark 4: At one limit, the α-segment width can be set to 0, making the α-segment and approximation error components equal to zero, and thus allowing the β-segment bandwidth to be increased until $\bar{\sigma}(\mathbf{Q}_\beta \mathbf{W})$ approaches 1. Therefore, an LTI Q-filter is a special case of the α-/β-segment class of Q-filters where β-segment performance is maximized. Alternatively, reducing the β-segment bandwidth will decrease $\bar{\sigma}(\mathbf{Q}_\beta \mathbf{W})$ and allow the α-segment width and bandwidth to be increased, thereby allocating higher performance to the α-segments.

Remark 5: When $Q_\beta(z)$ is causal, $\mathbf{Q}_\beta$ is lower triangular Toeplitz, and hence, $\mathbf{Q}_\beta \mathbf{W}$ is also lower triangular Toeplitz. Thus, from Theorem 1, the frequency-domain bound $\bar{\sigma}(\mathbf{Q}_\beta \mathbf{W}) \leq \|Q_\beta(z)W(z)\|_\infty$ can be used. Additionally, matrix calculations in (24) use $\mathbf{Q}_{\alpha,h}$ and $\mathbf{Q}_{\alpha,h}\mathbf{W}_{\mathbf{FIR},h}$, rather than $\mathbf{Q}$ and $\mathbf{W}$ used in (12). Since calculations of maximum singular value are $O(N^3)$ [17], significant computational savings are gained when $N_{Qa}, N_{Qb}, N_W \ll N$.

Example 2: Consider $W(z) = 0.51 - 0.51z^{-1}$. For β-segments, use the low-pass filter $Q_\beta(z) = 0.9 + 0.1z^{-1}$, and for α-segments, use the identity filter $Q(z) = 1$. Then, $N_{Qa} = 1$ and $N_{Qb} = 0$. Let $N = 5$ and assume one α-segment with $N_{\alpha,1} = 2$ that begins at $T_1 = 3$. Then

$$\mathbf{W} = \begin{bmatrix} 0.51 & 0 & 0 & 0 & 0 \\ -0.51 & 0.51 & 0 & 0 & 0 \\ 0 & -0.51 & 0.51 & 0 & 0 \\ 0 & 0 & -0.51 & 0.51 & 0 \\ 0 & 0 & 0 & -0.51 & 0.51 \end{bmatrix}$$

and

$$\mathbf{Q} = \begin{bmatrix} 0.9 & 0 & 0 & 0 & 0 \\ 0.1 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.1 & 0.9 \end{bmatrix}.$$

$\mathbf{Q}$ is separated into the two matrices

$$\mathbf{Q}_\beta = \begin{bmatrix} 0.9 & 0 & 0 & 0 & 0 \\ 0.1 & 0.9 & 0 & 0 & 0 \\ 0 & 0.1 & 0.9 & 0 & 0 \\ 0 & 0 & 0.1 & 0.9 & 0 \\ 0 & 0 & 0 & 0.1 & 0.9 \end{bmatrix}$$

and

$$\mathbf{Q}_\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1 & 0.1 & 0 & 0 \\ 0 & 0 & -0.1 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the α-segment of $\mathbf{Q}_\alpha$ is

$$\mathbf{Q}_{\alpha,1} = \begin{bmatrix} -0.1 & 0.1 & 0 \\ 0 & -0.1 & 0.1 \end{bmatrix}.$$

$W(z)$ is FIR, so $N_W = 2$, $\mathbf{W}_{\mathbf{FIR}} = \mathbf{W}$, $\mathbf{W}_\varepsilon = \mathbf{0}_{5x5}$, and

$$\mathbf{W}_{\mathbf{FIR},1} = \begin{bmatrix} -0.51 & 0.51 & 0 & 0 \\ 0 & -0.51 & 0.51 & 0 \\ 0 & 0 & -0.51 & 0.51 \end{bmatrix}.$$

The β-segment norm is $\|Q_\beta(z)W(z)\|_\infty = 0.816$, the α-segment norm is $\|\mathbf{Q}_{\alpha,1}\mathbf{W}_{\mathbf{FIR},1}\|_2 = 0.161$, and the $W(z)$ FIR approximation error is 0. Therefore, the total norm is $0.977 < 1$, so the ILC is robustly monotonically convergent.

V. Conclusion

In this note, we have investigated robust monotonic convergence of the ILC algorithms incorporating LTV Q-filters. Frequency-domain uncertainty models were assumed to capture unmodeled and neglected dynamics, and hence, converted into time-domain uncertainty models for analysis. A sufficient condition for robust monotonic convergence was developed. A class of LTV Q-filters that has value for precision motion control was presented. The structure of this class of Q-filters was used to separate the robust monotonic convergence result into a dominant block diagonal form and an approximation error. This result provides insight into design tradeoffs associated with the presented class of LTV Q-filters.

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