Spring 2021

Prediction intervals for fractionally integrated time series and volatility models

Rukman Ekanayake

PREDICTION INTERVALS FOR FRACTIONALLY INTEGRATED TIME SERIES AND VOLATILITY MODELS

by

EKANAYAKE MUDIYANSELAGE RUKMAN SUMEDHA BANDARA EKANAYAKE

A DISSERTATION

Presented to the Graduate Faculty of the

MISSOURI UNIVERSITY OF SCIENCE AND TECHNOLOGY

In Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

in

MATHEMATICS WITH STATISTICS EMPHASIS

2021

Approved by:

Dr. V. A. Samaranayake, Advisor
Dr. Akim adekpedjou
Dr. Gayla Olbricht
Dr. Robert Paige
Dr. Gregory Gelles

PUBLICATION DISSERTATION OPTION

This dissertation consists of the following four articles, formatted in the style used by the Missouri University of Science and Technology:

Paper I, found on pages 17–47, is intended for submission to *Journal of Statistical Computations and Simulations*.

Paper II, found on pages 48–75, is intended for submission to *Journal of Forecasting*.

Paper III, found on pages 76–104, is intended for submission to *International Journal of Forecasting*.

Paper IV, found on pages 105-132, is intended for submission to *Mathematics and Computers in Simulation*.

ABSTRACT

The two of the main formulations for modeling long range dependence in volatilities associated with financial time series are fractionally integrated generalized autoregressive conditional heteroscedastic (FIGARCH) and hyperbolic generalized autoregressive conditional heteroscedastic (HYGARCH) models. The traditional methods of constructing prediction intervals for volatility models, either employ a Gaussian error assumption or are based on asymptotic theory. However, many empirical studies show that the distribution of errors exhibit leptokurtic behavior. Therefore, the traditional prediction intervals developed for conditional volatility models yield poor coverage. An alternative is to employ residual bootstrap-based prediction intervals. One goal of this dissertation research is to develop methods for constructing such prediction intervals for both returns and volatilities under FIGARCH and HYGARCH model formulations.

In addition, this methodology is extended to obtain prediction intervals for autoregressive moving average (ARMA) and fractionally integrated autoregressive moving average (FARIMA) models with a FIGARCH error structure. The residual resampling is done via a sieve bootstrap approach, which approximates the ARMA and FARIMA portions of the models with an AR component. AIC criteria is used to find order of the finite AR approximation on the conditional mean process. The advantage of the sieve bootstrap method is that it does not require any knowledge of the order of the conditional mean process. However, we assume that the order of the FIGARCH part is known. Monte-Carlo simulation studies show that the proposed methods provide coverages closed to the nominal values.

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my advisor Dr. Samaranayake for his continues guidance, inspiration and advice given me during my PhD program. I would like to thank my graduate committee for their valuable inputs and suggestions. I am grateful to all the members of the Department of Mathematics and Statistics, Missouri University of Science and Technology, for their help and advice in many ways to bring my academic pursuits to success. I would like to thank all my teachers and especially, Dr. Kanthi Perera who encouraged me to pursue higher studies. Furthermore, I would like to thank my parents S.B. Ekanayake and Yasodara Tennekoon, for giving me the strength and the support throughout this journey. Finally, I would like to thank my siblings Wathsala, Pandula, Dharshika, Anuradha, Janaka, and my fiancée Chathuri and all my friends who always there with me when needed.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

PAPER III

PAPER IV

LIST OF TABLES

1. INTRODUCTION

1.1. BACKGROUND

Time series analysis is a specialized area in Statistics which deals with data observed over time. It has applications in the fields such as astronomy, economics, engineering, environmental sciences, hydrology, and physics. Basically, there are two major aspects of time series analysis, namely modelling and forecasting. Many statistical approaches are utilized to both model and forecast empirical time series. However, the major emphasis in this study is on time series forecasting.

The standard time series formulations developed to model the conditional mean of a process, such as the autoregressive moving average (ARMA) models, and for the conditional variance, such as the generalized conditional heteroscedastic (GARCH) models, have autocorrelations that decay geometrically. These are sometimes known as short-memory processes and cannot effectively approximate long-term dependance with a parsimonious model. Time series with such long-term dependence are often termed long-memory processes. In early 90s, many researchers not only found long memory behaviour in the mean process of empirical time series, but also in squared residuals of empirical time series in domains such as finance and economics (deLima et al. (1994), Ding et al. (1993) and Harvey (1993)). This phenomenon necessitated the development of volatility models with long range dependence. Some of the models that addressed this need are the integrated GARCH (IGARCH, Engle & Bollerslev (1986)) and fractionally integrated GARCH (FIGARCH, Baillie et al. (1996)) formulations. More recently, the hyperbolic GARCH (HYGARCH, Davidson (2004)) models were introduced, that address some of the shortcomings of the FIGARCH model. While such models are widely available, bootstrap-based methodologies for obtaining prediction intervals for them is lacking. In this dissertation, prediction intervals for FIGARCH and HYGARCH volatility models, as well as for Autoregressive Fractionally Integrated Moving Average (FARIMA) processes whose conditional volatility exhibits long memory behavior, will be discussed. Distribution free bootstrap techniques are adapted in order to construct the prediction intervals.

It is important to discuss some key concepts and terminologies used in time series analysis, before moving on to describing the methodologies used to obtain prediction intervals.

1.1.1. Autocovariance Function. If $\{X_t\}_{t\in\mathbb{Z}}$ is a time series such that $Var(X_t)<\infty$ for each $t\in T$, then the covariance function $\gamma_X(\cdot,\cdot)$ of $\{X_t\}$ is defined by

$$\gamma_X(r,s):=Cov(X_r,X_s)=E\left[(X_r-EX_r)(X_s-EX_s)\right],\ r,s\in T.$$

1.1.2. Stationary Time Series. A time series $\{X_t\}_{t\in\mathbb{Z}}$ is said to be (covariance) stationary if

(i) $E|X_t|^2<\infty$ for all $t\in\mathbb{Z}$,

(ii) $E[X_t]=m$ for all $t\in\mathbb{Z}$,

(iii) $\gamma_X(r,s)=\gamma_X(r+t,s+t)$ for all $r,s,t\in\mathbb{Z}$.

Further, the autocorrelation function of a stationary processes $\{X_t\}_{t\in\mathbb{Z}}$ is defined as $\rho_X(h):=\dfrac{\gamma_X(h)}{\gamma_X(0)}$ for all $h\in\mathbb{Z}$, where $\gamma_X(h)=\gamma_X(t,t+h)$ for $t,h\in\mathbb{Z}$.

1.1.3. White Noise Process. The process $\{X_t\}$ is said to be white noise (WN) with mean 0 and variance σ^2, if the autocovariance function, γ_X , satisfies

$$\gamma_X(h) = \begin{cases} \sigma^2 & if\ h = 0 \\ 0 & if\ h \neq 0 \end{cases}.$$

1.2. CONDITIONAL MEAN PROCESSES

Here we present the definitions of some conditional mean processes. These are sometimes referred to as Box and Jenkins models. Autoregressive Moving Average (ARMA) formulations are widely used to model time series that exhibit short term dependence where the autocorrelation function decay to zero at an exponential rate. Therefore, the memory of the past event decays fast and its impact becomes negligible after short a period of time.

1.2.1. ARMA Process. A real valued process $\{X_t\}_{t\in\mathbb{Z}}$ is said to be an ARMA$(p,\ q)$ process if $\{X_t\}$ is stationary and satisfies

$$\Phi(L)(X_t - \mu) = \theta(L)\varepsilon_t,\ t \in \mathbb{Z},$$

where Φ and θ are the p^{th} and q^{th} degree polynomials with $\Phi(z) = 1 - \Phi_1 z - ... - \Phi_p z^p$ and $\theta(z) = 1 + \theta_1 z + ... + \theta_q z^q$ and having no common roots. The innovations are assumed to be white noise, that is: $\{\varepsilon_t\} \sim WN(0, \sigma^2)$ .The mean of the process is $\mu = E(X_t)$ for all t. The lag operator, L is defined by $L^k X_t = X_{t-k}$, for $k \in \mathbb{N}$.

The ARMA process becomes an Autoregressive process with order p $\left(\text{AR}\left(p\right)\right)$ when $q = 0$ and further, it becomes a Moving Average process with order q $\left(\text{MA}\left(p\right)\right)$

when $p = 0$. Figure 1.1 shows the sample path of a simulated ARMA(1, 1) process with length of 200. The sample autocorrelation function of it is given in the Figure 1.2.

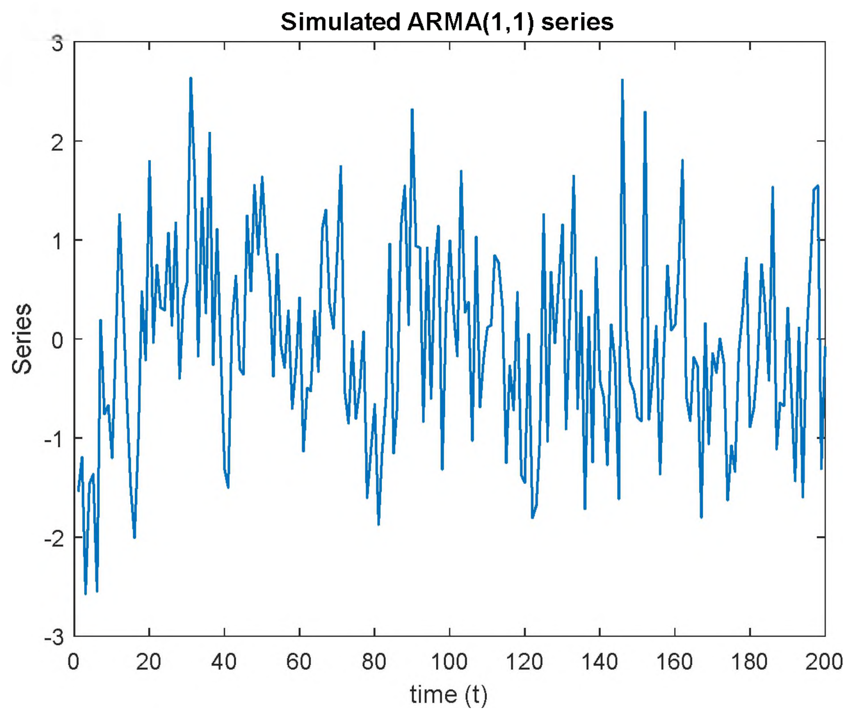

Figure 1.1. Simulated $(1-0.7L)X_t = (1-0.5L)\varepsilon_t$ time series

1.2.2. FARIMA Process. A real valued process $\{X_t\}_{t\in\mathbb{Z}}$ is said to be an Fractionally Integrated Autoregressive Moving Average (FARIMA(p,d,q)) process if the process $Y_t = \nabla^d (X_t - \mu)$ is a ARMA(p,q) process, where the difference parameter $d \in (-0.5, 0.5)$, $\left(\nabla^d = (1-L)^d\right)$.

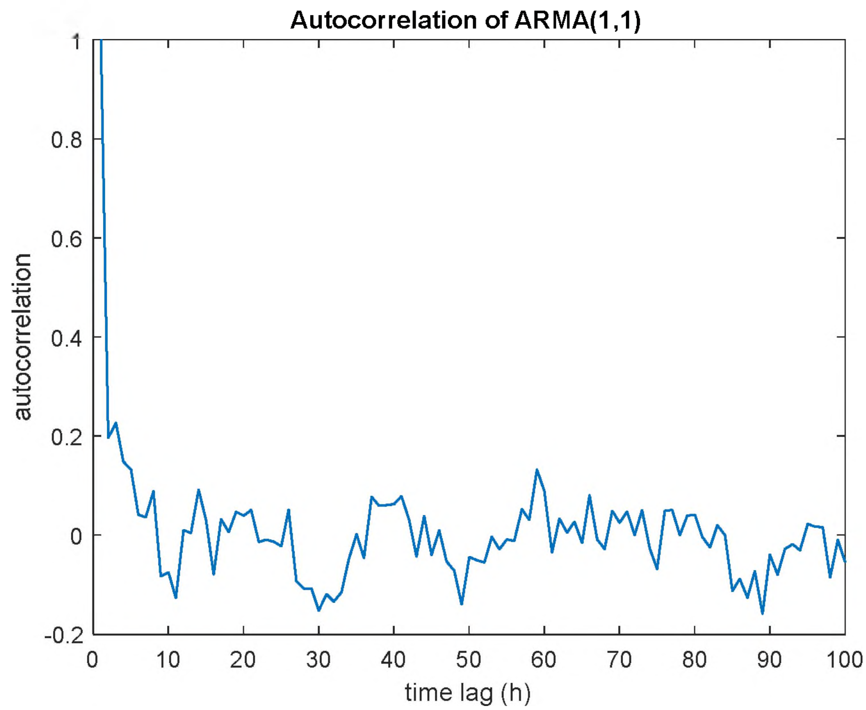

Figure 1.2. Sample ACF of $(1-0.7L)X_t = (1-0.5L)\varepsilon_t$

The FARIMA model is widely used in many areas such as geophysics, econometrics, and hydrology to model time series with a long memory property. Autocorrelation function of FARIMA decays to zero in slow hyperbolic rate rather than a fast-exponential rate and therefore, it is capable of modelling empirical time series with a long memory property. Note that the stationarity of a FAIRMA(p,d,q) process defined as in 1.2.2 depends on the fractional difference parameter d. If $d<0.5$, then the process is stationary and otherwise, it is not stationary. Furthermore, FARIMA(p,d,q) is said to be an intermediate-memory process when $-0.5<d<0$, and it is said to be a long-memory process if $0<d<0.5$, according to the definition given in Brockwell and Davis (2009). Figure 1.3 shows the sample path of the FARIMA process with the fractional difference parameter, $d=0.3$.

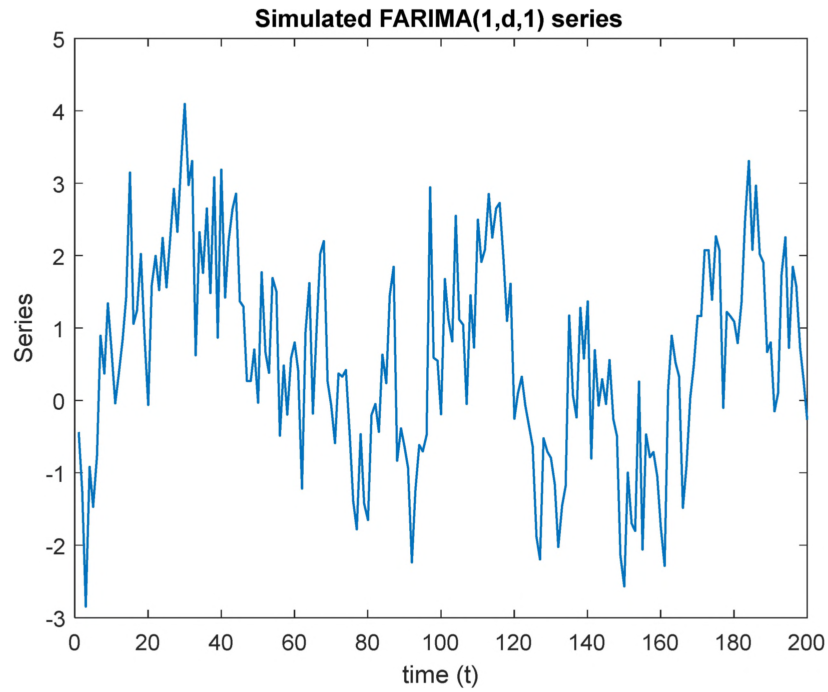

Figure 1.3. Simulated $(1-0.7L)(1-L)^{0.3}X_t = (1-0.5L)\varepsilon_t$ time series

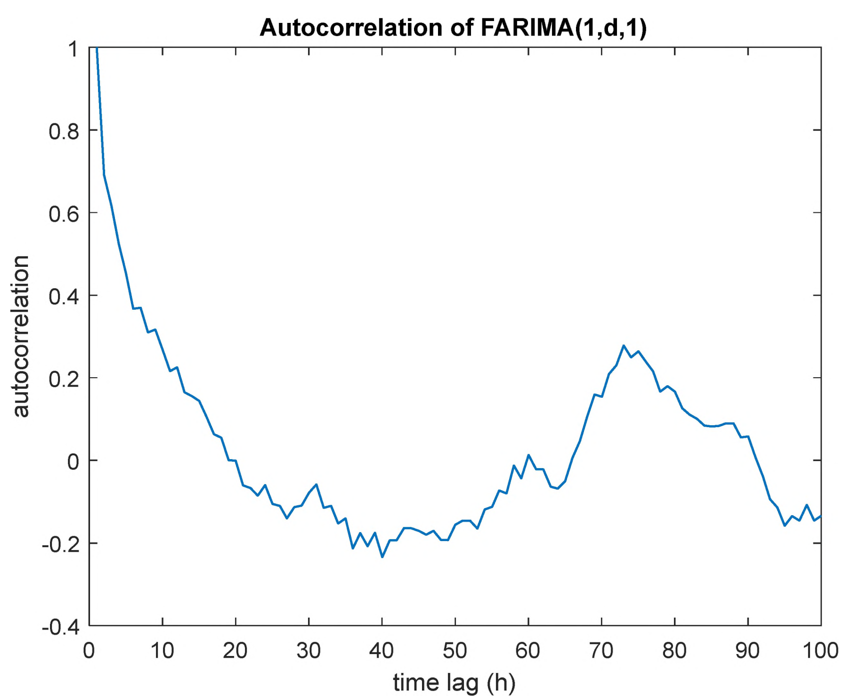

Figure 1.4. Sample ACF of $(1-0.7L)(1-L)^{0.3}X_t = (1-0.5L)\varepsilon_t$

The sample ACF of the simulated FARIMA process is given in the Figure 1.4. The ACF exhibits a long dependence here, not like in the simulated ARMA process with same AR and MA parameters.

In next section we will discuss time series whose variance change over time.

1.3. CONDITIONAL VARIANCE PROCESSES

Volatility (variance) of asset prices in financial markets often change over time. However, the asset prices themselves are uncorrelated but are not independent. Autoregressive Conditional Heteroscedastic (ARCH) model introduced by Engle (1982), is one formulation that is widely used to model such behavior.

1.3.1. ARCH(q). Let $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ be a real-valued discrete tim time series and let $\Im_t$ denote the sigma field generated by the collection of variables $\{\varepsilon_i : i < t\}$. Then the time series $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is said to be an ARCH(q) process (Autoregressive Conditional Heteroscedastic) if

$$E(\varepsilon_t \mid \Im_{t-1}) = 0 \text{ and } Var(\varepsilon_t \mid \Im_{t-1}) = \sigma_t^2 \text{ with}$$

$$\begin{aligned}\sigma_t^2 &= \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2, \\ &= \omega + \alpha(L)\varepsilon_t^2,\end{aligned}$$

where $q > 0,\ \omega > 0,\ \alpha_i \geq 0, i = 1,2,...,q$. Here L is the lag operator, with $\alpha(L) = 1 - \alpha_1 L - ... - \alpha_q L^q$.

Bollerlev (1986) introduced the generalized ARCH (GARCH) model which a natural generalization of ARCH(q) formulation and is capable of modeling a wider class of volatility processes with a limited number of lag terms when compared to ARCH models.

1.3.2. GARCH (*p, q*) Process. Given a real valued discrete time stochastic process $\{\varepsilon_t\}_{t\in\mathbb{Z}}$, with $\Im_t$ defined as before, $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is said to be a GARCH(p,q) process (Generalized Autoregressive Conditional Heteroscedastic) if it satisfies

$$\mathrm{E}(\varepsilon_t \mid \Im_{t-1}) = 0 \text{ and } \mathrm{Var}(\varepsilon_t \mid \Im_{t-1}) = \sigma_t^2,$$

$$\begin{aligned}\sigma_t^2 &= \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 \\ &= \omega + \alpha(L)\varepsilon_t^2 + \beta(L)\sigma_t^2,\end{aligned}$$

where $p \geq 0,\ q > 0,\ \omega > 0,\ \alpha_i \geq 0, i = 1,2,\ldots,q$ and $\beta_j \geq 0, j = 1,2,\ldots,p$. Here L is the lag operator, with $\alpha(L) = 1 - \alpha_1 L - \ldots - \alpha_q L^q$ and $\beta(L) = \beta_1 L + \ldots + \beta_p L^p$.

Note that when $\alpha(1) + \beta(1) = 1$, the GARCH process defined above becomes an Integrated GARCH (IGARCH) process, which was first introduced by Engle and Bollerslev (1986). The GARCH (*p*, *q*) process reduces to the ARCH(q) process when $p = 0$ and becomes a white noise process when $p = q = 0$.

As mentioned earlier, some studies show that the squared returns of some empirical financial time series exhibit long-range dependence. The Fractionally Integrated GARCH (FIGARCH) and Hyperbolic GARCH (HYGARCH) are introduced by Baillie et al. (1996) and Davidson (2004), respectively, are often used in modelling long memory in squared returns. The definition of the FIGARCH is given below based on the definitions given by Baillie at el. (1996) and Tayefi & Ramanathan (2016).

1.3.3. FIGARCH (*p, d, q*) Process. A real valued discrete time stochastic process $\{\varepsilon_t\}_{t\in\mathbb{Z}}$, with $\Im_t$ defined as before, is said to be a FIGARCH(p,d,q) process (Fractionally Integrated GARCH) if it satisfies

$$\mathrm{E}(\varepsilon_t \mid \Im_{t-1}) = 0 \text{ and } \mathrm{Var}(\varepsilon_t \mid \Im_{t-1}) = \sigma_t^2,$$

$$[1-\beta(L)]\sigma_t^2 = \omega + \left[1-\beta(L)-\phi(L)(1-L)^d\right]\varepsilon_t^2$$

where $0 < d < 1$, with $\phi(L) = 1-\phi_1 L - ... - \phi_q L^q$ and $\beta(L) = \beta_1 L + ... + \beta_p L^p$, where L is the lag operator, and all roots of $\phi(L)$ and $[1-\beta(L)]$ lie outside the unit circle.

1.3.4. HYGARCH (*p, d, q*) Process. A real valued discrete time stochastic process $\{\varepsilon_t\}_{t\in\mathbb{Z}}$, with $\Im_t$ defined as before, is said to be a HYGARCH(p,d,q) process (Hyperbolic GARCH) if it satisfies

$$\varepsilon_t = z_t\sigma_t,$$

$$\sigma_t^2 = \omega + \left\{1-\beta(L)-\phi(L)\left[1+\alpha\left((1-L)^d-1\right)\right]\right\}\varepsilon_t^2 + \beta(L)\sigma_t^2$$

where $0 < d < 1$ and $\alpha > 0$ with $\phi(L) = 1-\phi_1 L - ... - \phi_q L^q$ and $\beta(L) = \beta_1 L + ... + \beta_p L^p$ (L is the lag operator).

In addition to the above formulations, it is important to define an infinite order AR process introduced by Robinson (1991). We employed the definition used by Giraitis et al. (2009) to define this process.

1.3.5. ARCH(∞) Process. A real valued discrete time stochastic process $\{\varepsilon_t\}_{t\in\mathbb{Z}}$, is said to be an ARCH$(\infty)$ process if there exists a sequence of standard (zero mean and unit variance) independent and identically distributed random variables $\{z_t\}$ and a deterministic sequence $b_j \geq 0$, $j = 0,1,...$, such that

$$\varepsilon_t = z_t \sigma_t,$$

$$\sigma_t^2 = b_0 + \sum_{j=1}^{\infty} b_j \varepsilon_{t-j}^2 .$$

Moreover, assume that $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a causal, i.e for any t, ε_t has a representation as a measurable function of the present and past values ε_s, $s \le t$.

The GARCH(p,q) and HYGARCH(p,d,q) with $\alpha < 1$ can be represented as ARCH(∞) processes as defined above. Coefficients of the ARCH(∞) representation for GARCH(p,q) processes decay exponentially. For the GARCH representation, the weights b_j are defined by the generating function $\alpha(z)/(1-\beta(z)) = \sum_{j=1}^{\infty} b_j$; $b_0 = (1-\beta(1))^{-1}\alpha_0$, where $\alpha(z)$ and $\beta(z)$ are the polynominals associated with the GARCH process. Similarly, both IGARCH(p,q) and FIGARCH(p,d,q) processes can be represented as an integrated ARCH(∞) process with $\sum_{j=1}^{\infty} b_j = 1$. Therefore, both IGARCH and FIGARCH possesses have infinite unconditional variance. This is not the case with HYGARCH process, which is one reason why these processes are sometimes preferred over IGARCH and FIGARCH formulations.

The focus of this dissertation are the volatility models with long range dependence. Therefore, it is useful to examine the behavior of the autocovariance function of the squared returns $\{\varepsilon_t^2\}$ for such processes and compare that with the behavior associated with short-memory models. The squared returns $\{\varepsilon_t^2\}$ in ARCH(∞) representation of GARCH(p,q) has finite fourth moments with absolute summable autocovariance function,

$\gamma_{\varepsilon}(h) = Cov(\varepsilon_h^2, \varepsilon_0^2)$, and exponentially decaying coefficients, which implies short memory in returns squares $\{\varepsilon_t^2\}$. In contrast to the GARCH model, the autocovariances $Cov(\varepsilon_h^2, \varepsilon_0^2)$ of HYGARCH decay to zero at hyperbolic rate h^{-1-d}, with $d > 0$ (Giraitis et al. (2009)). Therefore, HYGARCH possesses intermediate memory in squared returns $\{\varepsilon_t^2\}$ (Brockwell and Davis (2009)). On the other hand, the finite fourth moment do not exist in IGARCH and FIGARCH models since they do not possess finite second moments. However, the FIGARCH model is frequently used in the literature to fit long range dependence in squared returns.

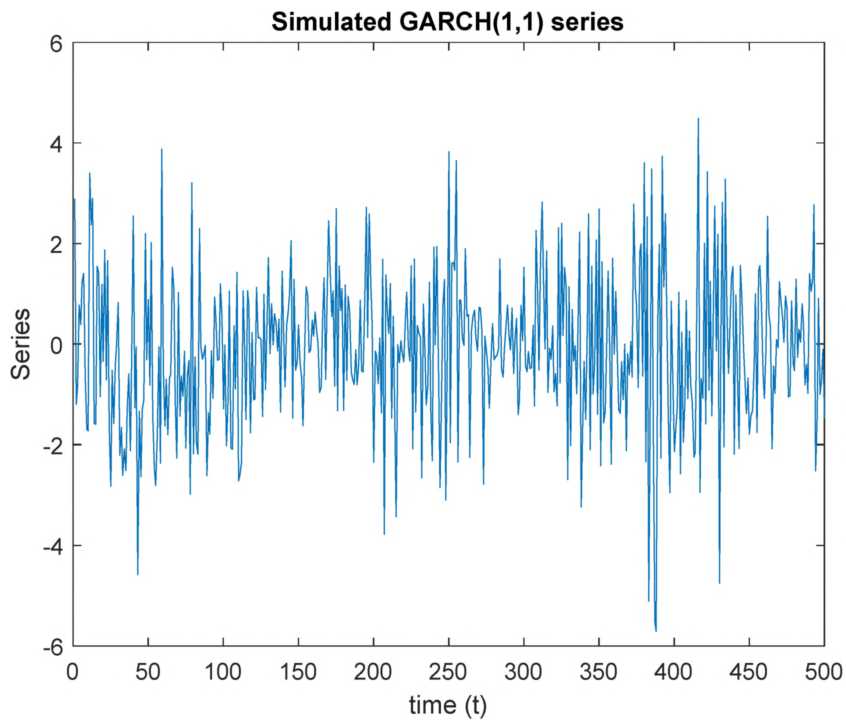

Figure 1.5. Simulated $GARCH(1,1)$ time series with $\sigma_t^2 = 0.1 + 0.1L\varepsilon_t^2 + 0.85L\sigma_t^2$

Figures 1.5 and 1.6 show that the sample paths of simulated $\mathrm{GARCH}(1,1)$ and $\mathrm{FIGARCH}(1,d,0)$. The sample ACF's of $\mathrm{GARCH}(1,1)$ and $\mathrm{FIGARCH}(1,d,0)$ are given in Figures 1.7 and 1.8. As seen in the graphs of ACF's, they are uncorrelated. However, squared returns given in Figures 1.9 and 1.10 shows that the returns are not independent in both models.

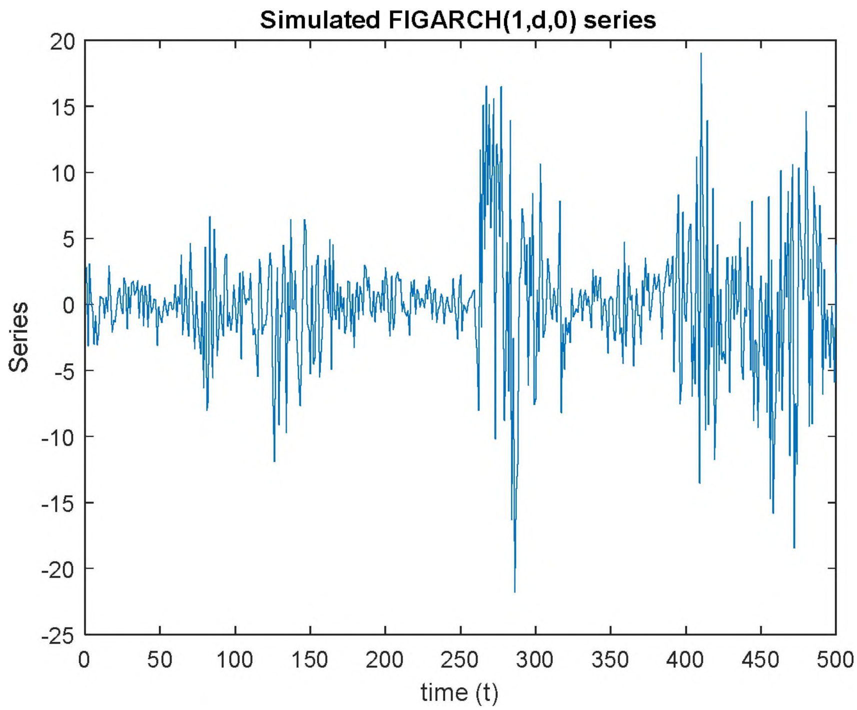

Figure 1.6. Simulated $\mathrm{FIGARCH}(1,d,0)$ time series with
$\sigma_t^2 = 0.1 + \left(1 - 0.45L - (1-L)^{0.75}\right)\varepsilon_t^2 + 0.45L\sigma_t^2$

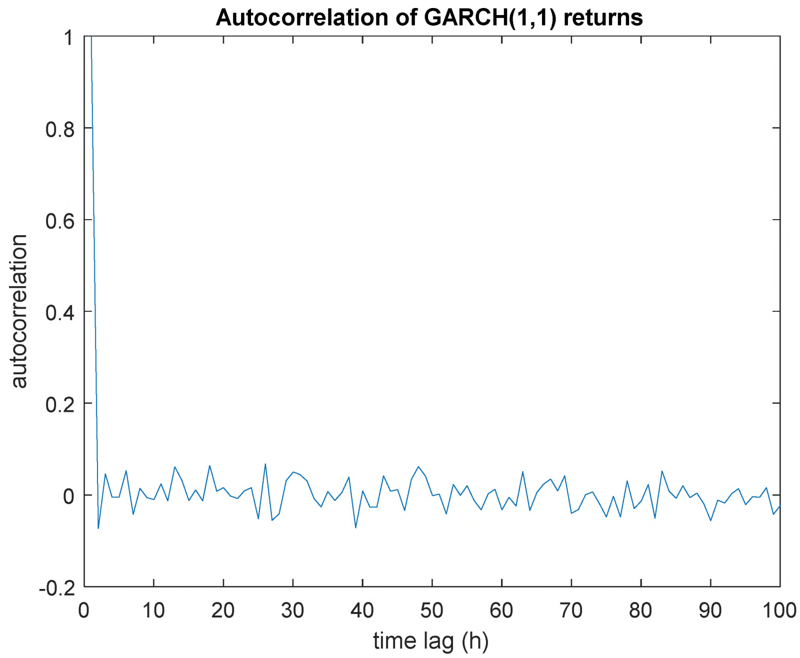

Figure 1.7. Sample ACF of $\text{GARCH}(1,1)$ returns

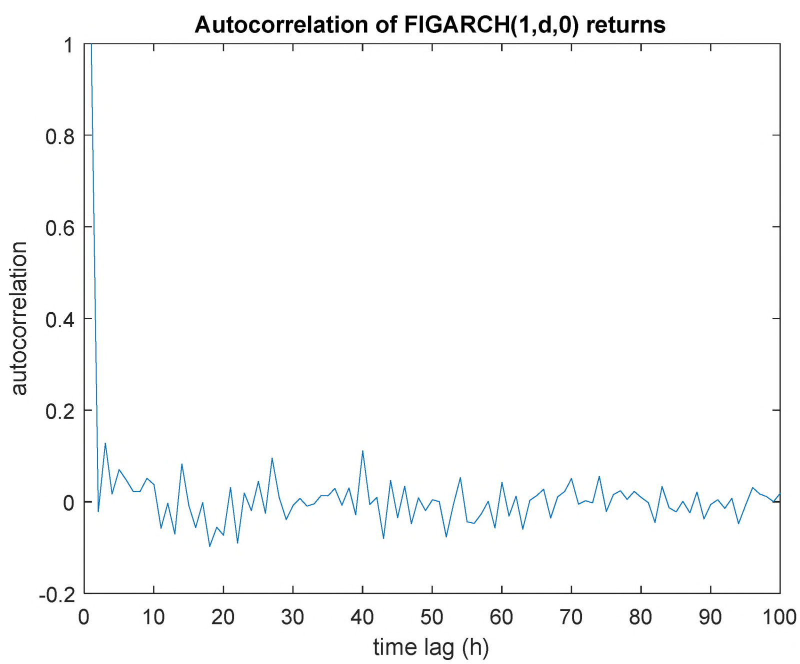

Figure 1.8 Sample ACF of $\text{FIGARCH}(1,d,0)$ returns

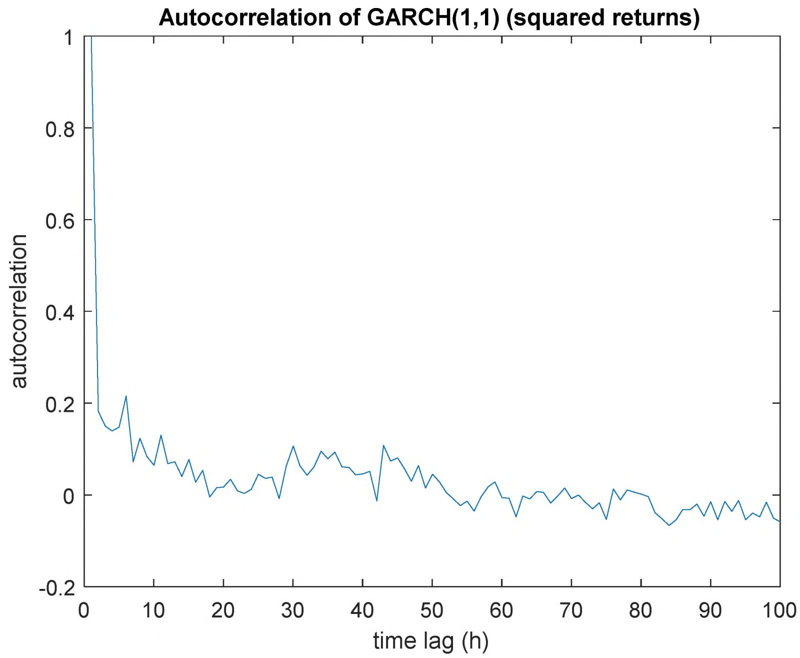

Figure 1.9. Sample ACF of $\mathrm{GARCH}(1,1)$ squared returns

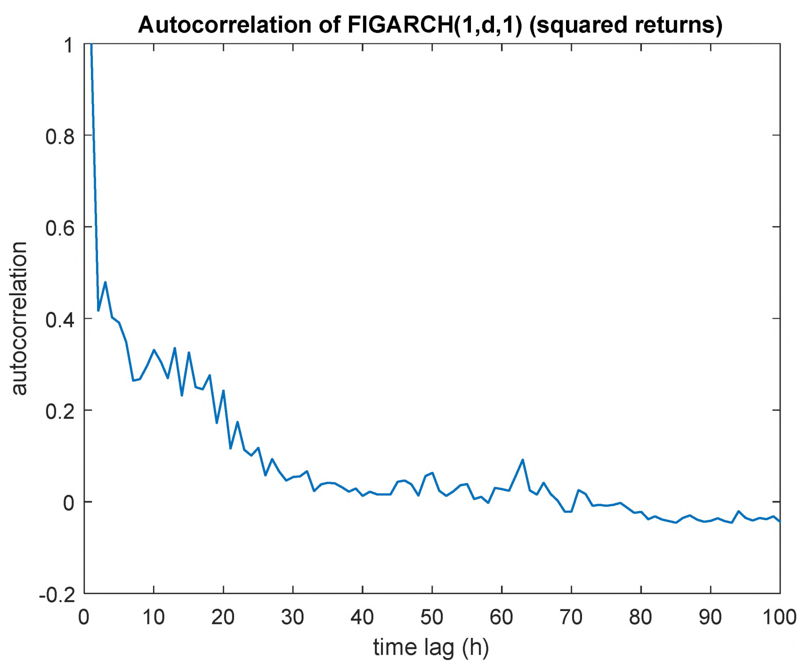

Figure 1.10. Sample ACF of $\mathrm{FIGARCH}(1,d,0)$ squared returns

1.4. THE ADAPTION OF THE BOOTSTRAP PROCEDURE

The bootstrap is a non-parametric resampling technique which was developed by Efron (1979) and can be utilized to make statistical inferences about unknown distributions. Freedman (1984), Stine (1982, 1987) and Findley (1986) applied bootstrap technique to time dependent data by resampling the residuals obtained by fitting a model to the data. Stine (1987) used bootstrap prediction mean squared errors to forecast future values of AR processes. Thombs & Schucany (1990) employed backward representation method (By expressing the current value of the time series as a linear function of future values) to construct bootstrap prediction intervals for AR models with known order p. The use of backward representation is limited to AR processes, and cannot be used for the processes with a MA component. Cao et al. (1997) developed a computationally faster conditional bootstrap without using the backward resampling method. This method was also restricted to AR processes with known orders.

Miguel & Olave (1999) extended the construction of prediction intervals to ARMA models with ARCH errors. Pascual, Romo & Ruiz (2006) developed prediction intervals for returns and volatilities using GARCH type models. Here after we referred method used by Pascual, Romo & Ruiz as PRR. In contrast to Miguel & Olave's method they incorporated the uncertainty of parameter estimations for the intervals. All the above bootstrap methods assume that the order of the process is known.

The foundation of implementing the bootstrap methods for time series with unknown order was laid by Kunsch (1989), Kreiss (1992). Further, these type of time series can be written as an infinite autoregressive process. Buhlmann (1997) extended this approach to a general class of time series that can be represent as an infinite order moving average and use the term "sieve bootstrap". Alonso et al. (2002, 2003) applied the sieve bootstrap approach to construct prediction intervals for linear processes. Those processes can be written as an invertible, and an infinite order moving average with absolute summable coefficients. Mukhopadhyay and Samaranayake (2010) improved the coverages of sieve bootstrap method used Alonso et al. (2002, 2003) by rescaling the residuals. Rupasinghe and Samaranayake (2012, 2013) extended sieve bootstrap prediction intervals for FARIMA models. They incorporate Poskitt's (2006) results to establish the theoretical validation, because the coefficients of infinite moving average representation are not absolute summable in FARIMA models.

The outline of this dissertation as follows. Paper I and Paper IV discuss the bootstrap prediction intervals for returns and volatilities in FIGARCH and HYGARCH models. Prediction intervals for ARMA models with FIGARCH errors are discussed in Paper II. Order of the AR approximation of ARMA component is carried out using approaches adopted in Alonso et al. (2003) and Thilakaratne & Samaranayake (2014). Finally, bootstrap prediction intervals for FARIMA models with FIGARCH errors is presented in Paper III. Here the order order of the AR approximation of FARIMA component is motivated by the approach taken in Rupasinghe and Samaranyake (2012, 2013). We used both sieve bootstrap approach as well as PRR method to construct the intervals in Papers II and III.

PAPER

I. BOOTSTRAP PREDICTION INTERVALS FOR FRACTIONALLY INTEGRATED GENERALIZED AUTOREGRESSIVE CONDITIONAL HETEROSCEDASTIC (FIGARCH) MODELS

ABSTRACT

The Generalized Autoregressive Conditional Heteroscedastic (GARCH) formulations are inadequate to model the persistent volatility found in certain financial assets. The integrated version of the GARCH formulation, namely the IGARCH model, was developed to handle such situations. Fractionally Integrated Generalized Autoregressive Conditional Heteroscedastic (FIGARCH) models, however, provide a more flexible alternative to modeling long-term dependence of volatility, providing a leptokurtic unconditional distribution for returns having long memory behavior. We propose a method based on the residual bootstrap to obtain prediction intervals for the returns and the conditional volatilities of FIGARCH processes. A Monte-Carlo simulation study, conducted using a variety of distributions for the error terms, show that the proposed intervals have good coverage probabilities in most cases.

Keywords: Fractional integration, Volatility modeling, Residual-based bootstrap, Long memory

1. INTRODUCTION

Time series literature is replete with many formulations developed to model the volatility of financial time series. Engle (1982) introduced the well-known Autoregressive Conditional Heteroscedastic (ARCH) model and Bollerslev (1986) extended the ARCH model to the Generalized ARCH (GARCH) model, which accommodate long-term dependence of volatility with a limited number of lag terms, compared to the ARCH formulation. Since the introduction of the ARCH and GARCH models, several variations have been developed by other authors. For example, the exponential GARCH or the EGARCH model (Nelson, 1991) was developed to allow asymmetric response to positive and negative shocks. A generally known fact about GARCH type models is their ability to model volatility clustering. Volatility clustering refers to the phenomenon where large returns tend to follow large returns and small returns tend to follow small returns. Highly persistent volatility, however, cannot be modeled well using the GARCH model or its alternatives such as the EGARCH. The Integrated GARCH (Engle and Bollerslev, 1986) formulation was developed to model time series with persistent volatility. Fractionally Integrated Generalized Autoregressive Conditional Heteroscedastic (FIGARCH) was introduced by Baillie et al. (1996) as an alternative to the IGARCH model, allowing the ability to model the long-memory nature of the conditional variance found in many financial time series, but without the assumption of a unit root in the model. In this paper, we introduce a residual bootstrap-based method of obtaining prediction intervals for the conditional volatility of FIGARCH processes as well as for future returns.

The conditional variance of a GARCH process can be written as an infinite sum of exponentially decaying terms containing squared past innovations. Similarly, the conditional variance of FIGARCH model can be expressed as a sum whose terms have a slower hyperbolic rate of decay. This provides the FIGARCH formulation the ability to model squared return processes having long memory. Thus, in the FIGARCH formulation, the effect of a past shocks (squared innovations) decay slowly to zero, unlike in the GARCH case where such effects decay at a faster exponential rate. In contrast to both the GARCH and FIGARCH processes, the effect of such shocks persists without decaying in the IGARCH process. Thus, the FIGARCH, while allowing for a past shocks to persist for a long period, assumes that eventually its effects become negligible, which is a more reasonable assumption.

There exist several publications on obtaining prediction intervals for the conditional mean of long memory processes as opposed to obtaining prediction intervals for the conditional variance. For example, Bisaglia and Grigoletto (2001) introduced bootstrap-based prediction intervals for Fractionally Integrated Autoregressive Moving Average (FARIMA) processes. Another example is Rupasinghe and Samaranayake, (2013) which established a sieve-bootstrap-based procedure to calculate prediction intervals using an algorithm that is computationally much faster than that proposed by Bisaglia and Grigoletto (2001). However, there are no published literature on obtaining prediction intervals for long memory GARCH type models, and there are only a few papers have discussed the construction of prediction intervals for short-memory ARCH and GARCH type models.

In order to construct prediction intervals, the underlying distribution of the point predictor or that of a pivotal statistic is needed. But this is not feasible in some situations, and in many instances, the asymptotic distribution of such statistics is used instead of the finite sample distribution, which is intractable. An alternative is the distribution free resampling approach, where a bootstrap-based technique is utilized. Reeves (2005) constructed prediction intervals for ARCH models using a bootstrap method and contrasted it with the traditional asymptotic prediction intervals. Reeves reported that the bootstrap-based method improved the coverage accuracy. Pascual et al. (2006) developed a bootstrap-based prediction intervals for both returns and volatilities for the $\text{GARCH}(1, 1)$ model which is referred to as Pascual-Romo-Ruiz (PRR) in the context. Their bootstrap method incorporated the uncertainty of parameter estimation when building the prediction intervals, which certainly improved the coverage. However, one drawback of this method is the time-consuming computational methodology required for the calculation of prediction intervals. Since GARCH model can be re-written as a linear ARMA type model, Chen et al. (2010) proposed computationally low-cost sieve bootstrap-based prediction intervals for returns and volatiles. Trucíos and Hotta (2016) constructed prediction intervals for returns and volatilities for EGARCH and the Glosten-Jagannathan-Runkle GARCH (GJR-GARCH) models by adapting the method used by Pascual et al. (2006). They found that volatility prediction could be poor when an additive outlier is present near the forecasting origin. Although there are a number of published literatures on bootstrap prediction intervals for the conventional volatility models, there are no such work available for long memory volatility models. Our paper presents an adaption of PRR algorithm developed to GARCH models, to construct the prediction intervals for FIGARCH models.

The sections of the paper are organized as follows: First, we introduced the FIGARCH model in Section 1, and then its properties in Section 2. Section 3 describes the residual based resampling technique and then in Section 4, Monte Carlo simulation results are reported. Section 5 presents an application of the proposed bootstrap-based prediction intervals for the FIGARCH model, and the conclusions are presented in Section 6.

2. THE FIGARACH MODEL

A real valued discrete time stochastic process $\{\varepsilon_t : t \in \mathbb{Z}\}$ is said to be an ARCH(q) process, if

$$\varepsilon_t = z_t \sigma_t, \tag{1}$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{i-1}^2,$$

where, $\omega > 0$ and $\alpha_i \geq 0, i = 1,\ldots,q$. In expression (1), it is assumed that $E(z_t) = 0$, $\text{var}(z_t) = 1$ and z_t's are uncorrelated. Thus, by the definition, $\{\varepsilon_t\}$ is an uncorrelated series with mean zero process with conditional variance σ_t^2, where the conditioning is done with respect to the $\sigma -$field $\Im_{t-1}$ generated by the set of random variables $\{z_k : k \leq t-1\}$. The conditional variance is a linear function of squared residuals up to q lags implying a Markovian dependence. The generalized version of ARCH (GARCH), introduced by Bollerslev (1986), gives a more flexible structure, compared to (1), with the conditional variance (volatility) σ_t^2 given by

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{i-1}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 = \omega + \alpha(L)\varepsilon_t^2 + \beta(L)\sigma_t^2, \tag{2}$$

where $p > 0,\ q > 0,\ \omega > 0,\ \alpha_i \geq 0,\ i = 1,...,q,\ \beta_j \geq 0,\ j = 1,...,p,\ \alpha(L)$ and $\beta(L)$ are such that $\alpha(L) = \alpha_1 L + \alpha_2 L^2 + ... + \alpha_q L^q$ and $\beta(L) = \beta_1 L + \beta_2 L^2 + ... + \beta_p L^p$, with L signifying the lag (or backshift) operator. The process defined in (2) is a stationary process and can be written as an $\text{ARMA}(p, q)$ formulation in ε_t^2:

$$[1 - \alpha(L) - \beta(L)]\varepsilon_t^2 = \omega + [1 - \beta(L)]v_t, \tag{3}$$

where $m = \max(p,\ q)$ and $v_t = \varepsilon_t^2 - \sigma_t^2$. The process $\{v_t\}$ can be shown to be uncorrelated and is interpreted as the innovations associated with the ARMA process. The formulation in (3) is said to be an IGARCH model if the autoregressive polynomial contains a unit root. Therefore, autoregressive representation of IGARCH can be given as

$$\phi(L)(1 - L)\varepsilon_t^2 = \omega + [1 - \beta(L)]v_t,$$

where $\phi(L) = [1 - \alpha(L) - \beta(L)](1 - L)^{-1}$ is of order $m - 1$.

Several studies have reported the presence of long memory in the autocorrelations of squared returns in financial asset prices. Thus, Baillie et al. (1996) adapted the idea of fractional integration in conditional mean models (FARIMA) in order to develop a FIGARCH process. The class of FARIMA(k, d, l) models for the discrete time real-valued process $\{y_t\}$ is defined as

$$a(L)(1 - L)^d y_t = b(L)z_t, \tag{4}$$

where $a(L)$ and $b(L)$ are polynomials in the lag operators of orders k and l, respectively. Here, $\{z_t\}$ is an uncorrelated process with mean zero. The fractional integration parameter, d, lies between -0.5 and 0.5 for the stationary FARIMA model. The fractional differencing operator $(1-L)^d$ has an infinite binomial expansion and can be written in terms of the hypergeometric function,

$$(1-L)^d = F(-d,1,1;L) = \sum_{k=0}^{\infty}\Gamma(k-d)\Gamma(k+1)^{-1}\Gamma(-d)^{-1}L^k,$$

where $\Gamma(.)$ denotes the Gamma function. Analogous to FARIMA(k, d, l) model for the mean process given in (4), Baillie et al. (1996) defined the FIGARCH model in the following manner:

$$\phi(L)(1-L)^d\varepsilon_t^2 = \omega + [1-\beta(L)]v_t. \quad (5)$$

where $0<d<1$, and all the roots of $\phi(L)$ and $[1-\beta(L)]$ lie outside the unit circle. Rearranging the terms in (5), an alternative representation for FIGARCH(p, d, q) can be obtained as

$$\sigma_t^2 = \omega + [1-\beta(L)-\phi(L)(1-L)^d]\varepsilon_t^2 + \beta(L)\sigma_t^2. \quad (6)$$

From (6), conditional variance of the $\{\varepsilon_t\}$ is obtained as:

$$\begin{aligned}\sigma_t^2 &= \omega[1-\beta(1)]^{-1} + \{1-[1-\beta(L)]^{-1}\phi(L)(1-L)^d\}\varepsilon_t^2 \\ &= \omega[1-\beta(1)]^{-1} + \lambda(L)\varepsilon_t^2\end{aligned}, \quad (7)$$

where $\lambda(L)=\sum_{k=1}^{\infty}\lambda_k L^k$. For the FIGARCH($p$, d, q) process given in equation (5) to be well-defined and the conditional variance in the ARCH(∞) representation in (7) to be positive, all the coefficient of ARCH representation in (7) must be non-negative. That is, each $\lambda_k \geq 0$ for $k \in \mathbb{N}$.

In equation (7), the conditional variance of FIGARCH(1, d, 1) can be written as follows:

$$\sigma_t^2 = \omega(1-\beta_1)^{-1} + [1-(1-\beta_1)^{-1}(1-\phi_1 L)(1-L)^d]\varepsilon_t^2, \quad (8)$$

where,

$$\lambda(L) = \sum_{k=1}^{\infty} \lambda_k L^k = 1 - [1 - (1-\beta_1)^{-1}(1-\phi_1 L)(1-L)^d].$$

Therefore, coefficients of the infinite ARCH model can be obtained by equating the coefficients of lag operator, thus obtaining

$$\lambda_1 = \phi_1 - \beta_1 + d,$$

$$\lambda_2 = (d-\beta_1)(\beta_1-\phi_1) + d(1-d)/2,$$

$$\lambda_3 = \beta_1\left[d\beta_1 - d\phi_1 - \beta_1^2 + \beta_1\phi_1 + d(1-d)/2\right] + d(1-d)/2\left[(2-d)/3 - \phi_1\right],$$

$$\vdots$$

$$\lambda_k = \beta_1\lambda_{k-1} + \left[(k-1-d)/k - \phi_1\right]\delta_{d,k-1},\ k \in \mathbb{N},$$

where $\delta_{d,k} = \delta_{d,k-1}(k-1-d)k^{-1},\ k \in \mathbb{N}$ refer to the coefficients in the series expansion of $(1-L)^d$, with $\delta_{d,0} = 1$ and $\delta_{d,1} = d$.

The FIGARCH formulation enables us to model a wide range of conditional volatility models. When $d = 0$, it becomes a GARCH(p,m) process where $m = \max(p,\ q)$. Similarly, when, $d = 1$ with $\beta(L) \neq 0$ and $\phi(L) = 1$, FIGARCH becomes a regular IGARCH model.

2.1. NON-NEGATIVITY OF THE CONDITIONAL VARIANCE

For the non-negativity of the conditional variance of the FIGARCH, all λ_k 's should be positive. Baillie et al. (1996) derived a set of sufficient conditions for the conditional variance to be non-negative. They are $0 \leq \beta_1 \leq \phi_1 + d$ and $0 \leq d \leq 1 - 2\phi_1$. We used this set of conditions in our study. Alternatively, Bollerslev and Mikkelsen (1996) state another set of sufficient inequality constraints $\beta_1 - d \le \phi_1 \le (2-d)/3$ and $d\left[\phi_1 - (1-d)/2\right] \le \beta_1(\phi_1 - \beta_1 + d)$. The latter conditions introduced by Bollerslev and Mikkelsen (1996) are less restrictive than the former conditions introduced by Baillie et al. (1996). Chung (1999) suggested another set of sufficient constraints given by $0 \le \phi_1 \le \beta_1 \le d < 1$. Finally, Conrad and Haag (2006) derived necessary and sufficient conditions for the non-negativity of the variance for the FIGARCH$(p,\ d,\ q)$ for $p \le 2$. According to their findings, conditional variance can be negative almost surely, even if all the original parameters of FIGARCH are positive and similarly conditional variance can be non-negative even if all the parameters are negative, except d. They also derived sufficient conditions for non-negativity of variance for $p > 2$.

2.2. ASYMPTOTIC NORMALITY OF THE PARAMETERS AND THE STATIONARITY OF THE PROCESS

Baillie *et al.* (1996) used a dominance type argument by extending the results available for IGARCH(1, 1), to claim the asymptotic normality of Q-MLEs of FIGARCH$(1,\ d,\ 0)$. They did not prove it theoretically, but their empirical study, however, suggests that parameter estimates are asymptotically normal. Robinson and Zaffaroni (2006) established conditions for consistency and asymptotic normality of Q-MLEs for class of ARCH(∞) under some general conditions, which also covers the FIGARCH type processes. According to their findings strong consistency requires $0 < d < 1$ and asymptotic normality requires $d > 0.5$.

By construction, FIGARCH with $\{\varepsilon_t\}$ defined as in equation (1) has the properties that $\text{cov}(\varepsilon_t, \varepsilon_{t-h}) = 0$ for $h > 0$ and $E(\varepsilon_t) = 0$. The hypergeometric function $F(-d,\ 1,\ 1;\ u)$, evaluated at $u = 1$, is 0 for $0 < d \leq 1$ and thus $\lambda(1) = 1$. Therefore, for $\omega > 0$, the second moment of the $\{\varepsilon_t\}$ does not exist. The implication is that the FIGARCH process is not covariance stationary. Giraitis et al. (2018) established the necessary and sufficient conditions for the FIGARCH to be covariance stationary with $\omega = 0$. Conrad and Haag (2006) suggested a way to obtain the covariance stationarity of $\{\varepsilon_t\}$ with $0 < d < 1$ by assuming $\text{var}(z_t) < 1$ in (1). However, it rules out long memory in ε_t^2 by indicating the absolute summability of auto-covariance function of ε_t^2, as shown in Zaffaroni (2004). Even though FIGARCH is not stationary, and its conditional variance is infinite, Baillie *et al.* (1996) truncated the coefficients in the infinite lag polynomial on conditional variance in order to simulate the FIGARCH process. They used larger truncation lag as 1000 in coefficients in $\lambda(L)$ lag polynomial, in order to incorporate the long run dependencies on the conditional variance. Furthermore, this truncated version of FIGARCH model has finite variance since it is using finite number of coefficients. Therefore, truncated FIGARCH is a covariance stationary with finite variance. Thus, it is feasible to use the model in order to construct bootstrap based prediction intervals.

3. BOOTSTRAP PREDICTION INTERVALS

In this section, we adapt the procedure proposed by Pascual et al. (2006) for the GARCH case to obtain prediction intervals for future values of returns and the future volatilities generated by a FIGARCH process.

1. Let $\{\varepsilon_t\}_{t=1}^{n}$ be a sequence of realizations of a FIGARCH(1, d, 1) process. Then estimate the parameters of the model $\hat{\theta} = (\hat{\omega}_1, \hat{\phi}_1, \hat{d}, \hat{\beta}_1)$ by using Quasi-Maximum Likelihood Estimation (Q-MLE) method.

2. Compute the residuals $\hat{z}_t = \varepsilon_t / \hat{\sigma}_t, t = 1,...,n$ where

$$\begin{aligned}\hat{\sigma}_t^2 &= \hat{\omega}(1-\hat{\beta}_1)^{-1} + [1-(1-\hat{\beta}_1)^{-1}(1-\hat{\phi}_1 L)(1-L)^{\hat{d}}]\varepsilon_t^2 \\ &\approx \hat{\omega}(1-\hat{\beta}_1)^{-1} + \hat{\lambda}_1\varepsilon_{t-1}^2 + \hat{\lambda}_2\varepsilon_{t-2}^2 + ... + \hat{\lambda}_k\varepsilon_{t-k}^2\end{aligned},$$

   and setting $\varepsilon_t^2 = n^{-1}\sum_{i=1}^{n}\varepsilon_i^2$, for $t = -k+1, ..., -1, 0$. Note that k is a suitably chosen truncation lag of the polynomial $\lambda(L)$.

3. Compute the centered residuals $\tilde{z}_t = \hat{z}_t - \bar{\hat{z}}_t$, where $\bar{\hat{z}}_t = n^{-1}\sum_{i=1}^{n}\hat{z}_i$.

4. Denote the empirical distribution function of the centered residuals by $\hat{F}_{\tilde{z}}(x) = n^{-1}\sum_{t=1}^{n} I_{(-\infty,x]}(\tilde{z}_t)$.

5. Draw a bootstrap sample with replacement from the above distribution and denote it by z_t^*, where $t = -m+1,...,-1,0,1,...,n$. We used m=2,000 in this study.

6. Generate the bootstrapped FIGARCH series ε_t^*, $t = -m+1,...,-1,0,1,...,n$ by first computing a bootstrapped conditional variance series, σ_t^{2*} using the FIGARCH parameters estimated in Step 1. Then use $\varepsilon_t^* = z_t^*\sigma_t^*$, $t = -m+1,...,-1,0,1,...,n$ to

generate ε_t^*. The non-positive lags represent 'burn-in' observations that are dropped to mitigate effects due to initial conditions.

7. Estimate the FIGARCH parameters $\theta^* = (\omega^*, \phi_1^*, d^*, \beta_1^*)$ for the bootstrapped series $\{\varepsilon_t^*\}$ using the Q-MLE method.

8. Use the new coefficients $\theta^* = (\omega^*, \phi_1^*, d^*, \beta_1^*)$ obtained in the previous step, compute the h-step ahead bootstrap forecasts of future returns and volatilities based on the following recursions:

$$\begin{aligned}\sigma_{n+h}^{2*} &= \omega^*(1-\beta_1^*)^{-1} + [1-(1-\beta_1^*)^{-1}(1-\phi_1^*)(1-L)^{d^*}]\varepsilon_{n+h}^{2*} \\ &\approx \omega^*(1-\beta_1^*)^{-1} + \lambda_1^*\varepsilon_{n+h-1}^{2*} + \ldots + \lambda_k^*\varepsilon_{n+h-k}^{2*},\end{aligned},$$

$\varepsilon_{n+h}^* = z_{n+h}^*\sigma_{n+h}^*$, for $h > 0$ and $\varepsilon_t^* = \varepsilon_t$ for $t \le n$.

9. Obtain the estimated bootstrap distribution of ε_{n+h}, denoted by $\hat{F}^*_{\varepsilon^*_{n+h}}(.)$, by repeating steps 5-8 B times $(B = 1000)$ in the simulation study. $\hat{F}^*_{\varepsilon^*_{n+h}}(.)$ is the estimate of the $F^*_{\varepsilon^*_{n+h}}(.)$, the bootstrap distribution function of ε_{n+h}^*, which is used to approximate unknown distribution of ε_{n+h} given the observed sample.

10. The $100(1-\alpha)\%$ bootstrap prediction interval for ε_{n+h} is then computed by $[Q^*(\alpha/2), Q^*(1-\alpha/2)]$, where $Q^*(.) = \hat{F}^{*-1}_{\varepsilon^*_{n+h}}$ are the percentiles of the estimated bootstrap distribution.

11. Similar to step 9, obtain the bootstrap distribution of future volatilities, σ_{n+h} and then compute the bootstrap prediction interval for volatility similar to step 10.

4. THE SIMULATION STUDY

To investigate the finite sample performance of the proposed bootstrap prediction intervals of the FIGARCH model, a Monte-Carlo simulation was carried out. The representations of $\{\varepsilon_t\}$ given in Equations (1) and (8) were used to simulate the FIGARCH process. This method become feasible due to the truncation of the infinite lag polynomial. The effect of the pre-sample values might have a higher impact than regular GARCH due to the long memory nature and the hyperbolic rate of decay of the response to a lagged squared innovation. Thus, as suggested by Baillie et al. (1996), truncating lag was selected at $k = 1{,}000$ to incorporate the long-run dependencies. The simulation results were compared with the conditional bootstrap method (CB) used by Miguel and Olave (1999) for ARMA model with ARCH innovations. They only focused on prediction intervals of returns and not volatilities. CB prediction interval method does not incorporate the uncertainty of the parameter estimates. Therefore, step 7 is discarded from the process and use the $\hat{\theta}$ in step 8 instead of θ^* while keeping the other steps the same as in the PRR method. Thus, following recursive equations are used to calculate CB prediction intervals:

$$\sigma_{n+h}^{2*} = \hat{\omega}(1-\hat{\beta}_1)^{-1} + [1-(1-\hat{\beta}_1)^{-1}(1-\hat{\phi}_1)(1-L)^{\hat{d}}]\varepsilon_{n+h}^{2*}$$
$$\approx \hat{\omega}(1-\hat{\beta}_1)^{-1} + \hat{\lambda}_1\varepsilon_{n+h-1}^{2*} + \ldots + \hat{\lambda}_k\varepsilon_{n+h-k}^{2*},$$

$$\varepsilon_{n+h}^{*} = z_{n+h}^{*}\sigma_{n+h}^{*}.$$

The Monte-Carlo simulation study was carried out for different error distributions, namely standard normal and t with 7 degrees of freedom. Centered exponential distribution was also considered to investigate the effect due to non-symmetric error distributions. Series of lengths 500 and 1500 were used. The t-distributed errors were generated as $z_t = 5^{1/2} z_{1,t}(z_{2,t}^2 + z_{3,t}^2 + \ldots + z_{8,t}^2)^{-1/2}$ by drawing independent and identically distributed standard normal $z_{i,t}$'s for $i = 1, 2, \ldots, 8$, as employed in Baillie et al. (1996). Here t -distributed errors also have a unit standard deviation. When generating the realizations, the first 6,000 were dropped to avoid the effects due to initial values.

We considered FIGARCH(1, d, 0) and FIGARCH(1, d, 1) models to simulate the data with $\omega = 0.1$, $d \in \{0.25, 0.50, 0.75, 0.95\}$, $\phi \in \{0, 0.20\}$, and $\beta \in \{0.10, 0.20, 0.45, 0.70, 0.90\}$. Note that out of these sets of parameter combinations, we only used the combinations which satisfied the sufficient conditions for non-negativity of the variance suggested by Baillie et al. (1996). For each combination of the model, sample size, nominal coverage probability, and error distributions, N = 500 independent time series were generated. Then bootstrap steps 1 through 10 were implemented for each time series generated. In each simulation R = 1,000 future values, $\{\varepsilon_{n+h}\}$ were generated. We estimated the coverage probabilities for future returns by calculating the proportion of those ε_{n+h} values falling between the lower and upper bounds of the bootstrap intervals. Therefore, the coverage for the i^{th} simulation run is given by $C(i) = R^{-1}\sum_{r=1}^{R} I_A[\varepsilon_{n+h}^r(i)]$ where $A = [Q^*(\alpha/2), Q^*(1-\alpha/2)]$ is the $100(1-\alpha)th$ bootstrapped prediction interval. $I_A(.)$ is the indicator function of the set A and $\varepsilon_{n+h}^r(i)$, $r = 1, 2, \ldots, 1000$ are the R future values generated at i^{th} simulation run. The theoretical and bootstrap lengths are obtained by using $L_T(i) = \varepsilon_{n+h}^r(1-\alpha/2) - \varepsilon_{n+h}^r(\alpha/2)$ and $L_B(i) = Q^*(1-\alpha/2) - Q^*(\alpha/2)$, respectively. $L_T(i)$ is the difference between $100(1-\alpha)th$ and the $100(\alpha/2)th$ percentiles generated from R future values of the underling model with known order and the coefficients. The mean coverage, standard error of mean coverage, mean bootstrapped prediction intervals, standard error of bootstrapped prediction intervals, and mean theoretical intervals are calculated as follows:

Mean coverage $\bar{C} = N^{-1}\sum_{i=1}^{N} C(i)$,

Standard error of mean coverage $SE_{\bar{C}} = \left\{[N(N-1)]^{-1}\sum_{i=1}^{N}[C(i)-\bar{C}]^2\right\}^{1/2}$,

Mean length (bootstrap) $\bar{L}_B = N^{-1}\sum_{i=1}^{N} L_B(i)$,

Standard error of mean length $SE_{\bar{L}_B} = \{[N(N-1)]^{-1}\sum_{i=1}^{B}[L_B(i)-\bar{L}_B]\}^{1/2}$,

Mean theoretical length $\bar{L}_T = N^{-1}\sum_{i=1}^{N} L_T(i)$.

In similar fashion, we obtained the coverage probability for future volatilities $\{\sigma_{n+h}\}$, $h = 1,\ 10,\ 20$. Thus proportion of those σ_{n+h} falling between upper and lower bound of bootstrapped volatilities was used as an estimator of the coverage probability for future volatility. The mean coverage, standard error of mean coverage, mean bootstrapped prediction intervals, standard error of bootstrapped prediction intervals, and mean theoretical intervals for future volatilities obtained were similar to those calculated for return series.

We investigated the type of model, nominal coverage probability, effect of the bootstrap truncation lag on coverage probabilities, and error distribution in this simulation study. We report the mean coverage, mean bootstrap length, mean theoretical length, standard error of mean coverage, and standard error of mean bootstrap interval length in Tables 1-9 for standard normal, centered exponential, and *t*-distributed innovations for both PRR and CB methods. Due to space limitation, we only report the behavior of 95%

intervals. The minimum value, percentiles (25^{th}, 50^{th}, 75^{th}), and maximum value of the (a) coverage probabilities, (b) the bootstrap interval bounds (upper and lower), and (c) the theoretical interval bounds (upper and lower), were computed for further investigation and results are available upon request.

4.1. PREDICTION INTERVALS FOR RETURNS

Tables 1-6 report the simulation results for future return series. Simulation results show that the coverage probabilities were close to the nominal value for the normal and the t distributed errors in both methods PRR and CB. Under these error distributions (Tables 1-4) if one rounded the coverage probabilities to two decimal places, 23 of the 24 cases would achieve 0.95 nominal level in PRR method while 12 of the 24 cases achieve 0.95 nominal level in CB method. On the other hand, the maximum and minimum coverage probabilities obtained using the centered and skewed exponential error distribution were 0.9286 and 0.9133 for FIGARCH(1, d, 0) with parameters $\omega = 0.10,\ \phi = 0, d = 0.25,\ \beta = 0.10$ and $\omega = 0.10,\ \phi = 0,\ d = 0.90,\ \beta = 0.10$ respectively. CB method gives even worse results in this case with most of the coverage probabilities ranging from 0.90 to 0.92. Note that the coverage probabilities get closer to the nominal value with increasing sample size n regardless of the error distribution used. However, the coverage probabilities obtained using PRR method decrease as sample size n increases for the first lag ahead prediction intervals for FIGARCH with exponentially distributed errors.

In most cases, the bootstrap lengths are less than the theoretical lengths for both PRR and CB methods when using the exponential error distribution as the distribution of the innovations. It is also noted that PRR interval width is slightly wider than the CB intervals. In few occasions not like PRR method interval width is narrower than the theoretical length in CB intervals. That leads to the conclusion that coverage probabilities obtained from CB method are somewhat liberal than those are obtained using PRR method. It can be noted that the higher value of β makes theoretical and bootstrap interval width wider, since β parameter is associated with the past lagged variance. Finally, the coverage probabilities obtained using PRR and CB intervals show only a little difference among them, although the PRR method is slightly ahead of CB method. This comparison is similar to the study done by Pascual *et al.* (2006) for their prediction intervals for GARCH(1,1) process.

Table 1. Coverage of 95% intervals for returns FIGARCH (1, *d*, 1) with parameters $\omega = 0.1$, $\phi = 0.2$, $d = 0.5$, $\beta = 0.45$, and normally distributed errors

Lead lag	Sample size	Theoretical length	PRR		CB	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	8.1927	0.9465 (0.0009)	8.2894 (0.2936)	0.9425 (0.0009)	8.1276 (0.2816)
	1,500	8.2054	0.9490 (0.0006)	8.2920 (0.2857)	0.9467 (0.0006)	8.2053 (0.2800)
10	500	8.6480	0.9471 (0.0010)	9.0534 (0.3151)	0.9420 (0.0009)	8.6682 (0.2746)
	1,500	8.6500	0.9508 (0.0007)	8.9662 (0.2733)	0.9474 (0.0007)	8.7502 (0.2570)
20	500	8.8316	0.9452 (0.0011)	9.3431 (0.3333)	0.9396 (0.0011)	8.8169 (0.2651)
	1,500	8.8119	0.9515 (0.0007)	9.2332 (0.2704)	0.9475 (0.0007)	8.9500 (0.2511)

Table 2. Coverage of 95% intervals for returns of FIGARCH (1, d, 0) with parameters $\omega = 0.1$, $\phi = 0$, $d = 0.75$, $\beta = 0.70$, and normally distributed errors

Lead lag	Sample size	Theoretical length	PRR		CB	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	13.7075	0.9486 (0.0008)	14.4595 (0.9551)	0.9438 (0.0008)	13.6482 (0.6366)
	1,500	13.6713	0.9487 (0.0006)	13.8366 (0.6438)	0.9466 (0.0006)	13.7280 (0.6399)
10	500	14.3548	0.9461 (0.0010)	16.1092 (1.7109)	0.9405 (0.0010)	14.2578 (0.6539)
	1,500	14.3354	0.9485 (0.0007)	14.6742 (0.6775)	0.9456 (0.0007)	14.4359 (0.6656)
20	500	14.8062	0.9440 (0.0011)	18.0165 (2.9435)	0.9380 (0.0011)	14.7725 (0.6877)
	1,500	14.8004	0.9488 (0.0008)	15.3545 (0.6949)	0.9457 (0.0008)	15.0135 (0.6744)

Table 3. Coverage of 95% intervals for returns of FIGARCH (1, d, 0) with parameters $\omega = 0.1$, $\phi = 0$, $d = 0.50$, $\beta = 0.45$, and t distributed errors

Lead lag	Sample size	Theoretical length	PRR		CB	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	7.5913	0.9472 (0.0009)	7.7956 (0.4959)	0.9439 (0.0010)	7.6657 (0.4888)
	1,500	7.6186	0.9498 (0.0006)	7.7323 (0.4924)	0.9475 (0.0006)	7.6445 (0.4856)
10	500	7.9504	0.9471 (0.0010)	8.3987 (0.5066)	0.9431 (0.0010)	8.1344 (0.4909)
	1,500	7.9451	0.9515 (0.0007)	8.3308 (0.4548)	0.9484 (0.0007)	8.1151 (0.4328)
20	500	7.9686	0.9461 (0.0011)	8.5280 (0.4788)	0.9415 (0.0010)	8.2087 (0.4777)
	1,500	7.9909	0.9512 (0.0007)	8.4872 (0.4493)	0.9475 (0.0007)	8.2109 (0.4238)

Table 4. Coverage of 95% intervals for returns of FIGARCH (1, *d*, 0) with parameters $\omega = 0.1,\ \phi = 0,\ d = 0.95,\ \beta = 0.90$, and *t* distributed errors

Lead lag	Sample size	Theoretical length	PRR Mean coverage (SE)	PRR Mean length (SE)	CB Mean coverage (SE)	CB Mean length (SE)
1	500	21.9002	0.9503 (0.0009)	23.1458 (2.1748)	0.9447 (0.0009)	22.3984 (2.1580)
	1,500	21.8846	0.9505 (0.0006)	22.5621 (2.1383)	0.9471 (0.0006)	22.1749 (2.1054)
10	500	22.4492	0.9514 (0.0009)	23.9179 (2.0273)	0.9444 (0.0009)	22.7487 (1.9510)
	1,500	22.5159	0.9513 (0.0007)	23.4620 (2.1716)	0.9472 (0.0007)	22.9138 (2.1757)
20	500	22.8628	0.9506 (0.0010)	24.6057 (1.9527)	0.9430 (0.0011)	23.3332 (1.9681)
	1,500	22.8517	0.9512 (0.0008)	24.2496 (2.2326)	0.9469 (0.0007)	23.4797 (2.1605)

Table 5. Coverage of 95% intervals for returns of FIGARCH (1, *d*, 0) with parameters $\omega = 0.1,\ \phi = 0,\ d = 0.25,\ \beta = 0.10$, and exponentially distributed errors

Lead lag	Sample size	Theoretical length	PRR Mean coverage (SE)	PRR Mean length (SE)	CB Mean coverage (SE)	CB Mean length (SE)
1	500	2.8962	0.9239 (0.0016)	2.9334 (0.0753)	0.9068 (0.0019)	2.8747 (0.0750)
	1,500	2.9102	0.9186 (0.0011)	2.9117 (0.0650)	0.9098 (0.0013)	2.8843 (0.0639)
10	500	3.0923	0.9250 (0.0015)	3.1341 (0.0658)	0.9187 (0.0015)	3.0655 (0.0678)
	1,500	3.0848	0.9266 (0.0010)	3.0751 (0.0557)	0.9238 (0.0010)	3.0377 (0.0535)
20	500	3.1763	0.9202 (0.0017)	3.1797 (0.0567)	0.9132 (0.0017)	3.0964 (0.0580)
	1,500	3.1707	0.9235 (0.0011)	3.1361 (0.0518)	0.9205 (0.0011)	3.0857 (0.0499)

Table 6. Coverage of 95% intervals for returns of FIGARCH (1, d, 0) with parameters $\omega = 0.1$, $\phi = 0$, $d = 0.90$, $\beta = 0.10$, and exponentially distributed errors

Lead lag	Sample size	Theoretical length	PRR		CB	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	2.9756	0.9224 (0.0016)	2.9320 (0.2449)	0.9069 (0.0021)	2.8753 (0.2370)
	1,500	2.9721	0.9133 (0.0012)	2.8937 (0.2275)	0.9060 (0.0015)	2.8732 (0.2245)
10	500	3.8716	0.9262 (0.0011)	3.1743 (0.1055)	0.9254 (0.0013)	3.1000 (0.0802)
	1,500	3.8581	0.9286 (0.0008)	3.1954 (0.1080)	0.9283 (0.0009)	3.1676 (0.1022)
20	500	4.1893	0.9172 (0.0011)	3.0880 (0.0645)	0.9164 (0.0014)	3.0415 (0.0557)
	1,500	4.1859	0.9202 (0.0008)	3.1110 (0.0684)	0.9197 (0.0013)	3.0753 (0.0553)

4.2. PREDICTION INTERVALS FOR VOLATILITIES

Here, we discuss the performance of PRR and CB prediction intervals for future volatilities. We apply the same Monte-Carlo design used for returns. Tables 7-9 report the results for normal, t and centered exponential error distribution, for lead lag 1, 10, 20 respectively. Further, we included lead lag 2 since the CB method is not able to provide prediction intervals for lead lag 1. In the FIGARCH formulation, one step ahead prediction for volatility is known because its condition on past returns and the uncertainty associated with σ^2_{n+h}, $(h=1)$ is only due to the parameter estimation. Consequently, volatility of one step ahead is completely determined at time t and therefore, the theoretical length of one step ahead prediction is zero. Similarly, CB method does not provide one step ahead prediction intervals because parameter estimates are fixed in every bootstrap step.

Table 7. Coverage of 95% intervals for returns of FIGARCH (1, d, 0) with parameters $\omega = 0.1$, $\phi = 0$, $d = 0.50$, $\beta = 0.45$, and normally distributed errors

Lead lag	Sample size	Theoretical length	PRR		CB	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	-	0.9320 (0.0113)	0.5237 (0.0218)	-	-
	1,500	-	0.9380 (0.0108)	0.2887 (0.0095)	-	-
2	500	0.2485	0.9591 (0.0063)	0.7267 (0.0302)	0.5481 (0.0184)	0.3005 (0.0160)
	1,500	0.2509	0.9545 (0.0057)	0.4775 (0.0160)	0.6454 (0.0151)	0.2794 (0.0112)
10	500	1.7380	0.9192 (0.0037)	1.8620 (0.0711)	0.8907 (0.0048)	1.6727 (0.0605)
	1,500	1.7573	0.9399 (0.0019)	1.8698 (0.0574)	0.9290 (0.0022)	1.7799 (0.0532)
20	500	2.1478	0.9135 (0.0041)	2.3364 (0.0938)	0.8901 (0.0049)	2.0811 (0.0783)
	1,500	2.1583	0.9410 (0.0021)	2.3814 (0.0742)	0.9302 (0.0023)	2.2447 (0.0692)

One step ahead predictions under the PRR method for normal and t errors provide the coverage below the nominal value for the different parameter combinations that is used. However, the exponential errors provide conservative probabilities with 0.9580 and 0.9540 for sample sizes 500 and 1500 respectively. Generally, as sample size increases the coverage of probabilities are close nominal value 0.95. When forecasting two or more lags ahead, the predicted coverages are well under the nominal value 0.95 in CB method. The differences are significant in contrast to the coverage probabilities in returns that we discussed in section 4.1, where under CB method coverage probabilities in returns are close to 0.95 in most cases. Table 7 shows the coverage probabilities under exponentially distributed errors. Coverage probabilities for first and second lags are close 0.95 while coverage probabilities for tenth and twentieth lags are well below 0.95. Overall, the PRR method provides a much better coverages for volatility than CB method for the parameters and the error distributions considered in this study.

Table 8. Coverage of 95% intervals for returns of FIGARCH (1, *d*, 0) with parameters $\omega = 0.1$, $\phi = 0$, $d = 0.90$, $\beta = 0.10$, and *t* distributed errors

Lead lag	Sample size	Theoretical length	PRR		CB	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	-	0.9420 (0.0105)	0.1979 (0.0127)	-	-
	1,500	-	0.9380 (0.0108)	0.1170 (0.0080)	-	-
2	500	1.3830	0.9550 (0.0029)	1.4776 (0.1055)	0.8546 (0.0067)	1.3949 (0.0995)
	1,500	1.3927	0.9495 (0.0029)	1.4476 (0.1011)	0.8761 (0.0055)	1.4074 (0.1000)
10	500	2.2723	0.9524 (0.0023)	2.3374 (0.0871)	0.9115 (0.0037)	2.2902 (0.0827)
	1,500	2.2640	0.9480 (0.0019)	2.3151 (0.0924)	0.9275 (0.0026)	2.2927 (0.0925)
20	500	2.2401	0.9510 (0.0023)	2.2820 (0.0465)	0.9136 (0.0036)	2.2359 (0.0432)
	1,500	2.2418	0.9475 (0.0018)	2.2717 (0.0414)	0.9294 (0.0025)	2.2465 (0.0393)

Table 9. Coverage of 95% intervals for returns of FIGARCH (1, *d*, 1) with parameters $\omega = 0.1$, $\phi = 0.20$, $d = 0.50$, $\beta = 0.45$, and exponentially distributed errors

Lead lag	Sample size	Theoretical length	PRR		CB	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	-	0.9580 (0.0090)	0.9054 (0.0797)	-	-
	1,500	-	0.9540 (0.0094)	0.4732 (0.0318)	-	-
2	500	1.0482	0.9546 (0.0051)	1.7733 (0.1317)	0.7062 (0.0146)	1.2328 (0.1043)
	1,500	1.0744	0.9437 (0.0048)	1.3333 (0.0796)	0.7422 (0.0127)	1.1250 (0.0713)
10	500	2.3655	0.9284 (0.0048)	3.0446 (0.2150)	0.8151 (0.0085)	2.4797 (0.1586)
	1,500	2.3745	0.9284 (0.0038)	2.6998 (0.1705)	0.8615 (0.0059)	2.4821 (0.1644)
20	500	3.1114	0.9096 (0.0052)	3.5279 (0.2479)	0.8101 (0.0080)	2.9321 (0.1891)
	1,500	3.0982	0.9169 (0.0038)	3.2513 (0.2235)	0.8558 (0.0053)	2.9835 (0.2194)

5. APPLICATION TO A REAL DATA SET

The proposed method was applied to S&P 500 return data from November 5, 2010 through May 2, 2018, for a total of 2201 observations. Data was obtained from the website https://finance.yahoo.com. Following standard practice, daily percentage returns of closing prices i.e. $r_t = 100.\log(s_t / s_{t-1})$ for $t = 2, 3, ..., 2201$ were used. Here s_t denotes the closing price at day *t*. The following figure shows one-step ahead bootstrap prediction interval (95%) for S&P 500 returns.

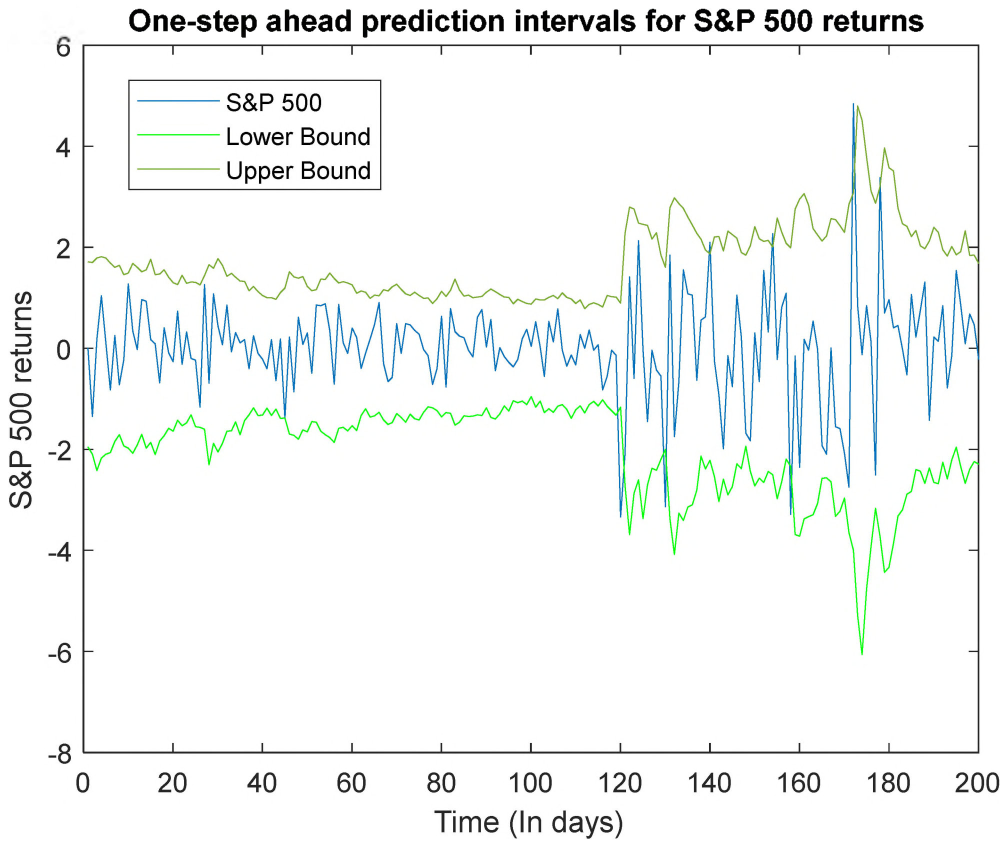

Figure 1. One-step ahead prediction intervals for S&P 500

Table 10. Estimated coverage probabilities for S&P 500 future returns

Lead lag	Coverage
1	0.9600
10	0.9424
20	0.9227

We analyzed the coverage probabilities for 1 step, 10th step, and 20th step ahead forecasts for the S&P 500. Table 10 reports coverage probabilities of the future lags.

6. CONCLUSIONS

In this paper we adapted the procedure proposed by Pascual *et al.* (2006) to construct bootstrap prediction intervals for GARCH realizations. Finite sample properties were investigated using a Monte-Carlo simulation study. In this study it is assumed that the order of the FIGARCH process is known. This is not a great limitation because in most empirical modeling situations, researchers have found that a GARCH process with orders $p = q = 1$ would suffice. Extending this argument, one would assume that FIGARCH$(1,\ d,\ 1)$ would suffice in most cases, as was demonstrated in our example with S&P 500 data. Simulation study shows that the proposed bootstrap-based prediction intervals perform well. The coverage probabilities obtained in the simulation study are close to the nominal values for symmetric error distributions, under varying sample sizes and parameter combinations. Further extension of obtaining prediction intervals for models such as Autoregressive-FIGARCH, FARIMA-FIGARCH using sieve bootstrap method is currently ongoing.

REFERENCES

Baillie, R. T., Bollerslev, T., & Mikkelsen, H. O. (1996). Fractionally integrated generalized autoregressive conditional heteroscedasticity. *Journal of Econometrics*, *74*(1), 3–30.

Bisaglia, L., & Grigoletto, M. (2001). Prediction intervals for FARIMA processes by bootstrap methods. *Journal of Statistical Computation and Simulation*, *68*(2), 185–201.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroscedasticity. *Journal of Econometrics*, *31*(3), 307–327.

Bollerslev, T., & Ole Mikkelsen, H. (1996). Modeling and pricing long memory in stock market volatility. *Journal of Econometrics*, *73*(1), 151–184.

Chen, B., Gel, Y. R., Balakrishna, N., & Abraham, B. (2010). Computationally efficient bootstrap prediction intervals for returns and volatilities in ARCH and GARCH processes. *Journal of Forecasting*, *30*(1), 51–71.

Chung, C.F., 1999. Estimating the fractionally integrated GARCH model. Working Paper, National Taiwan University.

Conrad, C. (2006). Inequality Constraints in the Fractionally Integrated GARCH Model. *Journal of Financial Econometrics*, *4*(3), 413–449.

Engle, R. F. (1982). Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation. *Econometrica*, *50*(4), 987.

Engle, R. F., & Bollerslev, T. (1986). Modelling the persistence of conditional variances. *Econometric Reviews*, *5*(1), 1–50.

Giraitis, L., Surgailis, D., & Škarnulis, A. (2017). Stationary Integrated ARCH (∞) and AR(∞) Processes with Finite Variance. *Econometric Theory*, *34*(6), 1159–1179.

Miguel, J. A., & Olave, P. (1999). Bootstrapping forecast intervals in ARCH models. *Test*, *8*(2), 345–364.

Nelson, D. B. (1991). Conditional Heteroscedasticity in Asset Returns: A New Approach. *Econometrica*, *59*(2), 347.

Pascual, L., Romo, J., & Ruiz, E. (2006). Bootstrap prediction for returns and volatilities in GARCH models. *Computational Statistics & Data Analysis*, *50*(9), 2293–2312.

Reeves, J. J. (2005). Bootstrap prediction intervals for ARCH models. *International Journal of Forecasting*, *21*(2), 237–248.

Robinson, P. M., & Zaffaroni, P. (2006). Pseudo-maximum likelihood estimation of ARCH(∞) models. *The Annals of Statistics*, *34*(3), 1049–1074.

Rupasinghe, M., Mukhopadhyay, P., & Samaranayake, V. A. (2013). Obtaining prediction intervals for FARIMA processes using the sieve bootstrap. *Journal of Statistical Computation and Simulation*, *84*(9), 2044–2058.

Trucíos, C., & Hotta, L. K. (2016). Bootstrap prediction in univariate volatility models with leverage effect. *Mathematics and Computers in Simulation*, *120*, 91–103.

Zaffaroni, P. (2004). Stationarity and Memory of ARCH (∞) Models. *Econometric Theory*, *20*(01), 147–160.

II. SIEVE BOOTSTRAP-BASED PREDICTION INTERVALS FOR ARMA MODELS WITH FRACTIONALLY INTEGRATED GENERALIZED AUTOREGRESSIVE CONDITIONAL HETEROSCEDASTIC (FIGARCH) ERRORS

ABSTRACT

In this paper, a sieve bootstrap-based prediction interval for Autoregressive (AR) and Autoregressive Moving Average (ARMA) models with Fractionally Integrated GARCH (FIGARCH) error structure is proposed. The order of the AR or ARMA component is assumed unknown but the order of the FIGARCH error structure is assumed to be known. The ARMA or AR parts of the models are approximated by a finite order AR process whose order $\hat{p}$ is estimated using the AIC criterion from among models of order one through $p_{\max}$, where $p_{\max}$ is determine by a criteria given later in this paper. Resampling is done on residuals obtained by employing one-step method which fits an $\mathrm{AR}(\hat{p})-\mathrm{FIGARCH}(1,d,1)$ to the data. Results are compared with a two-step method where resampling is done by first fitting an $\mathrm{AR}(\hat{p})$ and then fitting a $\mathrm{FIGARCH}(1,d,1)$ to the residuals of the $\mathrm{AR}(\hat{p})$ fit. A Monte-Carlo simulation study shows that both methods yield good intervals for most of the parameter combinations, with coverage probabilities reasonably close to nominal level. The one-step method, however, produces better coverage probabilities when the roots of the AR polynomial in the AR-FIGARCH model are closer to unity.

Keywords: Prediction intervals, ARMA-FIGARCH, Bootstrap resampling, Simulation

1. INTRODUCTION

Typically, distributions of economic and financial time series exhibit leptokurtic behavior and the errors show heteroscedasticity. Therefore, traditional Gaussian based prediction intervals produce poor coverage probabilities. An alternative is a non-parametric bootstrap approach, which does not require distributional assumptions about the underlying error. Some of the earliest work on bootstrap-based time series prediction intervals can be found in Stine (1982, 1987) and Findley (1986). They compute the prediction mean square errors for future forecasts using a bootstrap technique. Thombs and Schucany (1990) used a non-parametric approach to establish the forecast intervals for autoregressive (AR) time series models by incorporating the variability of parameter estimates into the bootstrap process. They used backward representation method to construct the prediction intervals, where the current value is written in terms of a linear combination of future values. The backward representation method is only limited for AR models, and it is not possible to use it for processes with a Moving Average (MA) component. Cao et al. (1997) proposed a computationally much faster bootstrap method, which does not require backward resampling. In contrast to former method presented by Thombs and Schucany, the latter method does not incorporate the uncertainty of the parameter estimates into the prediction intervals. Alonso et al. (2002, 2003) proposed a sieve bootstrap (SB) based approach to constructing prediction intervals for stationary and invertible Autoregressive Moving Average (ARMA) processes with unknown orders. All the bootstrapped based prediction intervals discussed above work fairly well, with coverage probabilities close to the nominal values, if the conditional variance of the error distributions display homoscedastic behavior. This is because the bootstrapping was done by resampling the residuals of the model. When errors indicate heteroscedasticity, direct resampling approach may destroy the heteroscedastic error structure, providing prediction intervals that are too narrow during times of high volatility. Hence modelling heteroscedastic error structure become important when constructing residual-based bootstrap. In the following we discuss some pioneering work done with respect to modeling conditionally heteroscedastic error in the time series context.

The Autoregressive Conditional Heteroscedastic (ARCH) formulation was introduced by Engle in 1982 to model empirical time series with heteroscedastic behavior in the conditional variance. Bollerslev (1986) extended the ARCH model by proposing the Generalized ARCH (GARCH) formulation, which accommodates a wider class of time series with a heteroscedastic error structure. Subsequently, the Integrated GARCH (IGARCH) representation was introduced by Engle and Bollerslev (1986) to model time series with persistent volatility in squared shocks (error terms), which cannot be adequately modeled using ARCH and GARCH formulations. Motivated by long memory behavior of the autocorrelations of squared or absolute residuals reported in studies by Ding, Granger & Engle (1993) and Harvey (1993), Baillie et al. (1996) introduced the Fractionally Integrated GARCH (FIGARCH) representation to model time series that exhibits long memory in the conditional variance with respect to squared shocks. One of the major advantages of the FIGARCH model is that it incorporates a wide range of conditional heteroscedastic models, including ARCH, GARCH and IGARCH formulations as special cases. Often, these models were employed to obtain point forecasts without considering the uncertainty associated with future values.

Miguel and Olave (1997) introduced a bootstrap-based interval forecasting method for $\text{ARMA}(p, q)-\text{ARCH}(r)$ models with known orders p, q and r. This prediction interval, however, did not incorporate the uncertainty associated with the parameter estimation. Pascual et al. (2006) developed a bootstrap-based prediction interval for returns and volatility for the $\text{GARCH}(1, 1)$ model. In contrast to Miguel and Olaves's intervals, these intervals incorporated the variability due to parameter estimates. Thilakaratne and Samaranayake (2014) used a sieve bootstrap technique to compute prediction intervals for Autoregressive-GARCH (AR-GARCH) processes. Although they allowed for conditional heteroscedasticity in error variance, it was assumed that the squares of the conditional heteroscedastic error terms have short memory. In contrasts, our proposed method can accommodate long memory in the heteroscedastic error components. More specifically, we propose a sieve bootstrap-based prediction interval for $\text{ARMA}(p, q)-\text{FIGARCH}(r, d, s)$ type models. As this the case with the sieve bootstrap technique, we approximate the ARMA-FIGARCH model by a finite order AR-FIGARCH process.

The proposed method is different from that adopted in Thilakaratne and Samaranayake (2014). The proposed method uses a one-step approach where the parameters in both the AR and the FIGARCH components of the model as estimated simultaneously while Thilakaratne and Samaranayake used a two-step estimation method. In the two-step estimation method, one first estimates the parameters of AR part, and then use the AR residuals to estimate the parameters of the FIGARCH part. By contrast, parameter estimation in one-step estimation is done by maximizing the quasi likelihood function of the complete AR-FIGARCH model.

This paper is organized as follows. We introduce the ARMA-FIGARCH model in Section 2. Then we discuss the sieve bootstrap technique in Section 3. Section 4 reports the Monte-Carlo simulation study and its results. An application to a real-world data set is presented in Section 5. Finally, we present our conclusions in Section 6.

2. ARMA-FIGARCH PROCESS

Let $\{X_t\}_{t\in\mathbb{Z}}$ be a real valued time series with $E(X_t)=\mu_x$, which follows $\text{ARMA}(p,\ q)-\text{FIGARCH}(r,\ d,\ s)$ model,

$$\begin{aligned}\tilde{X}_t &= \sum_{i=1}^{p} a_i \tilde{X}_{t-i} + \sum_{j=1}^{q} b_j \varepsilon_{t-j} + \varepsilon_t,\\ \varepsilon_t &= \sigma_t z_t,\\ \sigma_t^2 &= \omega + \beta(L)\sigma_t^2 + \left[1-\beta(L)-\phi(L)(1-L)^d\right]\varepsilon_t^2,\end{aligned} \tag{1}$$

where, $a_p \neq 0,\ b_q \neq 0,\ \omega > 0$, and $\tilde{X}_t = X_t - \mu_x$. The series $\{Z_t\}_{t\in\mathbb{Z}}$ is a white noise process with zero mean and unit variance. Note that L is the lag operator and $\beta(L)=\beta_1 L+\beta_2 L^2+...+\beta_r L^r$ and $\phi(L)=1-\phi_1 L-\phi_2 L^2-...-\phi_s L^s$ with $\beta_r \neq 0$ and $\phi_s \neq 0$. It is assumed that the fractional differencing parameter d in the expression for the conditional variance lies between 0 and 1.

Observe that the sequence $\{\varepsilon_t\}$ in the above expression follows a $\text{FIGARCH}(r,\ d,\ s)$ process. In our simulation study we restrict our analysis to the $\text{FIGARCH}(1,\ d,\ 1)$ error structure. This is the FIGARCH process that is most frequently employed to model empirical processes. The conditional variance of a FIGARCH(1, d, 1) process can be express as

$$\sigma_t^2 = \omega + \left[1 - \beta_1 L - (1-\phi_1 L)(1-L)^d\right]\varepsilon_t^2 + \beta\sigma_{t-1}^2,$$

$$\sigma_t^2 = \omega(1-\beta_1)^{-1} + \left[1-(1-\beta_1)^{-1}(1-\phi_1 L)(1-L)^d\right]\varepsilon_t^2. \qquad (2)$$

Furthermore, conditional variance in expression (2) can be written as $\sigma_t^2 = \omega(1-\beta_1)^{-1} + \lambda(L)\varepsilon_t^2$, where $\lambda(L) = \sum_{k=1}^{\infty}\lambda_k L^k$. The coefficients of λ_i, $i \in \mathbb{N}$ can be obtained by equating terms with common power in the equation

$$\lambda(L) = \lambda_1 L + \lambda_2 L^2 + \ldots + \lambda_k L^k + \ldots = \left[1-(1-\beta_1)^{-1}(1-\phi_1 L)(1-L)^d\right].$$

Thus,

$\lambda_1 = \phi_1 - \beta_1 + d$ and $\lambda_k = \beta_1\lambda_{k-1} + \left[(k-1-d)/k - \phi_1\right]\delta_{d,k-1}$, $k = 2,3,\ldots,$

where, $\delta_{d,k} = \delta_{d,k-1}(k-1-d)k^{-1}$, $k = 2,3,\ldots,$ refer to the coefficients in the series expansion of $(1-L)^d$, with $\delta_{d,0} = 1$ and $\delta_{d,1} = d$. Now, all λ_i for $i \in \mathbb{N}$ must be positive to ensure the non-negativity of the conditional variance σ_t^2. Following Baillie et al. (1996), the sufficient conditions for non-negativity of conditional variance are $0 \le \beta_1 \le \phi_1 + d$ and $0 \le d \le 1 - 2\phi_1$.

The above formulation of ARMA-FIGARCH permits us to model time series with short memory in the conditional mean and long memory in conditional variance.

3. THE SIEVE BOOTSTRAP-BASED PREDICTION INTERVAL METHOD

The sieve bootstrap (SB) technique based on residual resampling from a sequence of AR approximations originated from Buhlmann (1997) and Kreiss (1992,1988). The advantage of the method is that it does not require the prior knowledge about the order of the underlining process. Buhlmann (1997) used truncation of an infinite AR process to approximate a class of linear processes, which include stationary and invertible ARMA processes. The order of the truncation, $p = p(n)$ is assumed to advance to infinity at a smaller rate than the sample size n, as n approaches infinity. Kreiss (1988) originally proposed this finite AR approximation to a class of linear process that can be written as an infinite AR process. Akaike Information Criteria (AIC) was used by Buhlmann (1997) to choose the order of AR approximation. Alonso et al. (2002, 2003) formalized the above ideas and adapted the sieve bootstrap technique to construct the prediction intervals for ARMA type models with homoscedastic error distributions. Mukhopadhyay and Samaranayake, (2010) modified the SB method used by Alonso et al. (2002, 2003) to improve the coverage probabilities in ARMA type processes. Later Rupasinghe and Samaranayake (2012, 2013) established a computationally much faster SB prediction interval for Fractionally Integrated Autoregressive Moving Average (FARIMA) processes.

In our proposed approach, we approximate the ARMA part of ARMA-FIGARCH model by a finite AR process using the SB concept. The optimal order $\hat{p}$ of the finite AR approximation is found by using AIC values. Following Alonso et al. (2003) and Thilakaratne & Samaranayake (2014), the value for $\hat{p}$ is chosen from the values $1, 2, \ldots, p_{\max}$, using the AIC criteria, where, $p_{\max} = c\left(n / \log(n)\right)^{1/(2r+2)}$ with $r > 2$ and some

$c > 0$. This $p_{\max}$ is different from the ARMA process with homoscedastic errors used by Alonso et al. (2002), and Mukhopadhyay & Samaranayake (2010), where they used $p_{\max}$ as $n/10$ when constructing prediction intervals using the SB method.

The steps for the proposed SB based (one step estimation) procedure for obtaining prediction interval for ARMA-FIGARCH model is given below:

1. Select the maximum order $p_{\max}$ for the given realization $\{X_t\}_{t=1}^{n}$ of an AR-FIGARCH process. We used value of $p_{\max} = 22, 27$ for $n = 500,\ 1000$ respectively, according to the formula described in the previous paragraph. Then find the optimal order $\hat{p}$ from among the values $p = 1, 2, ..., p_{\max}$ using the AIC criteria.

2. Use the least-squares estimates, $\tilde{\varphi}_1, \tilde{\varphi}_2, ..., \tilde{\varphi}_{\hat{p}}$ of $AR(\hat{p})$ process as the initial values for AR part in maximum likelihood estimation of AR-FIGARCH model. Assignment of initial values for the FIGARCH parameters $\omega_1, \phi_1, d, \beta_1$ can be done using random values within a feasible region. Obtain the maximum likelihood estimates, $\hat{\varphi}_1, \hat{\varphi}_2, ..., \hat{\varphi}_{\hat{p}}, \hat{\omega}_1, \hat{\phi}_1, \hat{d}, \hat{\beta}_1$ of AR-FIGARCH by using these initial values.

3. Compute the $(n - \hat{p})$ residuals $\hat{z}_t$ by using $\hat{z}_t = \hat{\varepsilon}_t / \hat{\sigma}_t$, $\hat{\varepsilon}_t = -\sum_{j=0}^{\hat{p}} \hat{\varphi}_j (X_{t-j} - \bar{X})$ where

$$\hat{\sigma}_t^2 = \hat{\omega}(1-\hat{\beta}_1)^{-1} + \left[1 - (1-\hat{\beta}_1)^{-1}(1-\hat{\phi}_1 L)(1-L)^{\hat{d}}\right]\hat{\varepsilon}_t^2,$$
$$\approx \hat{\omega}(1-\hat{\beta}_1)^{-1} + \hat{\lambda}_1\hat{\varepsilon}_{t-1}^2 + \hat{\lambda}_2\hat{\varepsilon}_{t-2}^2 + ... + \hat{\lambda}_k\hat{\varepsilon}_{t-k}^2,$$

with $\hat{\varphi}_0 = -1$, $t \in (\hat{p}+1, ..., n)$ and $\bar{X}$ is the mean of the process $\{X_t\}_{t=1}^{n}$. Further, note that $\hat{\varepsilon}_t^2 = (n-\hat{p})^{-1}\sum_{i=\hat{p}+1}^{n} \hat{\varepsilon}_i^2$ for $t \le \hat{p}$ and k is the truncation lag of the polynomial $\lambda(L)$. Note that *k* should be select large enough to obtain a reasonable approximation to $\lambda(L)$. In our simulation study, we used *k*=1,000.

4. Compute the centered residuals $\tilde{z}_t = \hat{z}_t - \bar{\hat{z}}_t ; t \in (\hat{p}+1,...,n)$, where, $\bar{\hat{z}}_t = (n-\hat{p})^{-1} \sum_{i=\hat{p}+1}^{n} \hat{z}_i$.

5. Denote the empirical distribution function of centered residuals $\{\tilde{z}_t\}_{\hat{p}+1}^{n}$ as $\hat{F}_{\tilde{z}}(x) = (n-\hat{p})^{-1} \sum_{t=\hat{p}+1}^{n} I_{(-\infty,x]}(\tilde{z}_t)$.

6. Draw a bootstrap sample with replacement from the above distribution and denote it by z_t^*, for $t = -m-k+1,... -1,0,1,...,n$. m is chosen as 2,000 in this study.

7. Generate the bootstrapped FIGARCH series ε_t^*, $t = -m+1,..., -1,0,1,...,n$ by first constructing a bootstrapped conditional variance series, $\{\hat{\sigma}_t^{2*}\}$ using estimated FIGARCH parameters of the AR-FIGARCH model obtained in Step 2. Then use $\varepsilon_t^* = z_t^* \hat{\sigma}_t^*$, $t = -m+1, ..., -1,0,1, ..., n$ to generate $\{\varepsilon_t^*\}$. In our simulation study, m was chosen to be 2,000.

8. Then generate the bootstrapped AR-FIGARCH series, X_t^*, $t = -m+1, ..., 0,1, ..., n$ using the bootstrapped FIGARCH errors, $\{\varepsilon_t^*\}$ created in Step 7 and based on the recursion $X_t^* - \bar{X} = \sum_{j=1}^{\hat{p}} \hat{\varphi}_j \left(X_{t-j}^* - \bar{X}\right) + \varepsilon_t^*$. Set the initial $\hat{p}$ values, X_t^*, $t \le \hat{p}$ equal to $\bar{X}$. Drop the first m observations to eliminate the effect of the initial values.

9. Fit an AR-FIGARCH to the bootstrapped series $\{X_t^*\}_{t=1}^n$ and then estimate the parameters of it using the Q-MLE method. The parameter estimates of AR and FIGARCH coefficients are denoted by $\hat{\varphi}_1^*, \hat{\varphi}_2^*, ..., \hat{\varphi}_{\hat{p}}^*$ and $\hat{\omega}_1^*, \hat{\phi}_1^*, \hat{d}^*, \hat{\beta}_1^*$, respectively.
10. Compute the h-step ahead bootstrap forecasts of future values using the bootstrapped AR coefficients $\hat{\varphi}_1^*, \hat{\varphi}_2^*, ..., \hat{\varphi}_{\hat{p}}^*$ and FIGARCH coefficients $\hat{\omega}_1^*, \hat{\phi}_1^*, \hat{d}^*, \hat{\beta}_1^*$ as shown in the following recursions:

$$X_{n+h}^* - \bar{X} = \sum_{j=1}^{\hat{p}} \hat{\varphi}_j^* (X_{n+h-j}^* - \bar{X}) + \varepsilon_{n+h}^*, \quad \varepsilon_{t+h}^* = z_{n+h}^* \hat{\sigma}_{n+h}^*$$

$$\begin{aligned} \hat{\sigma}_{n+h}^{2*} &= \hat{\omega}^*(1-\hat{\beta}_1^*)^{-1} + \left[1-(1-\hat{\beta}_1^*)^{-1}(1-\hat{\phi}_1^*)(1-L)^{\hat{d}^*}\right]\varepsilon_{n+h}^{*2}, \\ &\approx \hat{\omega}^*(1-\hat{\beta}_1^*)^{-1} + \hat{\lambda}_1^*\varepsilon_{n+h-1}^{*2} + \ldots + \hat{\lambda}_k^*\varepsilon_{n+h-k}^{*2}, \end{aligned}$$

for $h > 0$ and $\varepsilon_t^* = \hat{\varepsilon}_t$ for $t \le n$.
11. Obtain the estimated bootstrap distribution of X_{n+h}, denoted by $\hat{F}_{X_{n+h}^*}^*(.)$ by repeating steps 6-10, $B = 1{,}000$ times in the simulation study. $\hat{F}_{X_{n+h}^*}^*(.)$ is the estimate of the $F_{X_{n+h}^*}^*(.)$, the bootstrap distribution function of X_{n+h}^*, which is used to approximate unknown distribution function of X_{n+h} given the observed time series.
12. The $100(1-\alpha)\%$ prediction interval for X_{n+h} is then computed as $\left[Q^*(\alpha/2), Q^*(1-\alpha/2)\right]$, where $Q^*(.) = \hat{F}_{X_{n+h}^*}^{*-1}$ are the percentiles of the estimated bootstrap distribution.

There is only a slight difference between the processes of constructing prediction intervals using the one-step estimation method and the two-step estimation method. To accommodate the process under the two-step estimation method, parameter estimations of Step 2 and Steps 9 must be changed to a two-steps estimation process while keeping the other steps fixed. By applying this change, we estimated the coverages under two-steps estimation method.

4. THE SIMULATION STUDY

We investigated the finite sample performance of the proposed SB prediction intervals for ARMA-FIGARCH model using a Monte-Carlo simulation. We compared the performance of the proposed method with that of the two-step estimation method used by Thilakaratne and Samaranayake (2014) for the AR-GARCH model. Note that they also used the SB technique and then applied the two-step estimation method. A Monte-Carlo simulation was carried out with three different error distributions: normal, t with 7-degrees of freedom, and centered exponential with mean 0 and variance 1. Sample sizes $n = 500$ and $n = 1,000$ were used in the simulation. The following 2 models were used to simulate the conditional variance of the FIGARCH error structure:

Model 1: $\sigma_t^2 = 0.05 + \left[1 - 0.45L - (1-L)^{0.5}\right]\varepsilon_t^2 + 0.45\sigma_{t-1}^2,$

Model 2: $\sigma_t^2 = 0.05 + \left[1 - 0.2L - (1-L)^{0.9}\right]\varepsilon_t^2 + 0.2\sigma_{t-1}^2.$

We used the $\text{AR}(1)$, $\text{AR}(2)$ and $\text{ARMA}(1,1)$ models along with the error structure of the heteroscedastic conditional variance, as defined in Models 1 and 2. The following AR and ARMA models were considered:

$$X_t = a_1 X_{t-1} + \varepsilon_t,$$
$$X_t = a_1 X_{t-1} + a_2 X_{t-2} + \varepsilon_t,$$
$$X_t = a_1 X_{t-1} + b_1 \varepsilon_{t-1} + \varepsilon_t,$$

with $\rho_1,\ \rho_2 \in \{0.3, 0.5, 0.7, 0.9\}$ and $b_1 \in \{-0.3, -0.5, -0.8\}$, where $a_1 = \rho_1 + \rho_2$, $a_2 = -\rho_1\rho_2$ for the $AR(2)$ model. We used $a_1 = \rho_1$ in the $AR(1)$ model as well as AR part of the ARMA model. Here $1/\rho_1$ and $1/\rho_2$ are the roots of autoregressive polynomial.

We generated $N = 500$ independent time series for each combination of the model, sample size, nominal coverage, and error distribution. Then Steps 1 through 12 were implemented. In each simulation run, $R = 1,000$ future values, $\{X_{n+h}\}$, for $h = 1,\ 10,\ 20$ were generated. Then, the coverage probabilities were estimated by calculating the proportion of those future values, X_{n+h}, that fall between the lower and upper bounds of the bootstrap intervals. Therefore, the coverage for the i^{th} simulation run is given by $C(i) = R^{-1}\sum_{r=1}^{R} I_A[X_{n+k}^r(i)]$, where $A = \left[Q^*(\alpha/2), Q^*(1-\alpha/2)\right]$ is the $100(1-\alpha)th$ bootstrapped prediction interval. $I_A(.)$ is the indicator function of the set A and $X_{n+k}^r(i),\ r = 1,\ 2, \ldots,\ 1,000$ are the R future values generated at the i^{th} simulation run. The theoretical and bootstrap lengths were obtained by using $L_T(i) = X_{n+k}^r(1-\alpha/2) - X_{n+k}^r(\alpha/2)$ and $L_B(i) = Q^*(1-\alpha/2) - Q^*(\alpha/2)$, respectively. $L_T(i)$ is difference between $100(1-\alpha)th$ and the $100(\alpha/2)th$ percentiles generated from R future values of the underling model with known order and coefficients. Similarly, $L_B(i)$ is the difference between $100(1-\alpha)th$ and the $100(\alpha/2)th$ bootstrapped percentiles calculated following steps 1-12. The mean coverage, mean bootstrap prediction interval length, mean theoretical interval length, and their standard errors were calculated as follows:

Mean coverage $\bar{C} = N^{-1}\sum_{i=1}^{N} C(i)$,

Standard error of mean coverage $SE_{\bar{C}} = \left\{\left[N(N-1)\right]^{-1}\sum_{i=1}^{N}\left[C(i)-\bar{C}\right]^{2}\right\}^{1/2}$,

Mean length (bootstrap) $\bar{L}_B = N^{-1}\sum_{i=1}^{N} L_B(i)$,

Standard error of mean length $SE_{\bar{L}_B} = \left\{\left[N(N-1)\right]^{-1}\sum_{i=1}^{B}\left[L_B(i)-\bar{L}_B\right]^{2}\right\}^{1/2}$,

Mean theoretical length $\bar{L}_T = N^{-1}\sum_{i=1}^{N} L_T(i)$.

In total, 120 different combinations of model type, sample size, nominal coverage probability $(1-\alpha)$ and error distributions were investigated in this study. However, we report only a representative sample of these results for 95% intervals to conserve space. These results are reported in Tables 1-9. The tables report mean coverage, mean interval length, mean theoretical length, standard error of mean coverage, and standard error of mean interval length.

To further investigate the behavior of the intervals for each of the 120 combinations, the minimum value, percentiles (25^{th}, 50th and 75^{th}), and maximum value of coverage probabilities, the bootstrap interval bounds (lower and upper), and theoretical interval bounds (lower and upper), were computed, based on the 1,000 values generated through simulation. The complete set of results from the simulation study are available upon request from the corresponding author.

The coverage probabilities given by the simulation results are close to nominal values under both one-step and two-step methods. However, the coverages produced by the one-step estimation method are slightly higher than that from the two-step estimation method for all parameter combinations. The reason could be that the one-step approach is able to capture the sampling variation better than the two-step method. The interval lengths reported for the one-step estimation method are wider than theoretical lengths, but in some instances the interval lengths reported in the two-step estimation method are narrower than the theoretical lengths. For example, one-step ahead bootstrap length given for sample size 500 in Table 1 is slightly less than the theoretical length. However, much larger deviation of lengths can be seen in Table 2 for future lags 10 and 20. In those cases, theoretical lengths are larger than the bootstrap lengths for the two-step estimation method by at least 2 units. As discussed above, theoretical lengths are constructed by using the known underlining error process. Thus, intervals shorter than the theoretical lengths point to some deficiency in the two-step method.

Table 1. Coverage of 95% intervals for $X_t = 0.7X_{t-1} + \varepsilon_t$, where ε_t follows a FIGARCH(1, *d*, 0) with Model 2 and normal errors : AR root : 1.429

Lead lag	Sample size	Theoretical length	One Step Method		Two Step Method	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	3.1207	0.9519 (0.0011)	3.1990 (0.2376)	0.9479 (0.0020)	3.1146 (0.1626)
	1,000	2.5847	0.9538 (0.0009)	2.6470 (0.096)	0.9495 (0.0014)	2.7026 (0.0921)
10	500	5.1077	0.9547 (0.0008)	5.5889 (0.2506)	0.9489 (0.0009)	5.3503 (0.2264)
	1,000	4.5459	0.9566 (0.0006)	4.9732 (0.0997)	0.9508 (0.0008)	4.8405 (0.1045)
20	500	4.9465	0.9557 (0.0008)	5.6401 (0.1814)	0.9504 (0.0009)	5.3929 (0.1621)
	1,000	4.6147	0.9568 (0.0006)	5.1307 (0.0603)	0.9524 (0.0008)	5.0192 (0.0797)

Table 2. Coverage of 95% intervals for $X_t = 1.6X_{t-1} - 0.63X_{t-2} + \varepsilon_t$, where ε_t follows a FIGARCH(1, *d*, 0) with Model 1 and normal errors : AR roots : 1.111, 1.429

Lead lag	Sample size	Theoretical length	One Step Method		Two Step Method	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	5.5912	0.9563 (0.0010)	5.9876 (0.1625)	0.9452 (0.0010)	5.7655 (0.1653)
	1,000	5.6954	0.9574 (0.0008)	6.0158 (0.1587)	0.9471 (0.0008)	5.7908 (0.1504)
10	500	33.4060	0.9531 (0.0012)	36.8860 (1.0886)	0.9021 (0.0017)	29.3163 (0.8593)
	1,000	33.8881	0.9565 (0.0009)	36.6710 (0.9366)	0.9237 (0.0014)	31.4069 (0.7942)
20	500	39.1981	0.9539 (0.0012)	45.2263 (1.5137)	0.9005 (0.0018)	34.6119 (1.1573)
	1,000	39.5086	0.9591 (0.0009)	44.3088 (1.1342)	0.9250 (0.0014)	37.0210 (0.9158)

Table 3. Coverage of 95% intervals for $X_t = 0.9X_{t-1} - 0.8\varepsilon_{t-1} + \varepsilon_t$, where ε_t follows a FIGARCH(1, *d*, 0) with Model 1 and normal errors : AR root : 1.111, MA root: 1.429

Lead lag	Sample size	Theoretical length	One Step Method		Two Step Method	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	5.9759	0.9533 (0.0010)	6.3194 (0.1901)	0.9414 (0.0011)	5.9588 (0.1739)
	1,000	5.9747	0.9568 (0.0007)	6.2752 (0.1842)	0.9482 (0.0008)	6.0130 (0.1694)
10	500	6.3270	0.9590 (0.0009)	7.0733 (0.2151)	0.9457 (0.0009)	6.4556 (0.1865)
	1,000	6.3406	0.9608 (0.0007)	6.9479 (0.1942)	0.9508 (0.0007)	6.4986 (0.1766)
20	500	6.4117	0.9603 (0.0010)	7.4858 (0.2377)	0.9455 (0.0010)	6.6595 (0.2019)
	1,000	6.3848	0.9636 (0.0007)	7.3055 (0.2060)	0.9516 (0.0007)	6.6685 (0.1788)

Table 4. Coverage of 95% intervals for $X_t = 0.9X_{t-1} + \varepsilon_t$, where ε_t follows a FIGARCH(1, d, 0) with Model 1 and t errors : AR root : 1.111

Lead lag	Sample size	Theoretical length	One Step Method		Two Step Method	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	5.1106	0.9566 (0.0009)	5.5571 (0.1558)	0.9459 (0.0010)	5.1851 (0.1441)
	1,000	5.2265	0.9596 (0.0007)	5.7647 (0.1814)	0.9497 (0.0008)	5.3398 (0.1450)
10	500	11.3347	0.9565 (0.0011)	12.9195 (0.3880)	0.9408 (0.0012)	11.4645 (0.3239)
	1,000	11.5331	0.9624 (0.0007)	13.3923 (0.4827)	0.9480 (0.0009)	11.9058 (0.3269)
20	500	12.3066	0.9573 (0.0012)	14.6911 (0.4549)	0.9393 (0.0013)	12.7528 (0.3819)
	1,000	12.4507	0.9647 (0.0008)	15.4522 (0.7160)	0.9488 (0.0009)	13.0985 (0.3607)

Table 5. Coverage of 95% intervals for $X_t = 1.4X_{t-1} - 0.45X_{t-2} + \varepsilon_t$, where ε_t follows a FIGARCH(1, d, 0) with Model 2 and t errors : AR root : 1.111, 2.000

Lead lag	Sample size	Theoretical length	One Step Method		Two Step Method	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	2.4276	0.9513 (0.001)	2.4921 (0.1158)	0.9446 (0.0026)	2.5931 (0.1303)
	1,000	2.4048	0.9539 (0.0008)	2.4817 (0.0889)	0.9483 (0.0017)	2.5409 (0.0917)
10	500	11.0727	0.9529 (0.001)	12.0464 (0.3524)	0.9327 (0.0015)	10.5856 (0.3157)
	1,000	10.9245	0.9561 (0.0007)	11.9027 (0.2618)	0.9437 (0.001)	10.9539 (0.2347)
20	500	12.4381	0.954 (0.0011)	14.1592 (0.2664)	0.9324 (0.0016)	12.1108 (0.2818)
	1,000	12.3034	0.9569 (0.0007)	13.7376 (0.1696)	0.9435 (0.0011)	12.5265 (0.2054)

Table 6. Coverage of 95% intervals for $X_t = 0.5X_{t-1} - 0.5\varepsilon_{t-1} + \varepsilon_t$, where ε_t follows a FIGARCH(1, *d*, 0) with Model 2 and *t* errors : AR root : 1.111, MA root : 2.000

Lead lag	Sample size	Theoretical length	One Step Method: Mean coverage (SE)	One Step Method: Mean length (SE)	Two Step Method: Mean coverage (SE)	Two Step Method: Mean length (SE)
1	500	2.6520	0.9517 (0.0011)	2.7394 (0.1714)	0.9471 (0.0013)	2.6918 (0.1338)
	1,000	2.6304	0.9539 (0.0010)	2.7353 (0.1722)	0.9494 (0.0011)	2.7028 (0.1379)
10	500	2.9769	0.9553 (0.0009)	3.2939 (0.1275)	0.9518 (0.0009)	3.2330 (0.1059)
	1,000	2.9988	0.9564 (0.0007)	3.2626 (0.1173)	0.9533 (0.0007)	3.2114 (0.0996)
20	500	2.9041	0.9561 (0.0010)	3.2939 (0.0699)	0.9539 (0.0008)	3.2893 (0.0792)
	1,000	2.9291	0.9569 (0.0007)	3.2447 (0.0615)	0.9544 (0.0007)	3.2427 (0.0768)

Table 7. Coverage of 95% intervals for $X_t = 0.7X_{t-1} + \varepsilon_t$, where ε_t follows a FIGARCH(1, *d*, 0) with Model 1 and exponential errors : AR root : 1.429

Lead lag	Sample size	Theoretical length	One Step Method: Mean coverage (SE)	One Step Method: Mean length (SE)	Two Step Method: Mean coverage (SE)	Two Step Method: Mean length (SE)
1	500	4.3982	0.9577 (0.0017)	5.0549 (0.2194)	0.9404 (0.0019)	4.5486 (0.1897)
	1,000	4.2552	0.9571 (0.0014)	4.8846 (0.3109)	0.9395 (0.0017)	4.3878 (0.1956)
10	500	5.9014	0.9651 (0.0010)	7.9496 (0.3075)	0.9486 (0.0010)	6.7102 (0.2593)
	1,000	5.7549	0.9657 (0.0008)	7.7161 (0.4764)	0.9504 (0.0008)	6.6018 (0.2868)
20	500	6.4687	0.9540 (0.0011)	8.4588 (0.3014)	0.9364 (0.0010)	7.0035 (0.2631)
	1,000	6.3700	0.9555 (0.0009)	8.3494 (0.4662)	0.9393 (0.0009)	6.9287 (0.3157)

Table 8. Coverage of 95% intervals for $X_t = 1.2X_{t-1} - 0.35X_{t-2} + \varepsilon_t$, where ε_t follows a FIGARCH(1, *d*, 0) with Model 1 and exponential errors : AR roots : 1.429, 2.000

Lead lag	Sample size	Theoretical length	One Step Method		Two Step Method	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	4.3843	0.9595 (0.0017)	5.0811 (0.2278)	0.9427 (0.0018)	4.5907 (0.1949)
	1,000	4.2555	0.958 (0.0013)	4.8964 (0.3152)	0.9411 (0.0016)	4.4265 (0.2112)
10	500	8.956	0.9729 (0.0009)	13.1102 (0.5022)	0.9572 (0.0009)	10.8066 (0.4138)
	1,000	8.7529	0.9743 (0.0007)	12.8744 (0.8299)	0.96 (0.0008)	10.7466 (0.4567)
20	500	9.9098	0.9612 (0.0010)	14.0963 (0.518)	0.9436 (0.0009)	11.3741 (0.4436)
	1,000	9.8245	0.9621 (0.0008)	14.0540 (0.8525)	0.9470 (0.0008)	11.3921 (0.5121)

Table 9. Coverage of 95% intervals for $X_t = 0.7X_{t-1} - 0.3\varepsilon_{t-1} + \varepsilon_t$, where ε_t follows a FIGARCH(1, *d*, 0) with Model 1 and exponential errors : AR roots : 1.429, MA root: 2.000

Lead lag	Sample size	Theoretical length	One Step Method		Two Step Method	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	4.3942	0.9582 (0.0019)	5.2341 (0.3726)	0.9417 (0.0021)	4.6110 (0.2553)
	1,000	4.3045	0.9602 (0.0012)	4.8589 (0.2157)	0.9447 (0.0015)	4.5371 (0.2104)
10	500	5.1263	0.9607 (0.0011)	7.2034 (0.7468)	0.9450 (0.0011)	5.7769 (0.3362)
	1,000	4.9859	0.9609 (0.0009)	6.2367 (0.2510)	0.9462 (0.0009)	5.5254 (0.2481)
20	500	5.5969	0.9506 (0.0013)	8.5814 (1.5762)	0.9321 (0.0012)	6.1023 (0.3961)
	1,000	5.4549	0.9524 (0.0010)	6.7032 (0.2648)	0.9360 (0.0009)	5.8147 (0.2653)

Tables 1-3 reports the coverages of $\text{AR}(1)-\text{FIGARCH}(1,\ d,\ 0)$, $\text{AR}(2)-\text{FIGARCH}(1,\ d,\ 0)$, and $\text{ARMA}(1,\ 1)-\text{FIGARCH}(1,\ d,\ 0)$ under error Model 2, Model 1, and Model 1, respectively with normal innovations. The computed coverages in both methods are close to 0.95 for the $\text{AR}(1)-\text{FIGARCH}(1,\ d,\ 0)$ model, as reported in Table 1. Coverages and lengths for a $\text{AR}(2)$ model with FIGARCH errors are reported in the Table 2, where one of the roots of autoregressive polynomial AR(2) is closer to unity (root equals 1.111) and the other root is 1.428. Again, coverages for one-lag ahead forecasts are closer to 0.95 in both methods. However, coverages for the h-step predictions for $h > 1$, including for lags which are not reported here, indicate liberal intervals with coverage probabilities less than 0.95 for the two-step estimation method. For example, coverages for 10^{th} and 20^{th} steps are 0.9021 and 0.9005, respectively, for samples size 500 for the two-step estimation method. However, as sample size increases to 1,000, these coverages increase to 0.9237 and 0. 9250 respectively. By contrast, coverages for the one-step estimation method lies in between 0.95 and 0.96, for all forecast lags between 1 and 20 (we only report coverages for lags 1, 10 and 20 in the above tables). Both methods provide reasonably good coverages for the $\text{ARMA}(1,\ 1)-\text{FIGARCH}(1,\ d,\ 0)$ model, where the AR root is 1.111 and MA roots is 1.125.

Relatively similar results can be seen for $\text{AR}(1)-\text{FIGARCH}(1,d,0)$, $\text{AR}(2)-\text{FIGARCH}(1,\ d,\ 0)$ and $\text{ARMA}(1,\ 1)-\text{FIGARCH}(1,\ d,\ 0)$ under t-distributed errors in Tables 4-6. The coverage probabilities for $\text{AR}(1)-\text{FIGARCH}(1,d,0)$, $\text{AR}(2)-\text{FIGARCH}(1,\ d,\ 0)$ and $\text{ARMA}(1,\ 1)-\text{FIGARCH}(1,\ d,\ 0)$ under Model 1 with exponentially distributed errors are reported in Tables 7-9. In the exponential case, we obtained coverage probabilities with conservative bootstrap lengths using the one-step estimation method, with coverages ranging from 0.95 to 0.98 in most of the cases (these are not reported here but can be obtained from the corresponding author). In fact, we observed that when the roots of the AR component are close to 1, the coverages are much larger than 0.95. Further, coverages given in one-step estimation method always larger than that of two-step estimation method.

5. APPLICATION TO A REAL DATA SET

Baillie et al. (1996) and Tayefi & Ramanathan (2016) discussed modelling exchange rates using a FIGARCH model. Borrowing from their idea, the proposed method was applied to logarithms of US Dollar to Japanese Yen (JPN/USD) exchange rates. Daily data was obtained for the period January 1, 2010 through December 15, 2021 from the website at URL: https://finance.yahoo.com. Figure 1 shows the calculated one-step ahead prediction intervals from March 11, 2020 to December 15, 2020. Furthermore, the estimated 95% coverage probabilities for lead lags l, 10 and 20 for the logarithm of exchange rates are reported in the Table 10. As the results show, the prediction intervals we obtained are quite conservative.

Table 10. Estimated coverage probabilities for future exchange rates

Lead lag	Coverage
1	0.9750
10	0.9948
20	1.0000

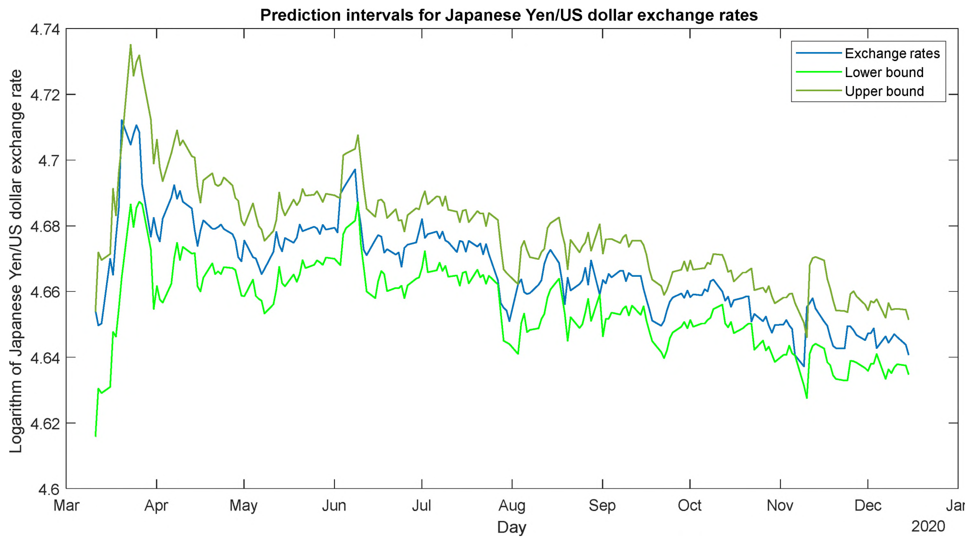

Figure 1. One-step ahead prediction interval for logarithm of Japanese Yen/US dollar exchange rates

6. CONCLUSION

Here we proposed a bootstrap-based method to obtain prediction intervals for AR and ARMA models with FIGARCH error structure. We extended the sieve bootstrap method proposed by Alonso et al. (2002,2003) to construct prediction intervals by incorporating long memory error variance, instead of an independent and identically distributed error structure. A Monte-Carlo simulation study is carried out to investigate finite sample properties. Furthermore, we assumed the order of the AR and ARMA part is unknown, and the order of FIGARCH part is known. Simulation results show that the proposed bootstrap method provide coverages closes to nominal under both parameter estimation methods in most of the cases. However, we recommend the one-step estimation method over the two-step estimation method when AR parameter closes to unity with weaker MA coefficient with FIGARCH errors.

REFERENCES

Alonso, A.’. M., Peña, D., & Romo, J. (2002). Forecasting time series with sieve bootstrap. Journal of Statistical Planning and Inference, 100(1), 1–11.

Alonso, A. M., Peña, D., & Romo, J. (2003). On sieve bootstrap prediction intervals. Statistics & Probability Letters, 65(1), 13–20.

Baillie, R. T., Bollerslev, T., & Mikkelsen, H. O. (1996). Fractionally integrated generalized autoregressive conditional heteroskedasticity. Journal of Econometrics, 74(1), 3–30.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. Journal of Econometrics, 31(3), 307–327.

Bühlmann, P., & Buhlmann, P. (1997). Sieve Bootstrap for Time Series. Bernoulli, 3(2), 123.

Cao, R., Febrerobande, M., Gonzalez-Manteiga, W., Prada-Sanchez, J. M., & Garcfa-Jurado, I. (1997). Saving computer time in constructing consistent bootstrap prediction intervals for autoregressive processes. Communications in Statistics - Simulation and Computation, 26(3), 961–978.

Ding, Z., Granger, C. W. J., & Engle, R. F. (1993). A long memory property of stock market returns and a new model. Journal of Empirical Finance, 1(1), 83–106.

Engle, R. F. (1982). Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation. Econometrica, 50(4), 987.

Engle, R. F., & Bollerslev, T. (1986). Modelling the persistence of conditional variances. Econometric Reviews, 5(1), 1–50.

Findley, D. F. (1986). On bootstrap estimates of forecast mean square errors for autoregressive processes. In: Allen D. M. ed. Computer Science and Statistics: Proceedings of the 17th Symposium on the Interface, pp. 11–17.

Harvey, A.C., (1993). Long memory in stochastic volatility, Manuscript (London School of Economics, London)

Kreiss, J. P. Asymptotic statistical inference for a class of stochastic processes [Habilitationsschrift]. Germany: University at Hamburg; 1988.

Miguel, J. A., & Olave, P. (1999). Bootstrapping forecast intervals in ARCH models. Test, 8(2), 345–364.

Mukhopadhyay, P., & Samaranayake, V. A. (2010). Prediction Intervals for Time Series: A Modified Sieve Bootstrap Approach. Communications in Statistics - Simulation and Computation, 39(3), 517–538.

Pascual, L., Romo, J., & Ruiz, E. (2006). Bootstrap prediction for returns and volatilities in GARCH models. Computational Statistics & Data Analysis, 50(9), 2293–2312. https://doi.org/10.1016/j.csda.2004.12.008

Rupasinghe, M., & Samaranayake, V. A. (2012). Asymptotic properties of sieve bootstrap prediction intervals for processes. Statistics & Probability Letters, 82(12), 2108–2114.

Rupasinghe, M., Mukhopadhyay, P., & Samaranayake, V. A. (2013). Obtaining prediction intervals for FARIMA processes using the sieve bootstrap. Journal of Statistical Computation and Simulation, 84(9), 2044–2058.

Stine, R. A. (1982). Prediction Intervals for Time Series. Ph.D. dissertation, Princeton University, Department of Statistics, Princeton, NJ, unpublished.

Stine, R. A. (1987). Estimating Properties of Autoregressive Forecasts. Journal of the American Statistical Association, 82(400), 1072–1078.

Tayefi, M., & Ramanathan, T. V. (2016). An Overview of FIGARCH and Related Time Series Models. Austrian Journal of Statistics, 41(3), 175–196.

Thilakaratne, M., Rupasinghe, M., Samaranayake, V. A.(2014) Sieve Bootstrap-Based Prediction intervals for Autoregressive Processes with GARCH Innovations . Joint Statistical Meetings. Proceedings, Boston, Massachusetts, 3284–3296.

Thombs, L. A., & Schucany, W. R. (1990). Bootstrap Prediction Intervals for Autoregression. Journal of the American Statistical Association, 85(410), 486–492.

III. SIEVE BOOTSTRAP-BASED PREDICTION INTERVALS FOR FARIMA-FIGARCH MODELS

ABSTRACT

A sieve bootstrap-based prediction interval method is proposed for Fractionally Integrated Autoregressive Moving Average (FARIMA) process with Fractionally Integrated Generalized Autoregressive Conditional Heteroscedastic (FIGARCH) errors. Here we assume the order of the FARIMA part is unknown and the order is determined by the AIC criterion. Resampling of the residuals is done by using two methods. First method fits an AR-FIGARCH process and then obtain the residuals of AR-FIGARCH model. Second method utilize the resamples of FIGARCH errors after fitting an AR model. A Monte-Carlo simulation study shows that both methods provide reasonably good coverage probabilities for most of the parameter combinations considered, but the coverages for intervals constructed by the two-step method deviate from the nominal value when the fractional integrated parameter d in the FARIMA part of the model is close to 0.5 and a root of the AR polynomial is close to one. Finally, the one-step method is applied to monthly consumer price index (CPI) inflation rates in Japan to provide an application to real-world data.

Keywords: Prediction interval, Fractional integration, Sieve bootstrap

1. INTRODUCTION

Modeling long memory processes have become important in areas such as hydrology, macroeconomics, geophysical sciences, and in modeling stock returns and exchange rates. Fractionally integrated ARMA models have been used extensively to model long memory processes. Weights of the autocorrelation function and the impulse response function of such models decay slowly at hyperbolic rate. The most widely used fractionally integrated model for long memory processes, the Autoregressive Fractionally Moving Average (FARIMA) model was independently introduced by Granger and Joyeux (1980) and Hosking (1991). Baillie et al., (1996) extend this model by adding time dependent error structure and modeled the Consumer Price Index (CPI) inflation of ten different countries. Apart from the long memory nature of the conditional mean of the inflation rates, Baillie et al. (2002) found out that the squared and absolute values of residuals obtained from the fractionally filtered inflation series also exhibit long memory. Therefore, they introduced the hybrid FARIMA-Fractionally Integrated Generalized Autoregressive Conditional Heteroscedastic (FIGARCH) formulation to model this dual long memory feature in both first and second moments of inflation rates.

Forecasting is an important aspect in long memory models. Generally, forecasting a process with short memory into the far future cannot be done accurately. On the other hand, forecastable horizon for long memory processes are much longer than that for short memory process, because explanatory power of the past observations decay at an exponential rate for short memory processes while it is slow hyperbolic decay for long memory processes. Brockwell and Davis, (1991) discussed the point prediction of future values in FARIMA process based on the innovations algorithm. Ray, (1993) suggested that approximating a FARIMA model to an AR(p) can be useful in long-range forecasting for long memory models. Discussions about forecasting with FARIMA and related processes can be found in the papers published by Crato and Ray (1996), Beran and Ocker (1999), Ramjee et al. (2002), and Baillie and Chung (2002).

There are plenty of articles available for FARIMA related point forecasting, however there is dearth of research articles about prediction intervals. Bisaglia and Grigoletto (2001) established prediction intervals for FARIMA processes by using bootstrap-based method. This bootstrap-based method involves parameter estimation of the fractional difference parameter, as well as the AR and MA coefficients jointly, using the Whittle approximation (Doukhan et al., 2003 and Fox and Taqque, 1986). However, the computational time required for the implementation of this method is high, but it performs well, providing the coverages close to the nominal value under normally distributed errors. Rupasinghe and Samaranayake, (2013) introduced a computationally much faster sieve bootstrap-based method of obtaining prediction intervals by approximating FARIMA by a finite order AR(p) model. This method provides coverages close to the nominal values under the normal errors as well as non-normally distributed errors such as exponential and a mixture of two normal distributions. The above authors, however, assumed that the error structure of the FARIMA process is homoscedastic. In practical situations, such as constructing prediction intervals for stock returns, exchange rates and inflation rates, errors often exhibit heteroscedastic behavior. Amjad et al. (2017) developed a bootstrap-based prediction interval for FARIMA-GARCH process to handle such a situation. The FARIMA-GARCH allows for long memory in the conditional mean process and short memory in the conditional variance process. As Baillie et al. (2002), and Conrad and Karanasos, (2005) suggested, the autocorrelations of both first and second moments of inflation rates decay in a slow hyperbolic rate, suggesting that FARIMA-FIGARCH model is a better fit for inflation rates. It can be argued that this phenomenon is not limited to exchange rates only. It is reasonable to assume that if long-memory exists in the mean process, then there is the possibility of such long memory to exhibit in the variance process as well. Thus, our aim is to introduce a bootstrap-based prediction interval for models that exhibit long memory in both the conditional mean and the conditional variance. Thus, our focus is to develop prediction intervals for the FARIMA-FIGARCH model using a residual-based bootstrap approach. The residuals for the bootstrap is obtained by fitting an AR-FIGARCH model, where the FARIMA component is approximated by a sequence of AR processes as employed in Rupasinghe and Samaranayake (2013). The resampling method based on such an approximation technique is generally called the sieve boostrap (SB).

The rest of the paper is as follows. Section 2 describe the FARIMA-FIGARCH model. A brief explanation about sieve bootstrap technique and the steps for obtaining the proposed bootstrap-based prediction intervals are given in Section 3. The results of a simulation study are reported in Section 4. A real-life application of the method is given in Section 5. Finally, conclusions are provided in Section 6.

2. FARIMA-FIGARCH PROCESS

2.1. THE FARIMA MODEL

Based on Granger (1980), Granger and Joyeux (1980), and Hosking (1981), the mathematical formulation of FARIMA(p, d_{FAR}, q) model is as follows. Let $\{X_t\}$ be a stationary process such that

$$a(L)(1-L)^{d_{FAR}} X_t = b(L)\varepsilon_t, \quad (1)$$

for some $-0.5 < d_{FAR} < 0.5$. Where $a(L) = 1 - a_1 L - \ldots - a_p L^p$, $b(L) = 1 - b_1 L - \ldots - b_q L^q$ are polynomials of degree p and q respectively, with $a_p \neq 0$ and $b_q \neq 0$, and with all their roots outside the unit circle. The innovations in $\{\varepsilon_t\}$'s are independent and identically distributed random variables with zero mean and unit variance. Then $\{X_t\}$ is called a stationary FARIMA process. According to the Wold decomposition, infinite-order moving-average representation of the above process is given by, $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$ and the infinite-order autoregressive representation is given by $\sum_{j=0}^{\infty} \pi_j X_{t-j} = \varepsilon_t$ provided the process is invertible. The coefficients in the infinite sums decay at a hyperbolic rate for large lags j. That is, $\psi_j \approx c_1 j^{d_{FAR}-1}$ and $\pi_j \approx c_2 j^{-d_{FAR}-1}$. Similarly, the autocorrelation function $\rho_j \approx c_3 j^{2d_{FAR}-1}$ decay at a hyperbolic rate for large lags j. Note that c_1, c_2, and c_3 in the above expressions for rate are constants. The parameter d_{FAR} represents the degree of the long memory present in the FARIMA(p, d_{FAR}, q) process. If $0 < d_{FAR} < 0.5$, then the process is said to be a long memory process and if $-0.5 < d_{FAR} < 0$, then it is an intermediate memory process. The process is stationary and invertible when $-0.5 < d_{FAR} < 0.5$. The process does not have finite variance when $0.5 \leq d_{FAR} < 1$. In this study we only considered processes with long range dependence and stationary, therefore, we only considere the cases where $0 < d_{FAR} < 0.5$.

2.2. FARIMA-FIGARCH MODEL

A FARIMA-FIGARCH model is quite useful when modeling features such as long memory in both the conditional mean and the conditional variance. Let $\{Y_t : t \in \mathbb{Z}\}$ be a real-valued process with following $\text{FARIMA}(p,\ d_{FAR},\ q) - \text{FIGARCH}(r,\ d_{FG},\ s)$ representation:

$$\begin{aligned} a(L)(1-L)^{d_{FAR}} X_t &= b(L)\varepsilon_t, \\ \varepsilon_t &= \sigma_t z_t, \\ \sigma_t^2 = \omega + \beta(L)\sigma_t^2 &+ [1-\beta(L)-\phi(L)(1-L)^{d_{FG}}]\varepsilon_t^2, \end{aligned} \tag{2}$$

where, $a(L)$, $b(L)$ and d_{FAR} are defined similar to FARIMA model definition in 2.1. The sequence $\{Z_t\}$ is a white noise process with zero mean and unit variance. L is the lag operator such that $\beta(L) = \beta_1 L + \beta_2 L^2 + \ldots + \beta_r L^r$ and $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_s L^s$ with $\beta_r \neq 0$ and $\phi_s \neq 0$. The fractional integrating parameter d_{FG} in the conditional variance expression is assumed to lie between 0 and 1. If $\sigma_t^2 = \omega$, a constant, then process reduces to a $\text{FARIMA}(p, d_{FAR}, q)$ model.

The sequence $\{\varepsilon_t\}$ in the above expression follows a $\text{FIGARCH}(r, d_{FG}, s)$ process. Our analysis in this study is restricted to the error structure of $\text{FIGARCH}(1,\ d_{FG},\ 1)$, which is commonly used to model conditional variance with long memory. The conditional variance of a FIGARCH(1, d_{FG}, 1) process can be express as:

$$\sigma_t^2 = \omega + [1 - \beta L - (1-\phi_1 L)(1-L)^{d_{FG}}]\varepsilon_t^2 + \beta\sigma_{t-1}^2,$$

$$\sigma_t^2 = \omega(1-\beta_1)^{-1} + [1-(1-\beta_1)^{-1}(1-\phi_1 L)(1-L)^{d_{FG}}]\varepsilon_t^2. \quad (3)$$

Furthermore, the conditional variance in expression (2) can be written as $\sigma_t^2 = \omega(1-\beta(1))^{-1} + \lambda(L)\varepsilon_t^2$, where $\lambda(L) = \sum_{k=1}^{\infty}\lambda_k L^k$. The coefficients λ_i'$s, i = 1,2,...$ can be obtained by equating terms with common power in the equation

$$\lambda(L) = \lambda_1 L + \lambda_2 L^2 + ... + \lambda_k L^k + ... = [1-(1-\beta_1)^{-1}(1-\phi_1 L)(1-L)^{d_{FG}}].$$

Thus, $\lambda_1 = \phi_1 - \beta_1 + d_{FG}$ and $\lambda_k = \beta_1\lambda_{k-1} + [(k-1-d_{FG})/k - \phi_1]\delta_{d_{FG},k-1}$, $k = 2,3,....$, where $\delta_{d_{FG},k} = \delta_{d_{FG},k-1}(k-1-d_{FG})k^{-1}$, $k = 2,3,...$, refer to the coefficients in the series expansion of $(1-L)^{d_{FG}}$, with $\delta_{d_{FG},0} = 1$ and $\delta_{d_{FG},1} = d_{FG}$. Now, all λ_i, for $i = 1,2,...$ must be positive to ensure the non-negativity of the conditional variance σ_t^2. Following Baillie et al. (1996), the sufficient conditions for non-negativity of conditional variance are $0 \le \beta_1 \le \phi_1 + d_{FG}$ and $0 \le d_{FG} \le 1 - 2\phi_1$. As stated earlier, for $-0.5 < d_{FAR} < 0.5$, FARIMA process is stationary and invertible with finite variance. Since the unconditional variance of FIGARCH process is infinite (Baillie et al., 1996), the unconditional variance FARIMA-FIGARCH process is infinite for $d_{FG} \ne 0$.

3. SIEVE BOOTSTRAP-BASED PREDICTION INTERVALS FOR THE FARIMA-FIGARCH MODEL

The sieve bootstrap is a method of approximating class of linear processes with sequence of finite AR processes, with the bootstrap applied to the residuals from fitting the approximate AR model. The order ($p(n)$) of the finite AR process is allowed to approach to infinity at a slower rate than the sample size *n,* as the sample size approaches infinity. The well-established sieve bootstrapped method was first introduced by Buhlmann (1997). The advantage of this technique is that it does not require the prior knowledge of the order of the underlying process such as the ARMA. Alonso et al. (2002, 2003) extended the sieve bootstrap technique to construct prediction intervals for a general class of linear processes which includes ARMA models. Later, Mukhopadhyay and Samaranayake (2010) modified the Alonso (2003) sieve bootstrap technique to improve the coverage probabilities.

Let $\{X_t\}$ be a zero mean FARIMA process as defined in equation (2.1). The process is invertible if the roots of the $b(L)$ polynomial lie out sided the unit circle and when $0 < d_{FAR} < 0.5$. In such situations (2.1) can be written as $\sum_{j=0}^{\infty} \pi_j X_{t-j} = \varepsilon_t,\ t \in \mathbb{Z}$ with $\pi_0 = 1$ and $\sum_{j=0}^{\infty} |\pi_j| < \infty$. With this representation, the sieve bootstrap method to obtain prediction intervals, proposed by Alonso et al. (2002, 2003) seems feasible. Some modifications, however, need to be applied before we proceed. To apply Alonso's method directly, coefficients of infinite moving average representation must satisfy $\sum_{j=0}^{\infty} j^r |\psi_j| < \infty$ for some $r \in \mathbb{N}$. However, the $\text{FARIMA}(p, d_{FAR}, q)$ model does not satisfy this condition. Poskit, (2006) overcame this issue by approximating much more general class of linear process, including FARIMA process, by finite order AR processes. After applying Poskit's results, Rupasinghe and Samaranayake, (2012, 2013) obtained a modified version of the sieve bootstrap prediction intervals introduce by Alonso et al. (2003). They also established asymptotic properties for the FARIMA prediction intervals using sieve bootstrap approach.

The FARIMA-FIGARCH model contains heteroscedastic FIGARCH errors instead of homoscedastic errors assumed in the FARIMA model. Similar to methodology discussed in Rupasinghe and Samaranayake's 2013 article, we approximated FARIMA-FIGARCH by a finite AR-FIGARCH model. Then the parameters of AR-FIGARCH are estimated by quasi maximum likelihood(Q-MLE) method. Hereafter, we will refer to this approach as the one-step estimation method. We compared this method with the two-step estimation method used by Thilakaratne and Samaranayake (2014) to compute prediction intervals for AR-GARCH models. Parameter estimation in the two-step estimation method is done by first fitting an AR approximation of the FARIMA component and then fitting a FIGARCH to the AR residuals. We used both one-step and two-steps estimation methods to construct the prediction intervals.

The AR-FIGARCH approximation is an important step in our proposed bootstrap technique. As suggested in Poskit (2006) and employed in Rupasinghe and Samaranayake (2013), the maximum truncation lag $p_{\max}$ is selected as $p_{\max} = [\log(n)]^{1.962}$ where n is the sample size. Along these lines, the optimal order p of the truncation of AR part from possible choices in $\{1, 2, ..., p_{\max}\}$ is selected using the AIC criteria. In our simulation study, we set $p_{\max}$ to 36 and 44 for sample sizes 500 and 1000 respectively. The steps for proposed

SB based one-step prediction interval for FARIMA-FIGARCH model, for the one-step procedure, is discussed as follows.

1. Select the maximum order $p_{\max}$ for the given realization $\{X_t\}_{t=1}^{n}$ of an FARIMA-FIGARCH process. We used value of $p_{\max} = 36,\ 44$ for $n = 500,\ 1000$ respectively. Then find the optimal order $\hat{p}$ among the possible values $p = 1, 2, ..., p_{\max}$ using the AIC approach. Here we assumed order of the order of the FIGARCH is known.

2. Use the least-squares estimates, $\tilde{\varphi}_1, \tilde{\varphi}_2, ..., \tilde{\varphi}_{\hat{p}}$ of the $\mathrm{AR}(\hat{p})$ parameters as the initial values for the AR part in maximum likelihood estimation of the AR-FIGARCH model. Assignment of initial values for the FIGARCH parameters $\omega_1, \phi_1, d, \beta_1$ are done by randomly selecting values in a feasible region. Obtain the maximum likelihood estimates, $\hat{\varphi}_1, \hat{\varphi}_2, ..., \hat{\varphi}_{\hat{p}}, \hat{\omega}_1, \hat{\phi}_1, \hat{d}, \hat{\beta}_1$ of AR-FIGARCH model by using these initial values.

3. Compute the $(n - \hat{p})$ residuals $\hat{z}_t$ by using $\hat{z}_t = \hat{\varepsilon}_t / \hat{\sigma}_t$, $\hat{\varepsilon}_t = -\sum_{j=0}^{\hat{p}} \hat{\varphi}_j (X_{t-j} - \overline{X})$ and

$$\begin{aligned}\hat{\sigma}_t^2 &= \hat{\omega}(1-\hat{\beta}_1)^{-1} + [1-(1-\hat{\beta}_1)^{-1}(1-\hat{\phi}_1 L)(1-L)^{\hat{d}}]\hat{\varepsilon}_t^2, \\ &\approx \hat{\omega}(1-\hat{\beta}_1)^{-1} + \hat{\lambda}_1\hat{\varepsilon}_{t-1}^2 + \hat{\lambda}_2\hat{\varepsilon}_{t-2}^2 + \ldots + \hat{\lambda}_k\hat{\varepsilon}_{t-k}^2,\end{aligned},$$

where $\hat{\varphi}_0 = -1, t \in (\hat{p}+1, ..., n)$ and $\overline{X}$ is the mean of the process $\{X_t\}_{t=1}^{n}$. $\hat{\varepsilon}_t^2 = (n-\hat{p})^{-1}\sum_{i=\hat{p}+1}^{n} \hat{\varepsilon}_i^2$ for $t \le \hat{p}$ and k the truncation lag of the polynomial $\lambda(L)$.

4. Compute the centered residuals $\tilde{z}_t = \hat{z}_t - \bar{\hat{z}}_t ; t \in (\hat{p}+1,...,n)$, where $\bar{\hat{z}}_t = (n-\hat{p})^{-1} \sum_{i=\hat{p}+1}^{n} \hat{z}_i$.

5. Denote the empirical distribution function of the centered residuals $\{\tilde{z}_t\}_{\hat{p}+1}^{n}$ as $\hat{F}_{\tilde{z}}(x) = (n-\hat{p})^{-1} \sum_{t=\hat{p}+1}^{n} I_{(-\infty,x]}(\tilde{z}_t)$.

6. Draw a bootstrap sample with replacement from the above distribution and denote it by z_t^*, for $t = -m-k+1,...-1,0,1,...,n$.

7. Generate the bootstrapped FIGARCH series ε_t^*, $t = -m+1,...,-1,0,1,...,n$ by first creating a bootstrapped conditional variance, $\hat{\sigma}_t^{2*}$, using estimated FIGARCH parameters of AR-FIGARCH obtained in Step 2. Then use $\varepsilon_t^* = z_t^* \hat{\sigma}_t^*$, $t = -m+1,...,-1,0,1,...,n$ to generate ε_t^*. Here m is chosen to be 2,000.

8. Then generate the bootstrapped AR-FIGARCH series, X_t^*, $t = -m+1,...,0,1,...,n$ using the bootstrapped FIGARCH errors, ε_t^* created in step 7 and using the recursion $X_t^* - \bar{X} = \sum_{j=1}^{\hat{p}} \hat{\varphi}_j \left(X_{t-j}^* - \bar{X}\right) + \varepsilon_t^*$ with the first $\hat{p}$ values of X_t^* set equal to $\bar{X}$. Drop the first m observations to eliminate the effect of the initial values. We used m=2000 in this study.

9. Fit an AR-FIGARCH for bootstrapped series $\{X_t^*\}_{t=1}^{n}$ and then estimate the parameters of it using the Q-MLE method and let the estimated AR and FIGARCH coefficients be denoted by $\hat{\varphi}_1^*, \hat{\varphi}_2^*, ..., \hat{\varphi}_{\hat{p}}^*$ and $\hat{\omega}_1^*, \hat{\phi}_1^*, \hat{d}^*, \hat{\beta}_1^*$ respectively.

10. Compute the h-step ahead bootstrap forecasts of future values using the bootstrapped AR coefficients, $\hat{\varphi}_1^*, \hat{\varphi}_2^*, \ldots, \hat{\varphi}_{\hat{p}}^*$ and FIGARCH coefficients $\hat{\omega}_1^*, \hat{\phi}_1^*, \hat{d}^*, \hat{\beta}_1^*$ using the following recursions:

$$X_{n+h}^* - \bar{X} = \sum_{j=1}^{\hat{p}} \hat{\varphi}_j^* (X_{n+h-j}^* - \bar{X}) + \varepsilon_{n+h}^*, \quad \varepsilon_{t+h}^* = z_{n+h}^* \hat{\sigma}_{n+h}^*,$$

$$\begin{aligned} \hat{\sigma}_{n+h}^{2*} &= \hat{\omega}^*(1-\hat{\beta}_1^*)^{-1} + [1-(1-\hat{\beta}_1^*)^{-1}(1-\hat{\phi}_1^*)(1-L)^{\hat{d}^*}]\varepsilon_{n+h}^{*2}, \\ &\approx \hat{\omega}^*(1-\hat{\beta}_1^*)^{-1} + \hat{\lambda}_1^* \varepsilon_{n+h-1}^{*2} + \ldots + \hat{\lambda}_k^* \varepsilon_{n+h-k}^{*2}, \end{aligned}$$

for $h > 0$, and letting $\varepsilon_t^* = \hat{\varepsilon}_t$ for $t \le n$.

11. Obtain the estimated bootstrap distribution of X_{n+h}, denoted by $\hat{F}_{X_{n+h}^*}^*(.)$, by repeating steps 6-10 $B = 1,000$ times in the simulation study. $\hat{F}_{X_{n+h}^*}^*(.)$ is the estimate of the $F_{X_{n+h}^*}^*(.)$, the bootstrap distribution function of X_{n+h}^*, and is used to approximate unknown distribution of X_{n+h} given the observed sample.

12. The $100(1-\alpha)\%$ prediction interval for X_{n+h} is then computed by $\left[Q^*(\alpha/2), Q^*(1-\alpha/2)\right]$, where $Q^*(.) = \hat{F}_{X_{n+h}^*}^{*-1}$ are the percentiles of the estimated bootstrap distribution.

There is only a slight difference between the process of obtaining prediction intervals under two-steps estimation method and one-step estimation method. In order to accommodate above steps under two-steps estimation method, parameter estimations in Step 2 and Step 9 need to change by estimation the parameters of the AR component first and the FIGARCH component using the residuals obtained by fitting the AR approximation. By applying this change, we estimated the coverages under two-steps estimation method.

4. THE SIMULATION STUDY

We investigated the finite sample properties of bootstrap prediction intervals for FARIMA-FIGARCH model by carrying out a Monte-Carlo simulation study. We used standard normal, t with 7 degrees of freedom, and centered exponential distributions with zero mean and unit variance, for FARIMA-FIGARCH innovations. The conditional variances of the FIGARCH error structure used in this study are given by

Model 1: $\sigma_t^2 = 0.05 + \left[1 - 0.45L - (1-L)^{0.5}\right]\varepsilon_t^2 + 0.45\sigma_{t-1}^2.$

Model 2: $\sigma_t^2 = 0.05 + \left[1 - 0.1L - (1-L)^{0.9}\right]\varepsilon_t^2 + 0.1\sigma_{t-1}^2.$

In addition, we considered FARIMA(0, d_{FAR}, 0), FARIMA(0, d_{FAR}, 1), FARIMA(1, d_{FAR}, 0) and FARIMA(1, d_{FAR}, 1) models along with FIGARCH error structures defined in Models 1 and 2. The parameters for AR part: $a_1 \in \{0, 0.5, 0.8, 0.9\}$, MA part: $b_1 \in \{0, -0.5, -0.8\}$ and fractionally integrated part: $d_{FAR} \in \{0.25, 0.4, 0.49\}$, were used to simulate the FARIMA process with FIGARCH errors. Some of these combinations for the FARIMA part were also employed in Rupasinghe et al. (2013). Sample sizes considered are 500 and 1,000.

We generated N = 500 independent time series for each combination of the model, sample size, nominal coverage, and error distribution. Then steps 1 through 12 were implemented. In each simulation run, $R = 1{,}000$ future values, $\{X_{n+h}\}$, $h = 1,\ 10,\ 20$ were generated. Then the coverage probabilities were estimated by calculating the proportion of those future values, X_{n+h}, falling between the lower and upper bounds of the bootstrap intervals. Therefore, the coverage for the i^{th} simulation run is given by $C(i) = R^{-1}\sum_{r=1}^{R} I_A[X^r_{n+k}(i)]$ where $A = \left[Q^*(\alpha/2), Q^*(1-\alpha/2)\right]$ is the $100(1-\alpha)th$ bootstrap prediction interval. $I_A(.)$ is the indicator function of the set A and $X^r_{n+k}(i),\ r = 1,\ 2,\ \ldots, 1{,}000$ are the R future values generated at the i^{th} simulation run. The theoretical and bootstrap lengths are obtained by using $L_T(i) = X^r_{n+k}(1-\alpha/2) - X^r_{n+k}(\alpha/2)$ and $L_B(i) = Q^*(1-\alpha/2) - Q^*(\alpha/2)$ respectively. $L_T(i)$ is difference between $100(1-\alpha)th$ and the $100(\alpha/2)th$ percentiles generated from R future values of the underling model with known order and coefficients. Similarly, $L_B(i)$ is the difference between $100(1-\alpha)th$ and the $100(\alpha/2)th$ bootstrapped percentiles calculated following the steps 1-12. The mean coverage, mean bootstrapped prediction interval length, mean theoretical interval length and their standard errors are calculated as follows:

Mean coverage $\bar{C} = N^{-1}\sum_{i=1}^{N} C(i)$,

Standard error of mean coverage $SE_{\bar{C}} = \left\{\left[N(N-1)\right]^{-1}\sum_{i=1}^{N}\left[C(i) - \bar{C}\right]^2\right\}^{1/2}$,

Mean length (bootstrap) $\bar{L}_B = N^{-1}\sum_{i=1}^{N} L_B(i)$,

Standard error of mean length $SE_{\bar{L}_B} = \left\{\left[N(N-1)\right]^{-1}\sum_{i=1}^{B}\left[L_B(i) - \bar{L}_B\right]^2\right\}^{1/2}$, and

Mean theoretical length $\bar{L}_T = N^{-1}\sum_{i=1}^{N} L_T(i)$.

In total, 60 different combinations of model type, sample size, nominal coverage probability $(1-\alpha)$ and error distributions were investigated in this study. However, we report only a representative sample of results for 95% intervals, in Table 1-8 due to space limitations. These tables report mean coverage, mean interval length, mean theoretical length, standard error of mean coverage and standard error of mean interval length. To further investigate the behavior of the intervals for each of the 60 combinations, the minimum value, percentiles (25^{th}, 50th and 75^{th}), and maximum value of coverage probabilities, the bootstrap interval bounds (lower and upper) and theoretical interval bounds (lower and upper), were computed, based on the 1,000 values generated through simulation. The complete set of results of the simulation study are available upon request from the corresponding author.

Tables 1-5 report the results under the normal error distribution. It can be seen that the coverages for normal errors and *t* distributed errors are similar. Therefore, we allocate only Table 6 to the *t* distributed errors. Finally, computed coverages and lengths under exponential errors are reported in Tables 7 and 8. Further, we compared the coverages and bootstrap lengths between the two estimation methods we used. One-step estimation method yields wider and conservative intervals with coverage probabilities larger than 0.95 (see in Tables 1-6). Out of 36 cases only in 4 occasions the coverage fell below 0.95. In contrast, the two-step estimation method yielded 14 cases with the coverages less than 0.95. Two-step estimation method provides poor coverages for FARIMA(1, d_{FAR}, 0) errors when an AR root is close to the unity and d_{FAR} is close to 0.5. However, coverages given under one-step method are close to 0.95 in such cases, as shown in Table 1. The coverages for the two-step procedure for lead lag one prediction are 0.9416 and 0.9469 for sample sizes 500 and 1,000 respectively in Table 1. However, as the prediction horizon increases, for example for lead lags 10 and 20, coverages are below 0.90 for the sample sizes 500 and 1,000 providing narrower intervals than the expected theoretical lengths. This is the only occasion that we can see a large contrast between the two estimation methods with respect to the coverages. When the forecast horizon is greater than 1, coverages for the one-step method are often greater than 0.96.

Two-step method do produce slightly conservative intervals but in general, intervals obtained using the one-step estimation method are wider than those obtained using the two-step estimation method. Only on two occasions the interval width of one-step estimation narrower than that from the two-step estimation method, when computing lead one coverages (see Tables 4 and 5). Also note that the theoretical and bootstrap intervals are narrower when the FIGARCH error structure follows Model 2. The reason could be that the parameter coefficient of lagged variance $(\beta_1 = 0.1)$ is smaller in Model 2 than in Model 1 $(\beta_1 = 0.45)$.

Table 1. Coverage of 95% intervals for $(1-0.9L)(1-L)^{0.49} X_t = \varepsilon_t$, where ε_t follows a FIGARCH(1, *d*, 0) with Model 1 and normal errors

Lead lag	Sample size	Theoretical length	One Step Method		Two Step Method	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	5.9674	0.9570 (0.0011)	6.4933 (0.2417)	0.9416 (0.0013)	6.2149 (0.2135)
	1,000	5.9305	0.9579 (0.0008)	6.3846 (0.2356)	0.9469 (0.0011)	6.2029 (0.2298)
10	500	33.9061	0.9514 (0.0013)	36.3476 (1.2598)	0.8601 (0.0025)	26.1094 (0.9134)
	1,000	33.7709	0.9550 (0.0009)	36.4494 (1.2863)	0.8959 (0.0019)	28.8851 (1.0850)
20	500	48.5217	0.9517 (0.0014)	53.8537 (1.8860)	0.8496 (0.0026)	36.1541 (1.1798)
	1,000	48.2148	0.9567 (0.0011)	54.1806 (1.8696)	0.8920 (0.0022)	41.1049 (1.4742)

Table 2. Coverage of 95% intervals for $(1-0.9L)(1-L)^{0.25} X_t = (1-0.8L)\varepsilon_t$, where ε_t follows a FIGARCH(1, *d*, 0) with Model 1 and normal errors

Lead lag	Sample size	Theoretical length	One Step Method		Two Step Method	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	5.5393	0.9563 (0.0010)	5.9123 (0.1356)	0.9443 (0.0011)	5.6024 (0.1276)
	1,000	5.5527	0.9581 (0.0007)	6.0727 (0.2752)	0.9492 (0.0008)	5.8572 (0.2955)
10	500	6.9764	0.9640 (0.0009)	8.0450 (0.1828)	0.9525 (0.0009)	7.4486 (0.1775)
	1,000	6.9949	0.9634 (0.0007)	8.2634 (0.5122)	0.9533 (0.0008)	7.6226 (0.3657)
20	500	7.4029	0.9661 (0.0009)	8.9589 (0.2145)	0.9531 (0.0011)	8.1382 (0.2118)
	1,000	7.3981	0.9670 (0.0007)	9.1603 (0.5559)	0.9546 (0.0009)	8.5427 (0.6602)

Table 3. Coverage of 95% intervals for $(1-0.8L)(1-L)^{0.25}X_t = (1-0.5L)\varepsilon_t$, where ε_t follows a FIGARCH(1, d, 0) with Model 1 and normal errors

Lead lag	Sample size	Theoretical length	One Step Method		Two Step Method	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	5.6821	0.9562 (0.0009)	6.0394 (0.1555)	0.9443 (0.0010)	5.7384 (0.1582)
	1,000	5.6681	0.9587 (0.0007)	6.0092 (0.1500)	0.9483 (0.0008)	5.7296 (0.1398)
10	500	8.7878	0.9631 (0.0009)	10.1004 (0.2602)	0.9507 (0.0010)	9.3140 (0.2445)
	1,000	8.7825	0.9633 (0.0007)	9.9058 (0.2368)	0.9524 (0.0008)	9.1760 (0.2076)
20	500	9.3226	0.9644 (0.0010)	11.2536 (0.2992)	0.9516 (0.0011)	10.1927 (0.2752)
	1,000	9.2921	0.9665 (0.0008)	11.0093 (0.2702)	0.9541 (0.0009)	9.9654 (0.2193)

Table 4. Coverage of 95% intervals for $(1-0.5L)(1-L)^{0.49}X_t = (1-0.8L)\varepsilon_t$, where ε_t follows a FIGARCH(1, d, 0) with Model 1 and normal errors

Lead lag	Sample size	Theoretical length	One Step Method		Two Step Method	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	2.4817	0.9443 (0.0019)	2.5803 (0.1607)	0.9469 (0.0021)	2.6339 (0.1205)
	1,000	2.4826	0.9491 (0.0014)	2.5786 (0.1667)	0.9506 (0.0019)	2.6028 (0.1154)
10	500	3.1492	0.9570 (0.0008)	3.6073 (0.1282)	0.9550 (0.0008)	3.5707 (0.1100)
	1,000	3.1524	0.9546 (0.0007)	3.4104 (0.1090)	0.9532 (0.0007)	3.4238 (0.1010)
20	500	3.1602	0.9599 (0.0008)	3.8609 (0.1069)	0.9583 (0.0008)	3.7999 (0.0980)
	1,000	3.1728	0.9580 (0.0006)	3.5950 (0.0716)	0.9568 (0.0007)	3.6386 (0.0848)

Table 5. Coverage of 95% intervals for $(1-L)^{0.4}X_t = \varepsilon_t$, where ε_t follows a FIGARCH(1, d, 1) with Model 2 and normal errors

Lead lag	Sample size	Theoretical length	One Step Method		Two Step Method	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	2.4293	0.9486 (0.0016)	2.5302 (0.1215)	0.9499 (0.0018)	2.6169 (0.1150)
	1,000	2.4446	0.9498 (0.0012)	2.5244 (0.1272)	0.9513 (0.0016)	2.6395 (0.1187)
10	500	3.6182	0.9595 (0.0008)	4.3147 (0.1240)	0.9551 (0.0010)	4.1976 (0.1278)
	1,000	3.5978	0.9578 (0.0006)	4.0506 (0.1119)	0.9538 (0.0007)	4.0196 (0.1179)
20	500	3.7269	0.9634 (0.0008)	4.8156 (0.1125)	0.9582 (0.0009)	4.5744 (0.1138)
	1,000	3.7084	0.9618 (0.0006)	4.4876 (0.0819)	0.9575 (0.0007)	4.3764 (0.1013)

Table 6. Coverage of 95% intervals for $(1-0.8L)(1-L)^{0.4}X_t = (1-0.5L)\varepsilon_t$, where ε_t follows a FIGARCH(1, d, 0) with Model 1 and t errors

Lead lag	Sample size	Theoretical length	One Step Method		Two Step Method	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	5.5716	0.9578 (0.0009)	6.1258 (0.2530)	0.9467 (0.0010)	5.7475 (0.2442)
	1,000	5.5764	0.9576 (0.0007)	6.0092 (0.2515)	0.9484 (0.0009)	5.6757 (0.2340)
10	500	10.8064	0.9634 (0.0010)	12.9725 (0.4994)	0.9511 (0.0011)	11.7676 (0.4659)
	1,000	10.8071	0.9643 (0.0008)	12.6592 (0.4937)	0.9527 (0.0010)	11.5684 (0.4507)
20	500	12.127	0.9679 (0.0011)	15.9896 (0.6263)	0.9538 (0.0013)	13.9553 (0.5401)
	1,000	12.1735	0.9694 (0.0009)	15.5293 (0.6006)	0.9552 (0.0012)	13.7483 (0.5394)

Table 7. Coverage of 95% intervals for $(1-0.5L)(1-L)^{0.25}X_t = \varepsilon_t$, where ε_t follows a FIGARCH(1, d, 0) with Model 1 and exponential errors

Lead lag	Sample size	Theoretical length	One Step Method		Two Step Method	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	4.4157	0.9567 (0.0019)	5.1837 (0.3271)	0.9401 (0.0021)	4.6393 (0.2548)
	1,000	4.4303	0.9608 (0.0013)	4.9921 (0.2539)	0.9457 (0.0015)	4.6091 (0.2376)
10	500	6.0561	0.9721 (0.0009)	9.4086 (0.8459)	0.9570 (0.0010)	7.5830 (0.4475)
	1,000	6.0734	0.9728 (0.0007)	8.6214 (0.4092)	0.9590 (0.0008)	7.4746 (0.3547)
20	500	6.3509	0.9660 (0.0009)	10.7496 (1.2349)	0.9515 (0.0009)	8.1160 (0.5002)
	1,000	6.4223	0.9655 (0.0007)	9.4076 (0.4484)	0.9513 (0.0007)	7.9657 (0.4028)

Table 8. Coverage of 95% intervals for $(1-0.5L)(1-L)^{0.49}X_t = (1-0.8L)\varepsilon_t$, where ε_t follows a FIGARCH(1, d, 0) with Model 2 and exponential errors

Lead lag	Sample size	Theoretical length	One Step Method		Two Step Method	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	2.2434	0.9443 (0.0028)	2.3534 (0.2429)	0.9441 (0.0031)	2.2217 (0.2115)
	1,000	2.2346	0.9472 (0.0022)	2.3116 (0.2367)	0.9482 (0.0027)	2.2682 (0.2235)
10	500	2.6754	0.9389 (0.0015)	2.7355 (0.1075)	0.9361 (0.0011)	2.5372 (0.0820)
	1,000	2.6561	0.9377 (0.0010)	2.5170 (0.0793)	0.9368 (0.0009)	2.4835 (0.0810)
20	500	2.9303	0.9330 (0.0015)	3.1060 (0.2822)	0.9291 (0.0011)	2.5508 (0.0511)
	1,000	2.9552	0.9318 (0.0009)	2.6019 (0.0484)	0.9302 (0.0009)	2.5149 (0.0454)

Results for FARIMA process under Model 1 and Model 2 errors generated with exponential white noise inputs are reported in Tables 7 and 8 respectively. Coverages are greater than 0.96 under the one-step estimation method except for lead lag 1 prediction for sample size 500 (see Table 7). On the other hand, the two-step estimation method provides coverages reasonably close to 0.95. However, when a moving average component is added to the FARIMA model, the coverages provided by both methods are well below the 0.95 for forecast horizons above lead one (see Table 8). Further, the bootstrap lengths given under both methods are less than the theorical lengths for lead lag 10 and 20 (see Table 8). It is possible that the AR approximation we employed is inadequate to model FARIMA with a moving average component with roots relatively close to unity and with conditionally heteroscedastic error with long memory. This is a phenomenon that needs further investigation.

As a summary, proposed bootstrap method works fairly well for the FARIMA-FIGARCH model. Both the estimation methods provide good coverages, but we recommend one step estimation method when the estimated AR parameter is close to 1 and fractionally integrated parameter (d_{FAR}) is close to 0.5, without or with weaker moving average parameter in the FARIMA part.

5. APPLICATION TO A REAL DATA SET

Baillie et al. (2002) modeled the monthly Consumer Price Index (CPI) inflation series for 8 different countries and found that it exhibits a long memory behavior in both first and second conditional moments. According to their investigation, this is the only economic variable that exhibit this property. So, they suggested the FARIMA-FIGARCH model to represent the underlying generating process of such series displaying dual phenomenon of long memory in inflation rates and squared inflation rates. Therefore, we used monthly Japanese CPI inflation series to construct the prediction intervals. The monthly CPI data were obtained from Federal Reserve Bank of St Louis https://fred.stlouisfed.org with data ranging from January 1960 to October 2020. We then computed the CPI inflation rates, which is defined as $r_t = 100 \cdot \Delta \log(\mathrm{CPI}_t),\ t = 1, 2, \ldots, 729$, where CPI_t is the monthly CPI. Here we used one-step estimation procedure to construct the prediction intervals. Table 9 reports the calculated coverage probabilities for 1st, 10th and 20th step ahead forecasts. The following figure shows the one-step ahead constructed sieve bootstrap prediction-based intervals for CPI inflation series.

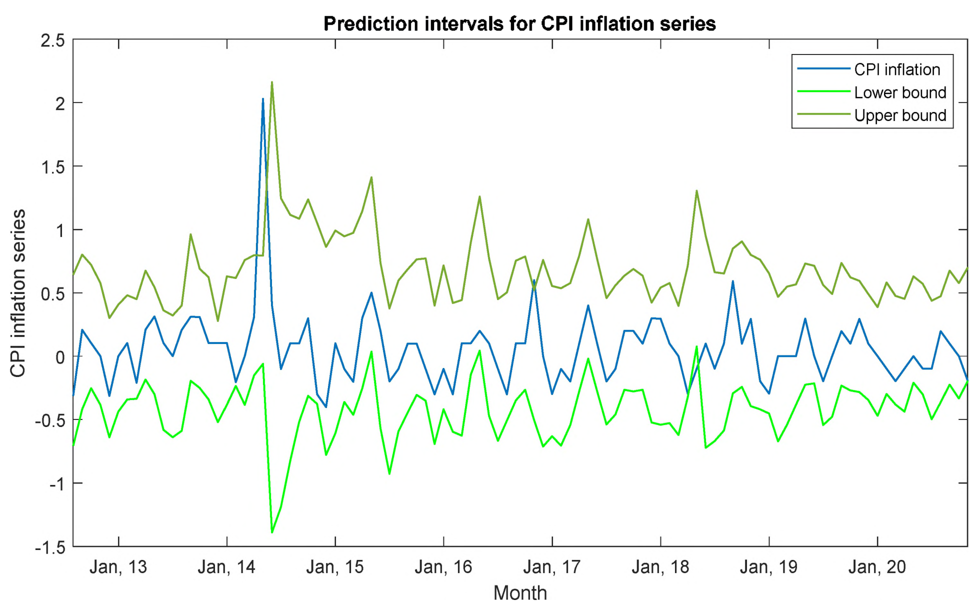

Figure 1. 95% sieve bootstrap-based prediction interval for CPI inflation rates

The Figure 1 reports the 95% upper and lower bounds. There are two future values that lie above the upper bootstrap bound and below the lower bootstrap bound, providing 96% of coverage, which is close to 0.95. Further, 10-step and 20-step intervals provide coverages slightly greater than the nominal value of 0.95. These results are consistent with the simulation results of FARIMA-FIGARCH under normal and *t* distributed white noise process. In general, intervals are narrower in most periods, with the widest interval seen in May 2014 with interval length 3.5511.

Table 9. Estimated coverage probabilities for future CPI inflation rates

Lead lag	Coverage
1	0.9600
10	0.9560
20	0.9753

6. CONCLUSIONS

The importance of modeling the dual presence of long memory in both the first two conditional moments is discussed by Baillie et al. (2002) and Conrad and Karanasos (2002). In this article we proposed a sieve bootstrap-based prediction interval for FARIMA-FIGARCH model. Finite sample performance was investigated using a Monte-Carlo study, which showed that the proposed intervals provide close to nominal coverage when the underlying white noise process that drives the innovations has a normal or a *t* distribution. Results when the underlying white noise process is exponential show good coverage only for lead one predictions. In this study we assumed the order of the FIGARCH component is known and in the simulation study the orders were set at $p = 1$ and $q = 1$ with d lying between 0 and 1. Note that this is not a great limitation because basic lower order GARCH type models provide better fit to empirical data as shown in most of the empirical studies. In addition, practitioners have routinely used GARCH(1, 1) and FIGARCH(1, *d*, 1) or FIGARCH(0, *d*, 1) to model empirically observed volatility. Further, coverages in application to CPI inflation rates confirms that the proposed bootstrap prediction interval appropriate for forecasting data with long memory found in the first and second conditional moments.

REFERENCES

Alonso, A.’. M., Peña, D., & Romo, J. (2002). Forecasting time series with sieve bootstrap. *Journal of Statistical Planning and Inference*, *100*(1), 1–11.

Alonso, A. M., Peña, D., & Romo, J. (2003). On sieve bootstrap prediction intervals. *Statistics & Probability Letters*, *65*(1), 13–20.

Amjad, A., Khan, S. A., Alamgir, Khalil, U., & Khan, D. M. (2017). Bootstrap prediction intervals for time series models with heteroscedastic errors. Pakistan Journal of Statistics, 33(1), 1-13

Baillie, R. T., Chung, C., & Tieslau, M. A. (1996). Analyzing inflation by the fractionally integrated ARFIMA-GARCH model. Journal of Applied Econometrics 11:23-30.

Baillie, R. T., Han, Y. W., & Kwon, T.-G. (2002). Further Long Memory Properties of Inflationary Shocks. *Southern Economic Journal*, *68*(3), 496.

Baillie, R. T., & Chung, S.-K. (2002). Modeling and forecasting from trend-stationary long memory models with applications to climatology. *International Journal of Forecasting*, *18*(2), 215–226.

Beran, J., & Ocker, D. (1999). SEMIFAR forecasts, with applications to foreign exchange rates. *Journal of Statistical Planning and Inference*, *80*(1–2), 137–153.

Bisaglia, L., & Grigoletto, M. (2001). Prediction intervals for farima processes by bootstrap methods. *Journal of Statistical Computation and Simulation*, *68*(2), 185–201.

Brockwell, P. J., & Davis, R. A. (1991). *Time Series: Theory and Methods*. Springer.

Bühlmann, P., & Buhlmann, P. (1997). Sieve Bootstrap for Time Series. *Bernoulli*, *3*(2), 123.

Conrad, C., & Karanasos, M. (2005). On the inflation-uncertainty hypothesis in the USA, Japan and the UK: a dual long memory approach. *Japan and the World Economy*, *17*(3), 327–343.

Crato, N., & Ray, B. K. (1996). Model selection and forecasting for long-range dependent processes. *Journal of Forecasting*, *15*(2), 107–125.

Doukhan, P., Oppenheim, G., & Taqqu, M. S. (2002). *Theory and Applications of Long-Range Dependence* (2003rd ed.). Birkhäuser.

Fox, R., & Taqqu, M. S. (1986). Large-Sample Properties of Parameter Estimates for Strongly Dependent Stationary Gaussian Time Series. *The Annals of Statistics*, *14*(2), 517–532.

Granger, C. W. J., & Joyeux, R. (1980). An introduction to long-memory timeseries models and fractional differencing. *Journal of Time Series Analysis*, *1*(1), 15–29.

Granger, C. W. J. (1980). Long memory relationships and the aggregation of dynamic models. *Journal of Econometrics*, *14*(2), 227–238.

Hosking, J. R. M. (1981). Fractional differencing. *Biometrika*, *68*(1), 165–176.

Mukhopadhyay, P., & Samaranayake, V. A. (2010). Prediction Intervals for Time Series: A Modified Sieve Bootstrap Approach. *Communications in Statistics - Simulation and Computation*, *39*(3), 517–538.

Poskitt, D. S. (2006). Autoregressive approximation in nonstandard situations: the fractionally integrated and non-invertible cases. *Annals of the Institute of Statistical Mathematics*, *59*(4), 697–725.

Ramjee, R., Crato, N., & Ray, B. K. (2002). A note on moving average forecasts of long memory processes with an application to quality control. *International Journal of Forecasting*, *18*(2), 291–297.

Ray, B. K. (1993). Modeling long-memory processes for optimal long-range prediction. *Journal of Time Series Analysis*, *14*(5), 511–525.

Rupasinghe, M., Mukhopadhyay, P., & Samaranayake, V. A. (2013). Obtaining prediction intervals for FARIMA processes using the sieve bootstrap. *Journal of Statistical Computation and Simulation*, *84*(9), 2044–2058.

Rupasinghe, M., & Samaranayake, V. A. (2012). Asymptotic properties of sieve bootstrap prediction intervals for processes. *Statistics & Probability Letters*, *82*(12), 2108–2114.

Thilakaratne, M., Rupasinghe, M., & Samaranayake, V. A. (2014) Sieve Bootstrap-Based Prediction intervals for Autoregressive Processes with GARCH Innovations . *Joint Statistical Meetings. Proceedings*, Boston, Massachusetts, 3284–3296.

IV. BOOTSTRAP PREDICTION INTERVALS FOR HYPERBOLIC GENERALIZED AUTOREGRESSIVE CONDITIONAL HETEROSCEDASTIC (HYGARCH) MODELS

ABSTRACT

It is well known that the volatility of certain financial assets exhibits long range dependence. Fractionally Integrated Generalized Autoregressive Conditional Heteroscedastic (FIGARCH) model was widely used to model such a behavior. However, the second moment of FIGARCH does not exit. Therefore, Hyperbolic Generalized Autoregressive Conditional Heteroscedastic (HYGARCH) model was developed and suits well for handling volatility with long memory dependence. Determining the uncertainty associated in predictions in long memory volatility models is useful for financial market participants. Conventional methods of constructing prediction intervals for such models provide poor coverages. A bootstrapped based method proposed by Pascual, Romo and Ruiz (PRR) was adapted to construct prediction intervals for returns and volatility for HYGARCH model. A Monte-Carlo simulation study was carried out and simulation results shown that the adaption of PRR method provide reasonable coverage probabilities for returns and volatility.

Keywords: Long memory, Prediction intervals, Volatility modeling

1. INTRODUCTION

Forecasting stock returns and volatility are important tasks for those studying financial markets. The Autoregressive Conditional Heteroscedastic (ARCH) and Generalized Autoregressive Conditional Heteroscedastic (GARCH) models introduced by Engel (1982) and Bollerslev (1986), respectively, are widely used to model and forecast financial returns and associated volatilities, which tend to evolve over time. Point estimates are widely used to forecast financial time series and their volatility. In contrast to point estimators, prediction intervals provide extra information about the uncertainty associated with future forecasts. Pascual et al. (2006) developed a bootstrap-based methodology for constructing prediction intervals for returns and volatility under a GARCH(1, 1) formulation. We will refer to this approach as the Pascual-Romo-Ruiz (PRR) method. Chen et al. (2010) proposed a computationally much faster, sieve bootstrap-based, prediction interval obtained by converting the GARCH formulation into an ARMA type model. Trucíos & Hotta (2016) adapted the PRR method and used it to construct prediction intervals for returns and volatilities for EGARCH processes (Nelson, 1991) and for the Glosten-Jagannathan-Runkle GARCH (GJR-GARCH, Glosten et al., 1993) models. They reported that the volatility prediction coverage could be poor in EGARCH and GJR-GARCH cases if an additive outlier is present near the forecasting horizon. The studies reported above only focus on prediction intervals for short memory volatility models. Yet, the phenomenon of long memory is not limited to the mean processes only, in which domain there have been considerable work (for example, Bisaglia & Grigoletto (2001) and Rupasinghe & Samaranayake (2013)).

The authors Ding et al. (1993), Ding and Granger (1996) and Harvey (1993) are the first to address the long-range dependence of squared or absolute returns of financial time series. Baillie et al. (1996, 2002) and Conrad & Karanasos (2005) also discussed useful applications of long memory volatility models. The GARCH formulation cannot model such a behavior as the weights of the autocorrelation function of squared returns, under the GARCH model, decay exponentially implying short-term dependence. Based on these observations, Baillie et al. (1996) introduced a Fractionally Integrated Generalized Autoregressive Conditional Heteroscedastic (FIGARCH) model, which permits the modeling of long memory behavior in squared or absolute returns. The conditional variance of FIGARCH can be written as an infinite lag polynomial of squared returns, and the coefficients of the lag polynomial have a slow hyperbolic decay. A similar behavior is associated with the autocorrelation function of squared returns under the FIGARCH formulation. However, coefficients of the infinite lag polynomial sum to one in such models implying that the unconditional variance of a FIGARCH process is infinite. Thus, unlike the GARCH, FIGARCH is a non-stationary process. Davidson (2004), introduced the hyperbolic GARCH (HYGARCH) model, which can be written as a combination of weighted GARCH and a FIGARCH components. The HYGARCH shares the covariance stationarity property of the GARCH component, while at the same time it contains the hyperbolic decaying impulse response coefficients found in the FIGARCH. Therefore, the HYGARCH formulation can model long run dependence of conditional variance without sacrificing the covariance stationarity property.

Ekanayake and Samaranayake (2020) introduced a method to construct bootstrap-based prediction intervals for returns and volatility for long memory FIGARCH processes.

However, the underlying infinite unconditional variance of the FIGARCH process hinders the development of asymptotic properties of the bootstrap estimates. An alternative HYGARCH model provides a solution because its unconditional variance is finite except in a limiting case. In this paper, we extend PRR algorithm to construct bootstrap prediction interval for returns and volatility for the HYGARCH model. The theoretical derivations of the asymptotic properties of the bootstrap-based prediction intervals for the HYGARCH process will be discussed in a separate paper.

The organization of the sections of this paper is as follows. We introduce the HYGARCH model in Section 2. Section 3 details the residual based bootstrap procedure employed to construct prediction intervals for HYGARCH processes. Section 4 presents the results of a detailed Monte-Carlo simulation study that examines the finite sample behavior of the intervals. The procedure is applied to NASDAQ stock return data with results presented in Section 5. Finally, Section 6 provides some concluding remarks.

2. THE HYPERBOLIC GARCH (HYGARCH) MODEL

The Integrated GARCH (IGARCH, Engle and Bollerlev, 1986) and the FIGARCH, formulations have several drawbacks, such as their infinite unconditional variance. Some of these issues connected with the IGARCH processes were addressed by the Davidson (2004) who developed the HYGARCH model. A time series $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is said to follow a HYGARCH$(p,\ d,\ q)$ process if it satisfies the conditions,

$$\varepsilon_t = z_t\sigma_t,$$

and

$$\sigma_t^2 = \omega + \left\{1 - \beta(L) - \phi(L)\left[1 + \alpha\left((1-L)^d - 1\right)\right]\right\}\varepsilon_t^2 + \beta(L)\sigma_t^2, \tag{1}$$

where, the sequence, $\{z_t\}$ is a white noise process with zero mean and unit variance. Note that L is the lag operator such that the polynomials $\beta(L) = \beta_1 L + \beta_2 L^2 + ... + \beta_p L^p$ and $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - ... - \phi_q L^q$ have no common roots, with $\beta_p \neq 0$ and $\phi_q \neq 0$. It is assumed that $\alpha \geq 0,\ d \geq 0$. The HYGARCH reduces to the FIGARCH or the GARCH when $\alpha = 1$ or $\alpha = 0$, respectively. Note that parameter d is unidentifiable when $\alpha = 0$. Further, when $d = 1$ in (1), depending on the value of α, the model becomes IGARCH or GARCH. In fact, when $\alpha < 1$ it becomes a GARCH model, and if $\alpha = 1$ then it becomes a IGARCH model.

In this study we construct prediction intervals for the HYGARCH$(1, d, 1)$ model by using the PRR bootstrap procedure. The procedure described herein can be directly extended to the general HYGARCH$(p,\ d,\ q)$ model. The formulation of HYGARCH$(1,\ d,\ 1)$ can be expressed as follows:

$$\begin{aligned} \sigma_t^2 &= \omega + \left\{1 - \beta_1 L - (1 - \phi_1 L)\left[1 + \alpha\left((1-L)^d - 1\right)\right]\right\}\varepsilon_t^2 + \beta_1 \sigma_{t-1}^2, \\ \sigma_t^2 &= \omega(1-\beta_1)^{-1} + \left\{1 - (1-\beta_1)^{-1}(1 - \phi_1 L)\left[1 + \alpha\left((1-L)^d - 1\right)\right]\right\}\varepsilon_t^2. \end{aligned} \tag{2}$$

The conditional variance of HYGARCH$(1, d, 1)$ can be written as

$$\sigma_t^2 = \omega\left(1 - \beta(1)\right)^{-1} + \lambda(L)\varepsilon_t^2, \tag{3}$$

where $\lambda(L) = \sum_{k=1}^{\infty} \lambda_k L^k$.

By equating terms with common powers in the equation

$\lambda(L)=\lambda_1 L+\lambda_2 L^2+...+\lambda_k L^k+...=1-(1-\beta_1)^{-1}(1-\phi_1 L)\left[1+\alpha\left((1-L)^d-1\right)\right]$, the coefficients λ_i can be easily obtained as follows:

$$\lambda_1=\phi_1-\beta_1+\alpha d,$$

$$\lambda_2=(d-\beta_1)(\beta_1-\alpha\phi_1)+\alpha d(1-d)/2,$$

$$\lambda_3=\beta_1\left[\alpha d\beta_1-\alpha d\phi_1-\beta_1^2+\beta_1\phi_1+\alpha d(1-d)/2\right]+\alpha d(1-d)/2\left[(2-d)/3-\phi_1\right],$$

$$\vdots$$

$$\lambda_k=\beta_1\lambda_{k-1}+\alpha\left[(k-1-d)/k-\phi_1\right]\delta_{d,k-1},\ k\in\mathbb{N},$$

where $\delta_{d,k}=\delta_{d,k-1}(k-1-d)k^{-1},\ \ k\in\mathbb{N}$ refer to the coefficients in the series expansion of $(1-L)^d$, with $\delta_{d,0}=1$ and $\delta_{d,1}=d$. The conditional variance defined in the equations (2) and (3), in the HYGARCH definition, must be positive. Therefore, non-negativity of λ_i's is required. Similar to the sufficient conditions for non-negativity of conditional variance of FIGARCH$(1,d,1)$ derived by Bollerslev and Mikkelsen (1996), the conditions for the non-negativity of conditional variance in the HYGARCH model are given by,

$$\beta_1-\alpha d\le\phi_1\le(2-d)/3 \text{ and } \alpha d\left[\phi_1-(1-d)/2\right]\le\beta_1\left(\phi_1-\beta_1+\alpha d\right). \tag{4}$$

These conditions are also applied in the studies by Dark (2005, 2010). The necessary and sufficient conditions for the non-negativity of conditional variance for general HYGARCH(1, d, q) were derived by Conrad (2010). Further, Conrad (2010) derived the sufficient conditions for non-negativity of conditional variance for the general HYGARCH(p, d, q), for $p\ge 2$. Please refer Theorem 2 and Theorem 3 in Conrad (2010) for details. These necessary and sufficient conditions are complex and therefore it is difficult to program into software. Thus, we employed sufficient conditions given in (4) in our study.

3. BOOTSTRAP PREDICTION INTERVALS

In this section, we apply the procedure proposed by Pascual et al. (2006) (PRR) to obtain prediction intervals for future returns and future volatilities when the underlying data generating process is HYGARCH.

1. Let $\{\varepsilon_t\}_{t=1}^n$ be a sequence of realizations from a HYGARCH(1, d, 1) process. Then estimate the parameters of the model $\hat{\theta}=\left(\hat{\omega}_1,\ \hat{\phi}_1,\ \hat{d},\ \hat{\beta}_1,\ \hat{\alpha}\right)$ by using the Quasi-Maximum Likelihood Estimation (Q-MLE) method.

2. Compute the residuals $\hat{z}_t=\varepsilon_t/\hat{\sigma}_t$, $t=1,\ldots,n$ where

$$\begin{aligned}\hat{\sigma}_t^2&=\hat{\omega}(1-\hat{\beta}_1)^{-1}+\left\{1-(1-\hat{\beta}_1)^{-1}(1-\hat{\phi}_1L)\left[1+\alpha\left((1-L)^{\hat{d}}-1\right)\right]\right\}\varepsilon_t^2\\&\approx\hat{\omega}(1-\hat{\beta}_1)^{-1}+\hat{\lambda}_1\varepsilon_{t-1}^2+\hat{\lambda}_2\varepsilon_{t-2}^2+\ldots+\hat{\lambda}_k\varepsilon_{t-k}^2\end{aligned},$$

   and setting $\varepsilon_t^2=n^{-1}\sum_{i=1}^n\varepsilon_i^2$, for $t=-k+1,\ldots,-1,\ 0$. Note that k is a suitably chosen truncation lag of the polynomial $\lambda(L)$. In the simulation study we used k=1,000.

3. Compute the centered residuals $\tilde{z}_t=\hat{z}_t-\bar{\hat{z}}_t$, where $\bar{\hat{z}}_t=n^{-1}\sum_{i=1}^n\hat{z}_i$.

4. Denote the empirical distribution function of the centered residuals by $\hat{F}_{\tilde{z}}(x)=n^{-1}\sum_{t=1}^n I_{(-\infty,x]}(\tilde{z}_t)$.

5. Draw a bootstrap sample with replacement from the above distribution and denote it by z_t^*, where $t = -m+1,...,-1,0,1,...,n$. m is chosen as 2,000 in this study.

6. Generate the bootstrapped HYGARCH series ε_t^*, $t = -m+1,...,-1,0,1,...,n$ by first computing a bootstrapped conditional variance series, σ_t^{2*}, using the HYGARCH parameters estimated in Step 1. Then use $\varepsilon_t^* = z_t^* \sigma_t^*$, $t = -m+1,...,-1,0,1,...,n$ to generate ε_t^*. The non-positive lags represent 'burn-in' observations that are dropped to mitigate effects due to the initial conditions.

7. Estimate the HYGARCH parameters $\theta^* = \left(\omega^*,\ \phi_1^*,\ d^*,\ \beta_1^*,\ \alpha^*\right)$ for the bootstrapped series $\{\varepsilon_t^*\}$ using the Q-MLE method.

8. Use the new coefficients $\theta^* = \left(\omega^*,\ \phi_1^*,\ d^*,\ \beta_1^*,\ \alpha^*\right)$ obtained in the previous step and compute the h-step ahead bootstrap forecasts of future returns and volatilities based on the following recursions:

$$
\begin{aligned}
\sigma_{n+h}^{2*} &= \omega^*(1-\beta_1^*)^{-1} + \left\{1-(1-\beta_1^*)^{-1}(1-\phi_1^*)\left[1+\alpha^*\left((1-L)^{d^*}-1\right)\right]\right\}\varepsilon_{n+h}^{2*}, \\
&\approx \omega^*(1-\beta_1^*)^{-1} + \lambda_1^*\varepsilon_{n+h-1}^{2*} + \ldots + \lambda_k^*\varepsilon_{n+h-k}^{2*},
\end{aligned}
$$

$\varepsilon_{n+h}^* = z_{n+h}^* \sigma_{n+h}^*$, for $h > 0$ and $\varepsilon_t^* = \varepsilon_t$ for $t \le n$.

9. Obtain the estimated bootstrap distribution of ε_{n+h}, denoted by $\hat{F}^*_{\varepsilon^*_{n+h}}(.)$, by repeating steps 5-8 B times $(B = 1{,}000)$ in the simulation study. $\hat{F}^*_{\varepsilon^*_{n+h}}(.)$ is the estimate of the $F^*_{\varepsilon^*_{n+h}}(.)$, the bootstrap distribution function of ε^*_{n+h}, which is used to approximate unknown distribution of ε_{n+h} given the observed sample.

10. The $100(1-\alpha)\%$ bootstrap prediction interval for ε_{n+h} is then computed by $[Q_{\varepsilon}^{*}(\alpha/2), Q_{\varepsilon}^{*}(1-\alpha/2)]$, where $Q_{\varepsilon}^{*}(.) = \hat{F}_{\varepsilon_{n+h}^{*}}^{*-1}$ are the percentiles of the estimated bootstrap distribution.

To construct the prediction intervals for volatility we followed the steps 1-8 and changed the rest of steps as below.

9.* Obtain the estimated bootstrap distribution of σ_{n+h}, denoted by $\hat{F}_{\sigma_{n+h}^{*}}^{*}(.)$, by repeating steps 5-8 B times $(B = 1{,}000)$ in the simulation study. $\hat{F}_{\sigma_{n+h}^{*}}^{*}(.)$ is the estimate of the $F_{\sigma_{n+h}^{*}}^{*}(.)$, the bootstrap distribution function of σ_{n+h}^{*}, which is used to approximate unknown distribution of σ_{n+h} given the observed sample.

10.*The $100(1-\alpha)\%$ bootstrap prediction interval for σ_{n+h} is then computed by $[Q_{\sigma}^{*}(\alpha/2), Q_{\sigma}^{*}(1-\alpha/2)]$, where $Q^{*}(.) = \hat{F}_{\sigma_{n+h}^{*}}^{*-1}$ are the percentiles of the estimated bootstrap distribution.

The performance of the PRR method is compared with the conditional bootstrap (CB) method for future observations used by Miguel and Olave (1999) to construct forecast intervals for the ARMA-ARCH model. Prediction intervals involved in CB method does not incorporate the uncertainty of the parameter estimates. The parameter estimation of bootstrapped series (i.e $\{\varepsilon_t^{*}\}$'s) in the PRR method are not carried out in the CB method. Instead, the CB method employs the parameters estimates obtained from the original observed series. Thus, step 7 is discarded, and a slight modification is included to step 8, by using parameter estimates of the observed realizations instead of the parameter estimates of bootstrapped series. Therefore, CB future values for returns ($\varepsilon^*_{n+h},\ h = 1,2,\ldots$) and volatilities ($\sigma^*_{n+h},\ h = 1,2,\ldots$) can be obtained as follows:

$$\sigma^{2*}_{n+h} = \hat{\omega}(1-\hat{\beta}_1)^{-1} + \left\{1-(1-\hat{\beta}_1)^{-1}(1-\hat{\phi}_1)\left[1+\alpha\left((1-L)^{\hat{d}}\right)-1\right]\right\}\varepsilon^{2*}_{n+h}$$
$$\approx \hat{\omega}(1-\hat{\beta}_1)^{-1} + \hat{\lambda}_1\varepsilon^{2*}_{n+h-1} + \ldots + \hat{\lambda}_k\varepsilon^{2*}_{n+h-k},$$

$$\varepsilon^*_{n+h} = z^*_{n+h}\sigma^*_{n+h}, \text{ for } h > 0 \text{ and } \varepsilon^*_t = \varepsilon_t \text{ for } t \le n.$$

By incorporating steps 9-10 and 9*-10* one can estimate CB prediction intervals for returns and volatilities.

4. THE SIMULATION STUDY

A Monte-Carlo simulation study was carried out to investigate the finite sample performance of the HYGARCH model. We simulate the HYGARCH series according to equations (1) and (2). The lengths of the time series considered here are $n = 500$ and $n = 1,500$. The truncation lag of the infinite lag polynomial on conditional variance is set to $k = 1,000$. The parameter combinations used in this study are $\omega = 0.1, \phi \in \{0, 0.2, 0.4\}$, $d \in \{0.40, 0.50, 0.75, 0.90\}$, $\beta \in \{0.10, 0.45, 0.70, 0.75\}$ and $\alpha \in \{0.85, 0.95\}$. Apart from these parameters, two special combinations are employed for the remaining parameters. One combination is $\omega = 0.1,\ \phi = 0.4,\ d = 0.4,\ \beta = 0.1, \alpha = 0.8$; employed by Kwan, W. et al. (2012) in their study. The other one is taken from the results based on the empirical study of exchange rate data in the seminal paper on HYGARCH by Davidson (2004); Inspired by estimates of exchange rates we employed the combination $\omega = 0.1,\ \phi = 0.2,\ d = 0.65,\ \beta = 0.75,$ and $\alpha = 0.98$. Along with these sets of parameter combinations, 3 different error distributions with mean 0 and standard deviation 1 were used to simulate the HYGARCH series. These are standard normal, t with 7 degrees of freedom and centered exponential with mean 1 (i.e. usual symmetric, a leptokurtic, and a skewed distribution). The t-distributed errors were generated as $z_t = 5^{1/2} z_{1,t}(z_{2,t}^2 + z_{3,t}^2 + \ldots + z_{8,t}^2)^{-1/2}$ by drawing independent and identically distributed standard normal $z_{i,t}$'s for $i = 1, 2, \ldots, 8$, as employed in Baillie et al. (1996). The parameters are estimated by maximizing the Gaussian likelihood function (Q-MLE) derive for the HYGARCH model. We employed constraints given in (4) to estimate parameters to ensure non-negativity of conditional variance. $N = 1{,}000$ independent series were generated for each combination of the model, sample size n, nominal coverage level, and error distribution. Steps 1-10 were implemented and for each simulation run and $R = 1{,}000$ future returns, ε_{n+h}, $h \in \mathbb{N}$ and future volatility σ_{n+h}, $h \in \mathbb{N}$ were generated using the original model. The coverages probabilities for future values, ε_{n+h} and future volatilities σ_{n+h} were estimated by calculating the proportion of those values falling into the bootstrap prediction intervals. The estimated coverage probability for future returns ε_{n+h}, at i^{th} simulation run is given by

$$C(i) = R^{-1} \sum_{r=1}^{R} I_A \left[\varepsilon_{n+h}^r(i) \right],$$

where, $A = \left[Q_\varepsilon^*(\alpha/2), Q_\varepsilon^*(1-\alpha/2) \right]$ is the $100(1-\alpha)th$ bootstrapped prediction interval. $I_A(.)$ is the indicator function of the set A. R future values generated at the i^{th} simulation run are denoted by $\varepsilon_{n+h}^r(i)$, $r = 1, 2, \ldots, 1{,}000$. The interval lengths were also investigated, and these bootstrap lengths and theoretical lengths for a i^{th} simulation run are computed using $L_B(i) = Q_\varepsilon^*(1-\alpha/2) - Q_\varepsilon^*(\alpha/2)$ and $L_T(i) = \varepsilon_{n+h}^r(1-\alpha/2) - \varepsilon_{n+h}^r(\alpha/2)$ respectively. $L_T(i)$ is the difference between $100(1-\alpha)th$ and the $100(\alpha/2)th$ percentiles generated from R future values of the underlying model with known order and known coefficients. The statistics: mean coverage, standard error of mean coverage, mean bootstrap length, standard error of mean bootstrap length, and mean theoretical length were computed as follows:

Mean coverage $\bar{C} = N^{-1}\sum_{i=1}^{N} C(i)$,

Standard error of mean coverage $SE_{\bar{C}} = \left\{[N(N-1)]^{-1}\sum_{i=1}^{N}[C(i)-\bar{C}]^2\right\}^{1/2}$,

Mean length (bootstrap) $\bar{L}_B = N^{-1}\sum_{i=1}^{N} L_B(i)$,

Standard error of mean length $SE_{\bar{L}_B} = \{[N(N-1)]^{-1}\sum_{i=1}^{B}[L_B(i)-\bar{L}_B]\}^{1/2}$,

Mean theoretical length $\bar{L}_T = N^{-1}\sum_{i=1}^{N} L_T(i)$.

In similar fashion to what is used for future return intervals, we estimated the coverage probability for intervals constructed for future volatilities. Then the mean coverage, standard error of the mean coverage, mean bootstrap length, standard error of the mean bootstrap length, and mean theoretical length were obtained for future volatility σ_{n+h}, $h = 1, 2, \ldots$ using equations similar to that we discussed above.

The performance of the intervals was investigated for future returns and volatilities by using the coverage probabilities and bootstrap lengths for different types of models with all the different parameter combinations, error distributions, and sample sizes. We explored the behavior of the coverage probabilities, bootstrap interval bounds (upper and lower) and theoretical interval bounds (upper and lower) of the minimum value, percentiles including

25th, 50th, 75th, 95th and maximum value for future returns and volatilities. We only report the simulation results for coverages and interval lengths (with corresponding standard errors) for 95% intervals, due to space limitation. The complete set of results are available upon request. All the simulations and computations have been carried out using MATLAB Mathematical Software.

4.1. PREDICTION INTERVALS FOR RETURNS

Tables 1-7 reported the Monte Carlo simulation results for future returns. These prediction results obtained for lead lag 1, lead lag 10 and lead lag 20, approximately corresponds to predictions of one day, two weeks and one month respectively under the scenario where each observation represents a trading day. Tables 1-7 reports the performance of HYGARCH$(1, d, 1)$ and HYGARCH$(1, d, 0)$ with normally distributed, $t-$ distributed, and exponentially distributed errors. If the coverages reported in Tables 1-3 are rounded to two decimal places, then one would achieve the nominal coverage of 0.95 in 11 and 9 cases out of 18 under PRR and CB methods, respectively. Apart from that, we cannot observe a significant difference between these two methods. Therefore, the incorporation of the variability in parameter estimation does not provide an improvement to the performance or the predictions intervals for returns under the HYGARCH model with normally distributed errors. Bootstrap intervals under both methods provide coverages close to nominal value 0.95 regardless of the parameter combinations used under $t-$ distributed errors as well. The performance under $t-$ distributed errors are similar to normally distributed errors, and they are in concordance with the results reported by Pascual et al. (2006) for GARCH prediction intervals. The interval lengths were also investigated, and the computed PRR and CB lengths are closer to each other and as well as to the theoretical lengths. Therefore, the performance of intervals computed for future returns does not depend on the uncertainty of the parameter estimation in symmetric distributions.

The performance of the intervals examined under exponentially innovations are reported in Tables 6-7 for both the PRR and CB methods. The lead lag one prediction intervals using the PRR method is clearly better than those constructed using the CB method. However, when the prediction horizon increases, the coverage differences get minimal. The bootstrap interval lengths reported for skewed exponential errors are slightly narrower than the theoretical intervals, in many cases, in contrast to the case with symmetric error distributions.

Table 1. Coverage of 95% intervals for returns of HYGARCH (1, d, 1) with parameters $\omega = 0.1,\ \phi = 0.4, d = 0.4,\ \beta = 0.1, \alpha = 0.80$, and normally distributed errors

Lead lag	Sample size	Theoretical length	PRR		CB	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	2.9176	0.9452 (0.0009)	2.8988 (0.0920)	0.9437 (0.0009)	2.9013 (0.0952)
	1,500	2.9139	0.9472 (0.0007)	2.9077 (0.0917)	0.9469 (0.0007)	2.9095 (0.0926)
10	500	3.1698	0.9438 (0.0009)	3.1465 (0.0457)	0.9435 (0.0009)	3.1549 (0.0489)
	1,500	3.1745	0.9478 (0.0007)	3.1766 (0.0395)	0.9478 (0.0007)	3.1830 (0.0409)
20	500	3.1836	0.9428 (0.0010)	3.1428 (0.0369)	0.9424 (0.0009)	3.1388 (0.0398)
	1,500	3.1796	0.9475 (0.0007)	3.1739 (0.0306)	0.9474 (0.0007)	3.1788 (0.0316)

Table 2. Coverage of 95% intervals for returns of HYGARCH (1, *d*, 0) with parameters $\omega = 0.1$, $\phi = 0.2$, $d = 0.65$, $\beta = 0.75, \alpha = 0.98$, and normally distributed errors

Lead lag	Sample size	Theoretical length	PRR		CB	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	8.2371	0.9443 (0.0009)	8.1720 (0.1452)	0.9438 (0.0009)	8.1686 (0.1467)
	1,500	8.2185	0.9481 (0.0006)	8.2131 (0.1401)	0.9479 (0.0006)	8.2207 (0.1428)
10	500	8.4222	0.9423 (0.001)	8.3183 (0.1424)	0.9430 (0.0009)	8.3442 (0.1440)
	1,500	8.4299	0.9469 (0.0007)	8.3909 (0.1317)	0.9463 (0.0006)	8.3935 (0.1363)
20	500	8.5479	0.9381 (0.0012)	8.3646 (0.1394)	0.9397 (0.0010)	8.4099 (0.1414)
	1,500	8.5434	0.9461 (0.0007)	8.5058 (0.1263)	0.9459 (0.0007)	8.5152 (0.1308)

Table 3. Coverage of 95% intervals for returns of HYGARCH (1, *d*, 0) with parameters $\omega = 0.1$, $\phi = 0$, $d = 0.9$, $\beta = 0.7, \alpha = 0.95$, and normally distributed errors

Lead lag	Sample size	Theoretical length	PRR		CB	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	5.0841	0.9452 (0.0009)	5.0524 (0.1030)	0.9446 (0.0009)	5.0639 (0.1071)
	1,500	5.0813	0.9477 (0.0006)	5.0756 (0.1011)	0.9472 (0.0006)	5.0760 (0.1027)
10	500	5.3677	0.9419 (0.0011)	5.2749 (0.0827)	0.9424 (0.0011)	5.3237 (0.0914)
	1,500	5.3998	0.9458 (0.0007)	5.3378 (0.0754)	0.9462 (0.0007)	5.3605 (0.0791)
20	500	5.4846	0.9386 (0.0011)	5.3383 (0.0730)	0.9397 (0.0011)	5.4089 (0.0807)
	1,500	5.4916	0.9454 (0.0007)	5.4459 (0.0613)	0.9459 (0.0007)	5.4782 (0.0652)

Table 4. Coverage of 95% intervals for returns of HYGARCH (1, *d*, 1) with parameters $\omega = 0.1,\ \phi = 0.2, d = 0.5,\ \beta = 0.45, \alpha = 0.85$, and *t* distributed errors

Lead lag	Sample size	Theoretical length	PRR		CB	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	3.3648	0.9459 (0.0009)	3.3678 (0.0555)	0.9448 (0.0010)	3.3636 (0.0568)
	1,500	3.3815	0.9480 (0.0006)	3.3739 (0.052)	0.9478 (0.0006)	3.3767 (0.0528)
10	500	3.4523	0.9441 (0.0009)	3.4405 (0.0394)	0.9435 (0.0010)	3.4445 (0.0422)
	1,500	3.4514	0.9480 (0.0006)	3.4616 (0.0348)	0.9480 (0.0006)	3.4652 (0.0354)
20	500	3.4623	0.9433 (0.0009)	3.4414 (0.0349)	0.9431 (0.0010)	3.4573 (0.0385)
	1,500	3.4695	0.9468 (0.0007)	3.4572 (0.0268)	0.9466 (0.0008)	3.4622 (0.0278)

Table 5. Coverage of 95% intervals for returns of HYGARCH (1, *d*, 0) with parameters $\omega = 0.1,\ \phi = 0, d = 0.75,\ \beta = 0.7, \alpha = 0.95$, and *t* distributed errors

Lead lag	Sample size	Theoretical length	PRR		CB	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	5.2819	0.9449 (0.0010)	5.2588 (0.0995)	0.9451 (0.0009)	5.2850 (0.1037)
	1,500	5.2618	0.9485 (0.0006)	5.2688 (0.0950)	0.9485 (0.0006)	5.2866 (0.0983)
10	500	5.4217	0.9417 (0.0010)	5.3108 (0.0889)	0.9426 (0.0010)	5.3510 (0.0929)
	1,500	5.4117	0.9466 (0.0006)	5.3818 (0.0843)	0.9470 (0.0006)	5.4092 (0.0872)
20	500	5.4751	0.9397 (0.0011)	5.3804 (0.0876)	0.9407 (0.0011)	5.4299 (0.0913)
	1,500	5.4872	0.9457 (0.0007)	5.4495 (0.0738)	0.9465 (0.0007)	5.5014 (0.0789)

Table 6. Coverage of 95% intervals for returns of HYGARCH (1, d, 1) with parameters $\omega = 0.1$, $\phi = 0.4$, $d = 0.4$, $\beta = 0.1, \alpha = 0.95$, and exponentially distributed errors

Lead lag	Sample size	Theoretical length	PRR		CB	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	3.2164	0.9233 (0.0019)	3.1800 (0.2250)	0.9059 (0.0023)	3.1422 (0.2360)
	1,500	3.2329	0.9174 (0.0014)	3.1643 (0.2211)	0.9082 (0.0016)	3.1529 (0.2249)
10	500	3.8428	0.9219 (0.0014)	3.3046 (0.1024)	0.9225 (0.0014)	3.3284 (0.1033)
	1,500	3.8079	0.9259 (0.0011)	3.2941 (0.0924)	0.9273 (0.0012)	3.3559 (0.1014)
20	500	4.0545	0.9124 (0.0016)	3.2721 (0.0781)	0.9135 (0.0017)	3.2943 (0.0774)
	1,500	4.0394	0.9195 (0.0013)	3.2965 (0.0663)	0.9210 (0.0014)	3.3547 (0.0721)

Table 7. Coverage of 95% intervals for returns of HYGARCH (1, d, 1) with parameters $\omega = 0.1$, $\phi = 0.4$, $d = 0.75$, $\beta = 0.45, \alpha = 0.85$, and exponentially distributed errors

Lead lag	Sample size	Theoretical length	PRR		CB	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	3.1117	0.9223 (0.0018)	3.0390 (0.1638)	0.9077 (0.0023)	3.0006 (0.1644)
	1,500	3.1166	0.9161 (0.0014)	3.0613 (0.1789)	0.9073 (0.0017)	3.0480 (0.1806)
10	500	3.6759	0.9212 (0.0014)	3.2675 (0.0868)	0.9230 (0.0015)	3.3485 (0.0972)
	1,500	3.6687	0.9276 (0.0010)	3.2918 (0.0846)	0.9291 (0.0010)	3.3407 (0.0883)
20	500	3.8649	0.9144 (0.0015)	3.2132 (0.0579)	0.9173 (0.0016)	3.3157 (0.0699)
	1,500	3.8754	0.9194 (0.0011)	3.2535 (0.0531)	0.9212 (0.0012)	3.3156 (0.0601)

4.2. PREDICTION INTERVALS FOR FUTURE VOLATILITIES

Tables 8-10 reports the performance of 95% PRR and CB prediction intervals for future volatilities when the series are generated with normal, student t, and exponential distributions respectively. Here we report the coverages and lengths for lead lag 1, 2, 10 and 20. One step ahead value in the volatility of GARCH type models (including HYGARCH) is completely determined by past observations. Therefore, only uncertainty associated with one-step ahead prediction when estimating an empirical time series using such models is due to parameter estimation. Consequently, theoretical volatility, σ_{n+1}^2 is same for $R = 1{,}000$ iterations for i^{th} simulation run and hence, the theoretical length is zero. Similarly, one-step ahead interval length under the CB bootstrap procedure is zero because the estimated parameters of each bootstrap run is fixed due to use of the original parameters estimates in each simulation run. On the other hand, PRR procedure provides different estimates in each bootstrap iteration, and hence one-step ahead predictions intervals under the PRR methods have non-zero lengths, as seen Tables 8-10.

We report results for HYGARCH parameter combinations and distributions as follows: $\omega = 0.1,\ \phi = 0, d = 0.75,\ \beta = 0.45, \alpha = 0.85$ with normally distributed errors and $\omega = 0.1,\ \phi = 0.40, d = 0.40,\ \beta = 0.10, \alpha = 0.95$ with t and exponentially distributed errors. The performance of prediction intervals under normal and t error (symmetric) distributions yield results similar to each other (based on the results not reported here). Mean coverages for one step ahead prediction intervals increase as sample size increases under both t and exponential error distributions and provide coverages close to nominal value 0.95 for sample size 1,500. However, the mean coverage for the parameter combination $\omega = 0.1,\ \phi = 0, d = 0.75,\ \beta = 0.45,\ \ \alpha = 0.85$, with normally distributed errors is 0.9220 when the sample size is 1,500. When forecasting for two steps into the future, the mean coverages come close to 0.95 in all models considered, all sample sizes, regardless of the error distribution for the PRR method. The mean coverage, however, is well under the nominal value 0.95 for the CB method. When predicting more than two steps, mean coverages improves as sample sizes increases for both methods, as expected. However, the intervals for volatility constructed under the CB method underperform (i.e. wider) for all future horizons $(h = 2,\ 10 \text{ and } 20)$. Therefore, the PRR method is recommended over the CB method for volatility prediction.

Table 8. Coverage of 95% intervals for volatilities of HYGARCH(1, d, 1) with parameters $\omega = 0.1,\ \phi = 0, d = 0.75,\ \beta = 0.45, \alpha = 0.85$, and normally distributed errors

Lead lag	Sample size	Theoretical length	PRR		CB	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	-	0.9020 (0.0133)	0.1857 (0.0054)	-	-
	1,500	-	0.9220 (0.0120)	0.1084 (0.0032)	-	-
2	500	0.3292	0.9533 (0.0041)	0.4420 (0.0109)	0.7754 (0.0109)	0.3445 (0.0089)
	1,500	0.3296	0.9560 (0.0036)	0.3895 (0.0081)	0.8190 (0.0090)	0.3413 (0.0071)
10	500	0.7934	0.9179 (0.0035)	0.7875 (0.0173)	0.8825 (0.0046)	0.7486 (0.0175)
	1,500	0.7952	0.9372 (0.0018)	0.7938 (0.0138)	0.9251 (0.0022)	0.7829 (0.0135)
20	500	0.8566	0.9153 (0.0036)	0.8495 (0.0173)	0.8817 (0.0047)	0.8109 (0.0175)
	1,500	0.8607	0.9373 (0.0018)	0.8593 (0.0112)	0.9267 (0.0021)	0.8488 (0.0110)

Table 9. Coverage of 95% intervals for volatilities of HYGARCH(1, d, 1) with parameters $\omega = 0.1$, $\phi = 0.40$, $d = 0.40$, $\beta = 0.10, \alpha = 0.95$, and t distributed errors

Lead lag	Sample size	Theoretical length	PRR		CB	
			Mean coverage (SE)	Mean length (SE)	Mean coverage (SE)	Mean length (SE)
1	500	-	0.9180 (0.0123)	0.3356 (0.0143)	-	-
	1,500	-	0.9420 (0.0105)	0.1974 (0.0089)	-	-
2	500	1.3657	0.9421 (0.0038)	1.4708 (0.0842)	0.8350 (0.0077)	1.3498 (0.0822)
	1,500	1.3683	0.9450 (0.0035)	1.4325 (0.0817)	0.8520 (0.0064)	1.3762 (0.0808)
10	500	1.8232	0.9371 (0.0034)	1.9229 (0.0721)	0.8763 (0.0057)	1.8412 (0.0754)
	1,500	1.8327	0.9438 (0.0027)	1.8725 (0.0627)	0.9069 (0.0039)	1.8621 (0.0682)
20	500	1.9176	0.9326 (0.0034)	1.9747 (0.0632)	0.8774 (0.0053)	1.8954 (0.0634)
	1,500	1.9033	0.9400 (0.0027)	1.9435 (0.0591)	0.9097 (0.0035)	1.9355 (0.0621)

Furthermore, the interval widths were also investigated for $h = 2$, 10 and 20. Interval width under CB method always less than the interval with under PRR method. The PRR interval lengths under normal and t distributed errors are always close to the theoretical lengths. However, the computed PRR lengths are narrower the theoretical lengths for lead lags 2, 10 and 20 as reported in Table 10.

Table 10. Coverage of 95% intervals for volatilities of HYGARCH(1, d, 1) with parameters $\omega = 0.1$, $\phi = 0.40$, $d = 0.40$, $\beta = 0.10, \alpha = 0.95$, and exponentially distributed errors

Lead lag	Sample size	Theoretical length	PRR Mean coverage (SE)	PRR Mean length (SE)	CB Mean coverage (SE)	CB Mean length (SE)
1	500	-	0.9240 (0.0119)	0.4062 (0.0376)	-	-
	1,500	-	0.9300 (0.0114)	0.2472 (0.0196)	-	-
2	500	1.3467	0.9416 (0.0040)	1.4675 (0.1270)	0.8059 (0.0093)	1.3941 (0.1409)
	1,500	1.3601	0.9435 (0.0036)	1.4586 (0.1287)	0.8357 (0.0075)	1.4164 (0.1301)
10	500	1.9106	0.9253 (0.0040)	1.7023 (0.0758)	0.8272 (0.0079)	1.6316 (0.0803)
	1,500	1.8889	0.9292 (0.0036)	1.6469 (0.0650)	0.8621 (0.0058)	1.6412 (0.0665)
20	500	2.2669	0.9096 (0.0043)	1.7175 (0.0656)	0.8131 (0.0077)	1.6304 (0.0598)
	1,500	2.2131	0.9131 (0.0038)	1.7111 (0.0583)	0.8522 (0.0056)	1.7179 (0.0600)

5. APPLICATION TO A REAL DATA SET

The proposed bootstrap prediction interval method was applied for daily NASDAQ stock data collected from the website https://finance.yahoo.com through time period: 4th of January 2010 to 27th of October 2020. Daily stock returns for closing prices were calculated by using $r_t = 100.\log(s_t / s_{t-1})$ for $t = 2,3,...,2751$, where, s_t denotes the observed daily closing price at day t. One-step-ahead prediction intervals was calculated for 250 data points starting from 20th of December 2019 through 27th of October 2020. Following figure displays the 95% one-step-ahead bootstrap prediction interval for NASDAQ stock returns.

The return data exhibit low volatility for around first 3 months and it becomes highly volatile during the period from the end of February until end of May of 2020. The interval captured this highly volatile period by providing a wider interval during this highly volatile period and narrower interval for the more tranquil period after the volatility subsides. The coverage probability calculated for one-step ahead prediction is 0.932.

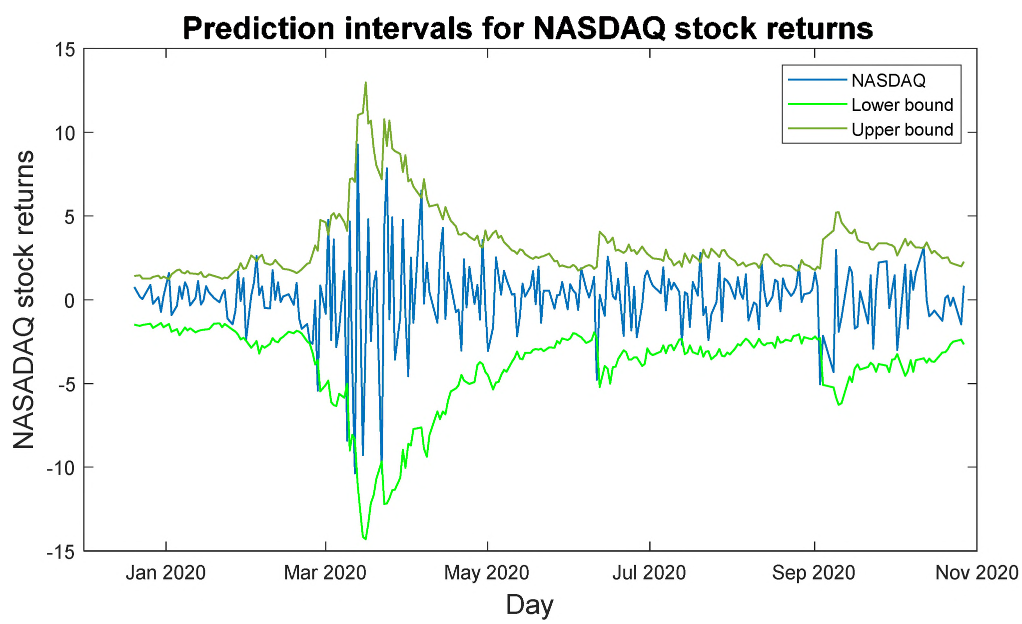

Figure 1. One step ahead prediction interval for NASADAQ stock returns

Table 11. Estimated coverage probabilities for future returns

Lead lag	Coverage
1	0.9320
10	0.8963
20	0.8788

Note that for longer forecast horizons, the coverage probabilities drop below 0.90. It is possible that for this time series, a higher order HYGARCH process may be warranted.

6. CONCLUSIONS

In this paper, we employed Monte-Carlo simulation to study the finite sample behavior of bootstrap prediction intervals for returns and volatilities under the $\text{HGARCH}(1,\ d,\ 1)$ model using the PRR algorithm (Pascual et al. 2006) and compared its performance with the conditional bootstrap (CB) method. The results show that the incorporation of the variability of the parameter estimates in the process of building the prediction intervals for returns, does not make any difference under symmetric error distributions. However, when constructing prediction intervals for future volatility it is necessary to incorporate the uncertainty of the parameter estimates to get coverage probabilities close to the nominal values. Under the skewed error distributions, we get poor coverage probabilities for returns under both methods.

Here we assumed the order of HYGARCH process is known. This is not a great limitation as almost all the studies involve with HYGARCH models are restricted to $\text{HGARCH}(1,\ d,\ 1)$. However, the extension of $\text{HGARCH}(1,\ d,\ 1)$ to $\text{HYGARCH}(p,\ d,\ q)$ is possible under the PRR method. Further extension is also possible, by adapting sieve bootstrap technique to construct prediction intervals for ARMA-HYGARCH and FARIMA-HYGARCH models.

REFERENCES

Baillie, R. T., Bollerslev, T., & Mikkelsen, H. O. (1996). Fractionally integrated generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, *74*(1), 3–30.

Baillie, R. T., Han, Y. W., & Kwon, T.-G. (2002). Further Long Memory Properties of Inflationary Shocks. *Southern Economic Journal*, *68*(3), 496.

Bisaglia, L., & Grigoletto, M. (2001). Prediction intervals for FARIMA processes by bootstrap methods. *Journal of Statistical Computation and Simulation*, *68*(2), 185–201.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, *31*(3), 307–327.

Bollerslev, T., & Ole Mikkelsen, H. (1996). Modeling and pricing long memory in stock market volatility. *Journal of Econometrics*, *73*(1), 151–184.

Chen, B., Gel, Y. R., Balakrishna, N., & Abraham, B. (2010). Computationally efficient bootstrap prediction intervals for returns and volatilities in ARCH and GARCH processes. *Journal of Forecasting*, *30*(1), 51–71.

Conrad, C., & Karanasos, M. (2005). On the inflation-uncertainty hypothesis in the USA, Japan, and the UK: a dual long memory approach. *Japan and the World Economy*, *17*(3), 327–343.

Conrad, C. (2010). Non-negativity conditions for the hyperbolic GARCH model. *Journal of Econometrics*, *157*(2), 441–457.

Dark, J. G. (2005). Modelling the conditional density using a hyperbolic asymmetric power ARCH model. *Mimeo, Monash University.*

Dark, J. G. (2010). Estimation of Time Varying Skewness and Kurtosis with an Application to Value at Risk. *Studies in Nonlinear Dynamics & Econometrics*, *14*(2).

Davidson, J. (2004). Moment and Memory Properties of Linear Conditional Heteroscedasticity Models, and a New Model. *Journal of Business & Economic Statistics*, *22*(1), 16–29.

Ding, Z., Granger, C. W. J., & Engle, R. F. (1993). A long memory property of stock market returns and a new model. *Journal of Empirical Finance*, *1*(1), 83–106.

Ding, Z., & Granger, C. W. J. (1996). Modeling volatility persistence of speculative returns: A new approach. *Journal of Econometrics*, *73*(1), 185–215.

Ekanayake, R., & Samaranayake, V. A. (2020). Bootstrap prediction intervals for Fractionally Integrated Generalized Autoregressive Conditionally Heteroscedastic Models. *Joint Statistical Meetings. Proceedings: 1312-1325.*

Engle, R. F., & Bollerslev, T. (1986). Modelling the persistence of conditional variances. *Econometric Reviews*, *5*(1), 1–50.

Engle, R. F., & Bollerslev, T. (1986). Modelling the persistence of conditional variances. Econometric Reviews, 5(1), 1–50.

GLOSTEN, L., JAGANNATHAN, R., & RUNKLE, D. (1993). On the Relation between the Expected Value and the Volatility of the Nominal Excess Return on Stocks. *The Journal of Finance*, *48*(5), 1779–1801.

Harvey, A.C., (1993), Long memory in stochastic volatility, Manuscript (London School of Economics, London).

Kwan, W., Li, W. K., & Li, G. (2012). On the estimation and diagnostic checking of the ARFIMA–HYGARCH model. *Computational Statistics & Data Analysis*, *56*(11), 3632–3644.

Miguel, J. A., & Olave, P. (1999). Bootstrapping forecast intervals in ARCH models. *Test*, *8*(2), 345–364.

Nelson, D. B. (1991). Conditional Heteroscedasticity in Asset Returns: A New Approach. *Econometrica*, *59*(2), 347.

Pascual, L., Romo, J., & Ruiz, E. (2006). Bootstrap prediction for returns and volatilities in GARCH models. *Computational Statistics & Data Analysis*, *50*(9), 2293–2312.

Rupasinghe, M., Mukhopadhyay, P., & Samaranayake, V. A. (2013). Obtaining prediction intervals for FARIMA processes using the sieve bootstrap. *Journal of Statistical Computation and Simulation*, *84*(9), 2044–2058.

Trucíos, C., & Hotta, L. K. (2016). Bootstrap prediction in univariate volatility models with leverage effect. *Mathematics and Computers in Simulation*, *120*, 91–103.

SECTION

2. CONCLUSION

In this study, we employed bootstrap-based prediction intervals for long memory volatility models and time series with long memory and heteroscedastic errors. Mainly, we utilized two bootstrap techniques where first one is PRR, which is introduced by Pascual et al. (2004) and the second one is sieve bootstrap method. We carried out Monte-Carlo simulation study for check the performance of the finite sample behavior.

PRR method and CB methods are applied to construct prediction intervals for returns and volatilities in long memory volatility models, FIGARCH and HYGARCH models as in Papers I and IV. Although both models are used to model long range dependence in conditional volatility, FIGARCH has infinite unconditional variance while unconditional variance in HYGARCH is finite. Bootstrap intervals constructed using PRR method include the uncertainty due to parameter estimates while intervals produced by CB method does not incorporate the uncertainty due to parameter estimates. Monte Carlo simulation shows that both methods produce good coverages for returns under symmetric error distributions, Gaussian, and *t*. However, when constructing prediction intervals for volatility PRR method dominates CB method in both models. Under skewed error distribution performance of prediction intervals for both returns and volatilities are not as good as symmetric error distributions.

Papers II and III investigated the finite sample performance of the prediction intervals of ARMA-FIGARCH and FARIMA-FIGARCH models. We incorporate the sieve bootstrap resampling technique to construct the prediction intervals. Two resampling techniques under sieve bootstrap method are used. Both methods provided better coverages close to the nominal value however, one step estimation method (Resampling done using the residuals of AR-FIGARCH) performs well under roots of the AR polynomial closes to one or /and fractional integration parameter d closes to 0.5.

APPENDIX

MATLAB ALGORITHM FOR FIGARCH PREDICTION INTERVALS

```
%% Initialization of parameters

parameters = [0.1 0.2 0.5 0.45]; % FIGARCH Parameters
w = parameters(1); phi = parameters(2); d = parameters(3);
beta = parameters(4);
N=500; n = 1500; k=1000; discard=6000; h=20; R=1000;
B=1000;
Para = zeros(N,4); BootPara = zeros(N,4);
%% Returns(defining matrices to store outputs of returns)
Blength     = zeros(N,h); Bcount     = zeros(N,h); Tlength
= zeros(N,h);  Bcount0_025 =  zeros(N,h); Bcount0_975 =
zeros(N,h); Bquantile  = zeros(N,2*h);
Blength99   = zeros(N,h); Bcount99   = zeros(N,h);
Tlength99 = zeros(N,h);  Bcount0_005 =  zeros(N,h);
Bcount0_995 =  zeros(N,h); Bquantile99= zeros(N,2*h);

Blengthp25  = zeros(N,h); Bcountp25  = zeros(N,h);
Tlengthp25  = zeros(N,h);
Blengthp50  = zeros(N,h); Bcountp50  = zeros(N,h);
Tlengthp50  = zeros(N,h);
Blengthp75  = zeros(N,h); Bcountp75  = zeros(N,h);
Tlengthp75  = zeros(N,h);
Blengthpmax = zeros(N,h); Bcountpmax = zeros(N,h);
Tlengthpmax = zeros(N,h);
Blengthpmin = zeros(N,h); Bcountpmin = zeros(N,h);
Tlengthpmin = zeros(N,h);

%% Volatilities(defining matrices to store outputs of
volatilities)

Hlength     = zeros(N,h); Hcount     = zeros(N,h); Vlength
= zeros(N,h);  Hcount0_025 =  zeros(N,h); Hcount0_975 =
zeros(N,h);  Hquantile  = zeros(N,2*h);
Hlength99   = zeros(N,h); Hcount99   = zeros(N,h);
Vlength99  = zeros(N,h);  Hcount0_005 =  zeros(N,h);
Hcount0_995 =  zeros(N,h);  Hquantile99= zeros(N,2*h);

%% constraints for the fmincon optimazation
```

```
init_para = [0.1 0.2 0.50 0.45]; % intialization of
parameters
A         = [0 -1 -1 1;0 2 1 0]; % This is for the FIGARCH
(1,d,1)
b         = [0 1];
lb        = [0.0001,0,0.0001,0.0001];
ub        = [inf,0.9999,0.9999,inf];
opts      = optimoptions('fmincon','Display','off');

%% defining the lambda's (coeficients of infinite ARCH
process)

lambda    = zeros(k,1);  delta    = zeros(k,1);
lambda(1) = phi-beta+d;  delta(1)= d;
tic
for l3=2:k
    delta(l3)  = (l3-1-d)/l3*delta(l3-1);
    lambda(l3) = beta*lambda(l3-1) + delta(l3)-
phi*delta(l3-1);
end

%%

parfor i=1:N
    n=1500; discard=6000;k=1000;
    %% Simulating the data
    rng(i);
    zt        = randn(n+k+discard,1);
    [ee1, ht] = figarchsim(parameters,n,k,discard,zt) ;

    %% Generating future values

    ee1_FV = zeros(R,n+h);
    hh_FV  = zeros(R,n+h);

    et1_fv       = zeros(n+h,1);
    et1_fv(1:n)  = ee1;
    et2_fv       = zeros(n+h,1);
    et2_fv(1:n)  = ee1.^2;
    hh_fv        = zeros(n+h,1);
    hh_fv(1:n)   = ht;

    rng shuffle;
    for l2=1:R
```

```
        zt_fv    = randn(h,1);

        for l=n+1:n+h

            hh_fv(l)  = w/(1-beta) + lambda'*et2_fv(l-1:-
1:l-k);
            et1_fv(l) = sqrt(hh_fv(l))*zt_fv(l-n);
            et2_fv(l) = (et1_fv(l))^2;
        end

        ee1_FV(l2,:) = et1_fv;
        hh_FV(l2,:)  = hh_fv;

    end

    fv   = ee1_FV(:,n+1:n+h); % Generated theoretical
future values
    fv_h = hh_FV(:,n+1:n+h); % Generated theoretical future
volatilities
    %% Percentiles of Return Theoritical Intervals
    etq = quantile(fv,[0.025 0.975]);  tql = etq(1,:); tqu
= etq(2,:);
    tlength = tqu-tql;                 Tlength(i,:) =
tlength;

    etq = quantile(fv,[0.005 0.995]);  tql = etq(1,:); tqu
= etq(2,:);
    tlength = tqu-tql;                 Tlength99(i,:) =
tlength;

    etq = quantile(fv,0);    Tlengthpmin(i,:) = etq;
    etq = quantile(fv,0.25); Tlengthp25(i,:)  = etq;
    etq = quantile(fv,0.50); Tlengthp50(i,:)  = etq;
    etq = quantile(fv,0.75); Tlengthp75(i,:)  = etq;
    etq = quantile(fv,1);    Tlengthpmax(i,:) = etq;

    %% Percentiles of Volatility Theoritical Intervals
    etq = quantile(fv_h,[0.025 0.975]);  tql = etq(1,:);
tqu = etq(2,:);
    vlength = tqu-tql;                 Vlength(i,:) =
vlength;

    etq = quantile(fv_h,[0.005 0.995]);  tql = etq(1,:);
tqu = etq(2,:);
    vlength = tqu-tql;                 Vlength99(i,:) =
vlength;
```

```
    %% Parameter estimation of FIGARCH (MLE approach)

    [para_hat, lgl] =
fmincon(@(x)l_figarch(x,ee1,k),init_para,A,b,[],[],lb,ub,[]
,opts);
    ht_hat           = cond_var(para_hat,ee1,k); % estimated
condtional variance (this function is similar to the
l_figarch)

    w_hat        = para_hat(1)*(1-para_hat(4)); % since
inthe l_garch function we tend to estimate w/(1-beta) not
w. therefore we did the correction below.
    para_hat(1)  = w_hat;
    Para(i,:)    = para_hat;

    e_hat1 = ee1./sqrt(ht_hat);
    e_hat  = e_hat1 - mean(e_hat1); % deameaned errors zt's


    %% Starting the bootstrap

    eB = zeros(B,h);
    hB = zeros(B,h);
    BPara =zeros(B,4);

    for j=1 : B
        discard = 2000; k=1000;
        v_star                  =
datasample(e_hat,n+k+discard);
        [e_star, ht_star]       =
figarchsim(para_hat,n,k,discard,v_star) ; % generating
bootstrap samples
        [para_star, lgl_star] =
fmincon(@(x)l_figarch(x,e_star,k),init_para,A,b,[],[],lb,ub
,[],opts);

        w_star    = para_star(1)*(1-para_star(4));
        phi_star  = para_star(2);
        d_star    = para_star(3);
        beta_star = para_star(4);

        BPara(j,:)     = [w_star phi_star d_star beta_star];
        lambda_star    = zeros(k,1);
        delta_star     = zeros(k,1);
        lambda_star(1)= phi_star-beta_star+d_star;
        delta_star(1)  = d_star;
```

```
        for m=2:k
            delta_star(m)   = (m-1-d_star)/m*delta_star(m-
1);
            lambda_star(m) =  beta_star*lambda_star(m-1) +
delta_star(m)-phi_star*delta_star(m-1);
        end

 %% 1. predicting future et and conditional variance using
original data (truncated lag k=1000)
v_ss              = datasample(e_hat,h);
ht_future         = zeros(n+h,1);
e1_future         = zeros(n+h,1);
e1_future(1:n)    = ee1;
e2_future         = zeros(n+h,1);
e2_future(1:n)    = ee1.^2;

        for r=n+1:n+h

            ht_future(r) = w_star/(1-beta_star) +
lambda_star(1:k)'*e2_future(r-1:-1:r-k);
            e1_future(r) = sqrt(ht_future(r))*v_ss(r-n);
            e2_future(r) = (e1_future(r))^2;

        end

        yh_b = e1_future(n+1:n+h); % future bootstrap
values
        hh_b = ht_future(n+1:n+h); % future conditional
variance values

        eB(j,:) = yh_b;
        hB(j,:) = hh_b; % future returns and future c.v

    end
    BootPara(i,:) = mean(BPara);
    %% Returns
    ebq    = quantile(eB,[0.025 0.975]);  bql =ebq(1,:);
bqu = ebq(2,:); Blength(i,:) = bqu - bql;
    Bquantile(i,:) = [bql bqu];
    bcount = (fv>bql&fv<bqu);    Bcount(i,:)       =
sum(bcount);
    bcount0_025 = (fv<bql);      Bcount0_025(i,:) =
sum(bcount0_025);
    bcount0_975 = (fv>bqu);      Bcount0_975(i,:) =
sum(bcount0_975);
```

```
    ebq    = quantile(eB,[0.005 0.995]);  bql =ebq(1,:);
bqu = ebq(2,:); Blength99(i,:) = bqu - bql;
    Bquantile99(i,:) = [bql bqu];
    bcount = (fv>bql&fv<bqu);    Bcount99(i,:)    =
sum(bcount);
    bcount0_005 = (fv<bql);      Bcount0_005(i,:) =
sum(bcount0_005);
    bcount0_995 = (fv>bqu);      Bcount0_995(i,:) =
sum(bcount0_995);

    ebq    = quantile(eB,0.25); Blengthp25(i,:) = ebq;
    bcount = (fv <= ebq);       Bcountp25(i,:)  =
sum(bcount);

    ebq    = quantile(eB,0.50); Blengthp50(i,:) = ebq;
    bcount = (fv <= ebq);       Bcountp50(i,:)  =
sum(bcount);

    ebq    = quantile(eB,0.75); Blengthp75(i,:) = ebq;
    bcount = (fv <= ebq);       Bcountp75(i,:)  =
sum(bcount);

    ebq    = quantile(eB,1);    Blengthpmax(i,:)= ebq;
    bcount = (fv <= ebq);       Bcountpmax(i,:) =
sum(bcount);

    ebq    = quantile(eB,0);    Blengthpmin(i,:) = ebq;
    bcount = (fv <= ebq);       Bcountpmin(i,:)  =
sum(bcount);
    %% Volatilities
    hbq    = quantile(hB,[0.025 0.975]);  bql =hbq(1,:);
bqu = hbq(2,:); Hlength(i,:) = bqu - bql;
    Hquantile(i,:) = [bql bqu];
    hcount = (fv_h>bql&fv_h<bqu);    Hcount(i,:)      =
sum(hcount);
    hcount0_025 = (fv_h<bql);      Hcount0_025(i,:) =
sum(hcount0_025);
    hcount0_975 = (fv_h>bqu);      Hcount0_975(i,:) =
sum(hcount0_975);

    hbq    = quantile(hB,[0.005 0.995]);  bql =hbq(1,:);
bqu = hbq(2,:); Hlength99(i,:) = bqu - bql;
    Hquantile99(i,:) = [bql bqu];
    hcount = (fv_h>bql&fv_h<bqu);    Hcount99(i,:)    =
sum(hcount);
```

```
    hcount0_005 = (fv_h<bql);      Hcount0_005(i,:) =
sum(hcount0_005);
    hcount0_995 = (fv_h>bqu);      Hcount0_995(i,:) =
sum(hcount0_995);

end

% Stored the above out put data to files
% FIGARCH Simulation
% Supported functions for the above algorithm

% Simulation of FIGARCH(1,d,1)

function [ee1, hh1]=figarchsim(parameters,n,k,discard,zt)
    % zt  (n+k+discard)x1 vector
    % parameters = [w phi d beta]; 4x1 vector;
    w=parameters(1); phi=parameters(2); d = parameters(3);
beta=parameters(4);
    et = zeros(k+discard+n,1);
    et2 = zeros(k+discard+n,1);
    ht = zeros(k+discard+n,1);
    lambda = zeros(k,1); delta = zeros(k,1);
    lambda(1)=phi-beta+d;  delta(1)=d;

    for i=2:k
        delta(i)=(i-1-d)/i*delta(i-1);
        lambda(i)=beta*lambda(i-1) + delta(i)-phi*delta(i-
1);
    end

   % bcast = zeros(k,1)+w/(1-beta);
    bcast =  zeros(k,1)+w/(1-sum(lambda)); %does not matter
    et2(1:k)=bcast;
    for j = k+1:length(zt)
        ht(j) = w/(1-beta) + lambda(1:k)'*et2(j-1:-1:j-k);
        et(j) = sqrt(ht(j))*zt(j);
        et2(j)=et(j)^2;
    end

ee1 = et(k+discard+1:k+discard+n);
hh1 = ht(k+discard+1:k+discard+n);


%% FIGARCH LIKELIHOOD FUNCTION
function y=l_figarch(x,ee1,k) % AIC = 2*(p+q+1) - 2* log(L)
or  AIC = 2*k - 2*log(L) % AICc = AIC + 2*k(k+1)/(n-k-1)
```

```
    % k = Truncation Lag
    w= x(1);
    phi=x(2);
    d = x(3);
    beta = x(4);

    lambda = zeros(k,1); delta = zeros(k,1);
    lambda(1)=phi-beta+d;  delta(1)=d;
    %l1 = fiweights(parameters);
    for i=2:k
        delta(i)=(i-1-d)/i*delta(i-1);
        lambda(i)=beta*lambda(i-1) + delta(i)-phi*delta(i-
1);
    end

    uSq = ee1.^2; N = length(ee1);
    epsilon2 = zeros(N+k,1);
    epsilon2(1:k)=mean(uSq);
    epsilon2(k+1:N+k)=uSq;

    tau = k+1:N+k;

    ht = zeros(size(epsilon2));
    likelihood = 0;
    for t = tau
        ht(t) = w + lambda'*epsilon2(t-1:-1:t-k);
        likelihood = likelihood + 0.5*(-log(2*pi)-
log(ht(t))-epsilon2(t)/ht(t));
    end

y = -likelihood;

%% ESTIMATED CONDITIOANL VARIANCE
function hh=cond_var(x,ee1,k) % Estimated conditioanl
variance
    % k = Truncation Lag
    w= x(1);
    phi=x(2);
    d = x(3);
    beta = x(4);

    lambda = zeros(k,1); delta = zeros(k,1);
    lambda(1)=phi-beta+d;  delta(1)=d;
    %l1 = fiweights(parameters);
    for i=2:k
        delta(i)=(i-1-d)/i*delta(i-1);
```

```
        lambda(i)=beta*lambda(i-1) + delta(i)-phi*delta(i-
1);
    end

    uSq = ee1.^2; N = length(ee1);
    epsilon2 = zeros(N+k,1);
    epsilon2(1:k)=mean(uSq);
    epsilon2(k+1:N+k)=uSq;

    tau = k+1:N+k;

    ht = zeros(size(epsilon2));
    %likelihood = 0;
    for t = tau
        ht(t) = w + lambda'*epsilon2(t-1:-1:t-k);
        %likelihood = likelihood + 0.5*(-log(2*pi)-
log(ht(t))-epsilon2(t)/ht(t));
    end
    hh=ht(tau);
```

BIBLIOGRAPHY

1. Alonso, A., Peña, D., & Romo, J. (2002). Forecasting time series with sieve bootstrap. *Journal of Statistical Planning and Inference*, *100*(1), 1–11.

2. Alonso, A. M., Peña, D., & Romo, J. (2003). On sieve bootstrap prediction intervals. *Statistics & Probability Letters*, *65*(1), 13–20.

3. Brockwell, P. J., & Davis, R. A. (2009). *Time Series: Theory and Methods (Springer Series in Statistics)* (2nd ed. 1991. 2nd printing 2009. Softcover reprint of the original 2nd ed. 1991 ed.). Springer.

4. Baillie, R. T., Bollerslev, T., & Mikkelsen, H. O. (1996). Fractionally integrated generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, *74*(1), 3–30.

5. Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, *31*(3), 307–327.

6. Bühlmann, P., & Buhlmann, P. (1997). Sieve Bootstrap for Time Series. *Bernoulli*, *3*(2), 123.

7. Cao, R., Febrerobande, M., Gonzalez-Manteiga, W., Prada-Sanchez, J. M., & Garcfa-Jurado, I. (1997). Saving computer time in constructing consistent bootstrap prediction intervals for autoregressive processes. *Communications in Statistics - Simulation and Computation*, *26*(3), 961–978.

8. Davidson, J. (2004). Moment and Memory Properties of Linear Conditional Heteroscedasticity Models, and a New Model. *Journal of Business & Economic Statistics*, *22*(1), 16–29.

9. deLima, P., Breidt, F., and Crato, N. (1994). Modelling long-memory stochastic volatility, Manuscript (Department of Economics, Johns Hopkins University, Baltimore, MD).

10. Ding, Z., Granger, C. W. J., & Engle, R. F. (1993). A long memory property of stock market returns and a new model. *Journal of Empirical Finance*, *1*(1), 83–106.

11. Efron, B. (1979). Bootstrap Methods: Another Look at the Jackknife. *The Annals of Statistics*, *7*(1), 1–26.

12. Engle, R. F. (1982). Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation. *Econometrica*, *50*(4), 987.

13. Engle, R. F., & Bollerslev, T. (1986). Modelling the persistence of conditional variances. *Econometric Reviews*, *5*(1), 1–50.

14. Findley D.F. (1986). On Bootstrap estimates of forecast mean square errors for autoregressive processes. Computer Science and Statistics: Proceedings for the 17th Symposium on the Interface, 11-17.

15. Freedman, D. A., Peters, S. C. (1984). Bootstrapping a regression equation: some empirical results. *Journal of the American Statistical Association* 79:97–106.

16. Giraitis, L., Leipus, R., & Surgailis, D. (2009): ARCH(∞) Models and Long Memory Properties In: Andersen, T.G., Davis, R.A., Kreiss, J.-P. and Mikosch, T. (eds.): Handbook of Financial Time Series. Springer, New York.

17. Harvey, A. (1993). Long memory in stochastic volatility, Manuscript (London School of Economics, London).

18. Kreiss J.P. (1992), "Bootstrap procedures for AR(1)-processes," In: JEockel, K.H.,Rothe, G., Sendler, W. (Eds.), Bootstrapping and Related Techniques, Springer, Heidelberg, pp. 107-113.

19. Künsch, H. R. (1989). The jackknife and the bootstrap for general stationary observations. *The Annals of Statistics* 17:1217–1241.

20. Miguel, J. A., & Olave, P. (1999). Bootstrapping forecast intervals in ARCH models. *Test*, *8*(2), 345–364.

21. Mukhopadhyay, P., & Samaranayake, V. A. (2010). Prediction Intervals for Time Series: A Modified Sieve Bootstrap Approach. *Communications in Statistics - Simulation and Computation*, *39*(3), 517–538.

22. Pascual, L., Romo, J., & Ruiz, E. (2006). Bootstrap prediction for returns and volatilities in GARCH models. *Computational Statistics & Data Analysis*, *50*(9), 2293–2312.

23. Poskitt, D. S. (2006). Autoregressive approximation in nonstandard situations: the fractionally integrated and non-invertible cases. *Annals of the Institute of Statistical Mathematics*, *59*(4), 697–725.

24. Robinson, P. M. (1991). Testing for strong serial correlation and dynamic conditional heteroskedasticity in multiple regression. *Journal of Econometrics*, *47*(1), 67–84.

25. Rupasinghe, M., & Samaranayake, V. A. (2012). Asymptotic properties of sieve bootstrap prediction intervals for processes. *Statistics & Probability Letters*, *82*(12), 2108–2114.

26. Rupasinghe, M., Mukhopadhyay, P., & Samaranayake, V. A. (2013). Obtaining prediction intervals for FARIMA processes using the sieve bootstrap. *Journal of Statistical Computation and Simulation*, *84*(9), 2044–2058.

27. Stine R.A., 1982. Prediction intervals for time series. Unpublished Ph.D. dissertation, Princeton University, Dept of Statistics.

28. Stine, R. A. (1987). Estimating Properties of Autoregressive Forecasts. *Journal of the American Statistical Association*, *82*(400), 1072–1078.

29. Tayefi, M., & Ramanathan, T. V. (2016). An Overview of FIGARCH and Related Time Series Models. *Austrian Journal of Statistics*, *41*(3), 175–196.

30. Thombs, L. A., & Schucany, W. R. (1990). Bootstrap Prediction Intervals for Autoregression. *Journal of the American Statistical Association*, *85*(410), 486–492.

VITA

Ekanayake Mudiyanselage Rukman Sumedha Bandara Ekanayake was born in Kandy, Sri Lanka. He attended Trinity College Kandy, Sri Lanka from 1994 to 2007. He entered the University of Peradeniya, Sri Lanka and obtained Bachelor's degree specializing Statistics in 2013. After graduating, he worked as an Instructor in the same university for two and a half years. In 2015, he enrolled in the PhD in Mathematics with Statistics emphasis at Missouri University of Science and Technology. He received his PhD in Mathematics with Statistics emphasis from Missouri S&T in May 2021.