01 Jan 1990

A Differential Equation Approach to Swept Volumes

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Abstract

A novel approach to the analysis of swept volumes is introduced. It is shown that every smooth Euclidean motion, or sweep, can be identified with a first-order, linear, ordinary differential equation. This sweep differential equation provides useful insights into the topological and geometrical nature of the swept volume of an object. A certain class, autonomous sweeps, is identified by the form of the associated differential equation, and several properties of the swept volumes of the members of this class are analyzed. These results are then applied to generate swept volumes for a number of objects. Implementation of the sweep differential equation approach via computer-based numerical and graphical methods is also discussed.

1. Introduction

An object in space undergoing a continuous Euclidean motion sweeps out a region in Euclidean space called its swept volume. The geometrical analysis and modeling of swept volumes plays a vitally important role in several facets of manufacturing automation, including NC machining, robotics and spatial motion planning (see [1], [2], [4], [6], [8]-[10], [12], [13], and [15]-[18]). Not only is it necessary to use both existing and new mathematical tools to describe the topology and geometry of swept volumes; it is crucial to the utility of these mathematical techniques that they be readily adaptable to efficient, cost-effective computer implementation. In recent years, considerable research has been devoted to the discovery of mathematical methods for investigating swept volumes and to the design of algorithms and the development of associated computer software for the integration of these analytical techniques into actual automated systems. The work in [1], [2], [4],[6]-[13], [15]-[18], and [20] represents a sample of some of the more successful efforts.

In this paper we give a rather brief description of what appears to be a novel approach to the study of swept volumes - one which fully explores the Lie group structure of the set of Euclidean motions via the theory of differential equations. We begin by giving precise mathematical definitions of sweep and swept volume which are couched in the language of Lie group theory (see [19]). From this there follows a natural identification of a smooth sweep with a system of first-order, linear, ordinary differential equations which we call the sweep differential equation. It follows from the theory of differential equations that the form of the sweep differential equation and the initial position of an object completely determine the swept volume of the object. Consequently, it is logical to classify sweeps according to the properties of their sweep differential equations, as certain types of differential equations are likely to produce swept volumes with particularly simple features. In this vein, we next identify and analyze the class of autonomous sweeps which are the sweeps having an autonomous differential equation. We follow this with several results concerning the geometry of autonomous swept volumes which are applied to a number of specific instances. In the penultimate section, we indicate how our differential equations approach is especially well-suited to numerical and graphical implementation with the aid of a computer. We conclude with some pertinent remarks on our method and its possible extensions and generalizations.

2. Sweep Differential Equations

The swept volume of an object in Euclidean n-space R^n is generated by a 1-parameter family of Euclidean motions of the form $\xi + Ax$ (translation plus rotation), where x is a generic, and ξ is a fixed vector in R^n, and A is a matrix in the special orthogonal group:

$$SO(n) = \{ A: A \text{ is a real, orthogonal, } n\times n \text{ matrix with } \det A = 1\}$$

SO(n) is a real analytic Lie group of dimension (n/2)(n-1). See [19] for details. Let Euc(n) be the Lie group of Euclidean motions in R^n. It is clear from the form of

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Euclidean motions that the Euclidean group Euc(n) can be identified with $R^n \times SO(n)$; hence it is a real analytic Lie group of dimension (n/2)(n+1).

2.1 Definition.

A sweep is a continuous mapping $\sigma : [0,1] \rightarrow Euc(n)$ such that $\sigma(0)$ is the identity. We say that the sweep is smooth if it has continuous derivatives of all orders. Every sweep can be written in the form

$$\sigma_t(x) = \xi(t) + A(t)\,x \qquad (1)$$

where $\xi(0) = 0$, $A(0) = I$, the identity matrix, $\xi(t) \in R^n$, $A(t) \in SO(n)$, and σ_t is the value of σ at t for every $0 \leq t \leq 1$.

We shall confine our attention, for the most part, to smooth sweeps. This is certainly not unreasonable since most sweeps encountered in practice are apt to be at least piecewise smooth.

2.2 Definition.

Let $R^n \supseteq M$ and σ be a sweep in R^n. The swept volume of M under σ is the subset of R^n defined by

$$S_\sigma(M) = U\{\sigma_t(M) : 0 \leq t \leq 1\}$$

Each of the sets $\sigma_t(M) = \{\sigma_t(x) : x \in M\}$ is a t-section of $S_\sigma(M)$.

Given a smooth sweep σ, let us find a differential equation having the solution $x = x(t)$ which generates the sweep. Setting $x = x(t,x^0) = \sigma_t(x^0) = \xi(t)+A(t)x^0$ and differentiating, we obtain

$$\dot{x} = \dot{\xi}(t) + \dot{A}\,x^0 \qquad (\dot{} = d/dt)$$

Solving $x = \xi + Ax^0$ for x^0 using the fact that $AA^T = A^TA = I$, where T denotes the transpose, and substituting in the above equation yields

$$\dot{x} = \dot{\xi}(t) + \dot{A}\,A^T(x - \xi(t))$$

It follows from this derivation that $x(t) = \sigma_t(x^0)$ is the unique solution of this differential equation satisfying the initial condition $x(0) = x^0$. This suggests the following concept.

2.3 Definition.

Let $\sigma_t(x) = \xi(t)+A(t)x$ be a smooth sweep in R^n. The smooth vector field

$$X_\sigma(x,t) = \dot{\xi} + B(t)\,(x - \xi(t))$$

where $B(t) = \dot{A}(t)A^T(t)$, is called the sweep vector field (SVF) of σ and

$$\dot{x} = X_\sigma(x,t) \qquad (2)$$

is called the sweep differential equation (SDE) of σ.

As (2) is linear, a solution such that $x(0) = x^0$ exists on the whole unit interval [0,1] (see [5]). This shows that there is a one-to-one correspondence between smooth sweeps and SDE's. Given this correspondence and the fact that the evolution of an object in a vector field is completely determined by the initial position of the object, it is quite logical to classify sweeps in terms of their SDE's. In this way it is to be expected that we will find special classes of sweeps which generate swept volumes exhibiting a variety of particularized geometric and topological features. We shall identify one such class in the next section.

3. Autonomous Sweeps

It is well-known that differential equations whose vector field does not explicitly depend on t, i.e. autonomous differential equations, have a raft of useful properties (see [5]). For example, their orbits in phase space generate a (local) flow in R^n which has additive group structure. Therefore, it is quite natural to single out the class of autonomous SDE's as one that should lead to readily computable topological and geometric features of the resulting swept volumes.

3.1 Definition.

A smooth sweep is said to be autonomous if its SDE is autonomous; i.e., X_σ in (2) does not depend on t. In the context of Lie group theory, a necessary and sufficient condition for $\sigma : [0,1] \rightarrow Euc(n)$ to be autonomous is that σ be what is called a 1-parameter subgroup (see [19]). There is, however, a far more elementary way of characterizing autonomous sweeps.

We take the partial derivative of X_σ with respect to t and set it equal to zero, whence

$$\partial_t X_\sigma = (\ddot{\xi} - B\dot{\xi} - \dot{B}\xi) + \dot{B}x = 0$$

The independence of x and t implies that this equation holds for all x and t if and only if $\dot{B} = 0$ and $\ddot{\xi} - B\dot{\xi} = d/dt\,[e^{-tB}\dot{\xi}] = 0$. This, in turn, is equivalent to $\dot{A} = BA$, with B constant, and $e^{-tB}\dot{\xi} = b$, with b constant. Here e^{-tB} is the usual matrix exponential (c.f. [3],[5], and [19]). But $\dot{A} = BA$ with $A(0) = I$ has unique solution $A(t) = e^{tB}$.

Moreover, since $AA^T = I$ we infer that $e^{tB}(e^{tB})^T = e^{t(B+B^T)} = I$ which implies $B + B^T = 0$, so $B \in o(n)$, where

$$o(n) = \{ B : B \text{ is a real, } n\times n \text{ skew-symmetric matrix}\}$$

We have now essentially proved the following result.

3.2 Theorem.

Let $\sigma_t(x) = \xi(t) + A(t)x$ be a smooth sweep. Then the following are equivalent:

(i) The sweep is autonomous.

(ii) $\dot{A}A^T = B$ is constant and $A^T\dot{\xi} = b$ is constant.

(iii) $A(t) = e^{tB}$, $B \in o(n)$ and $\dot{\xi} = e^{tB}b$ with $b \in R^n$.

(iv) The SDE of σ is $\dot{x} = Bx + b$, where $B \in o(n)$ and $b \in R^n$.

This result establishes an interesting link between Lie group and Lie algebras. It is a standard result that o(n) can be identified with the Lie algebra of SO(n) consisting of its left invariant vector fields. Consequently, one could use Theorem 3.2 to prove that the Lie algebra of Euc(n), which we denote by e(n), can be identified with the direct sum $R^n \oplus o(n)$.

It is easy to show that any sweep which is purely translational or rotational is autonomous. It is also not difficult to find examples of non-autonomous sweeps. For example,

$$\sigma_t(x) = t\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

is a smooth sweep in the plane which is not autonomous, as can be readily verified by using (ii) of Theorem 3.2 .

Here is a curious result. Although we have just established that smooth planar sweep need not be autonomous, we can also prove that autonomous sweeps are actually quite ubiquitous in the plane in a certain sense. If we define $y = x - \xi$, the SDE can be written in the form

$$\dot{y} = B(t)\,y$$

It can be shown that every $A(t) \in SO(2)$ may be written in the form

$$A(t) = e^{C(t)}$$

where

$$C(t) = \begin{pmatrix} 0 & -\alpha(t)\\ \alpha(t) & 0\end{pmatrix}$$

Hence, the differential equation for y is

$$\dot{y} = \dot{\alpha}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}y$$

Assuming that $\dot{\alpha}$ does not vanish, as we may do at least on subinterval of [0,1] unless A is the identity matrix, and introducing a change of independent variable via $d\tau/dt = \dot{\alpha}$ this equation is transformed into the following autonomous system:

$$\frac{dy}{d\tau} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}y$$

We shall now describe the phase portraits of the SDE's for autonomous sweeps in the plane (n = 2) and in 3-space (n=3). When n=2, it follows from Theorem 3.2 (iv) that the SDE is

$$\dot{x} = \begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & -\beta\\ \beta & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} + \begin{pmatrix}b_1\\b_2\end{pmatrix} \qquad (3)$$

If $b \neq 0$ and $\beta = 0$, the orbits are straight lines parallel to b (translation). Suppose $\beta \neq 0$, then (3) has just one stationary point at $x^* = \beta^{-1}(-b_2, b_1)$. A calculation shows that $(x_1 + \beta^{-1} b_2)^2 + (x_2 - \beta^{-1} b_1)^2$ = constant are integral curves of (3). Hence the orbits are all circles with center x^* (rotation). For the case n=3, we can convert the SDE into a simple normal form by a standard construction of linear algebra (c.f. [3] and [5]). As B is skew-symmetric, its eigenvalues have zero real parts and B has a unitary diagonalization. Using the real parts of the eigenvalues as columns, we construct a real, orthogonal matrix R such that

$$R^T B R = \Delta = \begin{pmatrix}0 & -\omega & 0\\ \omega & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

where $\omega = [B_{12}^2 + B_{13}^2 + B_{23}^2]^{1/2}$. Consequently, the change of variables $x = Ry$ transforms the SDE into the normal form

$$\dot{y} = \Delta y + c = \begin{pmatrix}0 & -\omega & 0\\ \omega & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix} + \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} . \qquad (4)$$

where $c = R^T b$. For the case of $\omega = 0$ and $c \neq 0$, the orbits are straight lines (translation). If $\omega \neq 0$, then define the line $L = \{ y : y_1 = -\omega^{-1} c_2, y_2 = \omega^{-1} c_1 \}$. In this case the circular cylinders $(y_1 + \omega^{-1} c_2)^2 + (y_2 - \omega^{-1} c_1)^2$ = constant are invariant surfaces for (4). If $c_3 = 0$, the motion of (4) is essentially two-dimensional, where every point of L is a stationary point of (4) and the orbits are circles with centers on L which lie in planes parallel to the y_1y_2 - plane (rotation about L). When $c_3 \neq 0$, L is an orbit and all other orbits are helices contained in the invariant circular cylinders (twist motion along L). This last case is illustrated in Fig. 1.

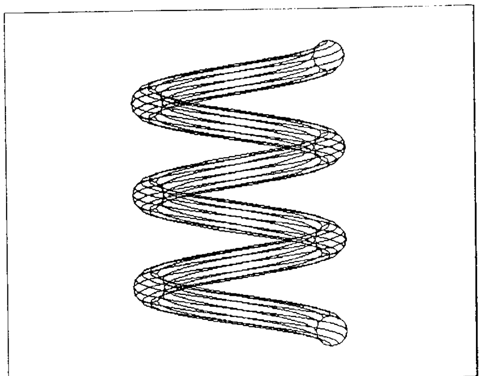

Fig. 1 An autonomous sweep in 3-space.

4. Geometrical Applications

Given an object M contained in R^n, the swept volume $S_\sigma(M)$ generated by a smooth sweep is completely determined by M and the flow induced by the SDE. Hence, at least in theory, we can determine all the topological and geometrical properties of $S_\sigma(M)$ from a thorough analysis of the trajectories of the SDE. We shall demonstrate how this line of reasoning can be applied for smooth autonomous 2-dimensional and 3-dimensional sweeps.

Let us assume that M is of finite extent and has a measure of regularity. In particular, we assume that M is a compact, n-dimensional submanifold of R^n whose boundary ∂M is piecewise-smooth (for example, say that M is diffeomorphic with a compact, n-dimensional polyhedron in R^n). A theorem of Weld and Leu [20] states that

$$S_\sigma(M) = M \cup S_\sigma(\partial M)$$

and this simplifies to $S_\sigma(M) = S_\sigma(\partial M)$ if $M \cap \sigma_t(M) = \emptyset$ for some $0 \leq t \leq 1$. A related result of Wang and Wang [18] shows that the swept volume is obtained from an envelope of the boundary. We infer from these results that $S_\sigma(M)$ can be determined by computing the swept volumes of (n-1)-dimensional manifolds. Even with this reduction of dimension, there are a number of possible pitfalls which must be taken into account. For example, the swept volume may develop singularities such as the degeneration of a ruled surface into a developable one [20], or the collapse to a lower dimensional object at some points.

In view of these dimension reduction results, we shall study the swept volume of Q, where Q is an (n-1)-dimensional submanifold of R^n which is smoothly diffeomorphic with a compact polyhedron in R^{n-1}. Consider $\Sigma : Q\text{x}[0,1] \to S_\sigma(Q)$, where $\Sigma(x,t) = \sigma_t(x)$, and define $\Sigma_t = \sigma_t$. Note that Σ_t is a Euclidean mapping of Q onto $\sigma_t(Q)$ for all $0 \leq t \leq 1$. A useful notion introduced in [20] is given in the following.

4.1 Definition.

The sweep σ is of <u>type 1 with respect to (w.r.t) Q</u> if $\sigma_t : Q \to S_\sigma(Q)$ maps interior points into interior points and boundary points into boundary points for all $0 \leq t \leq 1$. Otherwise σ is of <u>type 2 w.r.t. Q</u>.

If σ is of type 1 w.r.t. Q, the swept volume $S_\sigma(Q)$ cannot exhibit any of the singularities delineated above.

Our characterizations of the phase portraits for autonomous sweeps in R^2 and R^3 can be applied to extract a great deal of information about the swept volume of Q. We shall denote the interior of Q by int(Q), where $\text{int}(Q) = Q \setminus \partial Q$. The description of the phase plane for (3) together with standard properties of flows (see [5]) leads directly to a proof of the following.

4.2 Theorem.

Let σ be a smooth autonomous sweep in the plane, and let Q be smoothly diffeomorphic with the interval [0,1]. Suppose that X_σ, the SVF of σ, is transverse (i.e. not tangent) to int(Q). Then when σ is a translation, it is of type 1 w.r.t. Q. In fact, the mapping Σ is a smooth diffeomorphism. If σ is a rotation and $x^* \in Q$, then σ is of type 1 w.r.t. Q. More precisely, Σ is either a smooth diffeomorphism or it induces a smooth diffeomorphism of $Q\text{x}S^1$ onto $S_\sigma(Q)$, where S^1 is the unit circle. The analog of this result for n = 3 is obtained similarly from the phase space analysis of (4). Figures 2 and 3 illustrate a rotational sweep σ of Q which may be type 1 or type 2 depending on whether the SVF of σ is transverse to Int(Q).

4.3 Theorem.

Suppose that σ is a smooth autonomous sweep in R^3, Q is smoothly diffeomorphic with regular m-gon (m>2) in the plane, and X_σ is transverse to int(Q). Then when σ is a translation, σ is of type 1 w.r.t. Q and Σ is a smooth diffeomorphism. If σ is a rotation about L and $L \cap Q = \emptyset$, then either Σ is a smooth diffeomorphism or it induces a diffeomorphism of $Q\text{x}S^1$ onto $S_\sigma(Q)$.

When $L \cap Q = \emptyset$ and $\omega^{-1} | c_3 |$ is sufficiently large, Σ is a smooth diffeomorphism and a fortiori σ is of type 1 w.r.t. Q. The sweep σ is of type 2 when $\omega^{-1} | c_3 |$ is small enough. Figures 4 and 5 illustrate a helical sweep which may be type 1 or type 2 depending on whether $\omega^{-1} | c_3 |$ is large or small.

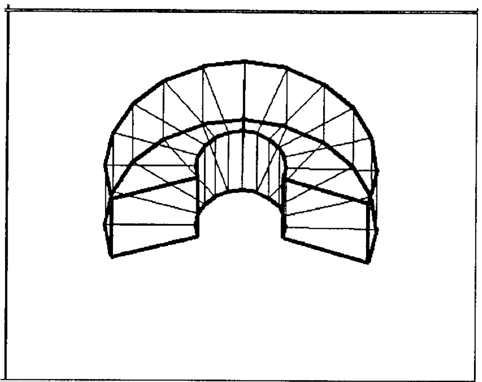

Fig. 2 A rotational sweep of a square polygon which is type 1 since SVF is transverse to Int(Q).

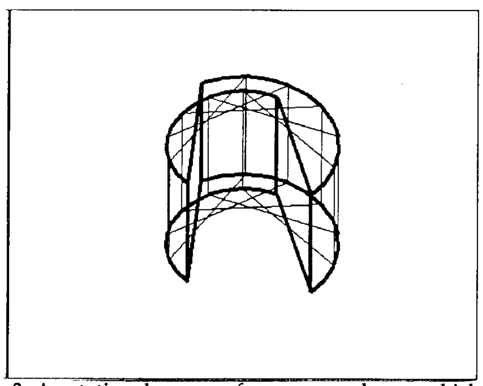

Fig. 3 A rotational sweep of a square polygon which is type 2 since SVF is not transverse to Int(Q).

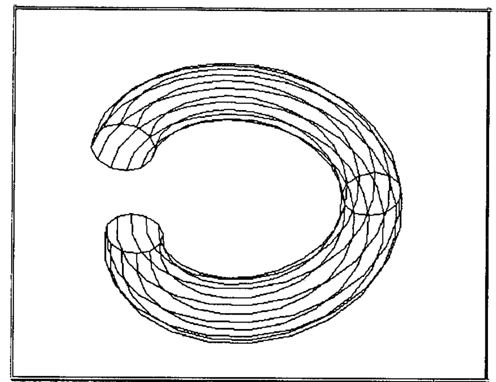

Fig. 4 A helical sweep of a circular disk which is type 1.

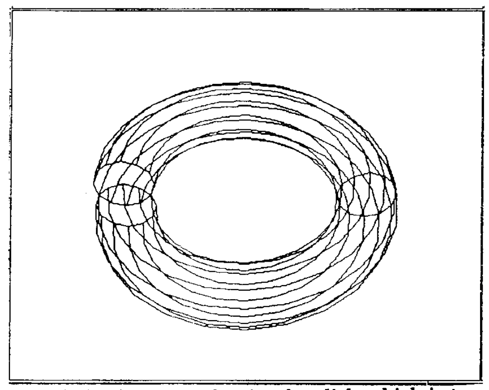

Fig. 5 A helical sweep of a circular disk which is type 2.

5. Computer Implementation

The differential equation approach is ideally suited to computer-based numerical and graphical analysis. There are a large number of standard codes based on methods such as Runge-Kutta, Adams-Moulton, Adams-Bashforth and Milne, or some combination thereof, which produce accurate (errors typically of order h^4), robust (stable), fast and efficient (typically of polynomial time) numerical solutions of differential equations as well as graphical displays of the trajectories (see [14]). It should therefore be relatively easy to develop software for analyzing and modeling swept volumes. In such an endeavor, one expects to be able to take advantage of the fact that SDE's generate isometric flows in order to produce faster programs.

In certain situations one is confronted with what may be termed an inverse problem. A sweep is observed and it is required that data be collected in order to describe the SDE analytically. This can be done to any desired degree of accuracy as follows. Choose a base point x^0 in the object and n-1 additional points $x^1,\ldots,x^{n-1}$, each lying on a different member of a set of n-1 mutually orthogonal straight lines through x^0. Partition the unit interval by introducing time $0 = t_0 < t_2 \cdots < t_m = 1$ (on a normalized time scale). Then mark the locations of the points as they move in the sweep at times $t_0, t_1, \ldots, t_m$. This determines $\xi(t_k)$ and $A(t_k)$ for $k = 0, 1, \ldots, m$. Using a standard interpolation scheme, we can approximate $\xi(t)$, $A(t)$, $\dot{\xi}(t)$, and $\dot{A}(t)$ for all t. Thus, we may approximate the sweep and SDE to any prescribed degree of accuracy by using sufficiently fine partition. This procedure can be carried out algorithmically.

6. Concluding Remarks

The method of sweep differential equations appears to have great potential as a tool for analyzing swept volumes and their intersections. We have only scratched the surface of this potential here, and we expect to generalize our result and explore other applications of this approach.

So far, we have confined the geometrical applications of our method to autonomous swept volumes. Of course, it is easy to verify that a smooth sweep can be approximated to any degree of accuracy by an autonomous sweep by sufficiently restricting the t-interval. But this does not really make an effective case for studying only autonomous sweeps, since the partitioning process needed to implement such an approach would tend to compromise the utility of the SDE method. Hence, it is necessary to demonstrate the applicability of the SDE method to more general swept volumes.

We have presented a general idea of how to implement our method with the aid of a computer. Some of the ideas were incorporated into the program which produced the figures presented in this paper. It is our intention to develop software for further SDE implementation and to test it in working automated manufacturing system.

Acknowledgements

We wish to thank Jack Schwartz and Paul Wright for some helpful conversations. Thanks are due also to Jr-Jyun Jang for developing the graphics program used for several of the figures in this paper. During the course of this research, M. C. Leu was funded by the National Science Foundation under grant MSM 8796277.

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