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Survivor bond models for securitizing longevity risk

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SURVIVOR BOND MODELS FOR SECURITIZING LONGEVITY RISK

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ABSTRACT

Longevity risk is the risk that a reference population's mortality rates deviate from what is projected from prior life tables. This is due to discoveries in biological sciences, improved public health measures, and nutrition, which have dramatically increased life expectancy. Longevity risk raises life insurers' liability, increasing product costs and reserves. Securitization through longevity derivatives is a way of dealing with this risk.

To enhance the pricing of life contingent products, we present an additive type mortality model in the style of the Lee-Carter. This model incorporates policyholder covariates. By using counting processes and martingale machinery, we obtain close form representations for the model's unknowns. We use the bond pricing approach from Wills and Sherris (2010) to price longevity bonds with this mortality model. Numerical studies suggest that asymptotic properties of model parameter estimators provide a close approximation of the true.

Pricing longevity derivatives uses a no-arbitrage approach by risk-adjusting the mortality and/or interest rate risks. There are various ways to calibrate the risk-adjusted probability measure. The risk neutral approach and the Wang transform are among the popular methods. In this work, we employ a mean-reverting Hull-White model with a moving target which was recently proposed by Zeddouk and Devolder (2020) for the mortality model and the Vasicek model for evolution of interest rate. We detail how to develop the risk-neutral measure in pricing longevity bonds.

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1. INTRODUCTION

There have been numerous notable advancements in medical treatment throughout the course of time, such as the discovery of cures for diseases and the development of ways to live comfortably and for a longer period of time with some conditions that were formerly deadly. The dissemination of new information and the cultivation of a heightened awareness have had a profound impact on the way people live their lives. As a consequence, people today, among other things, adhere to the appropriate standards of hygiene and consume food that is rich in nutrients. These factors contribute to a longer lifetime than was anticipated by the life tables, which in turn leads to a decrease in the overall mortality rate, and has thereby created a risk known as *longevity risk*. For example, in 2005, the life expectancy for U.S. males aged 60 was more than 5 years higher than expected in mortality forecasts from the 1980s. A notable upward trend is seen in the expected life expectancy from 1990 to 2020 for the U.S population, as shown in Figure 1.2. Figure 1.1 shows an abridged life table for the U.S. population in 2019.

There are two components that make up longevity risk: an idiosyncratic component, which can be diversified by the pooling of a large number of policyholders, and a non-diversifiable component, also known as *systematic risk* or *market risk*. In a broader sense, longevity risk may be divided into two categories: macro and micro longevity risk. Longevity risk in a vast group of individuals, such as policyholders in a huge pension plan or policyholders in a large insurance provider's annuity scheme, is referred to as macro-longevity risk. The most significant risk for those exposed to macro-longevity risk is the risk of misconstruing improvements in life expectancy trends, the so- called *trend risk* which is also a systematic risk. On the other hand, micro-longevity risk is concerned with the longevity risk of a small number of individuals Blake *et al.* (2013).

Age	mx	qx	lx	dx	Lx	Tx	ex
0	0.00557	0.00554	100000	554	99522	7915655	79.16
1	0.00039	0.00039	99446	38	99427	7816133	78.6
2	0.00023	0.00023	99408	23	99396	7716706	77.63
10	0.00012	0.00012	99295	12	99289	6921962	69.71
20	0.00078	0.00078	98978	77	98939	5930049	59.91
30	0.00134	0.00134	97964	131	97899	4944944	50.48
40	0.00208	0.00208	96397	200	96297	3972597	41.21
50	0.0039	0.0039	93857	366	93674	3020006	32.18
60	0.00902	0.00898	88410	794	88013	2105122	23.81
70	0.01819	0.01803	77938	1405	77235	1268792	16.28
80	0.04611	0.04507	58733	2647	57410	575371	9.8
90	0.12479	0.11746	26900	3160	25320	139653	5.19
100	0.33498	0.28693	3230	927	2767	8367	2.59

Figure 1.1. Abridged life table for the United States population in 2019

Longevity risk poses a threat to the stability of the global financial system. The severity is highlighted in Oppers *et al.* (2012) by the fact that the amount of money that is potentially at risk, also known as *exposure*, in the defined benefits pension plans for the United States as projected in 2007 was to be approximately 2.2 trillion, comprising of 42 million policyholders. According to a calculation made by Swiss Re, the overall worldwide exposure to longevity risk is something in the neighborhood of $21 trillion Burne (2011). Longevity risk does not only impact insurance companies that provide policies but other stakeholders. Examples of parties affected are employers that sponsor defined benefit pension plans for their workers and governments, who are obligated to provide state and civil service retirement benefits. If all of the systems were to collapse in the end, the financial responsibility of funding pensions and other social benefits would fall squarely on the shoulders of taxpayers, who are also highly impacted by the situation Dawn (1999). Despite important changes, such as the decision to increase the retirement

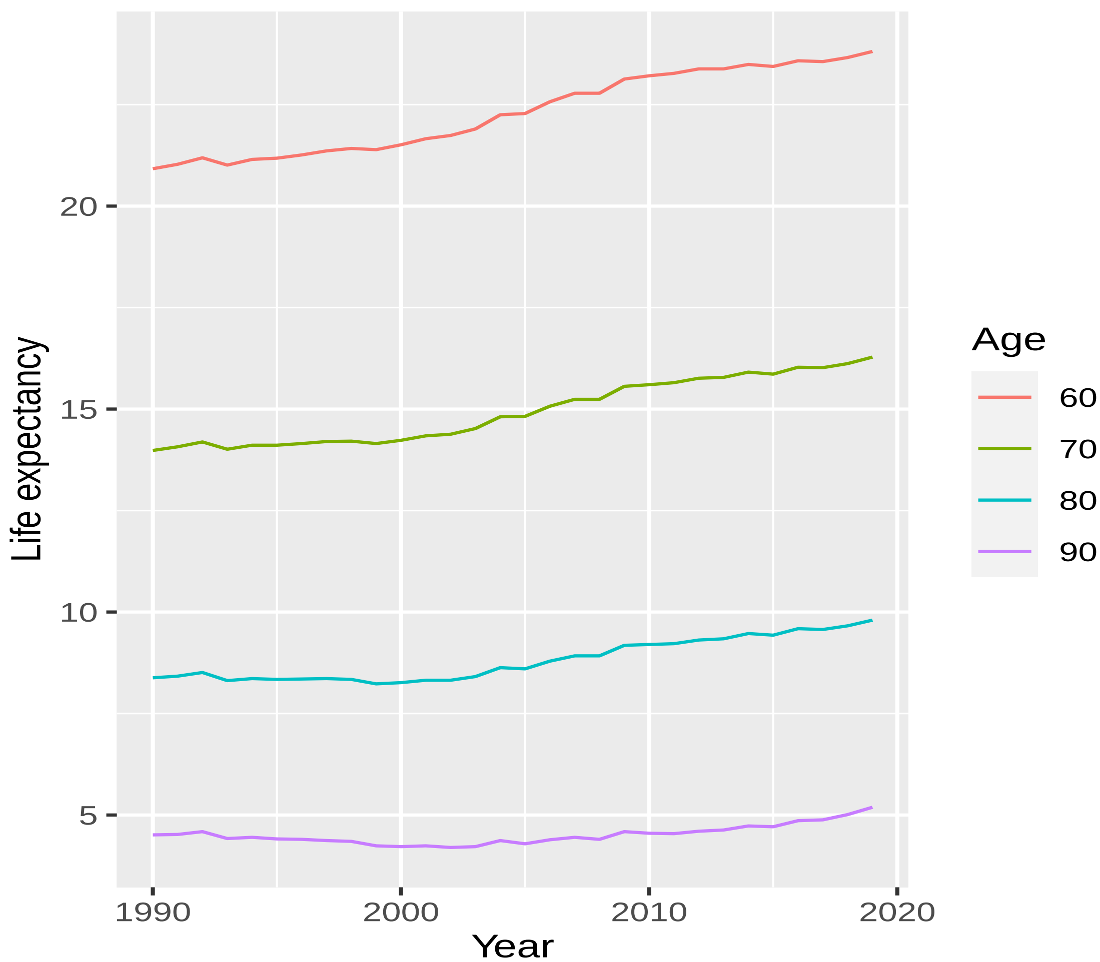

Figure 1.2. Life expectancy of the U.S population from 1990 to 2020

age from 65 to 67 by the year 2025 Lee and Skinner (1999), other institutions, such as the Social Security system in the United States, may continue to experience financial strain. In addition, increased life expectancy can result in annuitants outliving their own financial resources, especially when they are entitled to a lump sum benefit payment. Because of its magnitude and universality, longevity risk has become one of the most significant financial risk related to human life, posing a possible threat to the entire retirement income support structure. As a result, insurers and pension funds face the danger of not having enough assets to make annuity payments for an extended period of time than initially guaranteed by the insurance contract. Due to longevity risk, insurance companies are shifting from defined benefits pension plans to defined contribution plans in order to escape this risk but according to Creighton *et al.* (2005) and Lin and Cox (2005), longevity protection is mostly not provided by defined contribution pension plans. All these challenges exist because of the difficulty in accurately projecting mortality improvement over the years. Insurance models to mitigate this risk have therefore become critical for insurance firms.

There are a variety of ways of dealing with longevity risk. Many governments and pension plan providers, in an effort to counteract the unexpected improvements in life expectancy, try to mandate participants to make larger payments while in the work force or attempt to enforce the situation in which people have to work longer in their lifetime. For example, in the UK, the government extended the age of retirement for women from 60 to 65 between 2010 and 2018, and then for both gender to 66 in 2020. The UK Pension Act 2014 further proposes an increase to 67 by 2028, and 68 by 2046 Blake *et al.* (2013). When it comes to insurance firms, one of the options available to them is to sell the liability either through an insurance contract or a reinsurance contract, the so- called *pension buy-out*. Pension buy-outs, which have seen a surge in popularity in the United Kingdom since 2006 entails the transfer of all risks; interest rate and inflation risks, to the insurance market from different pension plans. Due to the difficulty in accurately quantifying longevity risk, with this difficulty being caused by the uncontrollable nature of systematic longevity risk, traditional reinsurance seems not to be so much of a viable solution. Some reinsurers are hesitant to take on this risk, with some even calling it "toxic" Wadsworth (2005). Another approach insurance companies have resorted to in managing longevity risk is *natural hedging*, that is, selling life insurance products to the same policyholders who have pension policies with them in order to balance out cash inflows and outflows to the company, hence keeping things stable.

Despite the existing situation that people's lifespans have increased, many people are simultaneously retiring earlier in life Wills and Sherris (2010). For Organisation for Economic Co-operation and Development (OECD) men aged 60-64, labor participation plummeted dramatically from 70 – 90 percent in the 1970s to 20 – 50 percent in 2005 Creighton *et al.* (2005), even though this rate has slowly risen to 75.3% as at 2021. Funding extended retirement duration will therefore need to greatly depend on income from sources like life annuities and other lifetime income guaranteed products, hence the need for very reliable retirement policies Wills and Sherris (2010). Moreover, the capital base of insurance and reinsurance companies have in recent times been deemed insufficient to absorb a substantial fraction of this risk, hence the need to locate a larger source of money capable of absorbing the risk. This leads to the idea of *securitization* of longevity risk.

Securitization is the conversion of illiquid assets and liabilities such as house mortgages, corporate loans and life insurance policies into marketable securities on the capital market. This approach has been used to deal with a wide range of risks, with an example of such being credit risk. Financial markets have the capacity to provide a risk pooling and risk management role in dealing with longevity risk. Securitization is currently an efficient alternative to the conventional insurance risk-transfer approaches such as reinsurance. Pure insurance risk securitization originated in the mid-1990s with the launch of Insurance-Linked Securitization (ILS) and the emergence of the catastrophe bond market Wills and Sherris (2010).

With respect to the capital market, longevity bonds are one of the most popular securities and we will restrict our discussion to these bonds since this work is based on securitizing longevity bonds. Blake and Burrows (2001) presented the concept of longevity bonds for the first time in 2001, and since then, this approach of managing longevity risk has gained popularity. In November 2004, the European Investment Bank (EIB) sold the first longevity bond which was a 25-year £540 million longevity bond. The bond was issued with an initial coupon of £50 million. Longevity bondholders receive coupon payments that decrease over time based on a survival index which is derived from a particular cohort or population. The survival index used for this EIB bond was from the population of 65 year old persons males who lived in England and Wales and retired the same year. There were several issues with this bond, including its design, pricing and inability to garner sufficient interest from investors, which made it unpopular on the market, thereby causing its withdrawal from the market a year after its launch. Other longevity related insurance linked securities (ILS) have since then been designed and issued including longevity swaps, longevity options, c.f Biffis and Blake (2009), Deng *et al.* (2012), Coughlan *et al.* (2007) and Cairns *et al.* (2008) for an overview.

We will now concentrate on longevity bonds as it is our preferred derivative for securitizing longevity risk in the chapters that follow. The *term to maturity* of a longevity bond, T, that is the duration of issue of the bond, can either be a fixed value or stochastic. The stochastic term to maturity reflects the death of the last policyholder in a given cohort. When T is stochastic, we have the so- called *survivor bond*. In reality, majority of the issued longevity bonds have a pre-defined term to maturity. The price of a typical longevity bond depends on a survival index which is derived from mortality projections with reference to a given cohort, and a model for interest rate.

Accurate mortality projections are a critical component of longevity securitization. However, researchers have had difficulty in delivering such projections. Mortality models can be deterministic or relational. Examples of these are the Gompertz, Makeham and Weibull mortality model which are constructed with the assumption that age-specific mortality rates satisfy a given function form. The downside of using deterministic mortality models is that even though they can provide good estimates of mortality, they do not do well for long-term predictions. Contrary to the behaviour of these deterministic models, longevity has been observed to be improving, albeit in an unpredictable way, hence warranting the introduction of stochastic models to help capture the random improvements in mortality. Since 1992, when the first stochastic mortality model, the Lee Carter model Lee and Carter (1992), was introduced, a plethora of stochastic mortality models have been proposed, several of which are variants to the Lee Carter model. Many of these models use time series to forecast mortality . They make use of data from several years and mostly incorporate an age-, time- or cohort dependent component or a combination of them in the model. Many of the existing models are good for predicting mortality rates but are not so useful for projecting mortality rates outside the range of ages attained, so- called *extrapolation* Su and Yue (2021). The Lee- Carter model uses singular value decomposition to decompose the log of the central death rates into an age component and a time component, and then a time series model is used to forecast mortality on the time component from the initial decomposition. In Lee and Miller (2001)'s extension of the aforementioned model, the time component was re-estimated in accordance with the life expectancy observed at birth. Plat (2009) proposed a stochastic mortality model that combines the desirable characteristics of current models while avoiding their shortcomings. The model is not complicated, fits historical data well, can be used across the full age spectrum and does a good job capturing the cohort effect and correlation between variables. Existing literature has also seen mortality models which incorporate either positive jumps, negative jumps or both. Positive jump components are used to model unforeseen spikes in mortality that may be caused by events such as hurricanes, tornadoes, famine, pandemics and so on. The negative jump component accounts for rapid improvements in mortality rates that can be attributed to, for example, the creation of new pharmaceuticals that serve as a treatment for diseases that were previously thought to be deadly, as well as other advancements in biomedical research Miguel Bravo (2021). There is room for considerable subjectivity in the choice of mortality models since none of the mortality models presented in literature today performs noticeably better than their other counterparts when it comes to predictions across all ages, population and time periods Tang and Li (2021).

According to Mayhew and Smith (2011), the time evolution of interest rates and inflation is an equally important risk to consider in an annuity contract. Interest rates can also be constant, continuous, or stochastic. An analysis of the evolution of interest rates over at least the past two decades also demonstrates a random pattern, warranting the introduction of stochastic interest rate models for the pricing of financial securities. Several stochastic interest rate models have been discussed in Chapter 2, which are all potential candidates for use for pricing longevity derivatives.

Prior to pricing a longevity risk associated security, it is necessary to have an underlying mortality risk process that is capable of being used as a risk adjusted pricing measure. To calculate risk adjustment in imperfect markets just like the longevity risk market, the equilibrium pricing theory may be utilized Kariya and Liu (2003). Wang (1996), Wang (2000a) and Wang (2002) devised a model for pricing risks with the intent to integrate both actuarial science and financial theory into a single framework. He proposed the use of a distortion parameter to transfom the probability of death before a given time, which results in a 'risk- adjusted' probability of death. This method is used by Lin and Cox (2005) and Liao *et al.* (2007) to obtain the risk-adjusted mortality measure. Other derivatives are priced under a "risk- neutral" measure instead of the "real-world" measure. The risk-neutral measure is deemed an equivalent probability measure to the actual probability measure. The Girsanov theorem can be used for the change of probability measure in the case of stochastic processes driven by Wiener processes.

The aims of this dissertation work are:

a. We propose the use of a general additive stochastic model with covariates for mortality. Since several studies have shown that various aspects of one's social and environmental circumstances, as well as their health and biological make-up, may have an impact on their likelihood of dying, it is vital to present a model that incorporates such considerations. We use martingale mechanisms to estimate the unkown parameters. We demonstrate pricing a longevity bond using a technique that is quite similar to the way that Collateralized Debt Obligations are structured. Specifically, we use the tranches that are calculated based on the percentage cumulative losses. The process for determining the price of the bonds will be exactly the same as the methodology that was utilized in Wills and Sherris (2010).

b. The goal of this project is to investigate in further depth how to predict the price of a longevity bond by combining a recently developed Hull-White mortality model Zeddouk and Devolder (2020) with a stochastic interest rate model. We present the theory on the risk-neutral technique, which will be enforced by the Girsanov theorem for use in pricing a longevity bond.

2. MATHEMATICAL PRELIMINARIES

The following theorems, definitions and discussion of concepts are gleaned from Shreve *et al.* (2004), Fleming and Harrington (2011), Kalbfleisch and Prentice (2011), Andersen *et al.* (2012) and Calin (2012).

2.1. REVIEW OF GENERAL CONCEPTS

Definition 1 *A cohort, in the context of this work, is a group of individuals who have comparable settings, health care systems, technological innovations, common demographical or statistical characteristics that influence their mortality and life expectancy.*

2.2. REVIEW OF SURVIVAL ANALYSIS

For any failure time T, we can define its probability distribution using the survivor function, $S(t)$, the probability density function, $f(t)$ and the hazard function, $h(x)$. T can be discrete or continuous. The following shows these distributions and some relationships between them in the continuous setting.

Definition 2 *A function, $f(t)$ is a probability density function of T if*

1. $f(t) \geq 0$,
2. $\int_0^\infty f(t)dt = 1$.

Definition 3 *The survivor function, generally denoted by $S(t)$ or $\bar{F}(t)$ is the probability that the random variable T is greater than t, that is:*

$$S(t) = P(T > t), \quad t \in (0, \infty).$$

In the case of the survival of an individual at age (x), the probability of survival of this individual for t more years is denoted by $S_x(t)$ or in actuarial notation, ${}_tp_x$

1. *$S(t)$ is a non- increasing function,*

2. $S(t) = \int_t^\infty f(t)dt$,

3. $lim_{t\to\infty} S(t) = 0$,

4. $lim_{t\to 0} S(t) = 1$.

In direct contrast to the survival function is the *cumulative distribution function*, generally denoted by $F(t)$.

Definition 4 *The cumulative distribution function is the probability that the failure time T is at most t units of time, that is:*

$$F(t) = P(T \leq t) = 1 - S(t) \quad t \in (0, \infty).$$

In the case of individuals aged (x), this is the probability of death of this individual by t time units and is denoted by $F_x(t)$ or in actuarial notation, ${}_tq_x$.
Note that

1. *$F(t)$ is a non- decreasing function,*

2. $F(t) = \int_0^t f(t)dt$,

3. $lim_{t\to\infty} F(t) = 1$,

4. $lim_{t\to 0} F(t) = 0$.

The functions $f(t)$, $S(t)$ and $F(t)$ are related as follows:

$$f(t) = \frac{d}{dt}F(t) = -\frac{d}{dt}S(t).$$

Definition 5 *The force of mortality is the instantaneous rate at which failure of subjects occur after their survival past time t, that is,*

$$\begin{aligned} h(t) &= \lim_{h \to 0^+} \frac{P(t \leq T < t + h | T \geq t)}{h} \\ &= -\left[\frac{d}{dt} log F(t)\right] \\ &= \frac{f(t)}{S(t)}. \end{aligned}$$

$F(t)$ can be obtained from $h(t)$ by

$$\begin{aligned} S(t) &= \exp\left[-\int_0^t h(s)ds\right] \\ &= \exp[-\Lambda(t)], \end{aligned}$$

where $\Lambda(t)$ is known as the *cumulative force of mortality* or the *integrated force of mortality*. From the previous expression of $S(t)$, the *probability density function* of T is given by:

$$f(t) = h(t)\exp[-\Lambda(t)].$$

2.3. REVIEW OF STOCHASTIC PROCESSES

The purpose of this part is to provide a reading aid for the rest of this dissertation by providing a review of certain ideas related to stochastic processes.

Definition 6 *A stochastic basis is called complete if the filtration $\mathcal{G}$ contains any subset of a P-null set, that is is $\mathcal{G}$ is complete, and if each $\mathcal{G}_t$ contains all P-null sets of $\mathcal{G}$.*

Completeness is a requirement necessary for proving some theorems in the general theory such as the Doob - Meyer decomposition for submartingales. For instance, completeness is also needed to show that a martingale can be modified so that the set of paths with both left-hand limits and right- hand limits at each t has probability 1, and that the modified process is indistinguishable from the original martingale.

We define $(\Omega, \mathcal{G}, \mathbb{P})$ as a complete probability space. Also, let T be the random variable that represents the failure time of an individual such that $T = [0, \tau] \subset \mathbb{R}$, where τ is the monitoring time for each individual in the cohort.

Definition 7 *A stochastic process,* $(X_t)_{t\in\mathbb{R}+}$ *on the probability space* $(\Omega, \mathcal{G}, \mathbb{P})$ *is a family of random variables which are parameterized by time* $t \in T$ *. A discrete time stochastic process has the support of T being* $\{0, 1, 2, \dots\}$ *and a continuous time stochastic process has a support that is a subset of* $\mathbb{R}_+$.

Definition 8 *A Filtration,* $\{\mathcal{G}_t\}_{t\in\mathbb{R}+}$, *refers to an increasing family of* σ *- algebras which is defined on the sample space* Ω, *that is* $(\mathcal{G}_s \subseteq \mathcal{G}_t \subseteq \mathcal{G}, \textit{if } s \leq t)$. *It refers to the time-accumulated information or history of the process by time* t.

Theorem 1 *Suppose that for every* $t \geq 0, \omega \in \Omega$, *there exists a number* $\mathcal{E}_{t,\omega} > 0$ *such that*

$$X_{t+s,\omega} = X_{t,\omega}, \quad 0 \leq s < \mathcal{E}_{t,\omega},$$

then the history of X_t *is a right- continuous filtration.*

An example of a right- continuous filtration is a counting process N_t which will be discussed later.

Definition 9 *The* $\sigma-$ *algebra on* $[0, \infty) \times \Omega$ *generated by all sets of the form*

1. $[0] \times A$, $A \in \mathcal{G}_0$, *where* $\mathcal{G}_0$ *is the history of the process at time* 0,
2. $(a, b] \times A, 0 \leq a < b < \infty, A \in \mathcal{G}_a$.

is called the predictable $\sigma-$*algebra for the filtration,* $\mathcal{G}, t \geq 0$.

Definition 10 *Let* $\mathcal{G}_{t^-}$ *represent the history of a process at a time right before* t. *A stochastic process, is said to be adapted to the filtration,* $\mathcal{G}_t$ *if it is* $\mathcal{G}_{t^-}$ *predictable for any* $t \in T$. *In other words, every stochastic process is adapted, that is, a stochastic process is its natural history.*

Definition 11 *A stochastic basis is a probability space* $(\Omega, \mathcal{G}, \mathbb{P})$ *equipped with a right continuous filtration,* $\{\mathcal{G}_t : t \geq 0\}$*, and is denoted by* $(\Omega, \mathcal{G}, \{\mathcal{G}_t : t \geq 0\}, \mathbb{P})$*.*

Definition 12 *A stochastic process is said to be cadlag if its sample paths are right-continuous with left hand limits on* $\mathcal{G}$*. Cadlag is "continue à droite, limite à gauche" in French.*

Definition 13 *A stochastic process* $(X_t)_{t\in\mathbb{R}+}$ *is*

1. *Integrable if* $sup_{0\leq t<\infty} E|X(t)| < \infty$*,*

2. *Square integrable if* $sup_{0\leq t<\infty} E\{X(t)\}^2 < \infty$*,*

3. *Bounded if there exist a finite constant* Γ *such that*

$$P\{sup_{0\leq t<\infty}|X(t)| < \Gamma\} = 1.$$

Let $L^p(\Omega \times \mathbb{R}_+)$ *denote the space of p-integrable processes, that is the space of stochastic processes* $u : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$ *such that*

$$\mathbb{E}\left[\int_0^\infty |u_t|^p dt\right] < \infty.$$

Also let $L^p_{ad}(\Omega\times\mathbb{R}_+)$*,* $p \in [1, \infty)$ *denote the space of* $\mathcal{G}_t$*- adapted processes in* $L^p(\Omega\times\mathbb{R}_+)$*.*

For this work, we will be using stochastic cadlag processes.

Definition 14 *A stochastic process,* $M_t, t > 0$ *is called a martingale with respect to a filtration* $\mathcal{G}_t$ *if :*

1. M_t *is integrable for each* $t \in T$*, that is* $E(|M_t|) < \infty$ *for all* $t \in T$*,*

2. M_t *is adapted to the filtration,* $\mathcal{G}_t$*,*

3. $M_s = E[M_t|\mathcal{G}_s]$*,* $\forall s < t$*.*

A martingale can be considered as a pure noise process. Note that $\forall s < t$, if $E[M_t|\mathcal{G}_s] \geq M_s$, M_t is called a *sub-martingale* and M_t is called a *super-martingale* if $E[M_t|\mathcal{G}_s] \leq M_s$. Every martingale must be stated with respect to a specific filtration.

Definition 15 *Let τ be an arbitrary index set. Then a collection of random variables $X_t; t \in \tau$ is uniformly integrable if*

$$lim_{n\to\infty} supE(|X_t| I_{\{|X_t|>n\}}) = 0.$$

Proposition 1 *A collection of random variables $\{X_t; t \in \tau\}$, where τ is uniformly integrable if and only if the following conditions are satisfied:*

1. $sup_{t\in\tau} E|X_t| < \infty$,

2. *For every $\epsilon > 0$, there exists $\delta(\epsilon)$ such that for any set A with $P\{A\} < \delta(\epsilon)$,*

$$sup_{t\in\tau} \int_A |X_t| dP < \epsilon.$$

Definition 16 *A stochastic process, $X_t; t \geq 0$ is said to be stationary if for every s, $\{X_{t_1}, X_{t_2}, \dots, X_{t_n}\}$ and $\{X_{t_1+s}, X_{t_2+s}, \dots, X_{t_n+s}\}$ have the same joint distribution.*

Definition 17 *A stochastic process, $N_t; t \geq 0$ is called a counting process if it represents the total number of "events" that occur by time t. N_t has the following properties:*

1. $N_t \geq 0$,

2. $N_t \in \{0, 1, 2, \dots\}$,

3. $N_s \leq N_t$ *for* $s < t$,

4. *The total number of events that occur in a time interval $(s, t]$ is $N_t - N_s$, $\forall s < t$.*

The paths of a counting process have jump discontinuities, with jumps of size +1.

Definition 18 *A counting process is said to have independent increments if* $\forall s < t$*, the distribution of the number of events that occur within* (s, t) *depends only on* $t - s$*.*

Definition 19 *A multivariate counting process is a p- variate process* $\{N_j; j = 1, 2, \ldots, p\}$ *such that:*

1. *Each* N_j *is a counting process,*
2. *No two component processes jump at the same time.*

Definition 20 *A stochastic process* $X_t, t \geq 0$ *which is measurable with respect to the sigma-algebra created by all left-continuous adaptive processes is said to be a predictable process if the time t behaviour of a predictable process is determined by the information before time* t*.*

When working with integrals with respect to a martingale process, the idea of predictability inevitably comes up.

Proposition 2 *Let* $X_t, t \geq 0$ *be an* $\mathcal{G}_t$ *predictable process. Then* X_t *is* $\mathcal{G}_{t-}$*- measurable.*

Many stochastic processes can be written as the sum of a *local martingale*; the random part and a *finite variation predictable process*; the systematic part. We now discuss the Doob-Meyer decomposition theorem for submartingales.

Theorem 2 *Doob-Meyer Decomposition, Fleming and Harrington Fleming and Harrington (2011)*
Let $M_t^\dagger, t \geq 0$ *be a right continuous non- negative sub-martingale with respect to a filtration* $\mathcal{G}$ *and* A_t *be a compensator, a smooth varying predictable increasing process which is also unique. The Doob- Meyer decomposition theorem states that the sub- martingale can be uniquely decomposed into the sum of a right- continuous martingale,* M_t *and the unique compensator,* A_t*, such that* $E\{A_t\} < \infty$ *and* $M_t^\dagger = M_t + A_t$ *a.s.*

Corollary 1 *Let $\{\mathcal{G}_t; t \geq 0\}$ be a right continuous filtration. Suppose $\{N_t; t \geq 0\}$ be a counting process which is $\mathcal{G}_t$- adapted and $E\{N_t\} < \infty$ for any t. Then, there exists a unique increasing right- continuous $\mathcal{G}_t$- predictable process A_t such that $A_0 = 0$ almost surely, $E\{A_t\} < \infty$ for any t, and $M_t = N_t - A_t; t \geq 0$ is a right- continuous $\mathcal{G}_t$- martingale.*

For any submartingale $M_t^\dagger$ with Doob- Meyer decomposition $M_t^\dagger = M_t + A_t, M_0 = 0$ implies that $E\{M_t^\dagger\} = E\{A_t\}$. It is worth noting that if M_t is a martingale, $E\{M_t^2\} < \infty$, and by Jensen's inequality, we can show that $(M_t^\dagger)^2$ is a submartingale and hence can be decomposed into a martingale and a compensator, which has a form discussed in the next corollary.

Corollary 2 *For $t \geq 0$, suppose M_t be a a cadlag martingale with respect to $\mathcal{G}_t; t \geq 0$. Assuming that $E\{M_t^2\} < \infty$, then there exists a unique right-continuous increasing predictable process denoted by $\langle M, M\rangle(t)$, known as a predictable quadratic variation of M_t such that $\langle M, M\rangle(0) = 0$ a.s. $E\langle M, M\rangle(t) < \infty$ for each t and $\{M_t^2 - \langle M, M\rangle(t); t \geq 0\}$ is a right- continuous martingale.*

Corollary 3 *Predictable variation of a martingale, $\{M = M_t; t \geq 0\}$ is a compensator of the process, M_t^2. As t increases, M_t^2 also increases.*

Definition 21 *A Brownian motion process is a stochastic process $B_t, t \in \mathbb{R}_+$, which satisfies the following:*

1. *The process starts at the origin; $B_0 = 0$ almost surely,*
2. *B_t has independent increments; that is for any times sequence $t_0 < t_1 < \cdots < t_n$, $B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}$, which are the increments, are independent,*
3. *B_t is continuous in t,*
4. *The increments $B_t - B_s$ follow a normal distribution with mean zero and variance, $t - s$, for $0 \leq t < s$.*

Brownian motion has applications in physics, finance, statistics, etc. Examples of applications of Brownian motion in finance is in stochastic interest rates and stochastic mortality models which will be utilized in this work.

A simple definition of a stochastic integral with respect to a Brownian motion is given as

$$\int_0^\infty f(t)dB_t = \int_0^\infty f(t)\frac{dB_t}{dt}dt,$$

but this definition is not applicable due to the fact that the sample paths of a Brownian motion are not differentiable even though it is continuous, that is

$$\frac{dB_t}{dt} = \frac{\pm\sqrt{dt}}{dt} = \pm\frac{1}{\sqrt{dt}} \simeq \pm\infty.$$

As a result, stochastic integrals will first be constructed as integrals of simple predictable processes.

Definition 22 *Let $\mathcal{P}$ denote the space of simple predictable processes $(u_t)_{t\in\mathbb{R}}$ of the form*

$$u_t = \sum_{i=1}^{n} F_i 1_{(t_{i-1},t_i]}(t), \quad t \in \mathbb{R}_+,$$

where $F_i \in L^2(\Omega, F_{t_{i-1}}, \mathbb{P})$ is $\mathcal{G}_{t_{i-1}}$- measurable, $i = 1, \ldots, n$. Then the stochastic integral with respect to Brownian motion $(B_t)_{t\in\mathbb{R}+}$ of this form is given by

$$\int_0^\infty u_t dB_t := \sum_{i=1}^{n} F_i(B_{t_i} - B_{t_{i-1}}),$$

and also extends to $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$ via the isometry formula

$$\mathbb{E}\left[\int_0^\infty u_t dB_t \int_0^\infty v_t dB_t\right] = \mathbb{E}\left[\int_0^\infty u_t v_t dt\right],$$

$u, v \in L^2_{ad}(\Omega \times \mathbb{R}_+)$.

Proposition 3 *For any $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$,*

$$\mathbb{E}\left[\int_0^\infty u_s dB_s \middle| \mathcal{G}_t\right] = \int_0^t u_s dB_s, \quad t \in \mathbb{R}_+.$$

In particular, $\int_0^t u_s dB_s$ is $\mathcal{G}_t$- measurable, $t \in \mathbb{R}_+$. In particular, $\mathbb{E}\left[\int_0^\infty u_s dB_s\right] = 0$.

Corollary 4 *The indefinite stochastic integral $\left(\int_0^t u_s dB_s\right)_{t\in\mathbb{R}_+}$ of $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$ is a martingale, that is*

$$\mathbb{E}\left[\int_0^t u_\tau dB_\tau \middle| \mathcal{G}_s\right] = \int_0^s u_\tau dB_\tau, \quad 0 \le s \le t. \tag{2.1}$$

It follows from (2.1) that

$$\mathbb{E}\left[\int_t^\infty u_\tau dB_\tau \middle| \mathcal{G}_t\right] = 0 \quad \textit{and} \quad \mathbb{E}\left[\int_0^t u_\tau dB_\tau \middle| \mathcal{G}_t\right] = \int_0^t u_\tau dB_\tau. \tag{2.2}$$

Particularly, $\int_0^t u_\tau dB_\tau$ is $\mathcal{G}_t$- measurable for all $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$.

Proposition 4 *Let $f \in L^2(\mathbb{R}_+)$. The stochastic integral $\int_0^\infty f(t)dB_t$ is a Gaussian random variable with mean 0 and variance $\int_0^\infty |f(t)|^2 dt$.*

We will now introduce the concept of quadratic variation of a Brownian motion.

Definition 23 *The quadratic variation of $(B_t)_{t\in\mathbb{R}+}$ is the process $\left(\langle B, B\rangle(t); t \in \mathbb{R}_+\right)$ defined by*

$$\langle B, B\rangle(t) = B_t^2 - 2\int_0^t B_s dB_s, \quad t \in \mathbb{R}_+.$$

Proposition 5 *Let $\pi = \{0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t\}$ be a family of subdivisions of $[0, t]$ such that $|\pi| := \max\limits_{i=1,2,\ldots,n} |t_i - t_{i-1}|$ converges to 0 as $n \to \infty$. Then it follows that*

$$\langle B, B\rangle(t) = \lim_{n\to\infty} \sum_{i=1}^n (B_{t_i} - B_{t_{i-1}})^2 \quad t \in \mathbb{R}_+, \tag{2.3}$$

where the limit exists in $L^2(\Omega)$ and is independent of the sequence $(\pi)_{n\in\mathbb{N}}$ of subdivisions chosen.

Definition 24 *A p-dimensional Brownian motion is a process,* ***B(t)*** $= \{B_1(t), B_2(t), \ldots, B_p(t)\}$ *which has the associated filtration $\mathcal{G}_t$, and has the following properties*

1. *Each $B_i(t)$ is a one dimensional Brownian motion,*

2. *$B_i(t), B_j(t); i \neq j$ are independent,*

3. *(Information accumulates) For $0 \leq s < t$, every set $\mathcal{G}_s$ is also in $\mathcal{G}(t)$,*

4. *(Adaptivity For each $t \geq 0$, the random vector $B(t)$ is $\mathcal{G}_s$- measurable,*

5. *(Independence of future increments) For each $0 \leq t < u$, the vector increments $B(u) - B(t)$ is independent of $\mathcal{G}_t$.*

2.4. REVIEW OF RELEVANT FINANCIAL TERMS

The definitions, lemmas, theorems and concepts that follow are from Calin (2012), Privault (2012) and Cairns (2018).

Definition 25 ***Risk- free rate of interest and short rate***
Let $R(t,T)$ be the risk- free rate of interest rate from time t to T. The instantaneous risk-free rate of interest is given as

$$r_t = \lim_{T\to t} R(t,T) = R(t,t).$$

The instantaneous risk- free rate is sometimes known as the short rate.

The value of a commodity or a financial security at a future date $t > 0$, V_t continuously compounded at a constant rate r is given as $V_t = V_0 \exp(rt)$ and this can be rewritten as a differential equation a $\dfrac{dV_t}{V_t} = r dt$. We can also have time- dependent interest rates so that the value of a commodity or a financial security at a future date is given as $V_t = V_0 \exp\left(\int_0^t r_s ds\right)$. r_s here is a time dependent random process known as *short term interest rate process or short rates*.

Instead of companies going for loans from the government and other lending agencies, they choose to borrow money from investors by issuing bonds.

Definition 26 ***Bonds***

1. *A bond is an investment in which the buyer lends an amount to the issuer in anticipation of an agreed series of payments.*

2. *These series of payments are determined based on some short term interest rate, $(r_t)_{t\in\mathbb{R}_+}$. The periodic stream of payments are called coupons of the bond.*

It is worth noting that not all bonds pay coupons.

Definition 27 *A zero-coupon bond is a contract which pays an amount of $B(T,T)$ at the maturity of the bond, T, where $B(t,T)$ is the price of this bond at time $t, 0 \leq t \leq T$. These bonds do not make coupon payments.*

For simplicity, we shall assume that $B(T,T) = \$1$. T is known as the *term to maturity of the bond*

2.4.1. Interest Rates. The underlying short- term interest rate for pricing derivatives (bonds in this case) may be fixed or variable, deterministic or stochastic. Below is the relationship between the price of a zero- coupon bond and the different types of interest rates.

(i) When the short term rate is deterministic and constant, $(r_t)_{t\in\mathbb{R}_+} = r$ for $0 \leq t \leq T$, the zero- coupon bond price at time t is given by:

$$B(t,T) = B(T,T)\exp\{-r(T-t)\}.$$

(ii) When the short rate is deterministic and time- dependent $(r_t)_{t\in\mathbb{R}_+} = r_t$ for $0 \leq t \leq T$, the zero- coupon bond price at time t is given as:

$$B(t,T) = B(T,T)\exp\left(-\int_t^T r(s)ds\right).$$

The pricing of a zero- coupon bond when the interest rate is a stochastic process is discussed in a later section.

Definition 28 ***Spot rate*** *The spot rate $R(r,T)$ at time t for maturity at time T is defined as the yield to maturity of the T- year bond. We have that*

$$R(t,T) = -\frac{\log P(t,T)}{T-t},$$

so that

$$P(t,T) = \exp\{-R(t,T)(T-t)\}.$$

This means a dollar investment at time t in a T- year bond will grow at an average rate of return of $R(t,T)$ for the remaining $T-t$ years.

2.4.2. Stochastic Interest Rates. We review stochastic interest rates in this section. There are one factor interest rate models and multi-factor interest rate. An assumption for the one-factor models is that they are driven by a one factor Wiener process. Short rate models are usually easy to work with analytically and that makes them very popular. Interest rates exhibit a variety of characteristics such as positivity, boundedness, and return to equilibrium and hence need the construction of particular models to account for these special features. Interest rate does not remain constant over a long period of time in the real world; it fluctuates in an unpredictable fashion, making the use of stochastic interest rates more appropriate in the design of financial securities. We will restrict our discussion to mean- reverting stochastic interest rate models.

Definition 29 *Mean reverting short rates are interest rate models that converge to an average level in the long run (that is, as t $\rightarrow \infty$).*

Let $B(t)$ is a standard Brownian motion and let $r_t, t \in \mathbb{R}_+$ be the spot rate at time t, that is the rate at which one can invest for a short period of time such as a second or a day. Below is a review of mean- reverting interest rate models which are very popular for use both in interest rate modeling and mortality modeling.

1. **Vasicek model** Vasicek (1977). This was the first model proposed which captures the mean reversion property in interest rates. This model was created as an extension of the Ornstein- Uhlenbeck process. For $r_t, t \in \mathbb{R}_+$, the Vasicek model satisfies the stochastic differential equation:

$$dr_t = a(b - r_t)dt + \sigma dB(t),$$

where a, b, σ are positive constants. The parameter a indicates the rate at which the interest rate adjusts to b. The solution of this equation is

$$r_t = b + (r_0 - b)e^{-at} + \sigma e^{-at} \int_0^t e^{as} dB(s),$$

r_t is a Gaussian process with mean and variance.

$$\begin{aligned} E[r_t] &= b + (r_0 - b)e^{-at}, \\ Var[r_t] &= \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right). \end{aligned}$$

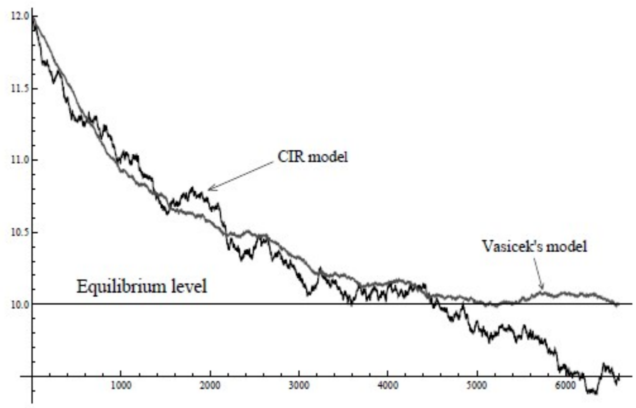

Figure 2.1. Vasicek model with $r_0 = 12, a = 3, \sigma = 0.15, b = 10$ Calin (2012)

The long run mean and variance respectively are such that

$$\begin{aligned}\lim_{t\to\infty} E[r_t] &= \lim_{t\to\infty} \left(b + (r_0 - b)e^{-at}\right) \\ &= b, \\ \lim_{t\to\infty} Var[r_t] &= \lim_{t\to\infty} \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right) \\ &= \frac{\sigma^2}{2a}.\end{aligned}$$

The disadvantage of this model is that it allows for negative interest rates and unbounded large rates. Investors would prefer to hold on to cash and not invest if interest were negative, hence negative interest rates are not ideal. However, in the real world, negative interest rates rarely occur, making this still a very popular model. The short rate is expected to increase if $r_t < b$ and decrease if $r_t > b$.

2. **Cox- Ingersoll- Ross model** Cox *et al.* (1985b). The Cox-Ingersoll-Ross (CIR) model assumes that the spot rates verify the stochastic differential equation:

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dB_t,$$

where a, b, σ are positive constants. The solution of this equation is

$$r_t = r_0 + abt - a\int_0^t r_s ds + \sigma \int_0^t \sqrt{r_s} dW_s.$$

r_t is not a Gaussian process and cannot be negative. The mean and second moment of the CIR model respectively are

$$\begin{aligned} E[r_t] &= b + e^{-at}(r_0 - b), \\ E[r_t^2] &= r_0^2 e^{-2at} + \left(2ab + \sigma^2\right)\left[\frac{b}{2a}\left(1 - e^{-2at}\right) + \frac{r_0 - b}{a}\left(1 - e^{-at}\right) e^{-at}\right]. \end{aligned}$$

Interest rates under CIR are always positive and the volatility of the short rate rises as the interest rate increases.

Figure 2.1 shows both the Vasicek and the Cox- Ingersoll- Ross interest rate processes.

3. **Hull and White model** Hull and White (1990) The Hull and White model is a time dependent extension of the Vasicek model assumes that the spot rates verify the stochastic differential equation:

$$dr_t = (\theta(t) - a(t)r_t)dt + \theta(t)dB(t),$$

where a and σ are constants. The solution of this equation is

$$r_t = r_0 e^{-at} + \frac{\sigma^2}{2a^2}(1 - e^{at})^2 + \sigma e^{-at}\int_0^t e^{as} dB(s).$$

$$E[r_t] = r_0 e^{-at} + \frac{\sigma^2}{2a^2}(1 - e^{at})^2.$$

$$Var[r_t] = \sigma^2 e^{-2at}\int_0^t e^{2as} ds = \frac{\sigma^2}{2a}(1 - e^{-2at}).$$

Theorem 3 ***Fundamental Theorem of Asset Pricing***

Let $(\Omega, \mathcal{G}, \mathbb{P})$ *with* $\mathbb{P}$ *being the real- world probability measure. Define*

$$B(t) = B(0)exp\left(\int_0^t r(s)ds\right).$$

1. *Bond prices evolve in a way that is arbitrage free if and only if there exists a martingale measure* $\widetilde{\mathbb{P}}$*, equivalent to* $\mathbb{P}$*, under which, for each* T*, the discounted price process* $\frac{P(t,T)}{B(t)}$ *is a martingale for all* $t: 0 < t < T$.
2. *If* (i) *holds, then the market is complete if and only if* $\widetilde{\mathbb{P}}$ *is the unique measure under which the* $\frac{P(t,T)}{B(t)}$ *are martingales.*

$\widetilde{\mathbb{P}}$ *is also referred to as a risk- neutral measure or a risk- adjusted measure.*

Prior to discussing how to price bonds using the martingale method, it is necessary to understand the Girsanov theorem.

A Brownian motion can be interpreted informally as a random walk, using the infinitesimal increments in its sample path, $\Delta B_t = \pm\sqrt{dt}$ such that

$$\mathbb{P}(\Delta B_t = +\sqrt{dt}) = \mathbb{P}(\Delta B_t = -\sqrt{dt}) = \frac{1}{2}. \tag{2.4}$$

A Brownian motion can either be a standard Brownian motion, that is one that has mean 0, or can be a drifted process, $\nu t + B_t; \nu \in \mathbb{R}$ with

$$\mathbb{E}[\nu t + B_t] = \nu t + \mathbb{E}[B_t] = \nu t. \tag{2.5}$$

Equation (2.5) can be rewritten in terms of the infinitesimal increments of the Brownian motion paths as

$$\mathbb{E}[\nu t + B_t] = \frac{1}{2}(\nu dt + \sqrt{dt}) + \frac{1}{2}(\nu dt - \sqrt{dt}) = \nu dt \neq 0.$$

Figure 2.2 shows a Brownian motion with a drift. A drifted Brownian motion can be converted to a centered Brownian motion by changing the probabilities of ups and downs as defined in equation (2.4) and finding $p, q \in [0, 1]$ such that

$$\begin{cases} p(\nu dt + \sqrt{dt}) + q(\nu dt + \sqrt{dt}) = 0 \\ p + q = 1. \end{cases}$$

The solution can be derived as

$$p = \frac{1}{2}(1 - \nu\sqrt{dt}) \quad \text{and} \quad q = \frac{1}{2}(1 + \nu\sqrt{dt}). \tag{2.6}$$

Continuing to view Brownian motion as a discrete random walk with independent increments of $\pm\sqrt{dt}$, the associated probability density may be derived by multiplying the aforementioned probabilities and dividing by the normalization factor $\frac{1}{2^N}$ as follows

$$\begin{aligned} 2^N \prod_{0<t<T} \left(\frac{1}{2} \pm \frac{1}{2}\nu\sqrt{dt}\right) &= \prod_{0<t<T} \left(1 \pm \nu\sqrt{dt}\right) \\ &= \exp\left(\log \prod_{0<t<T} \left(1 \pm \nu\sqrt{dt}\right)\right) \\ &= \exp\left(\sum_{0<t<T} \log\left(1 \pm \nu\sqrt{dt}\right)\right) \\ &\simeq \exp\left(\nu \sum_{0<t<T} \pm\sqrt{dt} - \frac{1}{2} \sum_{0<t<T} (\pm \nu\sqrt{dt})^2\right) \\ &= \exp\left(\nu \sum_{0<t<T} \pm\sqrt{dt} - \frac{1}{2}\nu^2 \sum_{0<t<T} dt\right) \\ &= \exp\left(-\nu B_T - \frac{1}{2}\nu^2 T\right). \end{aligned}$$

With this, the drifted process is transformed into a standard Brownian motion under the probability the new probability measure $\widetilde{\mathbb{P}}$ defined by

$$d\widetilde{\mathbb{P}}(\omega) = \exp\left(-\nu B_T - \frac{1}{2}\nu^2 T\right) d\mathbb{P}(\omega),$$

where $\Omega = C_0([0,T])$ is the Wiener space and $\omega \in \Omega$ is a continuous function on $[0,T]$. As a matter of fact, the Girsanov theorem can be applied to shifts by adapted processes through the following formulation.

Define a non- negative random variable Z on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ satisfying $\mathbb{E}(Z) = 1$. We define a new probability measure $\widetilde{\mathbb{P}}$ as

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega) dP(\omega) \quad \text{for all A} \in \mathcal{G}. \tag{2.7}$$

Hence any random variable X will now have two expectations, th first under the initial probability measure $\mathbb{P}$, denoted by $\mathbb{E}(X)$ and the second under this new probability measure $\widetilde{\mathbb{P}}$, denoted by $\widetilde{E}(X)$ such that

$$\widetilde{E(X)} = \mathbb{E}(XZ). \tag{2.8}$$

Z is therefore the *Radon- Nikodým derivative* of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$ so that Z is a ratio of two probability measures

$$Z = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}. \tag{2.9}$$

Theorem 4 *Let $W(t)$ be a Brownian motion. The ito integral, $I(t) = \int_0^t \Delta(u) dW(u)$, which is a Brownian motion is a martingale.*

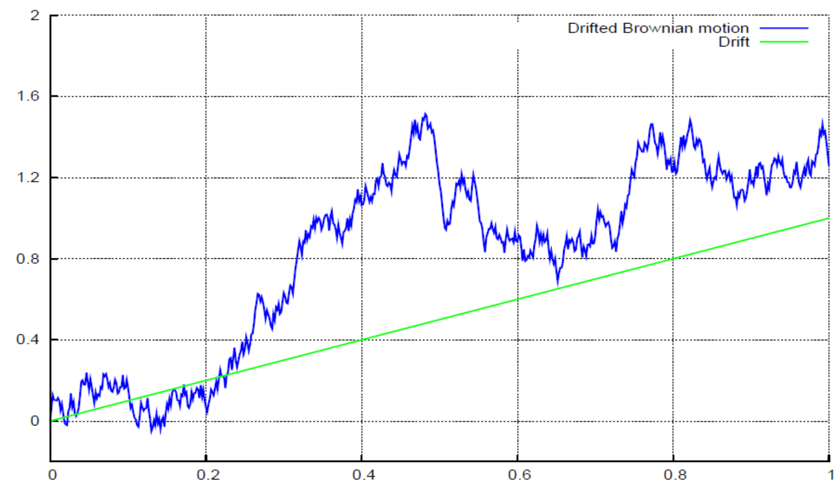

Figure 2.2. Brownian motion with a drift Privault (2012)

Proof

Let $0 \leq s \leq t \leq T$, suppose $t_l < t_k, t_l \leq s < t_{l+1}$ and $t_k \leq t < t_{k+1}$. Then we can write $I(t)$ as:

$$\begin{aligned}
I(t) &= \int_0^t \Delta(u)dW(u) \\
&= \sum_{j=0}^{l-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_l)[W(t_{l+1}) - W(t_l)] \\
&\quad + \sum_{j=l+1}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)] \\
\mathbb{E}(I(t)|\mathcal{G}(s)) &= \mathbb{E}\Big(\sum_{j=0}^{l-1} \Delta(t_j)\left[W(t_{j+1}) - W(t_j)\right]\Big|\mathcal{G}(s)\Big) \\
&\quad + \mathbb{E}\Big(\Delta(t_l)\left[W(t_{l+1}) - W(t_l)\right]\Big|\mathcal{G}(s)\Big) \\
&\quad + \mathbb{E}\Big(\sum_{j=l+1}^{k-1} \Delta(t_j)\left[W(t_{j+1}) - W(t_j)\right]\Big|\mathcal{G}(s)\Big) \\
&\quad + \mathbb{E}\Big(\Delta(t_k)\left[W(t) - W(t_k)\right]\Big|\mathcal{G}(s)\Big).
\end{aligned}$$

We need to show that $\mathbb{E}\left[I(t)|\mathcal{G}(s)\right] = I(s)$. Note that every random variable in $\sum_{j=0}^{l-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)]$ is $\mathcal{G}(s)-$ measurable and so

a. $\mathbb{E} \sum_{j=0}^{l-1} \Delta(t_j) \left[W(t_{j+1}) - W(t_j)|\mathcal{G}(s)\right] = \sum_{j=0}^{l-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)]$.

b. $\mathbb{E}\left(\Delta(t_l)\left[W(t_{l+1}) - W(t_l)|\mathcal{G}(s)\right]\right)$
$= \Delta(t_l)\mathbb{E}\left[W(t_{l+1})|\mathcal{G}(s)\right] - \Delta(t_l)\mathbb{E}\left[W(t_l)|\mathcal{G}(s)\right]$
$= \Delta(t_l)[W(s) - W(t)]$.

c. $\mathbb{E}\left(\Delta(t_j)\left[W(t_{j+1}) - W(t_j)\right]\Big|\mathcal{G}(s)\right)$
$= \mathbb{E}\left(\mathbb{E}\left\{\Delta(t_j)\left[W(t_{j+1}) - W(t_j)\right]\Big|\mathcal{G}(t_j)\right\}\Big|\mathcal{G}(s)\right)$
$= \mathbb{E}\Big(\left\{\Delta(t_j)\mathbb{E}\left[W(t_{j+1})\Big|\mathcal{G}(t_j)\right] - \Delta(t_j)\mathbb{E}\left[W(t_j)\Big|\mathcal{G}(t_j)\right]\right\}|\mathcal{G}(s)\Big)$
$= \mathbb{E}\big(\Delta(t_j)[W(t_j) - W(t_j)]\big|\mathcal{G}(s)\big)$
$=0$.

d. $\mathbb{E}\Big(\Delta(t_k)\left[W(t_{k+1}) - W(t_k)\right]|\mathcal{G}(s)\Big)$
$= \mathbb{E}\Big(\mathbb{E}[\Delta(t_k)\{W(t_{k+1} - W(t_k))|\mathcal{G}(t_k)\}]\Big|\mathcal{G}(s)\Big)$
$= \mathbb{E}\Big(\Delta(t_k)\{\mathbb{E}\left[W(t_{k+1})|\mathcal{G}(t_k)\right] - \mathbb{E}\left[W(t_k)|\mathcal{G}(t_k)\right]\}\Big|\mathcal{G}(s)\Big)$
$= \mathbb{E}\Big(\Delta(t_k)\left[W(t_k) - W(t_k)\right]\Big|\mathcal{G}(s)\Big)$
$= 0$.

Putting a, b, c, d together, we have

$$\begin{aligned}
\mathbb{E}(I(t)|\mathcal{G}(s)) &= \sum_{j=0}^{l-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_l)[W(s) - W(t)] \\
&= I(s). \qquad (2.10)
\end{aligned}$$

Theorem 5 ***Itô- Doeblin formula for Brownian motion***. *Let $f_(t,x)$ be a function for which the partial derivatives $f_t(t,x) = \frac{\partial f_t(t,x)}{\partial t}$, $f_x(t,x) = \frac{\partial f_t(t,x)}{\partial x}$ and $f_{xx}(t,x) = \frac{\partial^2 f_t(t,x)}{\partial x^2}$ are defined and continuous, and let $W(t)$ be a Brownian motion. Then, for every $T \geq 0$,*

$$\begin{aligned} f(T,W(T)) &= f(0,W(0)) + \int_0^T f_t(t,W(t))dt \\ &\quad + \int_0^T f_x(t,W(t))dW(t) + \frac{1}{2}\int_0^T f_{xx}(t,W(t))dt. \end{aligned} \tag{2.11}$$

Theorem 6 ***One dimensional Lévy's Theorem***. *Let $M(t)$, $t \geq 0$ be a martingale relative to a filtration $\mathcal{G}_t, t \geq 0$. Assume that $M(0) = 0$, $M(t)$ has continuous paths, and has the quadratic variation $\langle M, M\rangle(t) = t$ for all $t \geq 0$. Then $M(t)$ is a Brownian motion.*

Proof *A Brownian motion is a martingale which has independent increments which follow the normal distribution, therefore to show that $M(t)$ is a Brownian motion, we will show that its moment generating function is that of a normal random variable Applying the Itô- Doeblin formula for Brownian motion to this martingale which is also a Brownian motion , we have*

$$df(t,M(t)) = f_t(t,M(t))dt + f_x(t,M(t))dM(t) + \frac{1}{2}f_{xx}(t,M(t))dt, \tag{2.12}$$

which when integrated gives

$$\begin{aligned} f(t,M(t)) &= f(0,M(0)) + \int_0^t [f_t(s,M(s)) + \frac{1}{2}f_{xx}(s,M(s))]ds \\ &\quad + \int_0^t f_x(s,M(s))dM(s). \end{aligned} \tag{2.13}$$

For a fixed u, define a function

$$f(t,x) = exp\{ux - \frac{1}{2}u^2 t\},$$

so that

$$f_t(t,x) = -\frac{1}{2}u^2 t f(t,x); \quad f_x(t,x) = uf(t,x); \quad f_{xx}(t,x) = u^2 f(t,x).$$

a. $f(0, M(0)) = exp\{u(0) - \frac{1}{2}u^2(0)\} = 1,$

b. $\int_0^t \left[f_t(s, M(s)) + \frac{1}{2}f_{xx}(s, M(s))\right] ds = \int_0^t \left[-\frac{1}{2}u^2 f(t,x) + \frac{1}{2}u^2 f(t,x)\right] ds = 0,$

c. $M(t)$ *is a martingale making* $\int_0^t f_x(s, M(s))dM(s)$ *also a martingale. At t=0, the expectation of this integral is always 0 since it takes the value 0.*

It follows then, that

$$\begin{aligned} \mathbb{E}f(t, M(t)) &= \mathbb{E}f(0, M(0)) + \mathbb{E}\int_0^t [f_t(s, M(s)) + \frac{1}{2}f_{xx}(s, M(s))]ds \\ &\quad + \mathbb{E}\int_0^t f_x(s, M(s))dM(s) \qquad (2.14) \\ &= f(0, M(0)) \\ &= 1. \end{aligned}$$

Observe that

$$\mathbb{E}\left[exp\left(uM(t) - \frac{1}{2}u^2 t\right)\right] = 1,$$

$$E(exp\,(uM(t))) = exp\left(\frac{1}{2}u^2 t\right). \qquad (2.15)$$

By the uniqueness theorem of moment generating functions, it follows that $M(t) \sim N(0, t)$ *and hence is a Brownian motion.*

Theorem 7 ***One- dimensional Girsanov theorem*** *Let* $B(t), 0 \le t \le T$, *be a Brownian motion on a probability space* $(\Omega, \mathcal{G}, \mathbb{P})$*, and let* $\mathcal{G}_t, 0 \le t \le T$ *be a filtration for this Brownian motion. Let* $\Theta(t), 0 \le t \le T$*, be an adapted process. Define*

$$Z(t) = \exp\left\{ -\int_0^t \Theta(u)dB(u) - \frac{1}{2}\Theta^2(u)du \right\}, \tag{2.16}$$

$$\widetilde{B(t)} = B(t) + \int_0^t \Theta(u)du, \tag{2.17}$$

and assume that

$$\mathbb{E}\left(\int_0^T \Theta^2(u)Z^2(u)du \right) < \infty. \tag{2.18}$$

Set $Z = Z(T)$*. Then* $\mathbb{E}(Z) = 1$ *and under the probability measure* $\widetilde{\mathbb{P}}$ *given by*

$$\widetilde{\mathbb{P}(A)} = \int_A Z(\omega)dP(\omega) \quad \textit{for all } A \in \mathcal{G},$$

the process $\widetilde{B(t)}, 0 \le t \le T$*, is a Brownian motion.*

Proof

1. $\widetilde{B(t)} = B(t) + \int_0^t \Theta(u)du$.

2. $\left\langle \widetilde{B(t)}, \widetilde{B(t)} \right\rangle = \left\langle B(t), B(t) \right\rangle = t$ since $\int_0^t \Theta(u)du$ is constant and does not affect the quadratic variation of $\widetilde{B(t)}$.

3. Showing that $\widetilde{B(t)}$ is a martingale under $\widetilde{\mathbb{P}}$.

Proof of 3

Let

$$
\begin{aligned}
X(t) &= -\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)dW(u), \\
dX(t) &= -\Theta(t)dW(t) - \frac{1}{2}\Theta^2(t)dW(t), \\
[dX(t)]^2 &= \Theta^2(t)dt.
\end{aligned}
$$

Define $Z(t) = f(X(t)) = e^{X(t)}$, $f'(x) = e^{X(t)}$, $f''(x) = e^{X(t)}$.

$$
\begin{aligned}
d(f(X(t))) &= f'(X(t))dX(t) + \frac{1}{2}f''(X(t))[dX(t)]^2 \\
&= e^{X(t)}[-\Theta(t)dW(t) - \frac{1}{2}\Theta^2(t)dt] + \frac{1}{2}e^{X(t)}[\Theta^2(t)dt] \\
&= -e^{X(t)}\Theta(t)dW(t) \\
&= -\Theta(t)Z(t)dW(t). \qquad (2.19)
\end{aligned}
$$

Integrating both sides

$$
\begin{aligned}
f(X(t)) &= f(X(0)) - \int_0^t \Theta(u)Z(u)dW(u) \\
Z(t) &= Z(0) - \int_0^t \Theta(u)Z(u)dW(u). \qquad (2.20)
\end{aligned}
$$

The right hand side of Equation (2.20) is an Ito Integral. Define the Radon- Nikodým derivative after (2.9) as follows:

$$Z = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}.$$

Observe that

a. $\mathbb{E}[Z(t)] = \mathbb{E}[Z(0)] - \mathbb{E}\left[\int_0^t \Theta(u)Z(u)dW(u)\right] = \mathbb{E}[Z(0)] = 1,$

b. $Z(t) = E[Z(t)|\mathcal{G}_t] = E[Z|\mathcal{G}_t], \quad 0 \le t \le T.$

Next, we show that $\widetilde{W(t)}Z(t)$ is a martingale under $\mathbb{P}$

$$\begin{aligned} d(\widetilde{W(t)}Z(t)) &= \widetilde{W(t)}dZ(t) + Z(t)d\widetilde{W(t)} + d\widetilde{W(t)}dZ(t) \\ &= -\widetilde{W(t)}\Theta(t)Z(t)dW(t) + Z(t)dW(t) + Z(t)\Theta(t)dt \\ &\quad + (dW(t) + \Theta(t)dt)(-\Theta(t)Z(t)dW(t)) \\ &= (-\widetilde{W(t)}\Theta(t) + 1)Z(t)dW(t). \end{aligned} \tag{2.21}$$

Observe that the final equation in Equation (2.21) has no dt term and hence $\widetilde{W(t)}Z(t)$ is a martingale under $\mathbb{P}$. Let $0 \le s \le t \le T$. Since $\widetilde{W(t)}Z(t)$ is a martingale, we have

$$\begin{aligned} E(\widetilde{W(t)}|\mathcal{G}_s) &= \frac{1}{Z(s)}E[\widetilde{W(t)}Z(t)|\mathcal{G}_s] \\ &= \frac{1}{Z(s)}\widetilde{W(t)}Z(s) \\ &= \widetilde{W(t)}. \end{aligned} \tag{2.22}$$

This shows that $\widetilde{W(t)}$ is a martingale under $\widetilde{\mathbb{P}}$. We have shown that $W(0) = 0$, $W(t)$ has continuous paths and has quadratic variation of t and so by the Leóy theorem, $\widetilde{W(t)}$ is a martingale.

2.5. NO ARBITRAGE BOND PRICING

When pricing derivatives, the no-arbitrage principle is applied. This is to prevents traders from price differences to make profit . Derivatives are therefore priced under an equivalent measure or a risk- neutral measure, $\widetilde{\mathbb{P}}$ instead of a real world measure, $\mathbb{P}$.

Remark 1 *Following from Theorem (3), if $C(t)$ is some $\mathcal{G}_T$- measurable payoff of a derivative, and if $P(t,T)$ is the fair price of the derivative at time t, then the discounted derivative proud process $\frac{P(t,T)}{B(t)}$ is also a martingale under the risk neutral measure $\widetilde{P}$ and the price of* *a derivative at time t that has a payoff of $C(t)$ is given by*

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\left[C(t)\exp\left(-\int_t^T r(s)ds\right)\middle|\mathcal{G}_t\right]. \tag{2.23}$$

In the case of deterministic interest and constant interest rates (2.23) reduces to

$$\exp\left(-\int_t^T r(s)ds\right)\mathbb{E}_{\widetilde{\mathbb{P}}}\left[C(t)\middle|\mathcal{G}_t\right] \tag{2.24}$$

and

$$\exp\left(-(T-t)r(s)\right)\mathbb{E}_{\widetilde{\mathbb{P}}}[C(t)|\mathcal{G}_t]$$

respectively. Short rate processes can also be used to price derivatives and in that case, (2.23) would be the price of the derivative. We will focus the rest of the discussion on designing bonds using stochastic interest rates as the underlying short rate process for determing bond prices. These can be derived by finding the solution of stochastic differential equations of the form

$$dr_t = \alpha(t, r_t)dt + \sigma(t, r_t)dB(t), \tag{2.25}$$

where $B(t)_t \in \mathbb{R}+$ is a standard Brownian motion under the real world measure $\mathbb{P}$, and so $B(t) \sim N(0,t)$. The density process of this equivalent measure $\widetilde{\mathbb{P}}$ is given by

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left(-\int_0^t \Theta(s)dB(s) - \frac{1}{2}\int_0^t \Theta^2(s)ds\right).$$

We then obtain from (2.17) that

$$d\widetilde{B(t)} = dB(t) + \Theta(t)dt, \tag{2.26}$$

where $d\widetilde{B(t)}$ is a standard Brownian motion under the new measure $\widetilde{\mathbb{P}}$ and $\Theta(u)$ is an adapted process which is also known as the market price of the risk which can be obtained from market data. Substituting (2.26) into (2.25), we obtain

$$\begin{aligned} dr_t &= [\alpha(t,r_t) - \sigma(t,r_t)\Theta(t)]dt + \sigma(t,r_t)d\widetilde{B(t)} \\ &= \alpha^{\dagger}(t,r_t)dt + \sigma(t,r_t)d\widetilde{B(t)}. \end{aligned}$$

Theorem 8 ***Markov Property***. *The Markov property states that the future value after time t of a Markov process, $(X_s)_{s\in\mathbb{R}+}$ depends only on its present state t and not on the whole history of the process up to time t. This warrants us to write*

$$\mathbb{E}[f(X_{t_1}, X_{t_2}, \ldots, X_{t_n})|\mathcal{G}_t] = \mathbb{E}[f(X_{t_1}, X_{t_2}, \ldots, X_{t_n})|X_t],$$

for all $t_1, \ldots, t_n$ greater than t and all sufficiently integrable functions f on $\mathbb{R}^n$.

Theorem 9 *All solutions of stochastic differential equations such as (2.25) have the Markov property which reduces the arbitrage price (2.23) to*

$$P(t,T) = \mathbb{E}_{\widetilde{\mathbb{P}}}\left[\exp\left(-\int_t^T r(s)ds\right)\middle|r(t)\right]. \tag{2.27}$$

Definition 30 *The risk- neutral measure with maturity T with respect to the filtration $\mathcal{G}_t$ is the probability measure $\widetilde{\mathbb{P}}$ which is defined as*

$$\begin{aligned} \frac{d\widetilde{\mathbb{P}}|\mathcal{G}_t}{d\mathbb{P}|\mathcal{G}_t} &= \frac{\exp\left(-\int_t^T r(s)ds\right)}{P(t,T)}, \qquad 0 \le t \le T \\ &= \frac{\exp\left(-\int_t^T r(s)ds\right)}{\mathbb{E}_{\widetilde{\mathbb{P}}}\left[\exp\left(-\int_t^T r(s)ds\right)|\mathcal{G}_t\right]}, \end{aligned} \tag{2.28}$$

which is the same as the Radon- Nikodyḿ derivative in Equation (2.9).

PAPER

I. SECURITIZATION OF LONGEVITY RISK VIA TRANCHE- BASED SURVIVOR BONDS

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ABSTRACT

Due to advances in biomedical studies and improved lifestyle, longevity of humans has significantly increased. This results in insurance companies facing an increase in the likelihood of having to continue paying policyholders well beyond the life expectancy projected by life tables, leading to the so called longevity risk. Longevity risk is a prevalent problem in the life insurance industry. One way of dealing with longevity risk is by securitizing using longevity derivatives. Literature in securitization do not account for time varying covariates that are important predictors of future longevity. In this work, we propose an additive type mortality model in the flavor of Lee-Carter model that accounts for policyholders' concomitant covariates to better price life contingent products. We derive closed-form expressions of the model's unknowns using counting processes and martingale machinery. Numerical studies indicate that the asymptotic properties of model parameters' estimators when the size of the cohort of policyholders with fixed age is increasing, closely approximate the true. Those estimates are used to obtain survival probabilities and survivor bond payments, thereby helping to mitigate extended payments schedule. We use the proposed mortality model to determine the fair price off a survivor bind. This bond will be based on tranches with each tranche having a different price and expected return.

Keywords: Longevity risk; additive mortality; longevity derivatives; survivor bonds; securitization; tranches

1. INTRODUCTION

Over the years, there have been many significant advances in healthcare such as finding cures for diseases, as well as developing ways of living comfortably and longer with certain diseases which were once fatal. With much education and awareness creation, people have improved their lifestyle, for example, by practicing proper hygiene and eating healthy food, to name a few. These translate into longer lifetime than have been predicted by the life tables, hence causing a decline in mortality. The unpredictable nature of this decline causes a risk which is commonly known as *longevity risk*. Figure 1 shows an abridged life table for the U.S. population in 2019. Figure 2 also exhibits the increasing trend in life expectancy over the years from 1990 to 2020 for the U.S. population.

Any industry that makes payments to people depending on their survival is also prone to dealing with longevity risk. Among them are insurance companies since they provide life annuities to policyholders, private sector defined benefits pension plan providers and governments because they issue pension plans for the public sector workers. Longevity risk could even be of great concern to the insured especially for those with defined benefit schemes where they receive a lump sum amount, which is calculated based on the expected lifetime of the individuals; they may end up exhausting their retirement savings. Due to longevity risk, life insurance companies pay more benefits than budgeted, therefore they may run out of money to fulfill their commitment to policyholders. In recognition of this,

Age	mx	qx	lx	dx	Lx	Tx	ex
0	0.00557	0.00554	100000	554	99522	7915655	79.16
1	0.00039	0.00039	99446	38	99427	7816133	78.6
2	0.00023	0.00023	99408	23	99396	7716706	77.63
10	0.00012	0.00012	99295	12	99289	6921962	69.71
20	0.00078	0.00078	98978	77	98939	5930049	59.91
30	0.00134	0.00134	97964	131	97899	4944944	50.48
40	0.00208	0.00208	96397	200	96297	3972597	41.21
50	0.0039	0.0039	93857	366	93674	3020006	32.18
60	0.00902	0.00898	88410	794	88013	2105122	23.81
70	0.01819	0.01803	77938	1405	77235	1268792	16.28
80	0.04611	0.04507	58733	2647	57410	575371	9.8
90	0.12479	0.11746	26900	3160	25320	139653	5.19
100	0.33498	0.28693	3230	927	2767	8367	2.59

Figure 1. Abridged life table for the United States population in 2019

improved models for reserve estimation and pricing are called for. Even though longevity risk is one of the prevalent problems in the life insurance industry today, a lasting solution is yet to be determined.

To better manage this risk, life insurance companies either share it with reinsurance companies or issue longevity derivatives so as to transfer the risk to investors on the capital market. This approach is called *securitization of longevity risk*. Over the years, this has been done using derivatives such as survivor bonds, longevity swaps, longevity options and longevity forwards cf. Bauer *et al.* (2010). Insurers also sell life insurance products to the same insured as a way of hedging the risk. The price of these longevity derivatives are determined contingent on the survival of the insured. Blake and Burrows (2001) was the first to propose the use of longevity bonds to securitize longevity risk and this method has been well embraced and discussed by several authors, among which are Blake *et al.* (2006a), Blake *et al.* (2006b), Cairns *et al.* (2006a) and Cairns *et al.* (2006b).

In pricing longevity risk, mortality modeling plays a paramount role because the payoff of these bonds are dependent on the fluctuations in mortality. Because of this, an appropriate mortality model needs to be chosen. This can be taken as deterministic, or modeled in a dynamic fashion to better capture the time evolution of death. In literature today, stochastic mortality models are being used since in reality mortality is not deterministic, and mostly predictions made with deterministic mortality models end up not being so precise. The Lee- Carter model, introduced by Lee and Carter (1992) has been well celebrated. In the model, the logarithm of the central death rates was predicted by adopting singular value decomposition and time series methodologies. Since its introduction, there have been several extensions of the Lee Carter model, cf. Cairns *et al.* (2006a), Plat (2009) and Fung *et al.* (2017) and the references therein for detailed reviews of some existing stochastic mortality models, a few of which are Lee and Carter (1992) model extensions. Hunt and Blake (2014) suggests a step- by- step general approach for modeling mortality which is data- driven and requires the making of some intuitive decisions which may be subjective to each modeler. Dahl (2004) mentions the benefits of using stochastic intensity models since they produce more realistic premium and reserve calculations as well as they value the risk of the insurance company in relation to the force of mortality.

A wealth of current research demonstrates that mortality is affected by socioeconomic and environmental factors, health conditions and biological characteristics, therefore we deem it necessary to propose a model that will make use of variables related to these. In projecting the mortality rates for the purpose of dealing with longevity risk, we need a model that has good interpret-ability and also, one that factors in the improvements in individual lives, the biological characteristics of the individual and his health conditions, hence our choice of the additive model with time- varying covariates. We choose the additive model over the popular Cox- proportional hazards model since we are interested in the absolute effect of the covariates and not the relative effect. Our usage of an additive risk model with time- varying covariates for our force of mortality which will help us study the relationship between the important risk factors and the occurrence of an event; which in our case is the receipt of benefit. Receiving benefit corresponds to the event of being alive by the time of benefit payment. The mortality model since an individual's last benefit receipt would be:

$$\lambda_i(s) = \lambda_0(s) + \boldsymbol{\beta}'_{(\boldsymbol{x})}\boldsymbol{x_i(s)}$$

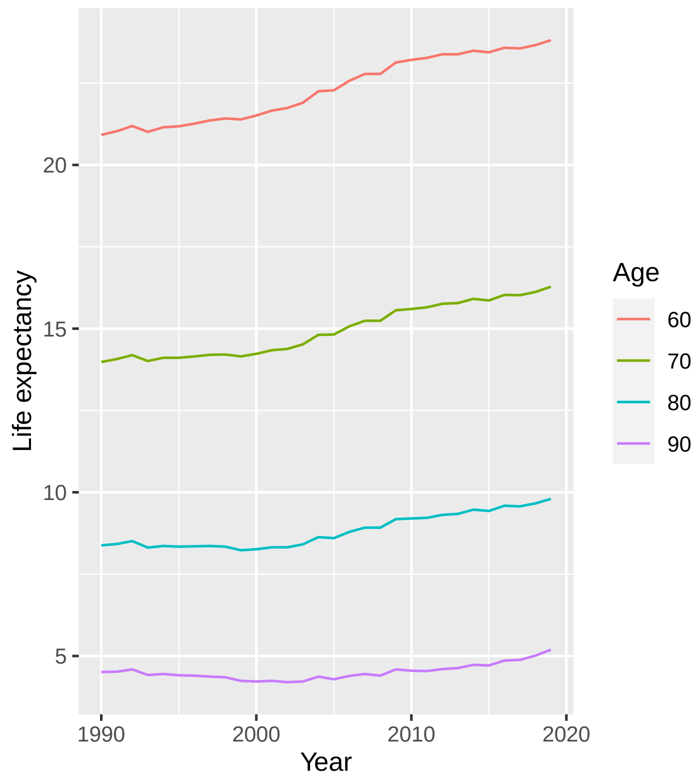

Figure 2. The graph shows an increasing trend in life expectancy from 1990 to 2020

where $\lambda_0(s)$ is the baseline hazard function, $\boldsymbol{x_i}$ is a p-dimensional vector of covariates and $\boldsymbol{\beta_{(x)}}$ are the age- specific regression coefficients corresponding to each covariate, $\boldsymbol{x_i(s)}$. As seen, the proposed model has the flavor of the Lee- Carter model. We will use the semi-parametric analysis approach developed by Lin and Ying (1994) to estimate the parameters $\lambda_0(s)$ and $\boldsymbol{\beta_{(x)}}$.

We use the mortality model proposed to derive the price of the longevity bond using tranches which will be based on percentage cumulative losses, an approach similar to how Collateralized Debt Obligations are designed. The bond pricing method will be the same as the approach considered in Wills and Sherris (2010).

This work is organized as follows. In section 2, we give a summary of some existing mortality models and introduce the mortality model we will use to design the longevity bond. In section 3, we obtain estimators for the unknown parameters in the model and investigate the asymptotic properties of these estimators. Section 4 houses the longevity bond pricing framework and the fair price of the bond is also presented. We investigate the finite sample properties through a simulation study in section 5. Some concluding remarks and acknowledgements are relegated to section 6.

2. MODEL AND UNDERLYING DYNAMICS

2.1. MORTALITY RATE BACKGROUND AND PROPOSED MODEL

Pricing of longevity risk begins with mortality modeling. Significant improvements in the duration of life have been observed in most countries in the last decade or so. As a consequence, a bad anticipation of this evolution threatens life insurers, that is it may lead to underestimated prices and reserves related to contract providing long-term living benefits Pitacco *et al.* (2002). To deal with this situation, a better model for mortality coupled with good pricing of underlying security are needed. To cope with the uncertainty in mortality trends, numerous models have been proposed since Gompertz published the mortality law in 1825. There are three general approaches to mortality forecasting namely *expectation*, *extrapolation* and *explanation*. Those encompass dynamic affine type processes, the 2-factor type model, the Lee-Carter model and its extensions for example Renshaw *et al.* (1996), Carter (1996), Lee and Miller (2001), Booth *et al.* (2002). One can cope with the uncertainty over mortality trends by modeling mortality as a dynamic model, as affine-type processes, cf. Lando (1998) and Biffis (2005), or as a generalized linear model using time series models. The generalized linear model has been shown to be a good model for mortality for UK male and UK annuitant and pensioner, Sithole *et al.* (2000) and can be viewed as an affine model. In this work, we focus on dynamic affine mortality models for the purpose of applying martingale and counting processes machinery for longevity risk applications and pricing. In affine model, the instantaneous mortality rate $\lambda_x(t)$ of a unit aged (x) is the sum of one deterministic component and one dynamic variable, Y_t, where Y_t represents random departure from the deterministic mortality table and has the form

$$\lambda_x(t) = \lambda_0^x(t) + Y_t^x(t).$$

The deterministic part $\lambda_0^x(t)$ can be taken to be of Gompert type, a Weibull type (with increasing mortality or decreasing mortality over time) or other type of infinite dimensional function that can be viewed as the best estimate assumption. The term $Y_t^x(\cdot)$ represents a random departure from the deterministic. The postulated model has the flavor of an affine type wherein we take the random departure from the deterministic to be a function of some unit specific concomitant variables that act additively on the deterministic. Specifically, consider a portfolio of units initially aged (x), that is all units in the portfolio have their age in $[x, x+1)$. Suppose n_x units are in the portfolio. At time t, a p-dimensional vector of covariates is recorded. We postulate that the instantaneous mortality rate, $\lambda_x(t)$ is given by

$$\lambda_x(t) = \lambda_0^x(t) + h(z(t), \boldsymbol{\beta}),$$

where $\boldsymbol{\beta}$, the regression coefficient and $\mathbf{z}$, a p-dimensional vector of coefficients can also be taken to be time dependent. The function $\mathrm{h}(\mathbf{z}(t), \boldsymbol{\beta})$ acts additively on the deterministic function $\lambda_0^x(t)$. The difference between our modeling here and the one in survival analysis is with respect to the modeling of the time of death when collection of benefits stops that can be viewed as a stopping time τ with respect to some filtration $\mathcal{G}_t$ that contains all payments and other actuarial information.

2.2. UNDERLYING DYNAMICS

Throughout the work, all random variables are defined on the probability space $(\Omega, \mathcal{G}, \mathrm{P})$. Consider a portfolio consisting of n_x independent policyholders who bought a policy at age x. In what follows, x denote the age of a policyholder i, so that, according to actuarial science parlance, (x) is any policyholder aged x. For any $i \in \{1, ..., n_x\}$, let S_{ij}, with $j = 1, 2, ..., m_i$ be the distinct calendar time sequence at which the benefits are received by individual i, m_i is the total number of benefits received by policyholder i before death. In the monitoring of the cohort, all policyholders may begin at the same time or enter the study at different times, however, the former approach will be taken here. Let τ_i be the random time of death of policyholder i. At calendar time S_{ij}, the remaining lifetime of policyholder i is $\bar{S}_{ij} = \tau_i - S_{ij}$. Let $\mathcal{O}_i$ be the σ-field with respect to which $\bar{S}_{ij}$ is a stopping time. Let $\{\mathcal{G}_t : t \geq 0\}$ be the filtration describing the evolution of the mortality rate dynamic as time elapses. So, we have two parallel filtrations describing the evolution of the entire cohort. The overall history, at time t of the entire cohort is contained in $\mathcal{H}_t$ given by

$$\mathcal{H}_t = \mathcal{G}_t \bigvee \vee_{i=1}^{n_x} \mathcal{O}_i$$

We now augment the probability space with the filtration, $\mathcal{H}_t$ so that the following random entities are defined on the filtered probability space $(\Omega, \mathcal{G}, \mathcal{H}_t, \mathrm{P})$. The portfolio is composed of a representative sample of a homogeneous cohort with same age and no significant difference in health status at time 0 when they enter the study. The term $\boldsymbol{\beta}' x$ acts additively on the baseline hazard and hence alters the health status and future likelihood of survival of each member of the cohort. The individual lifetimes of the cohort are assumed independent but not identically distributed since it is covariates dependent.

We now introduce the stochastic processes relevant for modeling. To that end, for $i \in \{1, ..., n_x\}$ and $j \in \{1, 2, ..., m_i\}$, we define a unit i and j^{th} collection time by the vector $(i, j) \in \{1, ..., n_x\} \times \{1, 2, ..., m_i\}$. At calendar time s, the time elapsed since the j^{th} benefit was received by the i^{th} policyholder is $\mathrm{R}_i(s) = s - S_{ij}$. The time between collections, so called gap times are $\{\mathrm{T}_{ij}, j = 1, 2, \ldots, m_i\}$ with $\mathrm{T}_{ij} = S_{ij} - S_{i,j-1}$. The gap times can be assumed to be independent and identically distributed, so called *renewal process*. Note that, in the insurance industry, payments of benefits could be regular or irregular, so that the former leads to a renewal process, whereas the latter can just be the independent case of payment schedule. The sequence of benefits received by individual (i, j) is denoted by $\{B_{ij}(s), j = 1, 2, \ldots, m_i\}$ which can be fixed or time dependent, that is increasing or decreasing over time. With $\mathrm{Y}_i(s) = \mathrm{I}\{\tau_i > s\}$ indicating whether or not an individual is alive by time s, the total number of payments made at time s is $\mathrm{Y}(s) = \sum_{i=1}^{n_x} Y_i(s)$, so that $n_x - \mathrm{Y}(s)$ is the number of policyholders who have died in the cohort at time s. Each policyholder collects benefits as long as he/she is alive, so that the counting process

$$\mathrm{N}_i(s) = \sum_{j=1}^{m_i} \mathrm{I}(S_{ij} \leq s < S_{i(j+1)}) = j\mathrm{I}(S_{ij} \leq s < S_{i(j+1)}),$$

tracks the number of collections of individual i. Using stochastic integration theory, the process

$$\mathrm{M}(s) = \mathrm{N}_i(s) - \int_0^s Y_i(u)\lambda_i(u)du; s \geq 0$$

is a zero-mean martingale with respect to the filtration $\mathcal{H}_s$. However, the natural time scale to be considered in this setting is the time between benefit collections. The processes with the aforementioned time are not martingale with respect to their corresponding filtration, but they are still zero-mean. Figure 3 shows a summary of the timeline of the ith policyholder.

To make inference about the parameter $\boldsymbol{\theta} = (\boldsymbol{\beta}, \lambda_0(t))$, the processes we could consider turn out to be those indexed by both calendar time and the lapse time between benefits collections. To that end, define $\mathcal{E}(t) = \mathrm{I}\{s - S_{ij} \leq t\}$ to be the time elapsed since the j^{th} collection. By virtue of the expressions of $\mathrm{N}_i(s)$ and $A_i(s) = \int_0^s Y_i(u)\lambda_i(u)du$, the

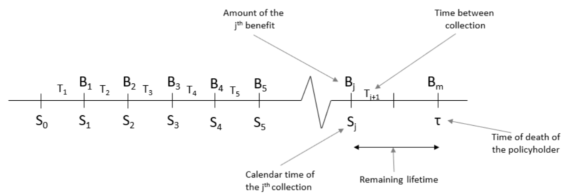

Figure 3. Timeline of ith policyholder

corresponding doubly-indexed processes are given by

$$\mathrm{N}_i(s,t) \quad \text{and} \quad A_i(s,t) = \int_0^t Y_i(s,u)[\lambda_0(u) + \boldsymbol{\beta}'\mathbf{x}(u+s)]du,$$

respectively where $\mathrm{N}_i(s,t)$ is the number of benefit collections before time t with gap time s. So that the process $\mathrm{M}_i(s,t) = \mathrm{N}_i(s,t) - \mathrm{A}_i(s,t)$ remains a martingale with respect to the filtration $\mathcal{H}_s$ for fixed t.

3. ESTIMATION AND LARGE COHORT PROPERTIES

3.1. ESTIMATING $\boldsymbol{\beta}$

We take the argument in the baseline mortality rate $\lambda_0(t)$ to be $s - S_{ij}$, so that the postulated model is $\lambda_i(s) = \lambda_0(s - S_{ij}) + \boldsymbol{\beta}'\mathbf{x}$. We have two unknowns here: the infinite dimensional parameter $\lambda_0(\cdot)$ and the regression coefficient $\boldsymbol{\beta}$. By virtue of the zero- mean martingale property of $\mathrm{M}_i(s,t)$, a method of moment estimate of $\Lambda_0(t)$ for fixed β is

$$\hat{\Lambda}_0(t|\boldsymbol{\beta}) = \int_0^t \frac{\sum_{i=1}^{n_x}[N_i(s,du) - Y_i(s,u)\boldsymbol{\beta}'\mathbf{x}_i(u+S_{ij})]}{\sum_{i=1}^{n_x} Y_i(s,u)} du.$$

Notice that $\hat{\Lambda}_0(t|\boldsymbol{\beta})$ is not yet an estimator since $\boldsymbol{\beta}$ is unknown. With $\nabla_{\boldsymbol{\beta}} = \left(\frac{\partial}{\partial \beta_i} : i = 1, ..., p\right)$, following Jacod (1975), the gradient of the likelihood process for estimating $\boldsymbol{\beta}$ is given by

$$U(\boldsymbol{\beta}; s, t) = \sum_{i=1}^{n_x} \int_0^t \nabla_{\boldsymbol{\beta}} \left[\log \lambda_i(u+s)\right] M_i(s, du). \tag{1}$$

Note that the score $U(\boldsymbol{\beta}; s, t)$ has an integrand that depends on the unknown baseline hazard. For $\boldsymbol{\beta}$ to be estimated using the score process $U(\boldsymbol{\beta}; s, t)$, the integrand has to be a predictable process. The integrand in the score depends on the infinite dimensional parameter $\lambda_0(t)$ that needs to be eliminated by substituting it with $\hat{\Lambda}_0(t|\boldsymbol{\beta})$. In doing so, an estimating function for $\boldsymbol{\beta}$ when taking the integrand to be a function EF(·;·) of the covariates is given by

$$\mathrm{EF}(\boldsymbol{\beta}; s, t) = \sum_{i=1}^{n_x} \int_0^t [\mathbf{x}_i(u+s) - \bar{\mathbf{x}}(u+s)][N_i(s, du) - Y_i(s, u)\boldsymbol{\beta}'\mathbf{x}(u+s)du], \tag{2}$$

where with

$$\bar{\mathbf{x}}(u+s) = \frac{\sum_{j=1}^{n} Y_j(s,u)\mathbf{x}_j(u+s)}{\sum_{j=1}^{n} Y_j(s,u)}.$$

Close form expression of $\boldsymbol{\beta}$ is obtained by setting $\mathrm{EF}(\boldsymbol{\beta}; s, t)$ to $\mathbf{0}$ leading to an estimate β given by

$$\begin{aligned} \hat{\boldsymbol{\beta}} &= \frac{1}{n_x} \sum_{i=1}^{n_x} \int_0^{\tau_i} Y_i(s,u) \left\{\mathbf{x}_i(u+s) - \bar{\mathbf{x}}(u+s)\right\}^{\otimes 2} du \\ &\quad \cdot \frac{1}{n_x} \sum_{i=1}^{n_x} [\mathbf{x}_i(u+s) - \bar{\mathbf{x}}(u+s)]\, N_i(s, du) \\ &= \hat{\mathbf{A}}^{-1} \hat{\mathbf{B}}. \end{aligned} \tag{3}$$

3.2. LARGE COHORT PROPERTIES

Before embarking on the pricing of longevity risk, it is important to address the properties of the estimators in this section when more policyholders are added to the cohort. We require consistency of the parameters for the purpose of pricing the derivative product.

We first deal with the cohort regression coefficient $\boldsymbol{\beta}$. Consistency and asymptotic normality of the parameters can be used to develop long term point-wise behavior of the price of the longevity risk and developing confidence intervals for the price. The latter is particularly important in the sense that a confidence interval allows for proper reserve planning to avoid the risk of not being able to fulfill their commitment with policyholders. The next theorem is on the consistency of $\boldsymbol{\beta}$ for a fixed cohort age.

Theorem 10 *Under the regularity conditions and as $n_x \to \infty$, $\hat{\boldsymbol{\beta}}$ converges almost surely to $\boldsymbol{\beta}_0$ and $\sqrt{n_x}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)$ converges in distribution to a p-dimensional zero-mean multivariate normal distribution with covariance matrix $\boldsymbol{\Xi} = \boldsymbol{A}^{-1}\boldsymbol{\Gamma}\boldsymbol{A}^{-1}$, where*

$$\boldsymbol{\Gamma} = \lim_{n_x \to \infty} \frac{1}{n_x} \sum_{i=1}^{n_x} \mathbb{E}(\boldsymbol{\Gamma}_i^{\otimes 2}),$$

with

$$\boldsymbol{\Gamma}_i = \int_0^{\tau_i} [\boldsymbol{x}_i(u+s) - \bar{\boldsymbol{x}}(u+s)]\, M_i(s, du).$$

Furthermore, $\boldsymbol{\Xi}$ can be consistently estimated by $\boldsymbol{\Xi} = \hat{A}^{-1}\hat{\boldsymbol{\Gamma}}\hat{A}^{-1}$ with

$$\hat{A} = \frac{1}{n_x} \sum_{i=1}^{n_x} \int_0^{\tau_i} Y_i(s,u)\, [\boldsymbol{x}_i(u+s) - \bar{\boldsymbol{x}}(u+s)]^{\otimes 2}\, du,$$

$$\hat{\boldsymbol{\Gamma}} = \lim_{n_x \to \infty} \frac{1}{n_x} \sum_{i=1}^{n_x} \mathbb{E}(\hat{\Gamma}_i^{\otimes 2}), \quad \hat{\boldsymbol{\Gamma}}_{\boldsymbol{i}} = \int_0^{\tau_i} [\boldsymbol{x}_i(u+s) - \bar{\boldsymbol{x}}(u+s)]\, \hat{M}_i(s, du),$$

with $\widehat{M}_i(s, du) = N_i(s, du) - Y_i(s,u)\{\hat{\lambda}_0(u+s) + \hat{\boldsymbol{\beta}}'\boldsymbol{x}_i(u+s)\}du$ being the martingale residual.

3.3. COHORT SURVIVOR FUNCTION ESTIMATE

Another very important result is the large cohort property of the survivor function. The true probability that an i^{th} member of the cohort who is alive at calendar time s, so with age $x + s$ survives to time T with $T \in (s, \tau)$ is

$$[{}_{T-s}p_{x+s}]_i = \exp\left[-\int_s^T \lambda_i(s+u)du\right] = \exp\left[-\int_s^T [\lambda_0(s+u) + \boldsymbol{\beta}'\mathbf{x}(u+s)]du\right].$$

An estimator of $[{}_{T-s}p_{x+s}]_i$ is given by

$$\begin{aligned}\widehat{[{}_{T-s}p_{x+s}]}_i &= \exp\left[-\int_s^T [\hat{\lambda}_0(s+u) + \hat{\boldsymbol{\beta}}'\mathbf{x}(u+s)]\right] du \\ &= \left[\hat{\bar{F}}_0([s,T])\right] \cdot \left[\exp\left(-\int_s^T \hat{\boldsymbol{\beta}}'\mathbf{x}(u+s)du\right)\right].\end{aligned}$$

For a cohort aged x, the mortality rate is given by

$$\hat{\Lambda}_{(x)}(s,t|\hat{\boldsymbol{\beta}}) = \hat{\Lambda}_0(s,t|\hat{\boldsymbol{\beta}}) + \int_x^{x+t} \hat{\boldsymbol{\beta}}\mathbf{x}(u)du,$$

whereas the survivor function is

$$\hat{\bar{F}}_{(x)}(s,t|\hat{\boldsymbol{\beta}}) = \hat{\bar{F}}_0(s,t|\hat{\boldsymbol{\beta}}) \exp\left[-\int_x^{x+t} \hat{\boldsymbol{\beta}}'\mathbf{x}(u)du\right].$$

Remark 2 *Two types of asymptotic properties can be investigated. One can fix the age of the cohort at* (x) *and investigate the behavior of the cohort as more and more policyholders join it. Another large sample behavior pertains to fixing the number of policyholders and assessing the impact of aging on the survivor function of the proposed model. We took the former route.*

Theorem 11 *For a fixed x, let $D[x, x+t]$ be the sup- norm on $[x, x+t]$. As $n_x \to \infty$, the estimator $\hat{\bar{F}}_{(x)}(s, t|\hat{\boldsymbol{\beta}})$ is a consistent estimator of the true survivor function of the cohort, that is*

$$\sup_{t\in[x,x+t]} |\hat{\bar{F}}_{(x)}(s, t|\hat{\boldsymbol{\beta}}) - \bar{F}(t)| = 0.$$

Proof: For a p-dimensional vector $\mathbf{a}$, $\mathbf{a}'$ is its transpose. Let $\mathbf{a}$ and $\mathbf{b}$ be two p-dimensional vectors. We shall denote by $\mathfrak{D}[x, x+t]$ the space of functions on $[x, x+t]$ which are right-continuous and with left-hand limits (the cadlag functions) and endow this with Skorohod's metric. We shall then denote by $\mathfrak{Q} = \mathfrak{D}^2[x, x+t] \times \mathbb{R}^p$ and endow this space with the product metric d_P formed from Skorohod $\mathfrak{D}[x, x+t]$ and Euclidean metrics for $\mathbb{R}^p$. Also, for $D \in \mathfrak{D}[x, x+t]$, denote by $||D||_\infty = \sup_{u\in[x,x+t]} |D(s)|$ and let $d : \mathfrak{Q} \times \mathfrak{Q} \to \text{Re}^+$ with

$$d[(U_1, U_2, \mathbf{b}), (V_1, V_2, \mathbf{r})] = \sqrt{||U_1 - V_2||_\infty^2 + ||U_2 - V_2||_\infty^2, |\mathbf{b} - \mathbf{r}|^2}.$$

Letting

$$Q \equiv Q_{n_x} = \left(\frac{1}{n_x} \sum_{i=1}^{n_x} \text{N}_i(s, t), \frac{1}{n_x} \sum_{i=1}^{n_x} \text{Y}_i(s, t), \hat{\boldsymbol{\beta}}_{n_x} \right),$$

observe that $Q \in \mathfrak{Q}$. Also, denote by

$$Q_0 = (\lambda(u), s(u), \boldsymbol{\beta}_0).$$

Then we have, $d(Q_{n_x}, Q_0) = 0$. We may view $\hat{\bar{F}}_{(x)}(s, t|\hat{\boldsymbol{\beta}})$ as a mapping H from $\mathfrak{Q}$ to $\mathfrak{D}[x, x+t]$ defined by

$$H : Q_{n_x} \mapsto \bar{F}(s, t).$$

It can also easily be shown that this mapping is continuous. Hence the consistency of $\hat{\bar{F}}_{(x)}(s, t|\hat{\boldsymbol{\beta}})$ is obtained by an application of the continuous mapping theorem.

4. PRICING OF LONGEVITY BOND

An alternative approach to reinsurance by insurance companies is *securitization* using *longevity derivatives* such as longevity bonds, cf. Blake and Burrows (2001) or options and swaps Lin and Cox (2005). These derivatives products are structured with longevity- dependent payoffs in order to help hedge longevity risk. Hedging is defined as the financial strategy in which one adopts certain approaches to limit financial risk. Longevity bond payments are made contingent on the proportion of a population cohort surviving to a certain age. A survivor swap is an agreement between two parties for one or more exchange of cash flows. The survivor swaps are more advantageous than survivor bonds in the sense that they can be arranged at a lower transaction cost, and they are more flexible and do not require the existence of a liquid market Dowd *et al.* (2006). However, survivor bonds have been shown to be one of the best ways in managing longevity risk.

In this work, we will use bonds as a means of securitizing longevity risk by providing its fair price. Various pricing techniques have been proposed in the literature. The most popular methods are the sharpe ratio rule Ludkovski and Young (2008), the risk neutral theory Biffis (2005), Dahl and Møller (2006), Dahl *et al.* (2008), and the Wang transform cf. Wang (2000) and Wang (2002). A review of the three methods is given in Bauer *et al.* (2010). Chen *et al.* (2010) discusses robustness of the various pricing methods.

For longevity bonds, payments are made to investors upon the survival of the policyholders, that is, investors receive reduced coupon payments when the actual number of annuitants alive at time t are more than the expected number of policyholders initially calculated. It is then evident that investors in longevity bonds usually receive less payments as compared to those who invest in straight bonds. Investors may have different risk preferences and so to cater for that and to attract a large pool of capital to fund this risk, longevity bonds can be priced using tranches. We will adopt this approach in the pricing of the bond. The bond will be customized for investors by classifying the pool of investors into groups based on their different risk preferences Kim and Choi (2011). A tranche is a proportion of the overall risk which is to be borne by a group of bondholders. Pricing bonds by tranching also helps to accommodate any customization of the bond to satisfy the preferences of each group of investors. As indicated earlier, we will adopt the methods proposed in Wills and Sherris (2010) for pricing longevity bonds using tranches. This bond will be issued via a Special Purpose Vehicle Wills and Sherris (2010), who will act as middlemen between the bond issuer and the investors.

To that end, consider a longevity bond that has a term to maturity of T. The value of T can be viewed as the time at which the last surviving member of the cohort dies, albeit, sometimes T may not be pre-specified. Since n_x is the number of policyholders in the cohort, n_{x+t} will live up to time t. We assume that the policyholders receive $\{B_i, i = 1, 2, ..., n_x\}$ as benefit payment at each time of payment. At the time of the payments, if more than the calculated expected policyholders are still alive, the bond holder does not receive the full pre-specified coupon or premium payment. Instead, they receive just a proportion of this payment or in some cases, they may have to forfeit premium payments entirely. The bond issuer receives the remainder of this coupon payment to help defray the liabilities they incur on the benefits payments of the extra policyholders. In some instances, they lose the entire coupon amount. Bonds of this type are known as coupon- at risk bonds Blake *et al.* (2006b).

Although most longevity bonds are issued in a single tranche, we propose a bond which will be structured using an approach similar to the one used for pricing Collateralized Debt Obligations (CDOs). The pool of the investors can be grouped into different tranches according to their risk preferences. To proceed, we will use the same 3 tranches as were used in Wills and Sherris (2010) labeled as junior tranche, mezzanine tranche, and the senior tranche. The tranches have been arranged in decreasing order of longevity risk exposure, that is the junior tranche is the more exposed, while the senior tranche is the least risky. Tranches can be formed based on the percentage cumulative losses (PCL) or total losses of the portfolio. We will use the former. Each tranche has a portion of the PCL allocated to it for absorption, also known as the tranche loss window. This is characterized by an open end, known as the attachment point and an exit known as the detachment point. The latter also serves as an open end for the next tranche. Each tranche is triggered when the PCL exceeds the attachment point of the tranche and this occurs when there are more of the insured alive than projected. A tranche can be depleted completely when the PCL attains or even exceeds the closing end of the said tranche. The expected losses and returns differ across the various tranches. Upon purchase of the bond, the bond holders make an initial payment to the SPV. This amount is pooled and referred to as the Principal or Face Value Z of the bond. A fraction of the face value Z is paid back to the insurer to serve as compensation for the liability incurred when more people than anticipated are alive during the payment period. Each tranche will be priced by equating the expected present value of all claim payments to the policyholders to the expected present value of all premium payments to investors. The annuity benefits payments $\{B_i, i = 1, 2, \ldots, n_x\}$ are assumed to be a whole life annuity, that is, individuals receive payments each period so long as they are alive. If $Y_i(t) = \mathrm{I}\{\tau_i > t\}$ indicates whether or not an individual is alive by time t, using actuarial notation, the survival probability of (x) to age $x + t$ denoted by ${}_tp_x$ is given by

$$ {}_tp_x \;=\; E(Y_i(t)) = \exp(-\Lambda_x(t)) = \exp\left(-\int_x^{x+t} \lambda(s)ds\right). \tag{4} $$

Note that n_{x+t} represents the number of survivors at time t who were initially aged x, so that $n_{x+t} = Y(t) = \sum_{i=1}^{n_x} Y_i(t)$. At time t, the loss on the annuity portfolio is

$$ \begin{aligned} L(t) &= \sum_{i=1}^{n_x} (B_iY_i(t) - E[B_iY_i(t)])^+ \\ &= \sum_{i=1}^{n_x} \max[0, B_iY_i(t) - E(B_iY_i(t))]. \end{aligned} $$

$E(B_iY_i(t)) = B_i({}_tp_x)$ where ${}_tp_x$ is estimated by

$${}_t\hat{p}_x = \exp\left[-\int_x^{x+t} \hat{\lambda}(s)ds\right]$$

Let Z be the face value of the bond. Then the Percentage Cumulative Loss (PCL) on the entire annuity portfolio by time t is obtained as

$$CL(t) = \frac{\sum_{j=1}^{t} L(j)}{Z}.$$

It is worth noting that

$$CL(t) = \begin{cases} < 1 & if \quad \sum_{j=1}^{t} L(j) < Z \\ = 1 & if \quad \sum_{j=1}^{t} L(j) = Z \\ > 1 & if \quad \sum_{j=1}^{t} L(j) > Z. \end{cases} \tag{5}$$

Following Wills and Sherris (2010), we will assume that $CL(t) \leq 1$ for the pricing of the bond. For a portfolio which is divided into H independent tranches, let $W_{A,h}, h \in \{1, 2, ..., H\}$ represents the open end of the tranche loss window, known as the tranche attachment point. Let $W_{D,h}$ represents the closing end of the tranche loss window, also known as the detachment point of the tranche. The tranche loss windows are a percentage of the face value of the bond. The attachment point for the most risky tranche is equal to 0, that is $W_{A,1} = 0$, while the detachment point for the senior tranche is 1, $W_{D,H} = 1$. For any two consecutive tranches, the detachment and attachment points are such that $W_{D,h-1} = W_{A,h}$, whereas for any particular tranche, its attachment point is less than the detachment point, that is $W_{A,h} < W_{D,h}$.

The percentage cumulative losses are allocated to the tranches starting with the junior tranche and ending with the senior tranche in decreasing order of risk exposure. If the PCL is allocated to a tranche, then it is within the attachment and detachment points of that tranche. Once the PCL exceeds the detachment point of a particular tranche, that tranche is depleted and rendered inactive. The excess of the new PCL and the detachment point of the recently inactive tranche is assigned to the next tranche. This goes on and on until the last annuitant in the portfolio dies or until the last tranche is retired. Under this scheme, the percentage cumulative loss incurred by the h^{th} tranche is given by:

$$CL_h(t) = \begin{cases} 0, & CL(t) < W_{A,h} \\ CL(t) - W_{A,h}, & W_{A,h} \le CL(t) < W_{D,h} \\ W_{D,h} - W_{A,h}, & CL(t) \ge W_{D,h} \end{cases}$$

with $CL(t) = \sum_{h=1}^{H} CL_h(t)$. For the h^{th} tranche, the expected percentage cumulative loss is

$$ECL_h(t) = \frac{\mathbb{E}\left(CL_h(t)\right)}{W_{D,h} - W_{A,h}},$$

while $1 - ECL_h(t)$ is the remainder of the face value on which new premium calculations are made. The price of the bond is calculated by setting the risk neutral expected present value of all future liability payments equal to the risk neutral expected present value of investor premiums as seen in (6) below.

The risk neutral expected present value of all claim payments, which are made from the h^{th} tranche to the insured, denoted by CP_h is given by

$$CP_h = V_{0,t}[ECL_h^*(t) - ECL_h^*(t-1)],$$

where $ECL_h^*(t)$ is the risk-adjusted expected percentage cumulative loss at time t, making $1 - ECL_h^*(t)$ the fraction of the bond principal value on which new premium calculations are made. The term $V_{0,t}$ represents the factor which is used to discount the future cashflows to time $t = 0$ and P_h represents the premium payments to the investors. The risk neutral expected present value of all premium payments to the investors from the h^{th} tranche, denoted by $PP_h(P_h)$ is then given by

$$PP_h(P_h) = \sum_{t=1}^{T} P_h V_{0,t-1}[1 - ECL_h^*(t-1)].$$

Hence, the fair price of the bond is obtained by setting the risk neutral expected present values of the losses incurred by the insurance company (or claim payment to the insured) and premiums paid to bondholders equal to each other as follows:

$$\sum_{t=1}^{T} P_h V_{0,t-1}[1 - ECL_h^*(t-1)] = \sum_{t=1}^{T} V_{0,t}[ECL_h^*(t) - ECL_h^*(t-1)] \tag{6}$$

From Equation (6), we obtain the fair price of the bond per tranche as:

$$P_h = \frac{\sum_{t=1}^{T} V_{0,t}[ECL_h^*(t) - ECL_h^*(t-1)]}{\sum_{t=1}^{T} V_{0,t-1}[1 - ECL_h^*(t-1)]}.$$

The fair price of the bond is a percentage of the bond principal that will be paid to the investors.

5. NUMERICAL STUDIES

5.1. DESCRIPTION AND SIMULATION SPECIFICATIONS

We illustrated our method through simulation studies. All annuitants were assumed to enter the study at age 65 with terminal age, $\tau_i \sim$ Uniform(65,100). The inter- event times were then simulated from the Weibull distribution with a shape parameter of 6 and a scale parameter of 1. For simplicity, we chose two covariates; **x**= (x_1, x_2) with $x_1 \sim$ Bernoulli (0.5) representing any categorical variable, and $x_2 \sim$ Uniform (0,1) representing any quantitative covariates. The baseline force of mortality was taken to be of Weibull form such that $\lambda_0(s) = \theta_1\theta_2(\theta_1 s)^{\theta_2 - 1}$ with θ_1 being the scale parameter and θ_2 being the shape parameter. For decreasing and increasing baseline force of mortality respectively, we take

$\theta_2 \in \{0.8, 2\}$. We performed 1000 simulations for each combination of sample sizes, $n_x \in \{30, 50, 100\}$ and $\theta_1 = 1$. Three tranches were used, with the junior tranche absorbing the first 15% of the percentage cumulative loss, followed by the mezannine tranche who were allocated the next 15% of the PCL and then the senior tranche which was set to absorb the last 70% of the PCL. The face value of the bond we used is $37,500,000. Benefit and premium payments correspond to the inter- event times whiles the term to maturity of the bond is determined by difference between the age of death of the last policyholder and the homogeneous starting age, 65. We assumed that policyholders receive $50,000 at each time of benefit collection. Risk- free interest rates from the U.S Department of treasury from 1985 to 2020 was used.

5.2. SIMULATION RESULTS

The results of the simulation are summarized in the tables below. Estimates of $\boldsymbol{\beta}_{(x)}$ and their corresponding standard errors are in Tables 1 and 2. As more policyholders join the annuity portfolio, the $\boldsymbol{\beta}_{(x)}$ estimates were just around the true value used. The amount of principal contributed by the various tranches is in Table 3. Tables 4 and 5 give the investor premium percentages. It was observed that for a specific age, the bond premiums reduce across the tranches in decreasing level of risk exposure. Also, for a specific value of θ_2, the investor premiums increase as sample size increases. This is due to the fact that the chance of liability increases with more policyholders. Premium payments to investors is lower for $\theta_2 = 0.8$ since declining mortality means more people actually living than the insurers anticipate, hence resulting in a reduced pay- out to investors. Tables 6 and 7 show the total premium amount in dollars received by investors at time 0 when the bond is issued.

Table 1. $\hat{\boldsymbol{\beta}}$ values and corresponding standard errors obtained for $\theta_1 = 1, \theta_2 = 0.8$

	n_x	$\hat{\beta}_1$	**se($\hat{\beta}_1$)**	$\hat{\beta}_2$	**se($\hat{\beta}_2$)**
1	30	0.984	0.434	1.000	0.758
2	50	0.995	0.327	0.996	0.569
3	100	1.000	0.227	0.993	0.393

Table 2. $\hat{\boldsymbol{\beta}}$ values and corresponding standard errors obtained for $\theta_1 = 1, \theta_2 = 2$

	n_x	$\hat{\beta}_1$	**se($\hat{\beta}_1$)**	$\hat{\beta}_2$	**se($\hat{\beta}_2$)**
1	30	0.995	0.517	0.982	0.906
2	50	0.986	0.396	1.003	0.688
3	100	0.988	0.277	0.988	0.481

6. CONCLUSIONS

We proposed a generalized additive model which captures improvements in individual lives. Individual risk profiles of policyholders were not neglected in determining their mortality. The mortality model is analytically tractable as a simulation study was successfully performed. Model parameters and factors can be easily interpreted, and this model is compatible with pricing of financial securities as we demonstrated this by pricing

Table 3. Principal contribution by the various tranches

Tr Index	**Tr name**	**Tr window**	**Notional Tr Principal ($)**
1	Junior	0% - 15%	5625000
2	Mezannine	15% - 30%	5625000
3	Senior	30% - 100%	26250000
	Total		37500000

Table 4. Bond premium percentages for $\theta_1 = 1, \theta_2 = 0.8$

	n_x	$\hat{\beta}_1$	$\hat{\beta}_2$	**Tr** 1 ($)	**Tr** 2 ($)	**Tr** 3 ($)
1	30	0.984	1.000	0.5516%	0.1335%	0.0203%
2	50	0.995	0.996	1.0528%	0.2653%	0.0678%
3	100	1.000	0.993	4.8184%	0.6537%	0.1694%

Table 5. Bond premium percentages for $\theta_1 = 1, \theta_2 = 2$

	n_x	$\hat{\beta}_1$	$\hat{\beta}_2$	**Tr** 1 ($)	**Tr** 2 ($)	**Tr** 3 ($)
1	30	0.995	0.982	0.6116%	0.1391%	0.0209%
2	50	0.986	1.003	1.2308%	0.2823%	0.0697%
3	100	0.988	0.988	4.8615%	0.6617%	0.1706%

Table 6. Bond premium amount at $t = 0$ for $\theta_1 = 1, \theta_2 = 0.8$

	n_x	$\hat{\beta}_1$	$\hat{\beta}_2$	**Tr** 1 ($)	**Tr** 2 ($)	**Tr** 3 ($)
1	30	0.984	1.000	193500	50062.5	5328.75
2	50	0.995	0.996	394800	99487.5	17797.5
3	100	1.000	0.993	1806900	245137.5	63525.0

Table 7. Bond premium amount at $t = 0$ for $\theta_1 = 1, \theta_2 = 2$

	n_x	$\hat{\beta}_1$	$\hat{\beta}_2$	**Tr** 1 ($)	**Tr** 2 ($)	**Tr** 3 ($)
1	30	0.995	0.982	458700	52162.5	7837.5
2	50	0.986	1.003	461550	105862.5	26137.5
3	100	0.988	0.988	1823063	248137.5	63975.0

a survivor bond. Designing longevity bonds using tranches makes the bond more attractive to a variety of investors since the bonds can be customized for them based on their risk preferences, thus expanding the pool of capital to fund the bond.

Mortality jumps can be incorporated into this model to capture sudden peaks in mortality which may be as a result of pandemics, natural disasters, and the like. The biggest challenge faced in this work was in securing real life data for application purposes. Due to the covariates, this data is seen to make policyholder easily identifiable and hence, not available from any of the insurance companies.

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II. LONGEVITY BOND PRICING WITH HULL-WHITE MORTALITY MODEL

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ABSTRACT

In recent decades, human life expectancy has risen dramatically as a result of scientific breakthroughs in the field of biomedicine and general improvements in people's health and standard of living, giving rise to *longevity risk*. This risk arises because insurers may end up paying benefits to policyholders much past the age at which they were originally expected to have ceased payments. The life insurance sector is currently facing a significant challenge in this area. Securitization with the use of longevity derivatives is one approach to dealing with the problem of extended life expectancy. Popular in literature is the use of the common stochastic interest rates to model mortality, so as to capture mortality improvements over time. Non- mean reversion models have been known to be better predictors of mortality than mean reversion models based on a proposal made by (Luciano and Vigna (2015)). Contrary to this, (Zeddouk and Devolder (2020)) demonstrates that mean reversion models with moving targets are better for modeling and predicting mortality. This paper explores in further detail how to estimate the price of a longevity bond by merging a Hull- White model proposed by (Zeddouk and Devolder (2020)) with the Vasicek model (Vasicek (1977)) for interest rate. The bonds will be priced using the risk neutral method.

Keywords: longevity risk, mean reversion, stochastic interest rate, stochastic mortality, Girsanov.

1. INTRODUCTION

Human lifespan, also known as life expectancy, refers to how long a person is expected to live on average based on their year of birth, gender, and other characteristics. The most common measure of life expectancy, known as *life expectancy at birth*, gives an estimate of mortality across all ages. Life insurance companies rely on mortality projections in designing life related policies, hence a reliable estimate of life expectancy is needed. Currently, insurance companies rely heavily on the deterministic mortality tables, which are constructed mostly based on historical data. The emerging problem with these is that due to advances and new discoveries in the healthcare system coupled with the improved lifestyle choices made by individuals, especially for those in developed countries, life expectancy has increased beyond the average lifetime projected by these mortality tables. Over the years, according to WHO records, the global life expectancy over almost two decades; that is the period between the years 2000 and 2016, was known to have increased by about 5.5 years, with Africa seeing as much as about a 10 year increase on average. The increase in life expectancy has been unsimulated and random, and hence poses a problem to all companies, governments, and stakeholders who rely on this information for different purposes. This uncertainty in the improvements in the evolution of life expectancy poses a risk, called *longevity risk*, and hence the need for this risk to be quantified and mitigated.

The uncertainty in mortality can be divided into two components; the systematic mortality risk and the unsystematic mortality risk. The unsystematic mortality risk is diversifiable since it can be completely eliminated based on the pooling of a large number of policies (Dahl *et al.* (2008)), that is, it is possible to avoid this risk by selling a sufficient number of similar insurance plans. The systematic risk, on the other hand, cannot be diversified by an increased number of policies in the portfolio, and so life insurance companies are in need of effective approaches for managing and minimizing this risk. The adverse effect of this risk on life insurance companies is that if it is underestimated, periodic annuity benefit payments need to be potentially made to the annuitants until death, even if death occurs at a later date than initially projected. This is a pressing issue in the life insurance industry as they incur huge unexpected losses, thereby putting a strain on the amount of profit realized annually, which in effect affects the market shares of the companies. Realistic and accurate reserve estimation models need to be developed in order to be able to cover such a liability. Moreover, overestimation of longevity risk by these companies will translate into high premiums, which will put a financial burden on the insured and potentially risk the companies' business with the insured. As a consequence, there is a need for an optimum estimate of the future longevity risk.

One practical approach of the life insurance industry in dealing with this risk is the application of the so- called *prudent man rule*. This involves setting the mortality rate at a high and safe value that will most likely result in a surplus, so as to issue this surplus as benefits to the insured who are still alive based on the so-called *contribution principle* c.f. (Dahl *et al.* (2008)). However, sometimes, this safe value is not accurate and "safe enough" and therefore does not yield the expected results. As a way to lessen the possibility of being saddled with hefty total claim payments due to longevity risk, insurers also take recourse to reinsurance by selling off sections of their risk portfolios to third parties. As another alternative, insurance companies make attempts to entirely eliminate this risk by offering a lump-sum to policyholders before or during retirement so as to sever their insurance relationship with these policyholders. These are known as *insurance buy-outs*. Buy-outs are becoming more common in the insurance industry. Another possibility is to convert this risk into derivatives such as longevity bonds, longevity options, and longevity swaps in order to attract investors who will essentially bear the liabilities and risk associated with the policyholders' lives, a process known as *securitization*. C.f. (Blake *et al.* (2006), Hunt and Blake (2015)) and the references therein for a review of various longevity- linked derivatives.

In the design and issuing of many financial derivatives, one form of economic and investment uncertainty that the security will be exposed to is the *interest rate risk*. Interest rate risk is the tendency that interest rate- dependent investments of an institution or individual will reduce or increase in value due to fluctuations in future interest rates. Insurance firms are also vulnerable to investment risks, particularly interest rate risk due to the fact that their investment portfolios are largely made up of fixed income securities. In securitizing longevity risk, these random fluctuations in interest rates affect their sources of funding for this highly prevalent longevity risk.

Several interest rate models, which can primarily be categorized as deterministic or stochastic, have been developed in literature. The deterministic models are usually either constant or time dependent, be it in continuous- or discrete- time. Stochastic interest rates are intended to capture the irregular changes in interest rates for the duration of the existence of interest rate sensitive financial instruments. There is a class of continuous-time stochastic interest rates which are solutions to stochastic differential equations, known as *diffusion processes*. These diffusion processes can be classified into mean-reverting with examples being the Vasicek model (Vasicek (1977)), Cox- Ingersoll Ross (Cox *et al.* (1985b)), Hull and White model (Hull and White (1990)), Black and Karasinski model (Black and Karasinski (1991)), and non mean- reverting processes proposed in (Uhlenbeck and Ornstein (1930)), (Pitts (1985)) and (Ho and Lee (1986)), to mention a few. The interest rate model we utilize in this work for pricing of the longevity bonds is the (Vasicek (1977)) model. Even though this model allows for negative interest rates, in practice, the likelihood of attaining negative values is insignificant.

In the structuring of longevity derivatives, as well as in designing retirement policies, insurers and policymakers also depend massively on mortality projections. Due to the fact that mortality modeling plays a paramount role in the pricing of longevity risk, mortality projections need to be as accurate as possible. Mortality can be modeled using either a deterministic or a stochastic approach. Many life insurance companies currently depend on deterministic life tables for mortality modeling. Other deterministic models are the Gompertz model and its extension, the Gompertz-Makeham model which partitions mortality as an exponential function of age (Gompertz (1825)) and a component that is independent of age (Makeham (1860)). Researchers have been leaning toward stochastic projections of mortality over the past few decades. This is because, in real life, there is too much uncertainty about death for it to be deterministic, which makes the earlier predictions less reliable. Incorporating a stochastic mortality intensity in pricing insurance products comes with some advantages. One of them is that premiums and reserves calculated based on these models better capture the dynamics and trends of improvement in mortality over the years, thereby leading to better pricing of insurance securities. Also, these models better quantify the risk that the insurance companies will eventually absorb as the risk is heavily dependent on the underlying force of mortality (Dahl (2004)).

Many stochastic mortality models have been proposed since 1992 when the first stochastic mortality model, the Lee Carter model (Lee and Carter (1992)) was introduced. The Lee Carter model is an additive model that forecasts the log of the central mortality rates by combining singular value decomposition and time series methods. (Plat (2009)) mentions some extensions to the Lee Carter model. These models sought to improve upon the Lee Carter model's inadequacies, however, they were only able to improve it in one or two areas; either they addressed the robustness problem (Renshaw and Haberman (2006)) and or accounted for correlation structure problems in the model extensions in (Currie (2006)) and (Renshaw and Haberman (2006)). Other extensions are the series of CBD models proposed by (Cairns *et al.* (2006a)). (Aro and Pennanen (2011)) proposes a user-friendly stochastic mortality model in which the yearly changes in survival probabilities for different age groups are modeled by a linear combination of basis functions that the user has to choose ahead of time. This allows the model to include important population-specific details. The stochastic mortality models that are commonly used are usually non-negative. However, despite the fact that negative mortality rates are incapable of having any meaningful interpretation, certain pieces of written work use them because the authors feel they provide more reliable mortality rate estimates. For example, in (Christiansen (2013)), the author justifies the use of Gaussian mortality models with additive noise in cases where the real distribution of future mortality and interest rates is made up of an affine drift term and a random noise term. (Hunt and Blake (2014)) proposed a systematic general method for modeling mortality, and their model necessitates making some decisions which may be subjective to the modeler. (Andrew Hunt (2015)) also has an extensive review that categorizes age-, period- and or cohort- dependent mortality models proposed before 2015 and compares and contrasts these models. They also provide information on important principles to consider when one wants to construct a model in their proposed classes c.f also (Plat (2009)). According to (Cairns *et al.* (2006a)), one advantage of using affine models for mortality is that it makes it easier to get a closed-form expression for the survival function. C.f (Cairns *et al.* (2006b), Fung *et al.* (2017)) and the references therein.

There are a number of striking similarities that can be drawn between interest rates and the force of mortality. In reality, both of these processes are stochastic by nature, are positive, and have term structures. As a consequence of this, researchers have adopted some common short rate diffusion processes for modeling mortality. (Dahl and Møller (2006)) utilized these connections to create approaches for modeling mortality risk based on the methodology established for interest rate modeling. (Biffis (2005)) likewise investigated the parallels between credit risk and mortality risk. In (Cairns *et al.* (2006a)), the authors examined how to model and price mortality risk and mortality- dependent securities based on these similarities between interest rates and force of mortality. (Milevsky and Promislow (2001), Dahl (2004), Schrager (2006), Denuit and Devolder (2006), Quittard-Pinon and Randrianarivony (2011), Shen and Siu (2013)) and several other authors also took advantage of this similarity, and some used them to price longevity- and mortality-linked derivatives. C.f (Christiansen (2013)) for more references.

As aforementioned, diffusion processes in the interest rate discussion and others not highlighted here in this paper can be classified into mean-reverting and non mean- reverting processes. Mean- reverting means that the mortality rate will, in the long run, converge to the process' long run average function. Mean reversion is a desirable property in stochastic interest rates because it enables the process to converge to an equilibrium level, the long term mean process. This is a feature that intuitively ought to be seen in mortality models, and has been shown in a number of affine stochastic mortality models. (Cairns *et al.* (2006b)) on the other hand, argued that the mean-reverting components should not be included in affine stochastic models. (Luciano and Vigna (2015)) also demonstrated that non- mean reverting affine models were better at fitting evolution of mortality than their mean reverting counterparts. (Zeddouk and Devolder (2020)) observed that the aforementioned discoveries were due to the fact that the long term averages for these mean- reverting processes were fixed, and this informed their conclusion that mean-reversion models are not appropriate for mortality modeling. In (Zeddouk and Devolder (2020)), they showed that incorporating moving targets in the mean- reverting short rate processes created models that were better for predicting mortality than the non- mean reverting ones, and inferably their mean- reverting counterparts with fixed targets. We will use the mean- reverting Hull and White model (Hull and White (1990)) with a moving target; the Gompertz function as proposed in (Zeddouk and Devolder (2020)) to determine the price of a longevity bond under the risk- neutral in this work.

Longevity bonds are one of the most common derivatives used to securitize longevity risk. A bond is a financial instrument that is usually issued by companies to borrow money from investors. The initial amount paid by the investors is known as the principal. With most bonds, investors receive coupons; a percentage of their principal periodically, and at maturity, they receive their principal. Generally, the price of longevity bonds is dependent on the survival of a cohort of individuals. This is why we need to model mortality by accounting for future improvements so we can be more accurate in our calculation of the survival probabilities, which is a function of the force of mortality. In coupon paying longevity bonds, the receipt of coupon payments and the amount of coupons to be received by investors are contingent on the survival of individuals in a cohort; that is, if actual survivors up to a particular time exceed what the company anticipated, a lesser coupon amount is paid to the investors, sometimes even to the extent of investors having to forfeit their coupon payments in cases of extreme improvements in mortality.

There are various methods one could adopt when pricing longevity- linked financial derivatives; in the context of this work, longevity bonds. A no-arbitrage framework is reasonable for use in the pricing of financial securities in incomplete and illiquid markets like the longevity market, so as to prevent the possibility of uniquely determining derivative prices. C.f (Björk (2009)) and (Leung *et al.* (2018)). The incomplete market's fundamental problem has been addressed by the introduction of a variety of techniques in the literature to help in determining the market risk premium and also perform a risk- adjustment. One of such techniques is the Wang transform (Wang (1996), Wang (2000), Wang (2002), and Wang (2003)). They proposed a distortion parameter which is used to attain risk-adjusted probabilities by transforming the predicted probability of death. They also provide a way to determine the risk premium using their approach. Though this approach has been criticized in the past (Cairns *et al.* (2006b), Bauer and Ruß (2006) and Pelsser (2008)), it still stands as one of the most popularly used by researchers. C.f. (Cox *et al.* (2006), Denuit *et al.* (2007), Chen and Cox (2009), Chen *et al.* (2010) and Bauer *et al.* (2010)), for further discussions on the Wang transform approach. Another commonly used method is the risk-neutral approach, in which derivative prices are determined under an equivalent martingale measure instead of the real world measure. The Girsanov theorem is applied for this change of measure. C.f (Lin and Cox (2005)) and (Liao *et al.* (2007)) for some applications. The equilibrium pricing theory may also be utilized to determine the risk adjustment as well (Hull (2003)).

The structure of this work is as follows: in sections 2 and 3, respectively, we explore the underlying mortality and interest rate models. The methodology for bond pricing under the risk neutral approach is discussed in section 4. The bond pricing method is illustrated with a simulation study and the results discussed in section 5. We give some detailed proofs in section 6.

2. UNDERLYING MORTALITY MODEL

In the following paragraphs, we will go through the mortality model that will be utilized when the price of the longevity bond is determined.

In this work, processes are defined on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$. $(\mathcal{G}_t)_{t\in\mathbb{R}+}$ will be the filtration representing the overall history of the processes, with $(\mathcal{G}_{t,r})_{t\in\mathbb{R}+}$ and $(\mathcal{G}_{t,\mu})_{t\in\mathbb{R}+}$ capturing evolution of the interest rate process, $(r_t)_{t\in\mathbb{R}+}$ and the mortality rate process $(\mu_x(t))_{t\in\mathbb{R}+}$ respectively. $\mathbb{P}$ is the probability measure. As mentioned earlier, we will use a special mean reverting Hull- White model with a moving target, proposed in Zeddouk and Devolder (2020) as the mortality process for an individual, initially aged x. This model will be the solution to the stochastic differential equation with the expression

$$d\mu_x(t) = [\gamma(t) - b\mu_x(t)]dt + \sigma_\mu dW_\mu(t), \tag{1}$$

where $W_\mu(t)$ is a standard brownian motion and follows a normal distribution with mean 0 and variance t. b and σ_μ are both positive constants with σ_μ representing the volatility of the mortality rate process. The target, $\gamma(t)$ is chosen to be the traditional Gompertz function which is known to be a good mortality model for older ages (Dickson *et al.* (2019) and expressed as

$$\gamma(t) = Ae^{Bt}, \tag{2}$$

where $A > 0$ and $B > 0$. $\frac{A}{b}$ can be seen as the baseline mortality model of an individual aged x, that is the mortality rate at $t = 0$, and B is the aging element in the model. We can therefore rewrite (1) as

$$d\mu_x(t) = b\left(\frac{A}{b}e^{Bt} - \mu_x(t)\right)dt + \sigma_\mu dW_\mu(t). \tag{3}$$

A potential candidate for the baseline mortality, $\frac{A}{b}$ is $\frac{A}{b} = \mu_x(0)$ so that (3) is equal to

$$d\mu_x(t) = b\left(\mu_x(0)e^{Bt} - \mu_x(t)\right)dt + \sigma_\mu dW_\mu(t). \tag{4}$$

For an individual policyholder initially aged x, the mortality rate process which is the solution to the stochastic mortality differential in (3) for $0 \leq s \leq t \leq T$ is

$$\mu_x(t) = \mu_x(s)e^{-b(t-s)} + \frac{A}{B+b}(e^{Bt} - e^{Bs-b(t-s)}) + \sigma_\mu e^{-bt}\int_s^t e^{bu}dW_\mu(u). \tag{5}$$

Conditional on the filtration up to time s, $\mathcal{G}_{s,\mu}$, $\mu_x(t)$ follows a normal distribution with mean and variance given as

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}(\mu_x(t)|\mathcal{G}_{s,\mu}) &= \mu_x(s)e^{-b(t-s)} + \frac{A}{B+b}(e^{Bt} - e^{Bs-b(t-s)}), \\ \mathbb{V}_{\mathbb{P}}(\mu_x(t)|\mathcal{G}_{s,\mu}) &= \frac{\sigma_\mu^2}{2b}(1 - e^{-2b(t-s)}). \end{aligned} \tag{6}$$

For simplicity in the subsequent derivations, we assume $s = 0$. The actual survival probability of an individual policyholder initially aged x for $(T - t)$ more years given that this individual lives up to time t can therefore be obtained from the model in (4) as:

$${}_{T-t}p_{x+t} = \mathbb{E}_{\mathbb{P}}\left(e^{-\int_t^T \mu_x(u)du}\middle|\mathcal{G}_{t,\mu}\right). \tag{7}$$

One disadvantage of the Hull and White model is its ability to take on negative values which appears to go against the intended attribute of having positive values for mortality. However, these models are still popular for stochastic mortality modeling because the probability of it being negative is insignificant in practice, due to the fact that mortality processes are normally not extremely volatile. C.f.(Chen *et al.* (2010) and Zeddouk and Devolder (2020)). Proofs of (5), (6), (7) are relegated to the Appendix.

3. INTEREST RATE MODEL

An interest rate model is another very important component of a bond pricing. In this work, we will employ the Vasicek interest rate model (Vasicek (1977)) which is usually represented by the following stochastic differential equation:

$$dr(t) = \kappa[\beta - r(t)]dt + \sigma_r dW_r(t), \tag{8}$$

where $W_r(t)$ is a standard brownian motion which is normally distributed with mean 0 and variance t. The parameters κ, β and σ_r are all positive constants with κ representing the speed of the mean reversion, β being the long run mean and σ_r representing the volatility of the interest rate process. For $0 \leq s \leq t \leq T$, it can be shown that

$$r(t) = r(s)e^{-\kappa(t-s)} + \beta(1 - e^{-\kappa(t-s)}) + \sigma_r e^{-\kappa t} \int_s^t e^{\kappa u} dW_r(u) \tag{9}$$

is the solution of the Vasicek model, which conditional on $\mathcal{G}_{s,r}$, the solution follows the normal distribution with mean and variance

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}(r(t)|\mathcal{G}_{s,r}) &= r(s)e^{-\kappa(t-s)} + \beta(1 - e^{-\kappa(t-s)}), \\ \mathbb{V}_{\mathbb{P}}(r(t)|\mathcal{G}_{s,r}) &= \frac{\sigma_r^2}{2\kappa}(1 - e^{-2\kappa(t-s)}). \end{aligned} \tag{10}$$

The long- run mean and variance follow from (10) when we take the $\lim_{t\to+\infty}$ yielding β and $\frac{\sigma_r^2}{2\kappa}$ respectively.

4. BOND PRICE UNDER THE RISK NEUTRAL METHOD

4.1. CONTEXT

In order to discuss risk neutral pricing, we need to understand what martingales and measures are. A measure is the unit in which the prices of derivatives are determined (Hull (2003)). We therefore need to find an adapted process known as the *market price* of the risk such that the price of the bond would be a martingale. Typically, a martingale is a stochastic process that has a zero drift, that is, in terms of the price of a derivative, the expected future price given its current price is the same as its current price.

The idea of risk- neutral valuation ensures that derivatives, in this case, bonds, are priced under a martingale framework such that, under the risk neutral- measure, the discounted price process of a bond is a martingale. According to the law of asset pricing, a no arbitrage pricing framework can be achieved if and only if the discounted price process is a martingale. Derivative security prices are calculated by discounting the expected value of future payoffs using a new measure $\widetilde{\mathbb{P}}$, also known as a risk-neutral measure. $\widetilde{\mathbb{P}}$ is considered to be equivalent to the actual or real- world measure $\mathbb{P}$. This means that $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ are identical on the sets with probabilities of 0 and 1, respectively, and just assign different probabilities to other possible events in $\mathcal{G}$, indicating that they agree on what is and is not possible (Shreve *et al.* (2004)).

The Girsanov theorem c.f. (Björk (2009), Calin (2012) and Shreve *et al.* (2004)), is the mechanism we will use to achieve this change of measure for both the mortality and interest rate processes. Since both the interest rate and mortality processes are diffusion processes, the theorem unifies them by providing an equivalent martingale measure. For the two processes $\mu_x(t)$ and $r(t)$ respectively, we define

$$Q_\mu(t) = \exp\left(-\int_0^t \Theta_\mu(u)dW_\mu(u) - \frac{1}{2}\int_0^t \Theta_\mu^2(u)du\right), \tag{11}$$

$$Q_r(t) = \exp\left(-\int_0^t \Theta_r(u)dW_r(u) - \frac{1}{2}\int_0^t \Theta_r^2(u)du\right), \tag{12}$$

where $\Theta_\mu(u)$ and $\Theta_r(u)$ are adapted processes obtained from $\mu_x(t)$ and $r(t)$ respectively.

4.2. GIRSANOV THEOREM

Let $\Theta(t) = \{\Theta_\mu(t), \Theta_r(t)\}$ be a vector of adapted processes obtained from the stochastic mortality and stochastic interest rate processes. $W(t) = \{W_\mu(t), W_r(t)\}$ is a two-dimensional Brownian motion with $W_\mu(t)$ and $W_r(t)$ being the Brownian motion for the mortality and interest rate processes respectively. Define

$$Z(t) = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left\{-\int_0^t \Theta(u)\cdot dW(u) - \frac{1}{2}\int_0^t \|\Theta(u)\|^2 du\right\}, \tag{13}$$

where $\|\cdot\|$ is the Euclidean norm. If the condition

$$\mathbb{E}\int_0^T \|\Theta(u)\|^2 Z^2(u)du < \infty, \tag{14}$$

is satisfied. Set $Z = Z(T)$, then $\mathbb{E}Z = 1$, so that under the new probability measure $\widetilde{\mathbb{P}}$, the process

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)du, \tag{15}$$

is a two-dimensional Brownian motion. C.f. (Björk (2009), Calin (2012) and Shreve *et al.* (2004)). Observe that

$$Z(t) = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = Q_\mu(t) \times Q_r(t). \tag{16}$$

The proof of the Girsanov theorem (c.f chapter 2 of this work), is based on the levy theorem which employs the fact that $\widetilde{W}(t)$ is a martingale under $\widetilde{\mathbb{P}}$.

Proposition 6 *(Privault (2012)) Considering a single process η, the risk- neutral measure with maturity T with respect to the filtration $\mathcal{G}_t$ is the probability measure $\widetilde{\mathbb{P}}$ which is defined as*

$$Q(T) = \frac{d\widetilde{\mathbb{P}}_{|\mathcal{G}_t}}{d\mathbb{P}_{|\mathcal{G}_t}} = \frac{\exp\left(-\int_t^T \eta(u)du\right)}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\int_t^T \eta(u)du\right)|\mathcal{G}_t\right]}, \quad 0 \le t \le T. \tag{17}$$

Substituting each of the processes for mortality and interest rate into (17), we obtain the Radon-Nikodyḿ derivatives defined in (11) and (12) to be the following:

$$\begin{aligned} Q_\mu(T) &= \frac{\exp\left(-\int_t^T \mu_x(u)du\right)}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\int_t^T \mu_x(u)du\right)|\mathcal{G}_{t,\mu}\right]} \\ &= \exp\left\{-\int_t^T \frac{\sigma_\mu}{b}(1-e^{-b(T-u)})dW_\mu(u) - \frac{1}{2}\int_t^T \left(\frac{\sigma_\mu}{b}(1-e^{-b(T-u)})\right)^2 du\right\}, \end{aligned} \tag{18}$$

$$\begin{aligned} Q_r(T) &= \frac{\exp\left(-\int_t^T r(u)du\right)}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\int_t^T r(u)du\right)|\mathcal{G}_{t,r}\right]} \\ &= \exp\left\{-\int_t^T \frac{\sigma_r}{\kappa}(1-e^{-\kappa(T-u)})dW_r(u) - \frac{1}{2}\int_t^T \left(\frac{\sigma_r}{\kappa}(1-e^{-\kappa(T-u)})\right)^2 du\right\}. \end{aligned} \tag{19}$$

Following from (16), we can now define $Z(T)$ as

$$\begin{aligned} Z(T) = \exp\bigg\{ &-\int_t^T \frac{\sigma_r}{\kappa}(1-e^{-\kappa(T-u)})dW_r(u) - \int_t^T (1-e^{-b(T-u)})dW_\mu(u) \\ &-\frac{1}{2}\int_t^T \left(\frac{\sigma_r}{\kappa}(1-e^{-\kappa(T-u)})\right)^2 + \left(\frac{\sigma_\mu}{b}(1-e^{-b(T-u)})\right)^2 du\bigg\}. \end{aligned} \tag{20}$$

By the Girsanov theorem it can be concluded that,

$$\widetilde{W}(t) = \left(W_r(t) + \int_t^T \frac{\sigma_r}{\kappa}(1 - e^{-\kappa(T-u)})du, W_\mu + \int_t^T \frac{\sigma_\mu}{b}(1 - e^{-b(T-u)})du \right) \tag{21}$$

is a two- dimensional Brownian motion.

Now that the new measure has been attained, we can determine the fair price of a $(T - t)$ year longevity bond at time t as

$$\begin{aligned} P(t, T, \mu, r) &= \mathbb{E}_{\widetilde{\mathbb{P}}} \left[e^{-\int_t^T \mu(u)+r(u)du} \middle| \mathcal{G}_t \right] \\ &= \mathbb{E}_{\mathbb{P}} \left[Z(T) e^{-\int_t^T \mu(u)+r(u)du} \middle| \mathcal{G}_t \right], \end{aligned} \tag{22}$$

where $\mathcal{G}_t = \mathcal{G}_{t,\mu} \vee \mathcal{G}_{t,r}$, and represents the overall history of the two processes which drive the price of the bond. Note that the Girsanov Theorem concludes independence of $\widetilde{W}_\mu(t)$ and $\widetilde{W}_r(t)$ under $\widetilde{\mathbb{P}}$ as well.

5. SIMULATION STUDIES AND APPLICATION

5.1. CALIBRATION OF PARAMETERS IN DIFFUSION PROCESSES

In the last decennium, parameter estimation in stochastic differential equations which depend on a sample of discrete data has garnered a significant amount of attention in the literature of finance. A variety of methods have been proposed in literature for the estimation of parameters in stochastic differential equations. Among these are the likelihood based methods such as simulated maximum likelihood and the discrete maximum likelihood, non- likelihood based methods like the general method of moments, the use of estimating functions and characteristic functions. c.f (Jeisman (2006)) and the references therein for a full review of these methods and more. We adopt the discrete maximum likelihood estimation approach in this work for the calibration of the parameters in both the mortality model and the interest rate model.

5.1.1. Calibration Of Parameters In Interest Rate Model. The discrete maximum likelihood estimation method used for calibration of the interest rate parameters integrates the Euler- Maruyama method (Iacus (2008)) and the distribition of the stochastic differential equation to obtain estimates for the various parameters in our models. The Euler- Maruyama method will be employed for the simulation as well.

Suppose Y_t is a stochastic process with the corresponding stochastic differential equation

$$dY_t = \alpha(Y_t, \boldsymbol{\theta})dt + \beta(Y_t, \boldsymbol{\theta})dW_t. \tag{23}$$

(23) can be approximated based on the Euler- Maruyama method as

$$Y_{t_{i+1}} = Y_{t_i} + \alpha(Y_{t_i}, \boldsymbol{\theta})\delta_i + \beta(Y_{t_i}, \boldsymbol{\theta})\gamma_i, \tag{24}$$

where $\delta_i = t_{i+1} - t_i$ and $\gamma_i \sim N(0, \delta_i)$. $Y_{t_{i+1}} - Y_{t_i}$ given Y_{t_i} is assumed to follow the normal distribution with mean $\alpha(Y_{t_i}, \boldsymbol{\theta})\delta_i$ and variance $\beta^2(Y_{t_i}, \boldsymbol{\theta})\delta_i$, so that the likelihood to be maximized would be

$$\mathcal{L}(\boldsymbol{\theta}, \boldsymbol{Y}) = \prod_{i=0}^{n-1} \frac{1}{\beta(Y_{t_i}, \boldsymbol{\theta})\sqrt{2\pi\delta_i}} \exp\left(-\frac{[Y_{t_{i+1}} - Y_{t_i} - \alpha(Y_{t_i}, \boldsymbol{\theta})\delta_i]^2}{2\beta^2(Y_{t_i}, \boldsymbol{\theta})\delta_i}\right). \tag{25}$$

In general, we will have varying values for the parameters depending on whatever dataset we are analyzing.

The vasicek model under consideration is represented with the stochastic differential equation in (8). The parameters κ, β and σ_r need to be estimated in this case. Let $\delta_i = t_{i+1} - t_i$ and $\boldsymbol{\theta} = \{\kappa, \beta, \sigma_r\}$. Given an observed interest rate process $r(0), r(1), \ldots r(n)$ at the times $t_0, t_1, \ldots, t_n$, the likelihood and negative log- likelihood are given by:

$$\mathcal{L}(\boldsymbol{\theta}|\boldsymbol{r}, \boldsymbol{t}) = \prod_{i=0}^{n-1} \frac{1}{\sqrt{2\pi\sigma_r^2\delta_i}} \exp\left(-\frac{[r(i+1) - r(i) - \kappa(\beta - r(i))\delta_i]^2}{2\sigma_r^2\delta_i}\right), \tag{26}$$

$$-\ell(\boldsymbol{\theta}|\boldsymbol{r},\boldsymbol{t}) = \frac{1}{2}\sum_{i=0}^{n-1}\frac{[r(i+1)-r(i)-\kappa(\beta-r(i))\delta_i]^2}{\sigma_r^2\delta_i} + \frac{n}{2}\log[2\pi\sigma_r^2\delta_i]. \tag{27}$$

The score processes can be obtained by taking the partial derivatives of (27) with respect to the parameters κ, β and σ_r to obtain

$$\begin{aligned}
-\frac{\partial\ell(\boldsymbol{\theta}|\boldsymbol{r},\boldsymbol{t})}{\partial\kappa} &= -\frac{1}{\sigma_r^2}\sum_{i=0}^{n-1}[r(i+1)-r(i)-\kappa(\beta-r(i))\delta_i][\beta-r(i)],\\
-\frac{\partial\ell(\boldsymbol{\theta}|\boldsymbol{r},\boldsymbol{t})}{\partial\beta} &= -\frac{\kappa}{\sigma_r^2}\sum_{i=0}^{n-1}[r(i+1)-r(i)-\kappa(\beta-r(i))\delta_i],\\
-\frac{\partial\ell(\boldsymbol{\theta}|\boldsymbol{r},\boldsymbol{t})}{\partial\sigma_r} &= -\frac{1}{\sigma_r^3}\sum_{i=0}^{n-1}\frac{[r(i+1)-r(i)-\kappa(\beta-r(i))\delta_i]^2}{\delta_i} + \frac{n}{\sigma_r}.
\end{aligned} \tag{28}$$

By setting these equations to 0, substituting the relevant estimators and solving, we obtain:

$$\begin{aligned}
\hat{\kappa} &= \frac{\sum_{i=0}^{n-1}[r(i+1)-r(i)][\hat{\beta}-r(i)]}{\sum_{i=0}^{n-1}[\hat{\beta}-r(i)]^2\delta_i},\\
\hat{\beta} &= \frac{r(n)-r(1)+\hat{\kappa}\sum_{i=0}^{n-1}r(i)\delta_i}{\hat{\kappa}[t(n)-t(1)]},\\
\hat{\sigma_r} &= \sqrt{\sum_{i=0}^{n-1}\frac{[r(i+1)-r(i)-\hat{\kappa}(\hat{\beta}-r(i))\delta_i]^2}{\delta_i}}.
\end{aligned} \tag{29}$$

5.1.2. Calibration Of Parameters In Mortality Model. (Shoji and Ozaki (1998)) proposed a method which is similar to the Euler method popularly known as the Shoji-Ozaki method. The approach approximates non- linear stochastic differential equations with a linear equation from a discrete observations. The non- linear stochastic differential equations allow the drift term to depend on time, just as the mortality model we will use in this work. The maximum likelihood estimation technique is then used to derive estimators for the various unknown parameters in the mortality model. Let X_t be a stochastic process which can be represented by the stochastic differential equation

$$dX_t = f(X_t, \boldsymbol{\theta}, t)dt + g(X_t, \boldsymbol{\theta})dW_t. \tag{30}$$

We let $\delta_i = t_{i+1} - t_i$. For this Shoji- Ozaki approach, $X_{t+1} - X_t|X_t$ follows $N(D(X_t, \boldsymbol{\theta}, t), G(X_t, \boldsymbol{\theta}, t))$ with

$$\begin{aligned} D(X_t, \boldsymbol{\theta}, t) &= \frac{f(X_t, \boldsymbol{\theta}, t)}{L_t}(e^{L_t\delta_i} - 1) + \frac{V_t}{L_t^2}(e^{L_t\delta_i} - 1 - L_t\delta_i), \\ G(X_t, \boldsymbol{\theta}, t) &= g^2(X_t, \boldsymbol{\theta})\frac{e^{2L_t\delta_i} - 1}{2L_t}, \quad L_t = \frac{\partial f(X_t, \boldsymbol{\theta}, t)}{\partial X_t}, \\ V_t &= \frac{g^2(X_t, \boldsymbol{\theta})}{2}\frac{\partial^2 f(X_t, \boldsymbol{\theta}, t)}{\partial X_t^2} f(X_t, \boldsymbol{\theta}, t) + \frac{\partial f(X_t, \boldsymbol{\theta}, t)}{dt}. \end{aligned} \tag{31}$$

Note that all the derivatives in (31) are to be evaluated at $t = t_i$ and $X_t = X_{t_i}$ to obtain estimates. In terms of the Hull-White model specified in (8), we can obtain the above equations as

$$\begin{aligned} L_t &= -b, \quad V_t = ABe^{Bt}, \\ D(X_t, \boldsymbol{\theta}, t) &= -\frac{Ae^{Bt} - b\mu(i)}{b}(e^{-b\delta_i} - 1) + \frac{ABe^{Bt}}{b^2}(e^{-b\delta_i} - 1 + b\delta_i), \\ G(X_t, \boldsymbol{\theta}, t) &= \frac{\sigma_\mu^2(1 - e^{-2b\delta_i})}{2b}. \end{aligned} \tag{32}$$

Following the same procedure outlined in Section 5.1.1, we have the following as our likelihood likelihood and score processes

$$\begin{aligned} \mathcal{L}(\boldsymbol{\theta}|\boldsymbol{\mu}, \boldsymbol{t}) = \prod_{i=0}^{n-1}\left(\frac{2\pi\sigma_\mu^2(1 - e^{-2b\delta_i})}{2b}\right)^{-1/2} \times \exp\Bigg[-\frac{b}{\sigma_\mu^2(1 - e^{-2b\delta_i})} \\ \times\left(\mu(i+1) - \mu(i) + \frac{Ae^{Bt_i} - b\mu(i)}{b}(e^{-b\delta_i} - 1) - \frac{ABe^{Bt_i}}{b^2}(e^{-b\delta_i} - 1 + b\delta_i)\right)^2\Bigg] \end{aligned} \tag{33}$$

The negative log- likelihood process which will be minimized to obtain the estimators is given by

$$-\ell(\boldsymbol{\theta}|\boldsymbol{\mu}, \boldsymbol{t}) = \frac{1}{2}\sum_{i=0}^{n-1}\left[\log\left(2\pi\sigma_{\mu}^{2}\right) + \log\left(1 - e^{-2b\delta_i}\right) - \log 2b\right] \quad + \frac{b}{\sigma_{\mu}^{2}}\sum_{i=0}^{n-1}\left[-\frac{1}{(1 - e^{-2b\delta_i})}\right.$$
$$\left.\times\left(\mu(i+1) - \mu(i) + \frac{(Ae^{Bt_i} - b\mu(i))(e^{-b\delta_i} - 1)}{b} - \frac{ABe^{Bt_i}}{b^2}(e^{-b\delta_i} - 1 + b\delta_i)\right)^2\right]. \tag{34}$$

We use numerical methods to minimize the negative log likelihood of both the interest rate and mortality processes to find the optimal parameters.

5.2. SIMULATION METHODS

Solutions to continuous stochastic differential equations are typically approximated using discrete methods, which are the foundation of the majority of simulation methodologies. The Euler (Iacus (2008)) and Shoji - Ozaki methods (Shoji and Ozaki (1998)) as discussed in sections 5.1.1 and 5.1.2 respectively, were initially developed to obtain solutions for deterministic differential equations, but it has since become one of the most used strategies for approximating stochastic differential equations. These are the methods we employed in our calibrations and also data simulation.

5.3. NUMERICAL ILLUSTRATION

U.S. mortality rates from www.humanmortality.org for the period of 1964 to 2019 was used in the calibration of the model. We obtained the mortality process for a reference population of indiviuals born in 1964, that is the individuals were aged 55 in 2019, and derived prices for longevity bonds with 10, 20 and 30 years of maturity. Risk- free interest rates from the U.S. department of Treasury from 1964 to 2019 was used for the calibration of the Vasicek model. It can be seen from Table 2 that in the long run, the interest rate process is expected to converge to about 5.78% at a rate of 0.01595.

Table 1. Hull- White parameter values calibrated using the Shoji- Ozaki method, from the U.S. mortality rates reported from 1964 to 2019

Parameter name	Parameter value	Standard error
A	0.050313	0.001164426
B	0.023045	0.000250915
b	0.814639	0.095581103
σ_μ	0.029852	0.000595274

Table 2. Parameter values for the Vasicek model calibrated using interest rates from the U.S. department of treasury from 1964 to 2019

Parameter name	Parameter value	Standard error
κ	0.01595	1.556468×10^{-4}
β	0.057716	1.032609×10^{-3}
σ_r	0.001255	3.083839×10^{-6}

Table 3. Different prices for a longevity bond with face value $1

T	10	20	30
Bond price	0.064632	0.002124998	0.000152

Using the parameters in Tables 1 and 2, we simulate both processes based on the varying terms to maturity of the bond. We considered a bond with face value of $1. Since this is a longevity bond, the payoffs are the series of survival probabilities at each time t. It is observed from the bond prices displayed in Table 3 that as the bond term increases, the price today of a bond with a face value of $1 reduces.

APPENDIX

The following proofs which rely heavily on results in Ito Calculus are needed for determining the price of the longevity bond.

A.1. PROOFS OF EQUATIONS (5) AND (9)

We need to derive the solution to the Vasicek model in (Vasicek (1977)) so as to use it to determine the discount process in (22)

$$\begin{aligned} d\mu_x(t) &= \left(Ae^{Bt} - b\mu_x(t)\right) dt + \sigma_\mu dW_\mu(t) \\ \frac{d\mu_x(t)}{dt} + b\frac{\mu_x(t)dt}{dt} &= \frac{Ae^{Bt}dt}{dt} + \sigma_\mu \frac{dW_\mu(t)}{dt} \\ \frac{d\mu_x(t)}{dt} + b\mu_x(t) &= Ae^{Bt} + \sigma_\mu \frac{dW_\mu(t)}{dt} \end{aligned} \tag{35}$$

The second equation in (35) is obtained by rearranging the first. Multiply both sides of the third equation in (35) by the integrating factor $e^{\int_s^t bdu} = e^{b(t-s)}$ and dt to obtain

$$e^{b(t-s)}[d\mu_x(t) + b\mu_x(t)dt] = e^{b(t-s)}Ae^{Bt}dt + e^{b(t-s)}\sigma_\mu dW_\mu(t) \tag{36}$$

Note that $W_\mu(t) \sim N(0,t)$. The left hand side of (36) is the differential of $d[e^{b(t-s)}\mu_x(t)]$. Suppose we have information for this process up to time s for $0 \leq s \leq t \leq T$, we can obtain the solution of the process $\mu_x(t)$ by integrating both sides of equation (36) from s to t, to obtain

$$\begin{aligned} \int_s^t d[e^{b(t-s)}\mu_x(t)] &= \int_s^t e^{b(u-s)}Ae^{Bu}du + \int_s^t e^{b(u-s)}\sigma_\mu dW_\mu(u) \\ e^{b(t-s)}\mu_x(t) - \mu_x(s) &= \frac{A}{B+b}e^{-bs}[e^{(B+b)t} - e^{(B+b)s}] + \sigma_\mu e^{-bs}\int_s^t e^{bu}dW_\mu(u) \end{aligned}$$

which we can rearrange to obtain

$$\mu_x(t) = \mu_x(s)e^{-b(t-s)} + \frac{A}{B+b}[e^{Bt} - e^{Bs-b(t-s)}] + \sigma_\mu e^{-bt}\int_s^t e^{bu}dW_\mu(u). \tag{37}$$

From literature Calin (2012) and Vasicek (1977), $\mu_x(t)$ is normally distributed with mean and variance given by

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(\mu_x(t)|\mathcal{G}_{s,\mu}) &= \mu_x(s)e^{-b(t-s)} + \frac{A}{B+b}(e^{Bt} - e^{Bs-b(t-s)}) \\ \mathbb{V}_{\mathbb{P}}(\mu_x(t)|\mathcal{G}_{s,\mu}) &= \sigma_\mu^2 e^{-2bt}\int_s^t e^{2bu}du \\ &= \frac{\sigma_\mu^2}{2b}(1-e^{-2b(t-s)})\end{aligned} \tag{38}$$

The expectation of the integral in (37) comes from the zero- mean property of ito integrals. The variance is obtained by the application of the isometry property of ito integrals.

To prove (9) and (10), we write the stochastic differential equation

$$\begin{aligned}dr(t) &= \kappa[\beta - r(t)]dt + \sigma_r dW_r(t) \\ \frac{dr(t)}{dt} + \kappa r(t)\frac{dt}{dt} &= \kappa\beta\frac{dt}{dt} + \sigma_r\frac{dW_r}{dt} \\ \frac{dr(t)}{dt} + \kappa r(t) &= \kappa\beta + \sigma_r\frac{dW_r}{dt}\end{aligned} \tag{39}$$

Given the history of the process up to time $s, 0 \leq s \leq t \leq T$, multiply both sides of the third equation in (39) by the integrating factor $e^{\int_s^t \kappa du} = e^{\kappa(t-s)}$ and dt. Intergrate both sides of the equation obtained to get

$$\begin{aligned}\int_s^t d(e^{\kappa(t-s)}r(t)) &= \int_s^t e^{\kappa(u-s)}\kappa\beta dt + \sigma_r\int_s^t e^{\kappa(u-s)}dW_r(u) \\ e^{\kappa(t-s)}r(t) - r(s) &= \beta e^{-\kappa s}[e^{\kappa t} - e^{\kappa s}] + \sigma_r e^{-\kappa s}\int_s^t e^{\kappa u}dW_r(u)\end{aligned} \tag{40}$$

Rearrange the last equation in (40) to obtain the solution of (8) as

$$r(t) = r(s)e^{-\kappa(t-s)} + \beta(1 - e^{-\kappa(t-s)}) + \sigma_r e^{-\kappa t}\int_s^t e^{\kappa u}dW_r(u) \tag{41}$$

Once again, given the information up to time s, $0 \leq s \leq t \leq T$, and applying the zero- mean and isometry properties of ito integrals to the last integral in (41), we find the mean and variance of $r(t)$ as

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(r(t)|\mathcal{G}_{s,r}) &= r(s)e^{-\kappa(t-s)} + \beta(1 - e^{-\kappa(t-s)})\\ \mathbb{V}_{\mathbb{P}}(r(t)|\mathcal{G}_{s,r}) &= \sigma_r^2 e^{-\kappa t}\int_s^t e^{2\kappa u}du \\ &= \frac{\sigma_r^2}{2\kappa}(1 - e^{-2\kappa(t-s)})\end{aligned} \tag{42}$$

For simplicity of the rest of the derivations, we assume s=0 so we write $0 \leq t \leq T$.

A.2. DISTRIBUTIONS OF $\int_t^T r(u)du$ AND $\int_t^T \mu_x(u)du$

We require the distributions of $\int_t^T r(u)du$ and $\int_t^T \mu_x(u)du$ so as to ascertain the adapted processes and equations in (18) and (19). We change the limits of integration to be from t to T since we intend to use these results to price a $(T - t)$ year longevity bond. Integrate both sides of (41) with $s = 0$ in the following:

$$\begin{aligned}\int_t^T r(u)du &= r(0)\int_t^T e^{-\kappa u}du + \beta\int_t^T (1 - e^{-\kappa u})\, du \\ &+ \int_t^T \sigma_r e^{-\kappa u}\left(\int_0^u e^{\kappa v}dW_r(v)\right)du \qquad (43)\\ &= \frac{r(0)}{\kappa}(e^{-\kappa t} - e^{-\kappa T}) + \beta\left(T - t + \frac{e^{-\kappa T} - e^{-\kappa t}}{\kappa}\right) \\ &+ \sigma_r\int_t^T e^{\kappa v}\left(\int_v^T e^{-\kappa u}du\right)dW_r(v) \qquad (44)\\ &= \frac{r(0)}{\kappa}(e^{-\kappa t} - e^{-\kappa T}) + \beta\left(T - t + \frac{e^{-\kappa T} - e^{-\kappa t}}{\kappa}\right) \\ &+ \frac{\sigma_r}{\kappa}\int_t^T \left(1 - e^{-\kappa(T-v)}\right)dW_r(v) \qquad (45)\end{aligned}$$

The second equation in (43) is as a result of an application of Fubini's theorem. (C.f. Calin (2012)). From literature, $\int_t^T r(u)du$ is normally distributed with mean and variance:

$$\begin{aligned}
\mathbb{E}\left(\int_t^T r(u)du \middle| \mathcal{G}_{t,r}\right) &= \frac{r(0)}{\kappa}(e^{-\kappa t} - e^{-\kappa T}) + \beta\left(T - t + \frac{e^{-\kappa T} - e^{-\kappa t}}{\kappa}\right) \\
\mathbb{V}\left(\int_t^T r(u)du \middle| \mathcal{G}_{t,r}\right) &= \frac{\sigma_r^2}{\kappa^2}\int_t^T \left(1 - e^{-\kappa(T-v)}\right)^2 dv \\
&= \frac{\sigma_r^2}{\kappa^2}\left(T - t - \frac{2(1 - e^{-\kappa(T-t)})}{\kappa} + \frac{(1 - e^{-2\kappa(T-t)})}{2\kappa}\right)
\end{aligned} \tag{46}$$

The explicit form for $\mathbb{E}_{\mathbb{P}}\left(e^{-\int_t^T r(u)du}\right)$ can be derived from using the moment generating function of normal distribution, $\mathbb{E}(e^{tX}) = e^{\mathbb{E}(X)t + \frac{1}{2}V(X)t^2}$,that is

$$\mathbb{E}_{\mathbb{P}}\left(e^{-\int_t^T r(u)du} \middle| \mathcal{G}_{t,r}\right) = \exp\left[-\mathbb{E}\left(\int_t^T r(u)du \middle| \mathcal{G}_{t,r}\right) + \frac{1}{2}\mathbb{V}\left(\int_t^T r(u)du \middle| \mathcal{G}_{t,r}\right)\right] \tag{47}$$

Dividing $e^{-\int_t^T r(u)du}$ by what we obtain in (47), we obtain the Radon- Nikodyḿ derivative (18).

We follow the same process we used in obtaining the distribution of $\int_t^T r(u)du$ above to get the distribution of $\int_t^T \mu_x(u)du$ through the following steps:

$$\begin{aligned}
\int_t^T \mu_x(u)du &= \mu_x(0)\int_t^T e^{-bu}du + \frac{A}{B+b}\int_t^T (e^{Bu} - e^{-bu})du \\
&+ \sigma_\mu \int_t^T e^{-bu}\left(\int_0^u e^{bv}dW_\mu(v)\right)du
\end{aligned} \tag{48}$$

$$\begin{aligned}
&= \frac{\mu_x(0)}{b}\left(e^{-bt} - e^{-bT}\right) + \frac{A}{B+b}\left(\frac{1}{b}(e^{-bT} - e^{-bt}) + \frac{1}{B}(e^{BT} - e^{Bt})\right) \\
&+ \frac{\sigma_\mu}{b}\int_t^T (1 - e^{-b(T-v)})dW_\mu(v)
\end{aligned} \tag{49}$$

Just as $\int_t^T r(u)du$, $\int_t^T \mu_x(u)du$ is known in literature to be normally distributed with mean and variance:

$$\begin{aligned}
\mathbb{E}\left(\int_t^T \mu_x(u)du \middle| \mathcal{G}_{t,\mu}\right) &= \frac{\mu_x(0)}{b}\left(e^{-bt} - e^{-bT}\right) + \frac{A}{B+b}\left(\frac{1}{b}(e^{-bT} - e^{-bt}) + \frac{1}{B}(e^{BT} - e^{Bt})\right) \\
V\left(\int_t^T \mu_x(u)du \middle| \mathcal{G}_{t,\mu}\right) &= \frac{\sigma_\mu^2}{b^2}\int_t^T (1 - e^{-b(T-v)})^2 dv \\
&= \frac{\sigma_\mu^2}{b^2}\left(T - t - \frac{2(1 - e^{-b(T-t)})}{b} + \frac{1 - e^{-2b(T-t)}}{2b}\right)
\end{aligned} \tag{50}$$

As in (47), we derive the equation in (19) through

$$\mathbb{E}_{\mathbb{P}}\left(e^{-\int_t^T \mu_x(u)du} \middle| \mathcal{G}_{t,\mu}\right) = \exp\left[-\mathbb{E}\left(\int_t^T \mu_x(u)du \middle| \mathcal{G}_{t,\mu}\right) + \frac{1}{2}\mathbb{V}\left(\int_t^T \mu_x(u)du \middle| \mathcal{G}_{t,\mu}\right)\right] \tag{51}$$

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SECTION

3. CONCLUSION

There is a need to assist insurance companies and all other stakeholders in dealing with the problem of longevity risk. To account for improvements in people's lives that lead to extended life expectancy, inherently causing the problem of longevity risk, we presented a generalized additive model in this work. This model takes into account policyholders' unique mortality risk profiles. Because of the effectiveness of the simulation study, we know that the mortality model is analytically tractable. This model's parameters have simple interpretations, and it is also suitable for pricing financial instruments. We also successfully applied the Hull-White model proposed by (Zeddouk and Devolder (2020)) to the pricing of longevity bonds under the risk neutral measure.

In future work, we plan to add jumps to the Hull-White model mortality model we used in the second paper and the additive mortality model we proposed in the first paper. This will account for sudden peaks in mortality caused by for instance pandemics and natural disasters and also sudden declines in mortality caused by for example, discovery of cures for fatal diseases.

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