The design, construction, and operation of a conducting field analog computer.

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THE DESIGN, CONSTRUCTION, AND OPERATION
OF A CONDUCTING FIELD ANALOG COMPUTER

BY

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A

THESIS

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ABSTRACT

The purpose of this investigation is to design and construct a simple conducting field analog to solve problems described by the two-dimensional Poisson or Laplace equations. Decisions concerning choice of conducting media, construction of problem models, and design and construction of a specialized power supply with instrumentation were based on the results of previous experiments. Several problems are solved using the designed apparatus. Partial solutions to these problems are displayed in graphical form.

The greatest deviation of the experimental results from iterative values was consistently less than 8%. In most cases the deviation was less than 6%. The console arrangement containing the specialized power supply and instrumentation was sufficiently flexible in design to allow the analog simulation of the various problems to be performed relatively quickly and easily.

ACKNOWLEDGEMENT

The author would like to express his appreciation to Dr. T. R. Faucett for the suggestions which led to this thesis. Additional thanks are due Professor C. L. Edwards for his interest and encouragement during the course of the work.

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I. INTRODUCTION

The expanding technology of the past few decades has created a need for detailed analysis of many problems previously avoided by use of rough approximations coupled with highly conservative design. In many cases the conservative approach to these problems is quite sufficient. However, the demand for lightweight, more efficient devices accentuates the need for detailed analysis in design.

Small facets of the ever increasing bulk of problems requiring this more detailed analysis are the systems which may be described mathematically by the two dimensional Poisson or Laplace equations. Since these equations may describe a great many physical phenomena, relatively simple methods which provide detailed and accurate solutions are desirable.

There are several methods which may be used to obtain solutions to these particular equations. These methods may be divided into three major categories: exact solutions, the numerical analysis approach, and the analogy method. Exact solutions are available only for a few special cases and thus do not provide a universal method of solution. With the aid of a digital computer the numerical analysis approach will yield a solution made up of discrete points in an arbitrary reference system. In most instances the analogy method will give a continuous solution that is independent of arbitrary reference systems.

This thesis contains the results of the investigation of one specific analogy that may be used to solve the two dimensional Poisson or Laplace equations. The analogy investigated was a uniformly conducting two dimensional electric field. The mechanism used to define this electric field was a sheet of conducting paper coupled to suitable generating and measurement equipment. The investigations conducted concerned:

a. Determination of the properties of the specific conducting paper used.

b. Design and construction of a suitable console to solve specific problems by use of the conducting paper.

c. Solution of several example problems to demonstrate the usefulness of the analogy.

d. Specific conclusions as to the limitations of the analogy.

II. REVIEW OF LITERATURE

The first practical use of an analogy to solve the two dimensional Poisson equation appears to be in the work of Taylor and Griffith (1,2,3,4). In 1917-18 they made extensive use of the thin membrane analogy for the solution of the plane elastic torsion problem. In particular they used a soap film membrane to determine torsional stresses in aircraft structural members. Their results were usually within 5% agreement of other solutions for the same problem.

After these initial experiments by Taylor and Griffith proved so fruitful, the membrane or soap film analogy became a popular technique for the solution of various physical problems described by the Poisson and Laplace equations. Den Hartog (5) gives a good discussion of the membrane analogy and its applications in determining the torsional shearing stresses in prismatic bars. Schneider (6) discusses the analogical solution of the Laplace equation in determining the temperature distribution in a two dimensional steady-state heat conduction problem.

The membrane or soap film analogy, however, is not the only analogy which may be used to solve these problems described by Poisson and Laplace equations. A uniformly conducting electric field can be shown to have a describing equation of the Poisson type. In fact, Kirchhoff (7) in 1845 performed experiments with an electrically conducting

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All references are in the bibliography.

sheet. Hetenyi (8) points out the similarity between the membrane and conducting field analogies and gives some discussion of each and their correlations.

It appears that the conducting field analogy has been in existence as long as the membrane analogy. However, its use was quite limited because of the difficulty in obtaining a good, inexpensive, uniformly conducting media. Fortunately, in the late 1940's the Western Union Telegraph Company developed an "electrosensitive recording paper for facsimile telegraph apparatus and graphic chart instruments" (9) called "Teledeltos". This paper has most of the qualities required for a good conducting field media. It is dry, light, easy to handle, and can be easily cut with common scissors to produce a two dimensional model of any shape. Unfortunately, it is slightly anisotropic; and its resistance properties are quite sensitive to changes in humidity. However, with careful use and preparation these drawbacks can usually be overcome.

Waner and Soroka (10) used Teledeltos Type L (2000 ohms/square) to determine stress concentrations in structural angles subjected to torsion. They neglected the anisotropic nature of the paper and obtained results within 5% of those predicted by other methods. Friedmann, Yamamoto, and Rosenthal (11) used Teledeltos Type H (20,000 ohms/square) to solve several torsional problems. This work differed from that of Waner and Soroka in that the

problem was first reduced from a Poisson equation with constant boundaries to a Laplace equation with variable boundaries. Karplus and Soroka (12) present good background and discussion concerning the theory of the conducting field analogy and its many applications including heat conduction problems and inviscid, noncompressible fluid flow problems.

Beadle and Conway (13) used Teledeltos to solve problems similar to those done by Waner and Soroka except they transformed the Poisson equation for torsion into a Laplace equation as did Friedmann, Yamamoto, and Rosenthal. Their results were comparable with those of Waner and Soroka. Beadle and Conway also neglected the anisotropic properties of the paper.

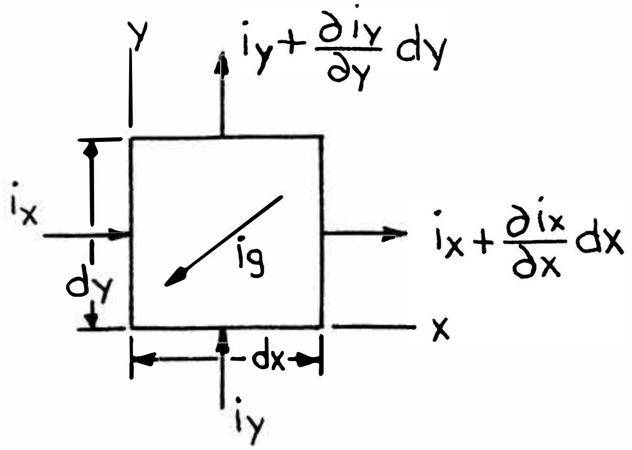
The work that has been done using the conducting field analogy for solution of specific problems is fairly extensive and has indicated that when properly used the analogy will yield quite acceptable results. The purpose of this thesis is not to simply solve another problem by use of this analogy, but to design and construct an instrumented power console which may be used in conjunction with Teledeltos models to provide repeatable solutions. In the literature surveyed the primary interest was in the problem to be solved and the equipment used was standard laboratory equipment, adapted when necessary to produce the desired results. For this reason it was felt that the

investigation of a specialized device, designed specifically for use in this analogy, would be desirable.

III. DISCUSSION

A. CONDUCTING MEDIA.

The describing equation for the voltage distribution over the surface of a uniformly conducting sheet may be determined in the following manner: Choose an element of the sheet dx by dy and of uniformly small thickness with the general currents shown.



Where: i_x = current crossing the left hand face in amperes per unit length of face.

i_y = current crossing the lower face in amperes per unit length of face.

i_g = a current uniformly distributed over the surface of the paper in amperes per unit length squared.

$\frac{\partial i_x}{\partial x}$ = the rate of change of i_x with respect to the x direction.

$\frac{\partial i_y}{\partial y}$ = the rate of change of i_y with respect to the y direction.

The magnitude of current entering a junction must be equal in magnitude to the current leaving the junction:

$$i_x dy + i_y dx + i_g dy dx = \left(i_y + \frac{\partial i_y}{\partial y} dy\right) dx + \left(i_x + \frac{\partial i_x}{\partial x} dx\right) dy. \quad (1)$$

Clearing and reducing,

$$\frac{\partial i_x}{\partial x} + \frac{\partial i_y}{\partial y} = i_g. \quad (2)$$

Also,

$$-\frac{\partial E}{\partial x} = i_x R_x, \text{ and } -\frac{\partial E}{\partial y} = i_y R_y. \quad (3)$$

Where: $\frac{\partial E}{\partial x}$ = voltage gradient across element in the x direction.

$\frac{\partial E}{\partial y}$ = voltage gradient across element in the y direction.

R_x = nominal resistance in the x direction.

R_y = nominal resistance in the y direction.

Solving equations (3) for i_x and i_y and substituting into (2) above:

$$-\frac{1}{R_x}\frac{\partial^2 E}{\partial x^2} - \frac{1}{R_y}\frac{\partial^2 E}{\partial y^2} = i_g.$$

Rewriting,

$$\frac{\partial^2 E}{\partial x^2} + \left(\frac{R_x}{R_y}\right)\frac{\partial^2 E}{\partial y^2} = -i_g R_x.$$

If

$$R_x = R_y = R$$

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} = -i_g R.$$

This equation is clearly of the Poisson form:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = K.$$

Where F is a function of x and y and K is a constant.

From this derivation it is apparent that the describing equation for the voltage distribution over the conducting sheet is indeed a form of Poisson's Equation and may be reduced to Laplace's Equation

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} = 0$$

by letting i_g be zero.

As previously mentioned Teledeltos fascimile paper is probably the most popular and convenient conducting media in use for this analogy. It has become so popular that the manufacturer prepares a special uncoated type specifically for analog field mapping.[1]

A thorough investigation of the properties of the paper was deemed necessary before any decisions could be made concerning design of a control console.

The paper is composed of a dry carbon base, which is the conductor, mixed with paper stock to provide body. The paper stock and carbon base are mixed in an aqueous compound with various binders and the paper produced from

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[1]This particular type (type L, one side coated) may be procured from: Artcote, 390 Coit Street, Irvington, New Jersey.

this. One side of the paper is then protected with an aluminized coating and the remaining side left black for ready contact with the carbon conductor.

This procedure produces a paper that is dry and easy to handle. It is non-polarizing thus allowing the use of either alternating or direct current. Reference marks on the paper may be made easily with a white non-conducting pencil on the black conducting side. A typist's white correction pencil is most suitable. Models of problems may be cut and trimmed easily with ordinary scissors.

Unfortunately, the paper has some disadvantages which must be carefully considered. Due to the manufacturing process the electrical resistance across the roll is greater than the resistance along the roll. Also, the paper is highly moisture sensitive, and variations in humidity change its nominal resistance of 2000 ohms per square significantly. The paper will withstand several hundred volts without breakdown. The power dissipation, however, cannot exceed .2 watts per square inch. This is due to the sensitivity of the resistance of the paper to its moisture content. When significant heating occurs the moisture is driven off causing large changes in resistance.

After experimentation it was decided to employ direct current rather than alternating current in performing tests. There were several reasons for this decision. First, it was felt that with a direct current power source it would

be easier to control the voltages and currents applied to a problem. Second, the relative accuracy and availability of suitable direct current instruments appeared to be superior to equivalent alternating current instruments. Also, no impedance matching problems would be encountered when using direct current.

When the describing partial differential equation was derived the current i_g was defined as uniformly distributed over the surface of the paper. Unfortunately, this condition was physically unrealizable since an infinite number of discrete current sources would have been required. However, there were three methods by which approximations to this situation could be obtained.

The first of these methods used a superposition technique. Waner and Soroka (9) employed this method which consisted of the following procedure:

1. The model of the problem was marked off into grid squares.
2. A single current conducting probe was placed in the center of one of the grid squares and a specific current passed into the paper model through it.
3. The voltages with respect to electrical ground were observed at all the grid intersections.
4. The first three steps were repeated until the current probe had been placed in the center of all grid squares and the corresponding voltages measured.

5. The voltage readings from the preceding parts were superimposed (added) at each grid intersection. The resulting voltages represented the value of the function E(x,y) at the grid intersections.

This method was a good approximation and produced very good results. However, for all but the simplest problems it was quite long and tedious and required the recording of an extremely large quantity of data. For this reason it was not considered suitable for use with the console arrangement to be designed.

The second method was employed by Friedmann, Yamamoto, and Rosenthal (11). Their approach was to reduce the special case Poisson equation

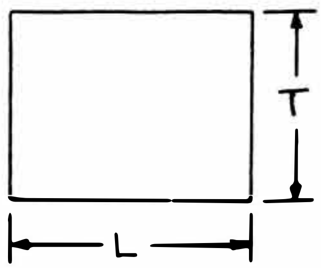
$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} = -i_g R$$

to a Laplace equation by changing the variable. This procedure eliminated the uniformly distributed current i_g and the problems associated with its physical achievement. However, the change in variable also changed the boundary conditions of the problem from the usual constant values to some variable condition. This introduced the completely new problem of approximating a continuously varying boundary condition. After experimentation it was decided that although this technique was practically possible, the labor and tedious adjustments required made it unfeasible for use in a console type arrangement.

Further experimentation indicated that the most practical method to supply i_g to the paper model was by placing a wire as a point source in the center of each grid square of an arbitrary grid system. By adjusting the current flow in each of these wires to the same magnitude, a good approximation of the uniform current distribution of i_g was obtained. Further investigation indicated that voltage measurements of reasonable accuracy could be made anywhere on the model except within a 3/4" radius of the point current sources. It was decided that this method would be the most feasible to employ with the console arrangement.

Detailed investigation indicated that the anisotropic nature of the Teledeltos paper could introduce significant errors when the point current source technique was used. To minimize this effect a "distortion" of the model was employed. The distortion was such that the equivalent resistance across the roll R_T, was the same as that along the roll R_L. A distortion relation was developed as follows:

A rectangle of the paper T units wide and L units long was chosen.



Remembering that the resistance of a normal conductor varies directly as the length and inversely with the area, the resistance R_T and R_L may be expressed:

$$R_T = R_t \frac{T}{L}, \qquad R_L = R_\ell \frac{L}{T}.$$

Where: R_t and R_ℓ are the transverse and longitudinal resistances respectively measured from a calibration square.

R_T and R_L are the transverse and longitudinal resistances of the T by L rectangle. But, it is desired that $R_T = R_L$. Thus,

$$R_t\left(\frac{T}{L}\right) = R_\ell\left(\frac{L}{T}\right),$$

$$\left(\frac{T}{L}\right)^2 = \frac{R_\ell}{R_t},$$

$$T = L\sqrt{\frac{R_\ell}{R_t}}.$$

For the particular roll of Teledeltos used $R_\ell = 1700$ ohms/square, $R_t = 2000$ ohms/square. Thus:

$$T = L\sqrt{\frac{1700}{2000}} = .922\,L.$$

Using this relation a model should be cut with its major axes in the transverse and longitudinal directions of the roll such that the units in the transverse direction are only .922 of those in the longitudinal direction.

In the process of determining the properties of the Teledeltos paper several techniques were investigated

concerning the preparation of models. The technique that produced the most stable, consistent models employed tempered masonite as a backing material. When suitably reinforced this masonite proved a rigid and durable backing. The paper model was then fastened to this backing with a very thin coat of good quality mucilage and all bubbles and wrinkles smoothed out before the glue set. This technique produced permanent models that were relatively free from irregularities and separations from the backing. The shrinking properties of the mucilage seemed to produce this tight bond between the paper and backing.

Constant boundary conditions were applied with fine gage stranded wire fastened directly to the model by 1/4" builder's staples driven into the masonite with a staple gun. The exact boundary lines and electrical connections were made with silver paint (General Cement No. 21-1.) It should be noted that additional paper was left on the model so that there was room to physically attach the boundary conditions.

The point application of internal current sources was accomplished by drilling a small hole in the center of each grid square on the model. A small loop was made in the end of a wire and the wire slipped through the drilled hole and fastened mechanically so that the small loop was pulled snugly against the paper surface. The electrical connection was completed with a drop or two of silver paint.

Marking the models was easily accomplished with the aid of a typist's white correction pencil (Carter's Excellence No. 4037 white). This pencil allowed grid squares to be drawn directly on the black conducting face of the model.

Figure 1 illustrates the application of constant boundary conditions, the point current sources, and the grid squares as marked with the white pencil.

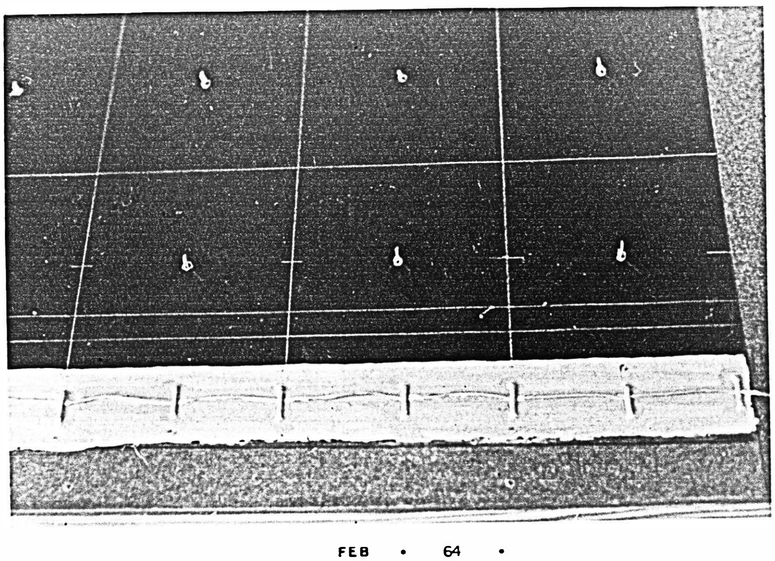


Figure 1

Application of Electrical Boundaries and Current Inputs

As might be expected it was discovered that relatively large models with small voltage and current gradients produced the most reliable results. Input grid systems for point current sources were based on the equivalent of a three inch net with the source at the center of each three inch grid square. Other grid systems could have been used,

but this one seemed to fit the problems quite satisfactorily.

B. CONSOLE DESIGN.

The design of a console to provide power and instrumentation for problem solution using the Teledeltos paper model required careful planning and forethought. Since the problems to be solved were electrically static (voltages and currents did not vary with time) a simple direct current power source appeared to be all that was necessary. However, the power supply not only had to supply voltage for boundary conditions but current for the simultaneous point current sources.

The simplest method of obtaining the equal current inputs would appear to be a number of potentiometers in parallel connected across the output of the power supply. The wiper of each potentiometer would be attached to the model as a point current source and the current from each source controlled by the corresponding potentiometer. However, if the supply is not an infinite source the supply voltage will vary as current requirements change. This indicates that when one current source is adjusted the supply voltage changes slightly and thus alters the current flow through all the remaining current paths. This characteristic would make accurate adjustment of the current sources very difficult if not impossible.

To minimize this problem a voltage regulated direct current power supply circuit was chosen. The regulation

feature of this supply allowed considerable fluctuation in the current drawn from the supply without significant output voltage variation. Experience in the investigation of the conducting paper indicated that a direct current supply that would regulate from 0 to 200 milliamps at any voltage from 1 to 300 volts was required. After considerable investigation a Heath Company circuit[2] was modified to supply the required currents and voltages in addition to incorporating the bulk of the measuring devices for the problem solution.

For convenience it was decided to include the regulated power supply and instrumentation all in one console, and the bank of parallel potentiometers needed to adjust point current inputs in another. To distinguish one console from the other the device containing the instruments and power supply was named the Power-Instrument console, and the separate bank of potentiometers was called the Current Regulator console.

The instrumentation selected consisted of one general purpose vacuum tube voltmeter and one direct current, 0-25ma milliammeter. The voltmeter was chosen for several reasons. Its very high input impedance would prevent it from affecting the problem. It could be used to measure resistances to the accuracy required by these experiments.

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[2]Heath Kit Regulated Power Supply Model PS-4.

The overall accuracy was commensurate with the other aspects of the problem and the initial cost of the instrument was relatively little.

The ammeter was chosen with the intention of making a shunt for it so that it could be used as a 0-250 ma. meter or the regular 0-25 ma. meter. The large range was for measuring the total output current of the supply and the smaller range was for measuring the currents in each point current source path.

Both the voltmeter and the ammeter were connected into the power supply circuit by the use of three position function switches. The voltmeter switch allowed the instrument to measure the output voltage of the power supply, or the voltage indicated by the measuring probe. The third switch position provided no input voltage so that the meter could be easily adjusted without having to disconnect any input devices.

The ammeter function switch engaged a shunt allowing power supply output currents of up to 250 ma. to be measured. A second position switched out the shunt and connected the 0-25 ma. range to the input terminals at the front of the console. This arrangement in conjunction with a switching arrangement in the Current Regulator console allowed this one meter movement to be switched into each point current source path. The remaining switch position provided no input and allowed the meter to be adjusted

without removing it from the circuit.

Figure 2 is the schematic diagram of the circuit for the Power-Instrument console and figures 3 and 4 show different views of the finished device.

The bank of parallel potentiometers (Current Regulator console) was separated from the Power-Instrument console for greater mechanical and electrical flexibility. In order to determine the current flowing in each potentiometer path a switching circuit was devised to use the ammeter mounted in the Power-Instrument console. This circuit employed a simple slide switch with each potentiometer. In the up position the switch placed the ammeter in series with the output of the wiper of the potentiometer for the determination of current flow in a particular path. In the down position the switch completed the output circuit leaving the ammeter free to be placed in the circuit of another point current source.

Twenty-four 1 megohm potentiometers were used in conjunction with an equal number of slide switches and spring wire clips. The spring clip arrangement for connecting the paper model to the console was chosen because it eliminated the need for special preparation of the wires from the model.

Figure 5 is the schematic diagram of the circuit for the Current Regulator console and figures 6 and 7 are front and rear views of the finished device.

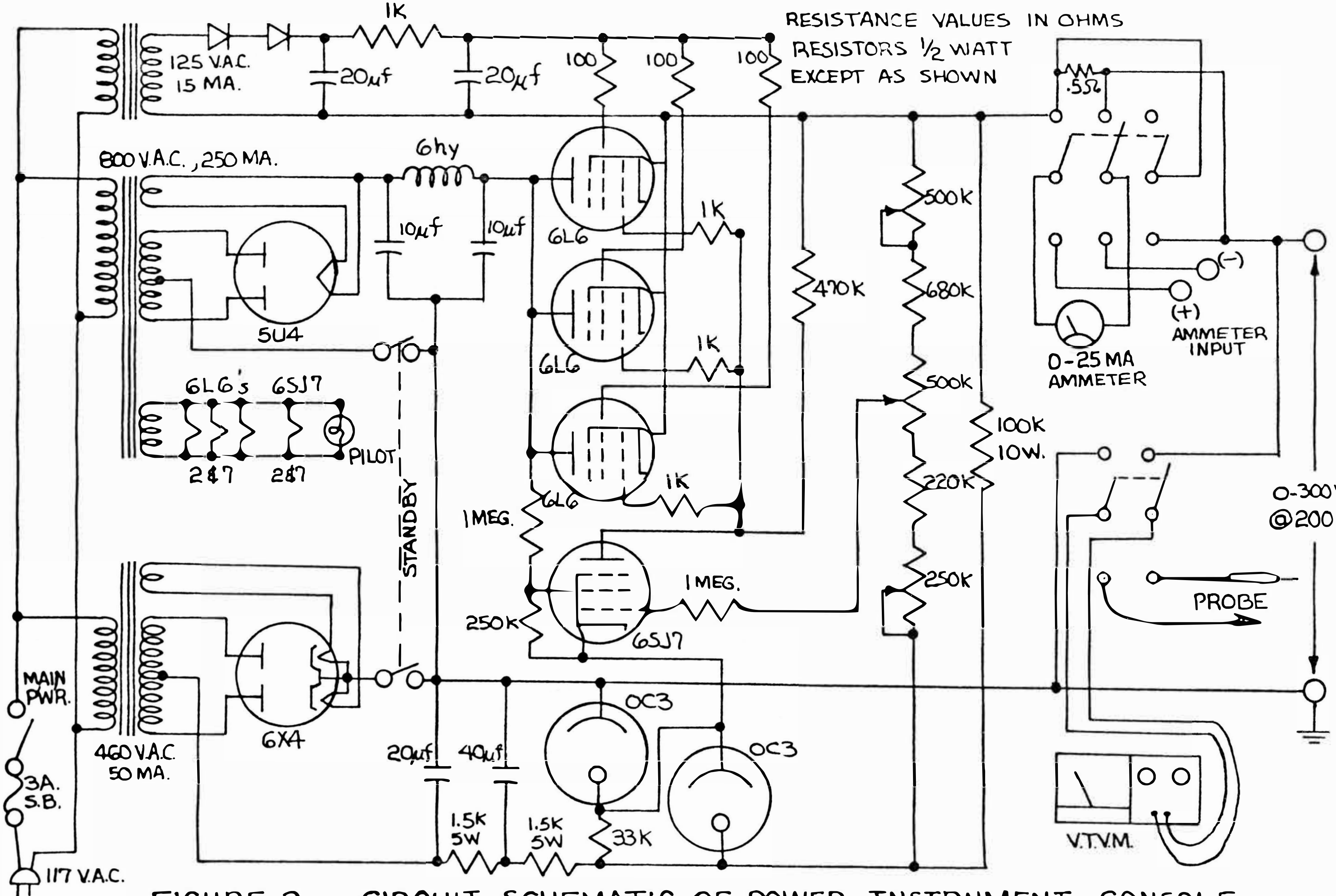

FIGURE 2 - CIRCUIT SCHEMATIC OF POWER-INSTRUMENT CONSOLE

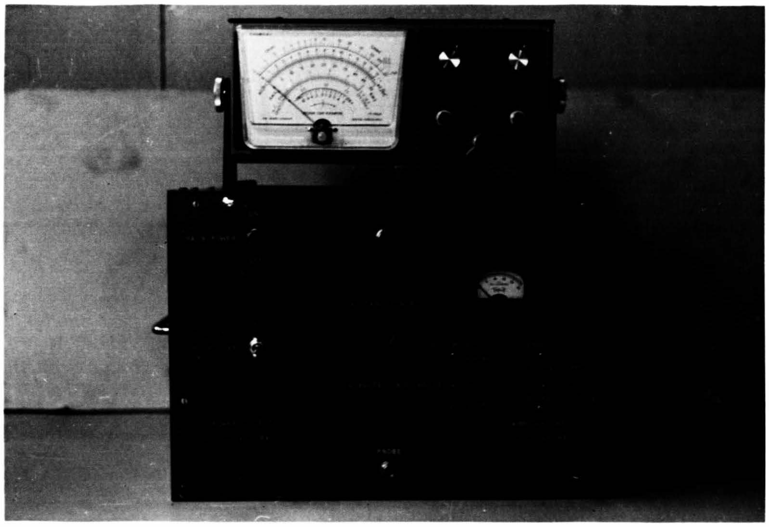

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Figure 3

Front View of Power-Instrument Console

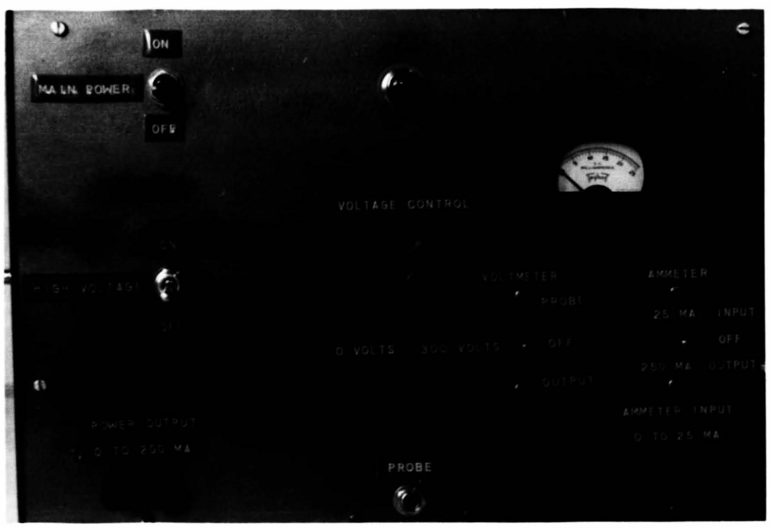

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Figure 4

Power Instrument Console Control Panel

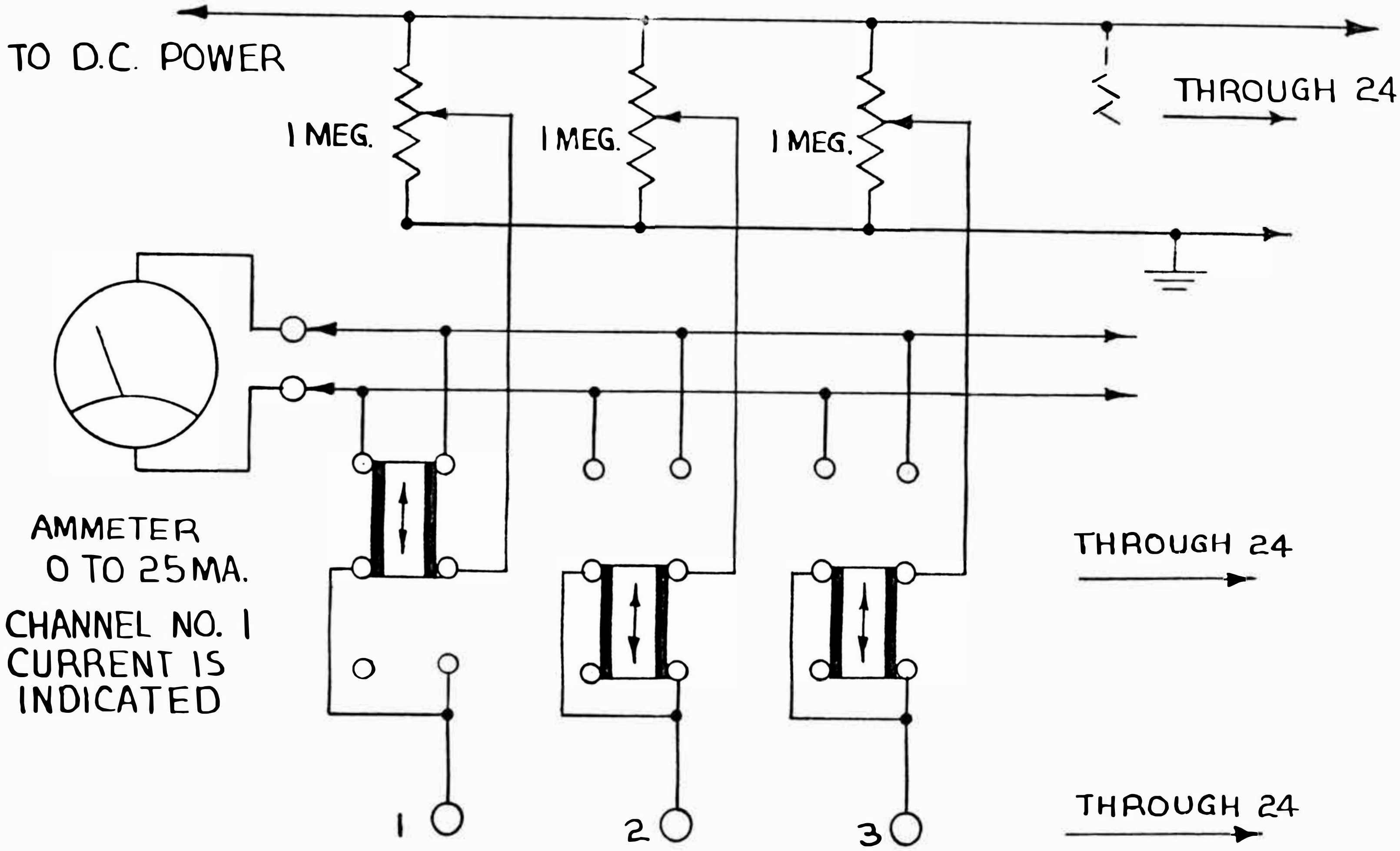

FIGURE 5 - CIRCUIT SCHEMATIC OF CURRENT REGULATOR CONSOLE

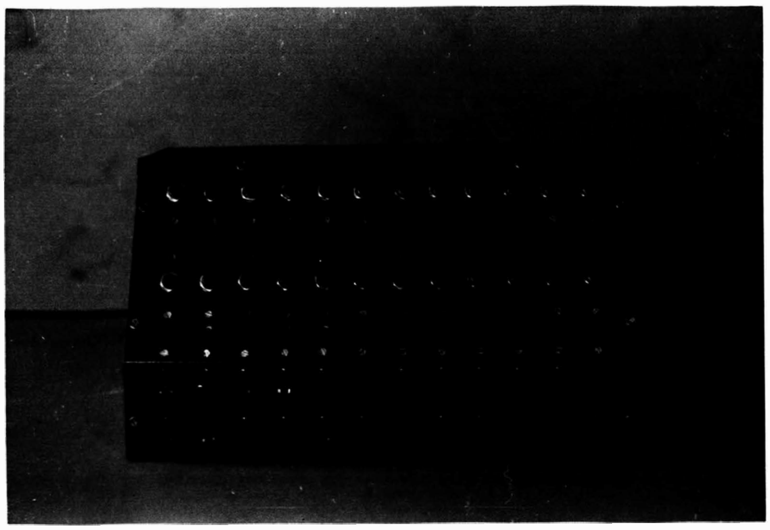

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Figure 6

Front View of Current Regulator Console

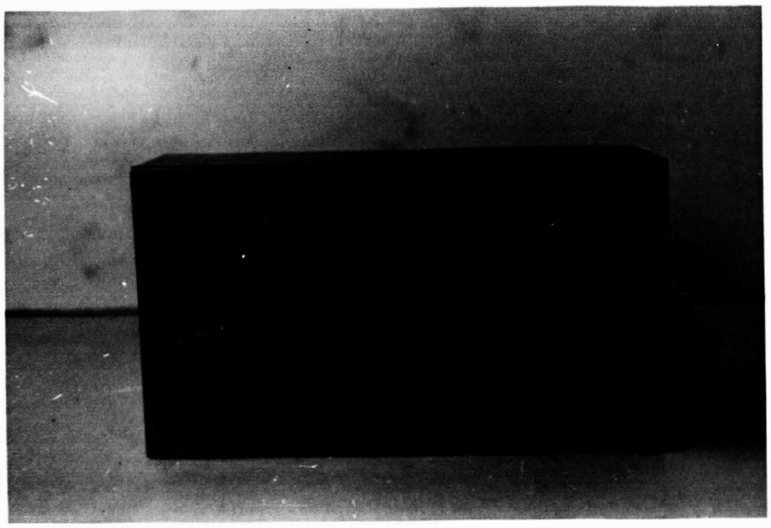

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Figure 7

Rear View of Current Regulator Console
Showing Power and Ammeter Connections

C. PROBE DESIGN.

In the experiments performed to determine the properties of the Teledeltos paper it was evident that the standard probe arrangement of the vacuum tube voltmeter was quite satisfactory. However, it was decided that some type of probe holding arrangement would allow more accurate placement of the probe by hand and allow the experimenter to be completely free to control the equipment and record data.

The probe holding device employed was simple and quite effective. A small block of wood was carved on one end to allow the plastic body of the probe to fit snugly against it. The probe body was fastened to the block by simply placing a metal strap over it and fastening the strap to the wooden block with two wing nuts. The body of the probe was attached to the block in a position that allowed the metal spike of the probe to extend beyond the wooden holding block by about one quarter of an inch. Two nylon furniture gliders were then fastened to the block in such a manner that with the tip of the probe they formed a stable three point contact arrangement.

The performance of this probe holding device was excellent. It was small enough so that no flexibility of movement was lost through its use. In addition to this, it caused the probe to always touch the paper at the same vertical inclination and with the same pressure thus eliminating possible sources of variation in obtaining data.

Figure 8 illustrates the probe and holder in position to measure the voltage on a model.

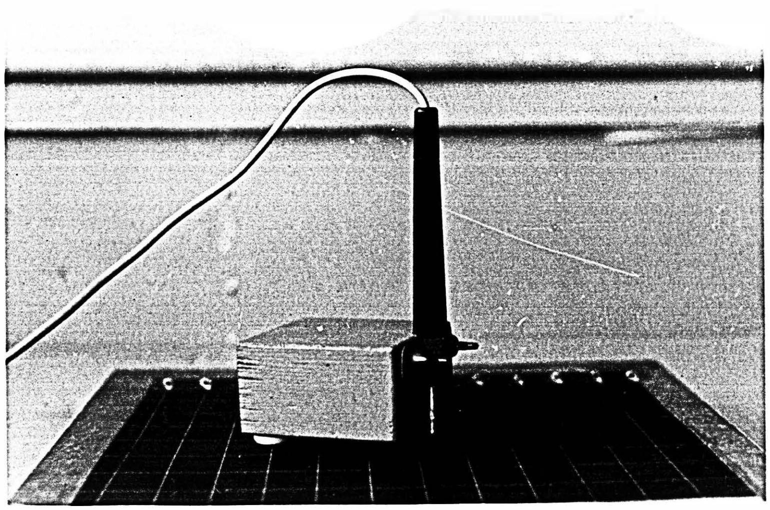

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Figure 8

Voltmeter Probe with Holder

D. CALIBRATION.

As with any experimental procedure the proper calibration of the instrumentation and fixed parameters was of utmost importance for good results. Unfortunately, there was little equipment available to properly calibrate the vacuum tube voltmeter. The best that could be obtained was the comparison of the voltmeter in the console to another voltmeter that was assumed to be accurate because it had provided good results in the past. The ammeter was calibrated with the aid of standard batteries and a precision resistor. It was found to be accurate within its smallest scale division over most of its range.

In order to compare the results of the analogy to the original equation the effective resistance of the paper had to be known. To determine the effective resistance, calibration squares were taken from the roll of Teledeltos paper and prepared so that the nominal resistances along and across the roll could be determined. From the calibration squares shown in figure 9 the ratio of the resistance along the roll R_ℓ, to the resistance across the roll R_t, was about 0.84. This ratio remained constant throughout the roll and from one day to the next even though changes in humidity caused the nominal resistance to change significantly. Once this ratio was determined, properly distorted calibration squares were made to measure the effective resistance of the paper. Figure 10 shows

the "equivalent" calibration squares that were used with properly distorted models. The effective resistance determined from these equivalent squares varied from 1600 ohms per equivalent square to 2200 ohms per equivalent square. The large difference in resistance between calibrations was due to changes in the relative humidity.

It was obvious from these results that the models would have to be calibrated with an effective resistance measurement each time a problem was solved. Solution of the same problem on several different days indicated that the results were consistent even though the effective resistance varied significantly. Therefore, the calibration technique for the models seemed quite sufficient.

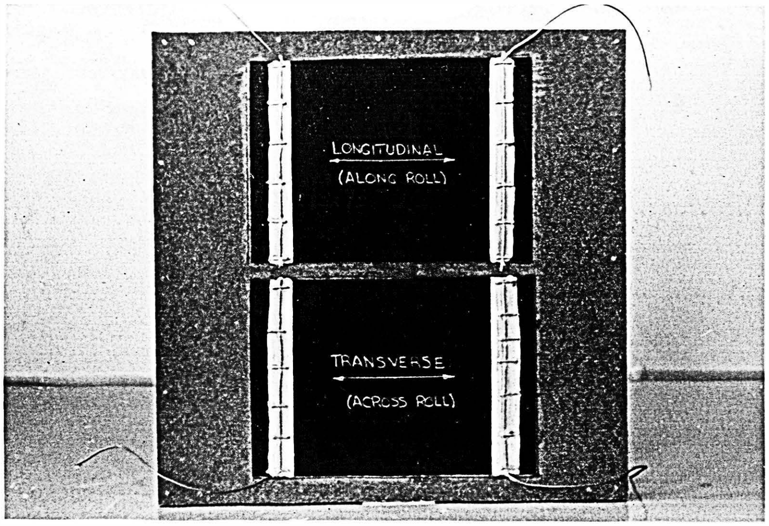

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Figure 9

Undistorted Calibration Squares

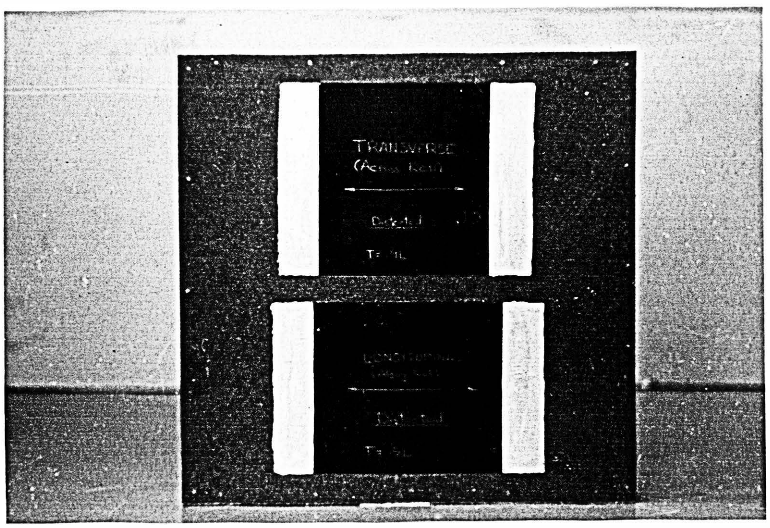

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Figure 10

Distorted "Equivalent" Calibration Squares

E. SOLVED PROBLEMS.

Four experiments were performed to demonstrate the use and versatility of the console and also to point out the applicability of the conducting field analogy. Two types of problems were solved. The first type was steady-state heat conduction to demonstrate the use of the analogy in solving the Laplace equation. In the second case elastic torsion problems were solved to illustrate the solution of the special case Poisson equation previously described.

1. Isothermals in a 90° Angle [2]

The first steady-state heat conduction problem solved was the location of isothermal lines in a symmetrical right angle of infinite length. The outside dimensions were 20 units by 20 units, and the thickness was 10 units. The outside edges of the angle were assumed to be perfectly insulated while a temperature difference of 50 degrees was imposed between the inner and outer boundaries. The physical situation this problem represents is shown in figure 11.

The describing equation for the temperature distribution on any cross section parallel to the xy plane is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

Where T, the temperature, is a function of the coordinates x and y. This equation is analogous to the equation derived for the conducting field if the distributed current

[2] Structural Angle.

i_g, is set equal to zero. Thus,

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} = 0.$$

E is the voltage distribution as a function of x and y. From the similarity of the two equations it is obvious that T is proportional to E. This relation makes possible the determination of many sets of isothermals from only one set of equipotential lines by variation of the arbitrary constant of proportionality between T and E.

The first step in the solution by analogy was the construction of a suitable model. Figure 12 shows the model that was constructed for this problem. In figure 13 the entire experimental set-up including the model, the Power-Instrument console, and the measuring probe is shown.

Fifty volts were applied to the inside boundary of the angle while the outside boundary was held at ground potential. By letting one volt be equivalent to one degree of temperature this voltage difference corresponded to a temperature difference of 50 degrees between the boundaries.

Figure 14 is a plot of the isothermal lines determined from the analogy. Point by point comparison of these data with that obtained by numerical means indicated a maximum deviation of less than 8%.

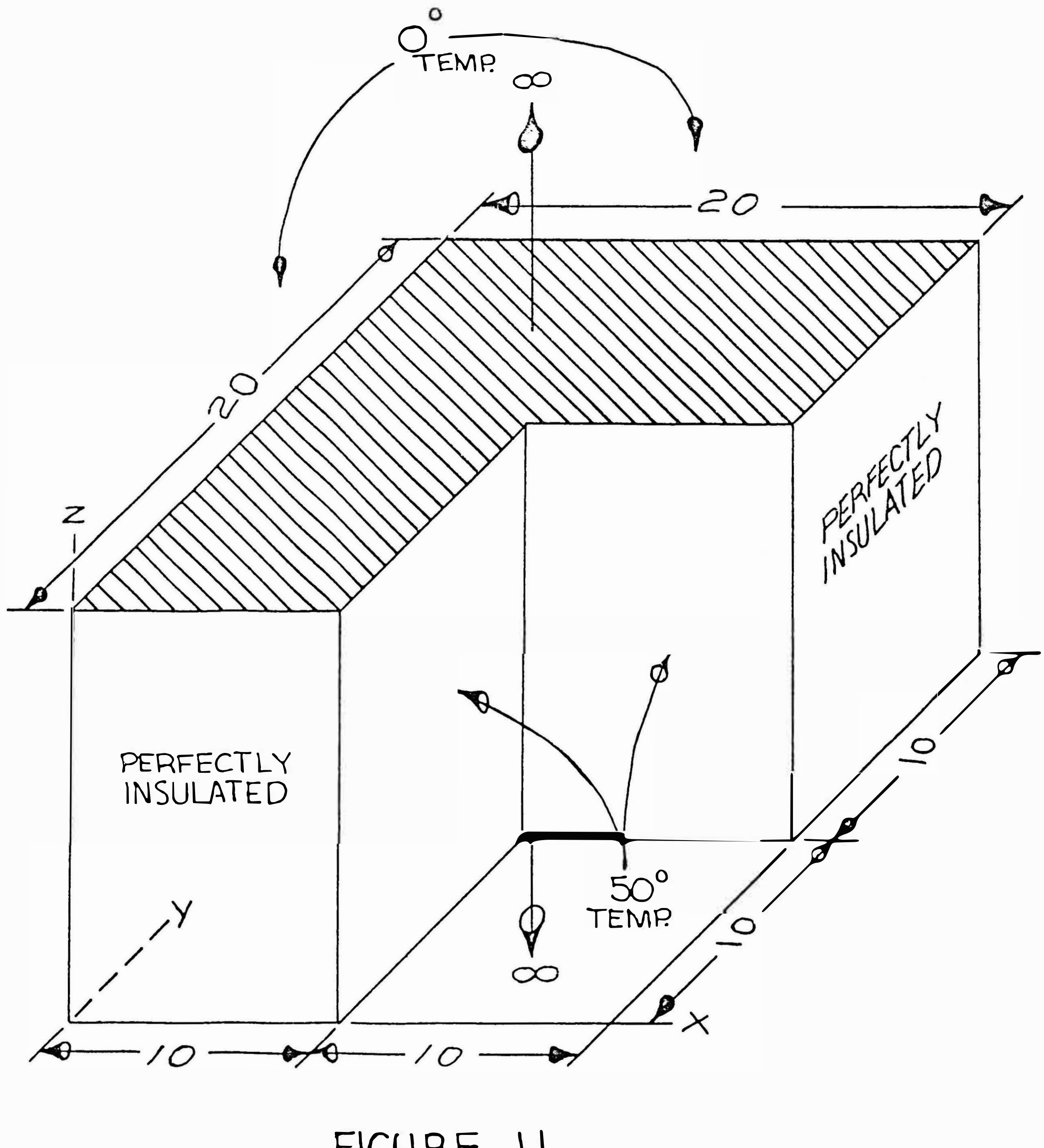

FIGURE 11
STEADY STATE HEAT CONDUCTION THROUGH A 90° ANGLE

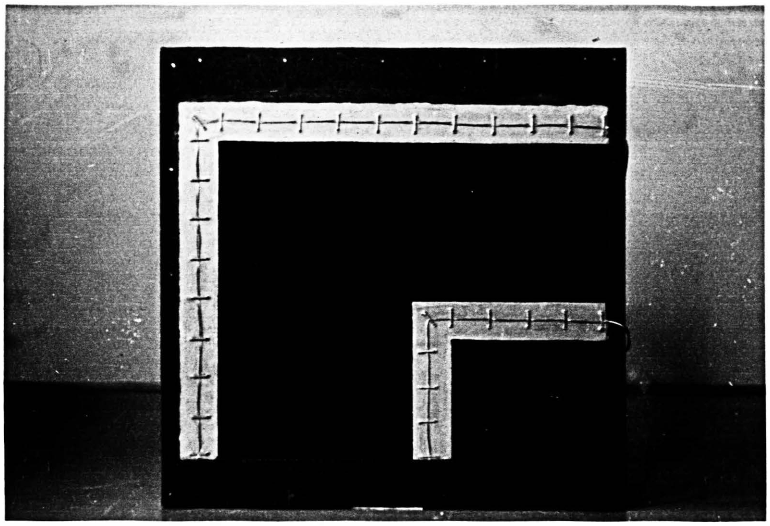

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Figure 12

Model for Steady-State Heat Conduction Through a 90° Angle

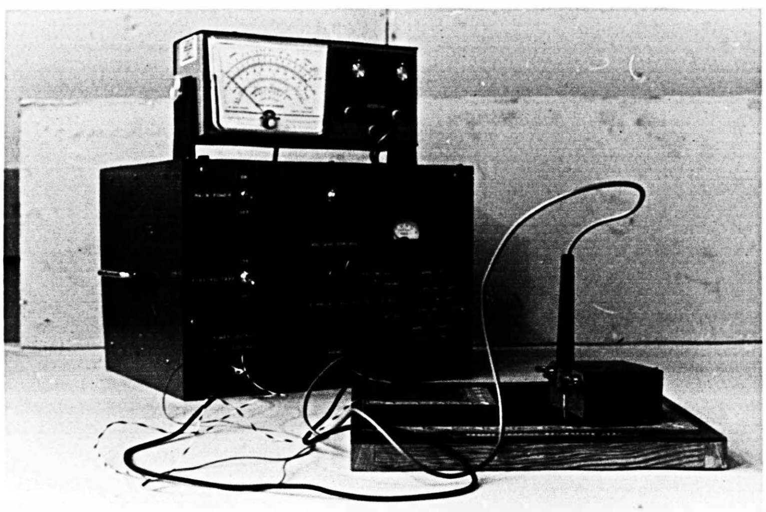

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Figure 13

Experimental Apparatus for Steady-State Heat Conduction Through a 90° Angle

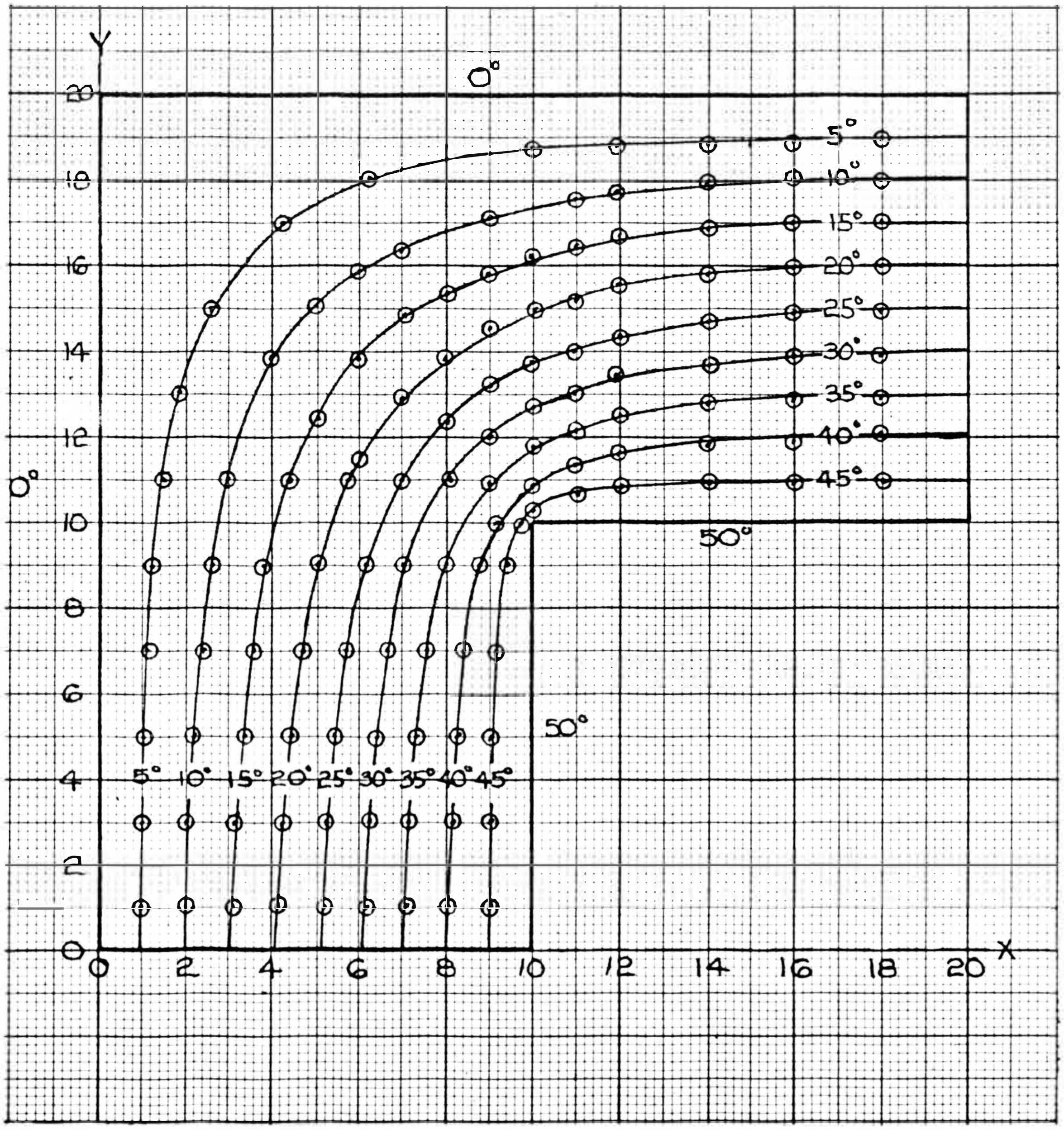

FIGURE 14
ISOTHERMAL LINES FOR STEADY STATE HEAT CONDUCTION THROUGH A 90° ANGLE

2. Isothermals in a Rectangular Plate with Variable Boundary Temperature

The preceding problem was quite easily adapted to solution by the conducting field analogy because the boundary conditions were constant. To illustrate that the console operated analogy has versatility, a variable boundary heat conduction problem was solved. This problem consisted of finding the location of isothermals in a homogeneous plate 12 units long by 8 units wide. One of the long edges of the plate was held at a constant temperature of zero degrees while the opposite edge was subjected to a uniformly changing temperature distribution which varied from zero degrees at one end to 48 degrees at the other end. The remaining ends and surfaces were perfectly insulated. Figure 15 illustrates the physical representation of the problem.

As in the preceding problem, the equation describing the temperature distribution in this plate is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0,$$

and the analogous electrical equation is

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} = 0.$$

The obvious difference is in the boundary conditions that are applied.

The model was prepared and the varying boundary condition approximated by attaching wires to the paper

along the boundary in a manner similar to that used to apply the point current sources. These wires were evenly spaced one for every unit of length as shown in figure 16. The completed model was connected to the Power-Instrument console and also to the Current Regulator console. The potentiometer circuits were used to vary the voltage at the contact points on the variable boundary. The voltage at each of these contact points was adjusted to be 4 volts greater than the voltage of the preceding point. This provided a distribution from 0 to 48 volts in 4 volt increments. The adjustment of these voltages had to be repeated several times before the desired distribution was obtained. This was caused by the coupling between the point sources that produced a change in all the point voltages when one was adjusted. About 4 to 5 cycles of adjustment seemed to be the most required to obtain the proper results. Checks with the measuring probe indicated that this approximation was quite good just a short distance away from the boundary. Figure 17 illustrates the completed experimental set-up.

Figure 18 is a plot of the isothermals determined from the analogy. Point by point comparison of this data with that obtained by numerical methods indicated a maximum deviation of less than 6%.

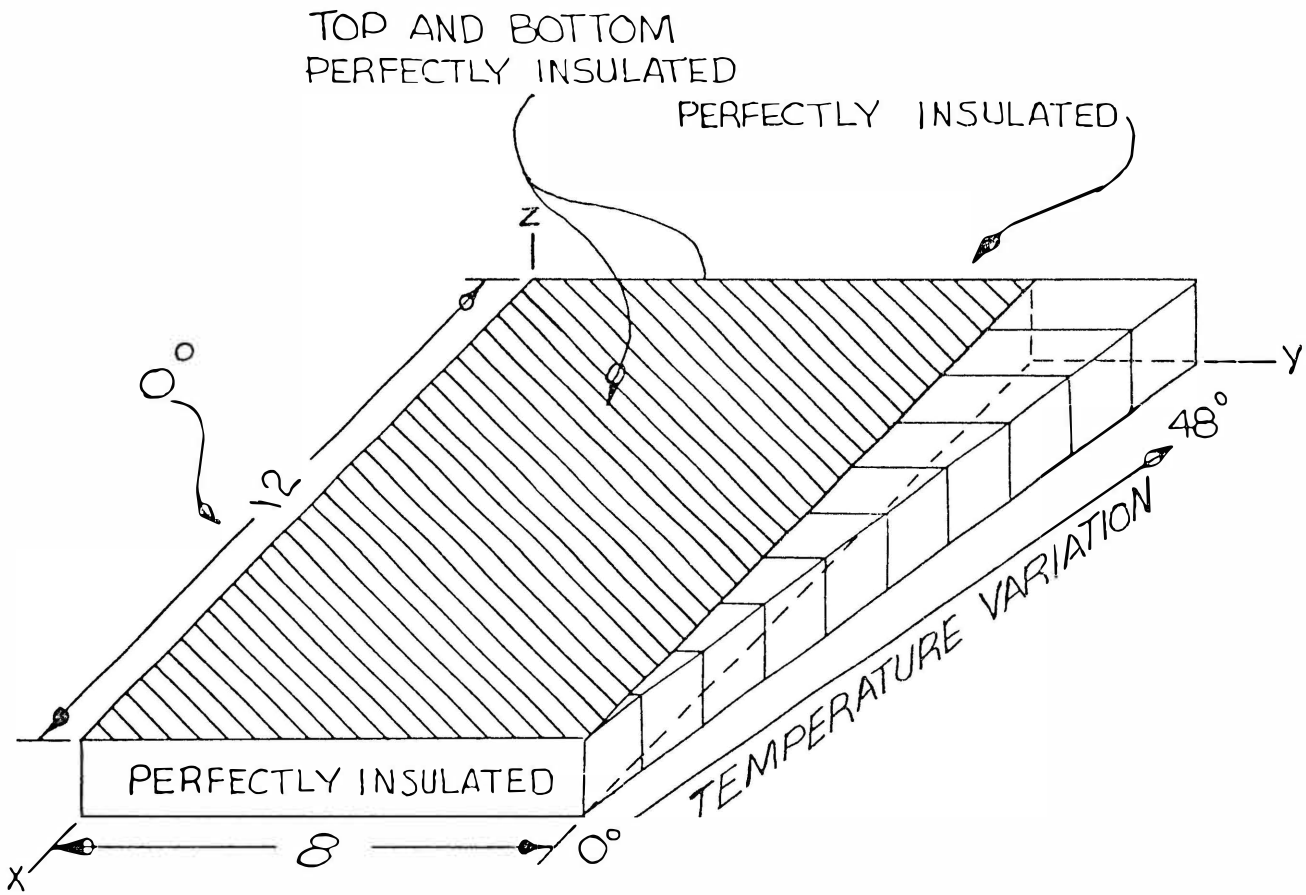

FIGURE 15
STEADY STATE HEAT CONDUCTION
THROUGH A SLAB WITH LINEAR
BOUNDARY TEMPERATURE DISTRIBUTION

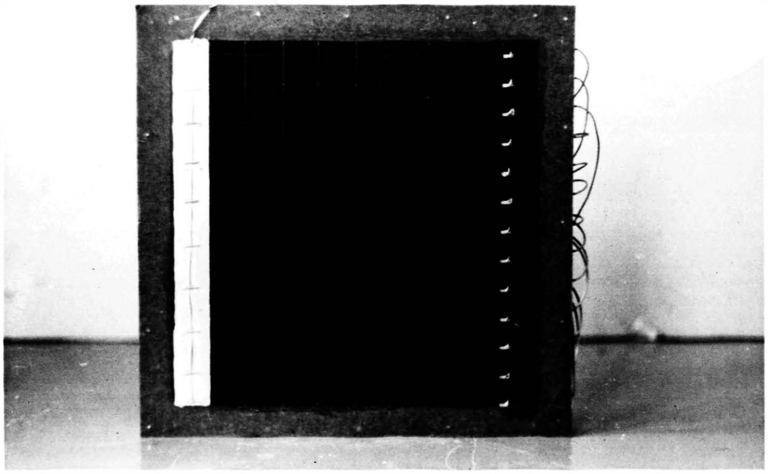

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Figure 16

Model for Steady-State Heat Conduction Through a Slab

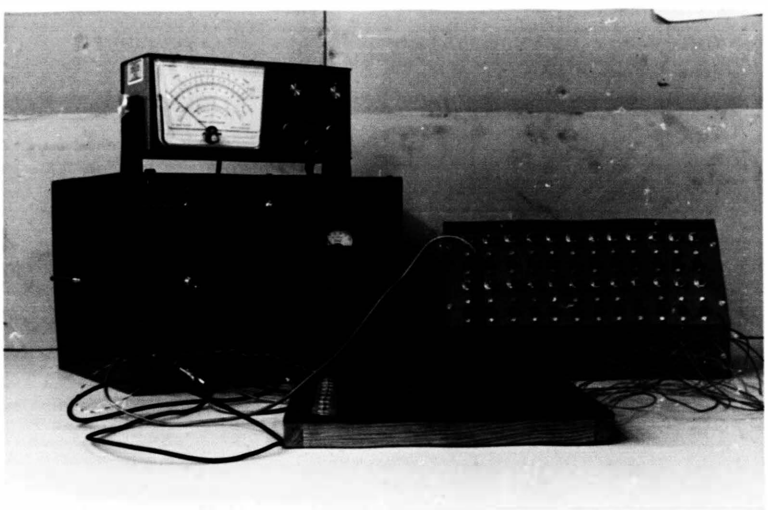

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Figure 17

Experimental Apparatus for Steady-State
Heat Conduction Through a Slab

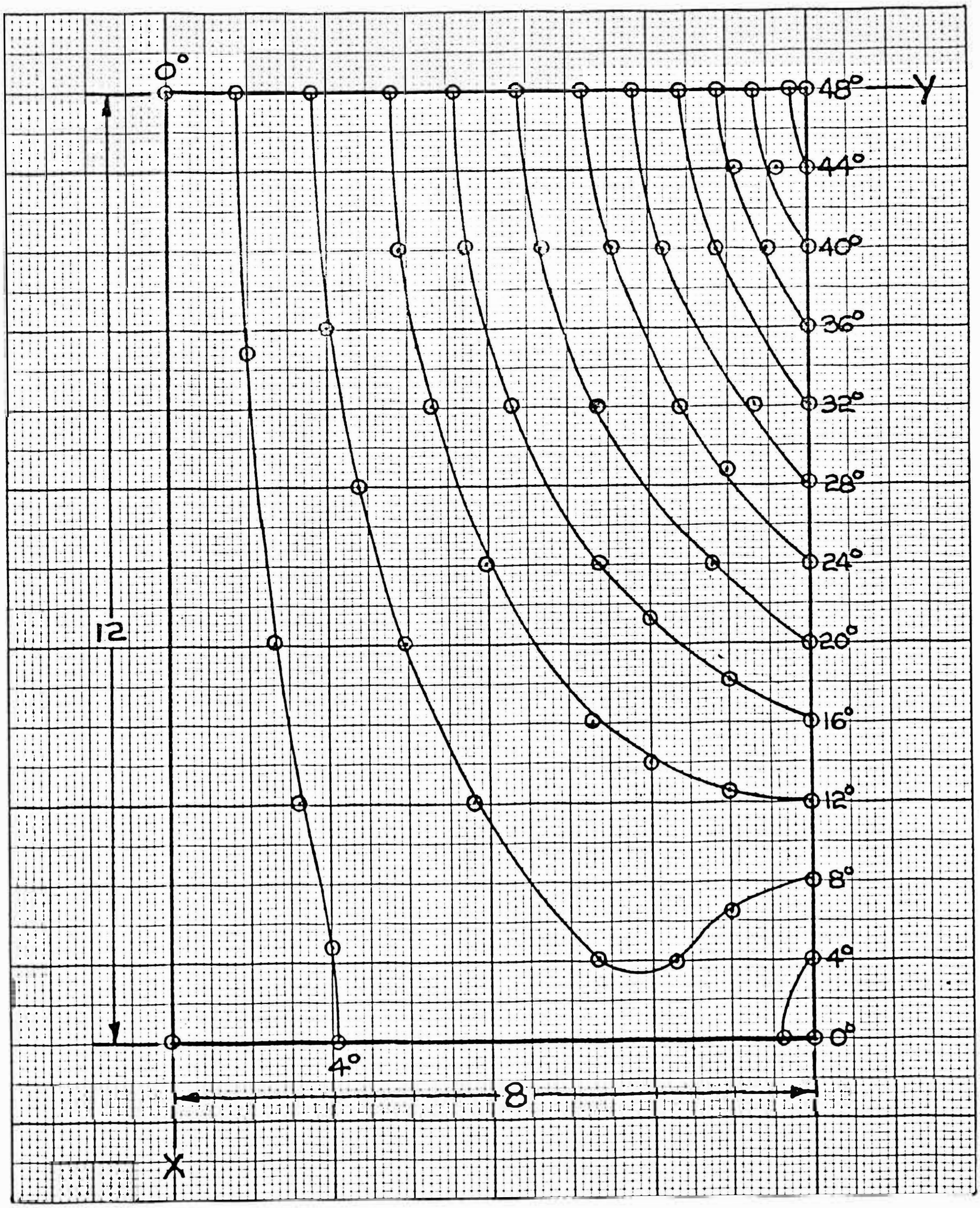

FIGURE 18
ISOTHERMAL LINES FOR STEADY STATE
HEAT CONDUCTION THROUGH A SLAB

3. Torsion of a Solid Rectangular Bar

The elastic torsion of a solid rectangular bar was chosen to illustrate the ability of the model and console arrangement in solving the Poisson type equation. As shown in figure 19, the particular problem selected was a homogeneous bar of cross section 13 units by 6 units subjected to some torque, T, which produced an angle of twist in the bar of θ radians per unit of length. The information desired was the values of the maximum shearing stress and the applied torque in terms of the angle θ and the modulus of rigidity of the material, G.

The describing equation for this problem is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta. \qquad (4)$$

Where ϕ is an arbitrary "stress" function such that the shearing stresses τ_{xz} and τ_{yz} are equal to $\frac{\partial \phi}{\partial y}$ and $-\frac{\partial \phi}{\partial x}$ respectively.

To aid in the understanding of these definitions it is convenient to plot the magnitude of the stress function ϕ versus x and y. This plot in three dimensions will produce a hill shaped surface covering the cross section in question. The conditions imposed upon ϕ are usually such that it is zero at the boundaries of the cross section. The shearing stresses τ_{xz} and τ_{yz} are equal in magnitude to the slope of this surface in the x or y directions respectively. From the Theory of Elasticity it can be

shown that the torque applied to the cross section is

$$T = 2\iint \phi \, dx\, dy \quad ,$$

which is twice the volume under the three dimensional surface described by the stress function. The usefulness of these relations is demonstrated when the results from the electrical analogy are to be compared with the physical problem described.

The analogous electrical field equation is

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} = -i_g R \qquad (5)$$

where R is the effective resistance of the paper and i_g the distributed current. Comparison of equations 4 and 5 reveals that R is analogous to 2G, i_g is analogous to θ, and E is analogous to ϕ. Since the value of the function ϕ may be arbitrarily assigned the value of zero at the external boundaries of the cross section, the electrical boundaries may be set at zero or ground potential.

Using the preceding information the model shown in figure 20 was prepared. Because of the symmetry of the stress function ϕ over the symmetrical cross section, only one fourth of the cross section was modeled. The point current sources were attached in the center of the effective 3 by 3-1/4 inch grid spaces and the model attached to the Power-Instrument and Current Regulator consoles. The experimental arrangement was similar to that shown in

figure 27. The effective resistance, R, was measured from calibration squares and found to be about 1800 ohms per square. Using this value and applying a distributed current of 2 milliamperes per square, data were taken and reduced.

In this particular type of cross section the maximum shearing stresses occur at the center outside fibers of the long sides. These shearing stresses were determined by the following procedure. The magnitude of the voltage function, E(x,y), was plotted versus x when y was zero as shown in figure 21. The maximum slope of the voltage function, $\frac{\partial E}{\partial x}$, was found graphically and related to the physical problem as follows:

$$K i_g R = 2G\theta,$$

$$K = \frac{2G\theta}{i_g R},$$

$$K \frac{\partial E}{\partial x} = \frac{\partial \phi}{\partial x},$$

where K is the proportionality constant between the electrical analogy and the physical system. Substitution of the values:

$$i_g = (2 \times 10^{-3}) \text{ amps},$$

$$R = 1800 \text{ ohms, and}$$

$$\frac{\partial E}{\partial x} = 10 \text{ volts/unit},$$

yielded

$$\tau_{yz} = -\frac{\partial\phi}{\partial x} = -\frac{2G\theta\ (10)}{(2\times10^{-3})(1800)} = -5.55G\theta.$$

This compared favorably with the value $\tau_{yz} = -5.66G\theta$ determined from the analytical method described on page 271 of Seely and Smith (13) for a cross section 13 units by 6 units. Data taken from several different runs of the experiment produced results for maximum stress with consistently less than 4% deviation from the calculated value.

Determination of the torque applied to the cross section proved to be more difficult. The determination of the volume enclosed by the voltage function, E(x,y), could only be determined by approximation. For purposes of comparison two different approximations were used. The first approximation required the arrangement of the data to cause the three dimensional representation of the voltage function to appear as a series of tapered slabs placed together as shown in figure 23. The volume of each slab was determined by measuring the areas of each side of the slab from figure 22, averaging these areas, and multiplying by the distance between the areas. The slab volumes in volt-units^2 determined in this manner were added together and multiplied by 8 to find twice the total volume under the voltage function curve. Multiplying this volume by the proportionality constant K, the torque in terms of G and θ was determined. The experimental Torque was 635 $G\theta$ compared

to 660 Gθ calculated using methods described in Seely and Smith (13), a deviation of 3.8%.

It was obvious that a portion of the deviation in the previous method was due to the approximation used in determining the volume. To obtain some idea of the inaccuracy caused by this approximation a second method of approximation was used to determine the volume. Since the first approximation was too small, the second approximation was chosen to be too large to find a range within which the actual volume would occur. To accomplish this the second method employed the slab technique used in the first method with one exception. The slabs were considered to be of uniform area as shown in figure 24 rather than tapered as in the preceding method. Using the same technique as in the first method of approximation the experimental torque determined was 692 Gθ compared to the calculated value of 660 Gθ. Thus, the true value of the torque was between the two approximations. Arbitrarily averaging the two approximations the deviation of the average from the calculated value was found to be consistently within 3%.

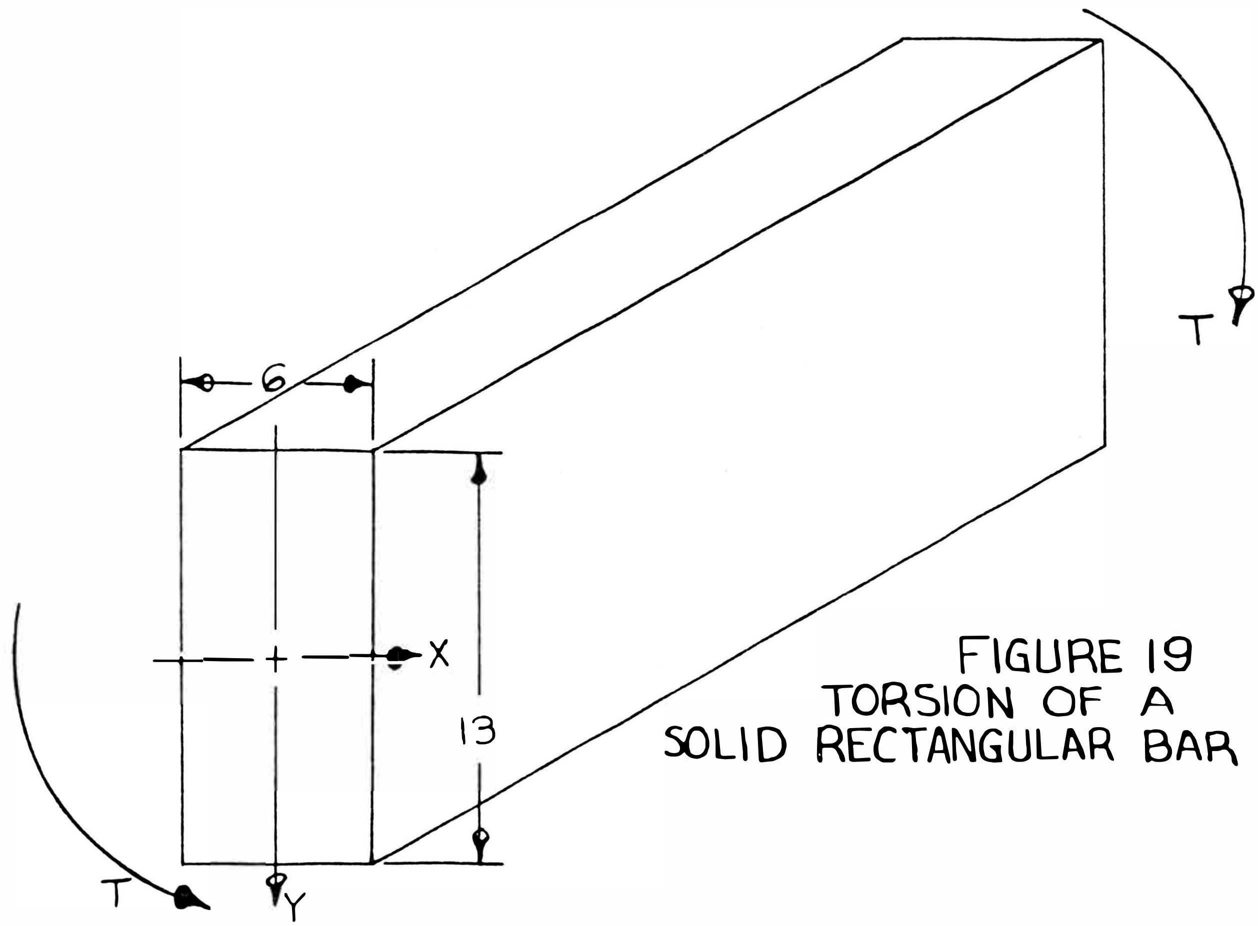

FIGURE 19
TORSION OF A
SOLID RECTANGULAR BAR

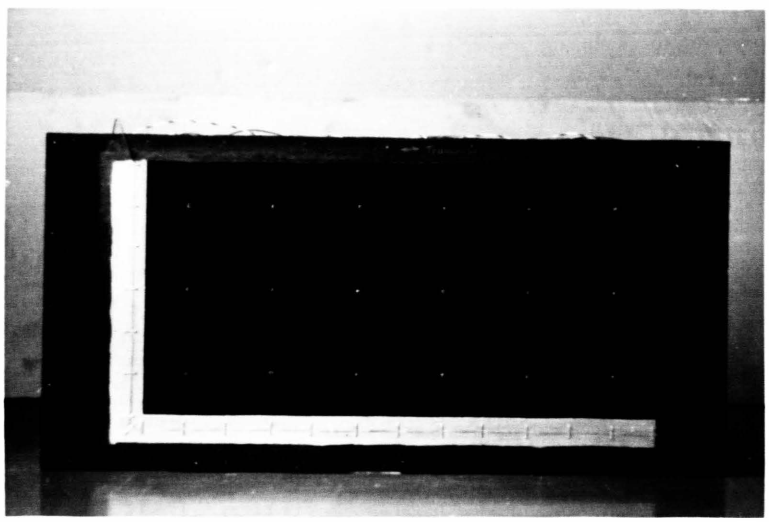

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Figure 20

Model for Torsion of a Solid Rectangular Bar
(Effective Grid Net 1 by 1.08 units)

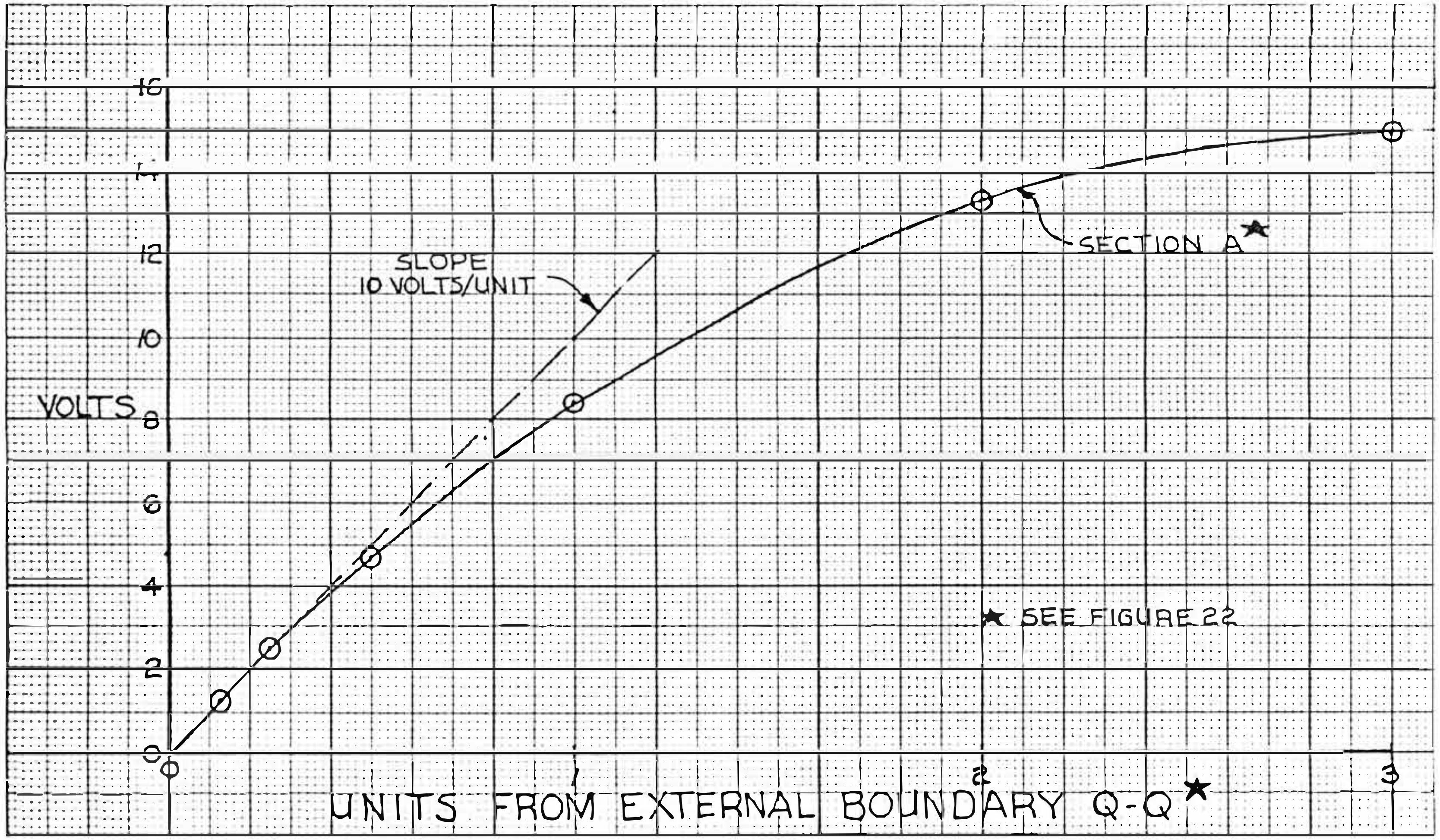
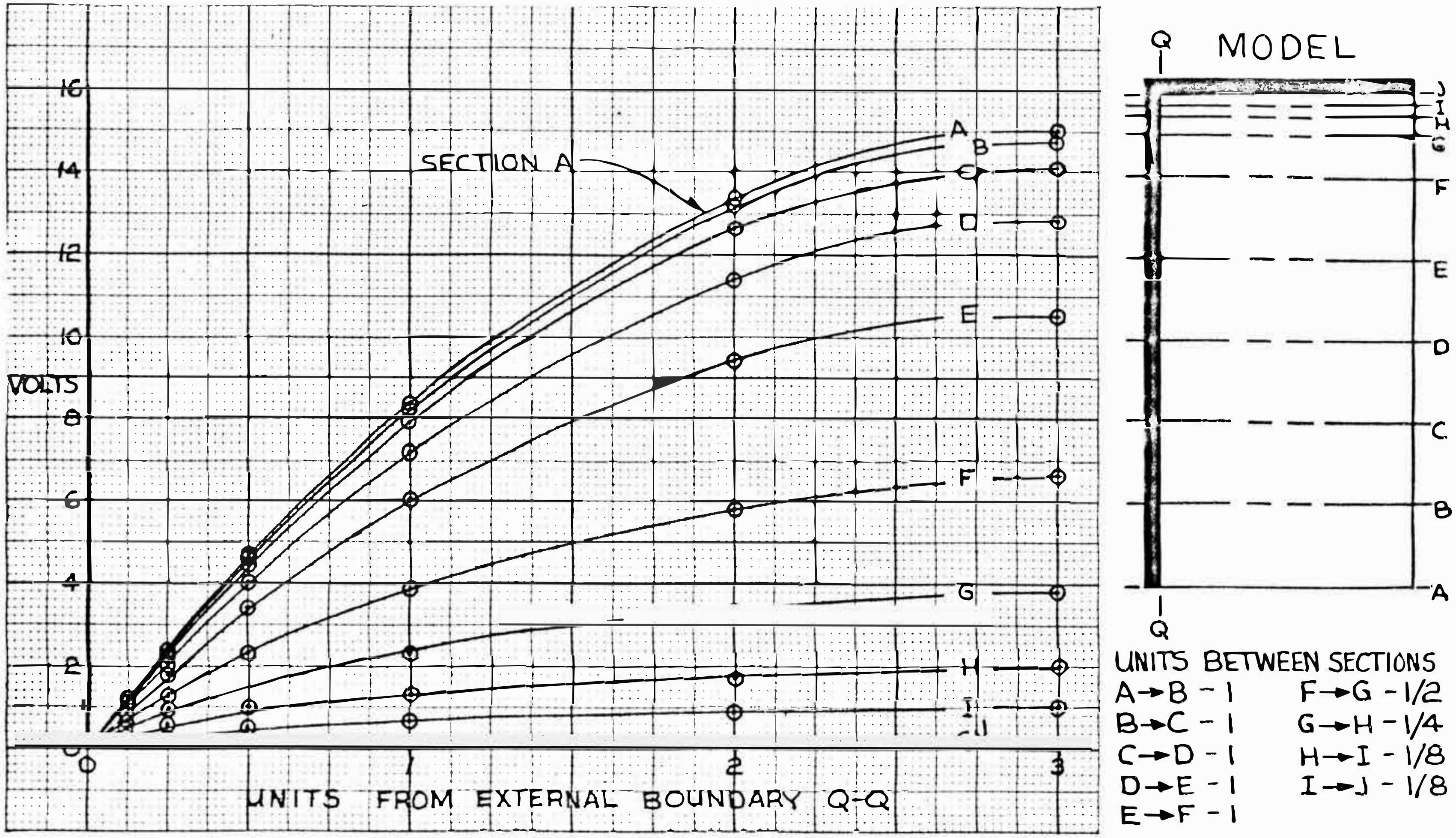


FIGURE 21
VOLTAGE FUNCTION NEAR POINT OF MAXIMUM STRESS FOR TORSION OF A SOLID RECTANGULAR BAR



UNITS BETWEEN SECTIONS

A→B - 1	F→G - 1/2
B→C - 1	G→H - 1/4
C→D - 1	H→I - 1/8
D→E - 1	I→J - 1/8
E→F - 1	

FIGURE 22
CROSS SECTIONS THROUGH VOLTAGE FUNCTION FOR TORSION OF A SOLID RECTANGULAR BAR

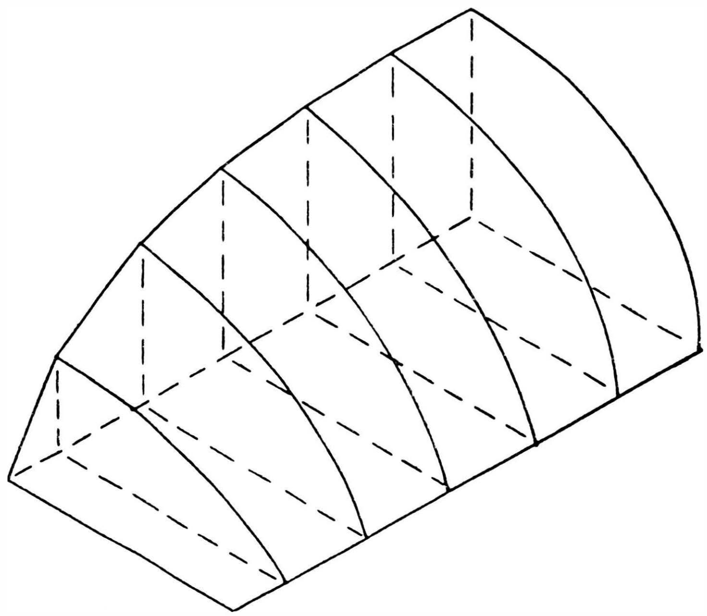


FIGURE 23
FIRST METHOD OF APPROXIMATION FOR THE THREE DIMENSIONAL VOLTAGE FUNCTION (TAPERED SLABS)

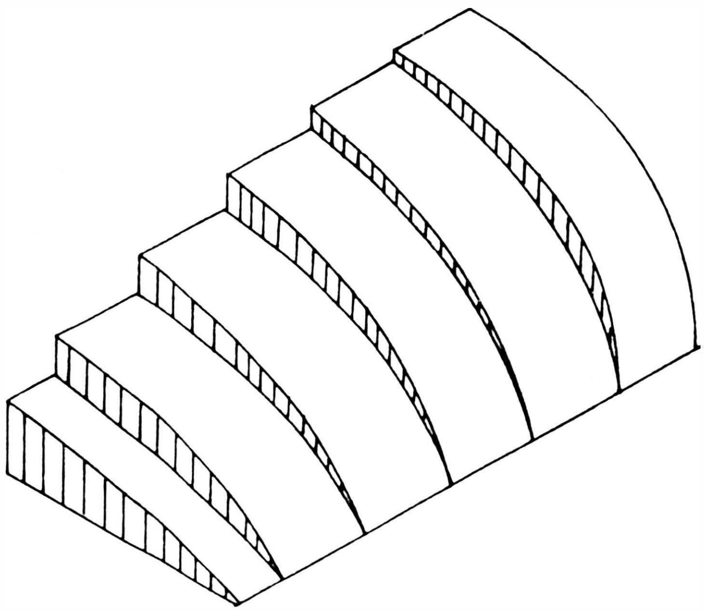


FIGURE 24
SECOND METHOD OF APPROXIMATION FOR THE THREE DIMENSIONAL VOLTAGE FUNCTION (FLAT SLABS)

4. Torsion of a Rectangular Bar with a Rectangular Hole

The last problem solved was the elastic torsion of a rectangular bar with a symmetrical rectangular hole as shown in figure 25. This problem was chosen to illustrate the use of the analogy in solving problems which are very difficult to describe and solve mathematically.

The physical concept of the stress function ϕ for this problem is the same as that for the solid shaft except for one modification. The ϕ surface over the hole must have zero slope since no stresses can be developed where there is no material. For the ϕ surface to have zero slope the value of ϕ over the area of the hole must be constant. This means that the surface over the hole must be a plane surface. This condition is accomplished easily with the electrical analogy by simply painting the area of the hole with conducting paint. Since the conducting paint has negligible resistance the voltage over the entire painted area must be the same, thus simulating the desired physical conditions.

Using these concepts the model shown in figure 26 was constructed. This model was then connected to the Power-Instrument and Current Regulator consoles as shown in figure 27. From this experimental set-up data similar to those for the solid bar were taken and reduced.

From figure 28 the maximum shearing stress of $4.20\ G\theta$ was found to occur at the outside fiber of the center of

the long side of the cross section. Of significant interest was the fact that at the outer fiber of the center of the short side a shearing stress of 3.51 $G\theta$ was found from figure 29. This represented a considerable increase in stress at this point over the solid bar with an equivalent torque applied. With the aid of figures 28 and 30 the average torque was found to be 340.5 $G\theta$ using the methods described in the previous problem.

Unfortunately, there was no simple mathematical solution available to determine the accuracy of the experimental results. However, a point by point comparison with an iterated solution yielded a deviation consistently less than 5% for several different experimental runs.

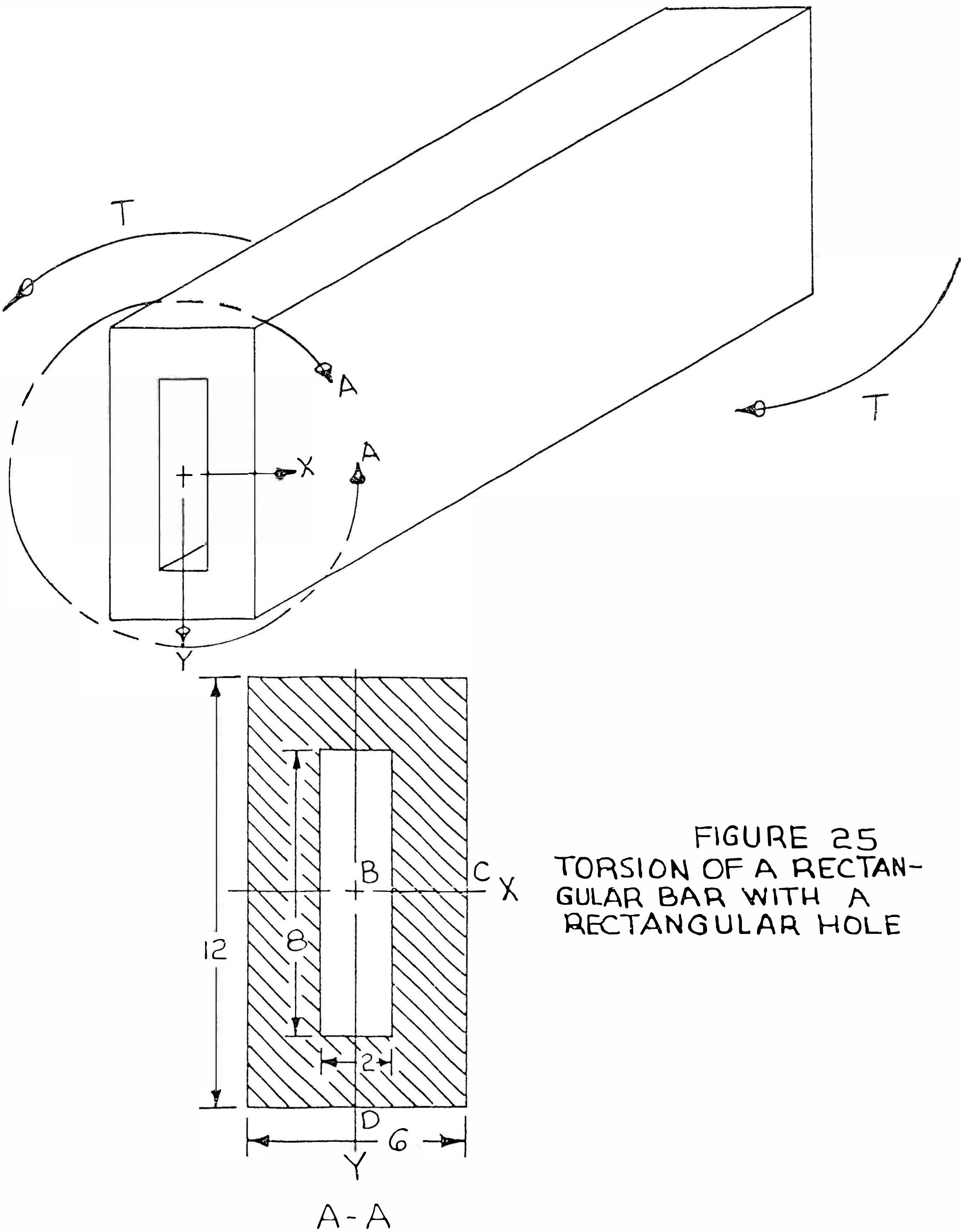


FIGURE 25
TORSION OF A RECTANGULAR BAR WITH A RECTANGULAR HOLE

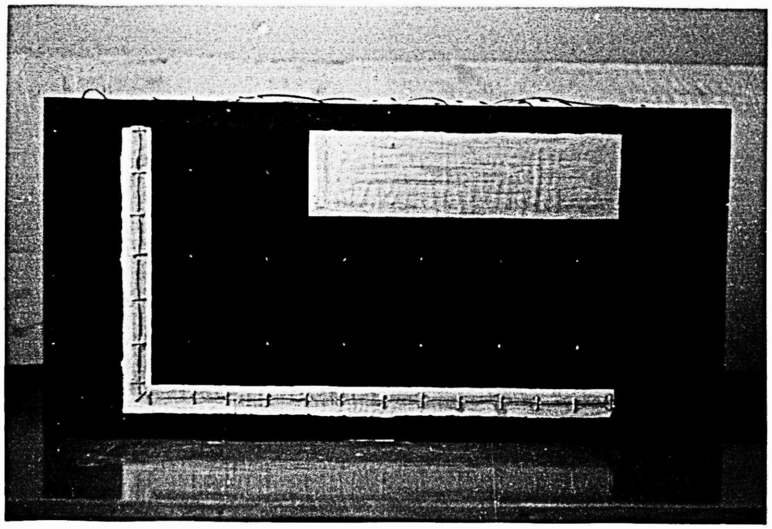


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Figure 26

Model for Torsion of a Rectangular
Bar with a Rectangular Hole

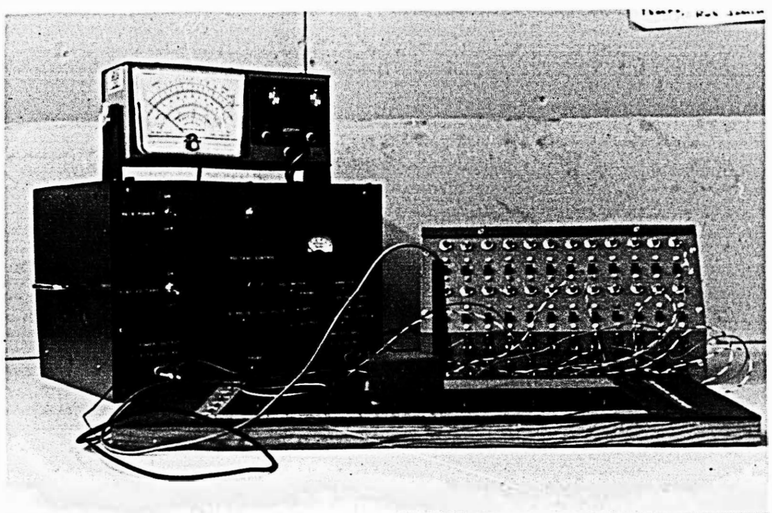

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Figure 27

Experimental Apparatus for Torsion
of Rectangular Bars

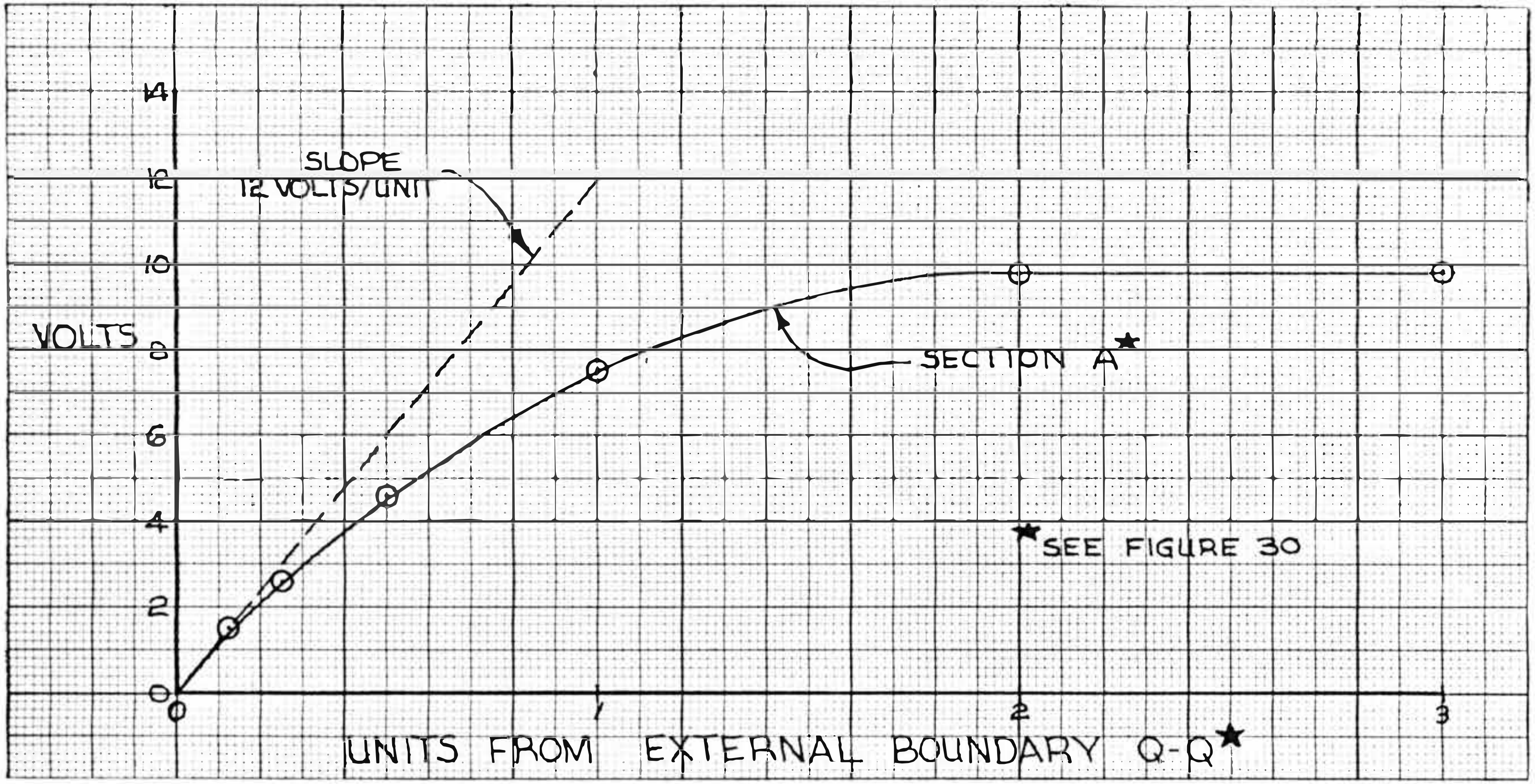

FIGURE 28
VOLTAGE FUNCTION NEAR POINT OF MAXIMUM STRESS
FOR TORSION OF A RECTANGULAR BAR
WITH A RECTANGULAR HOLE

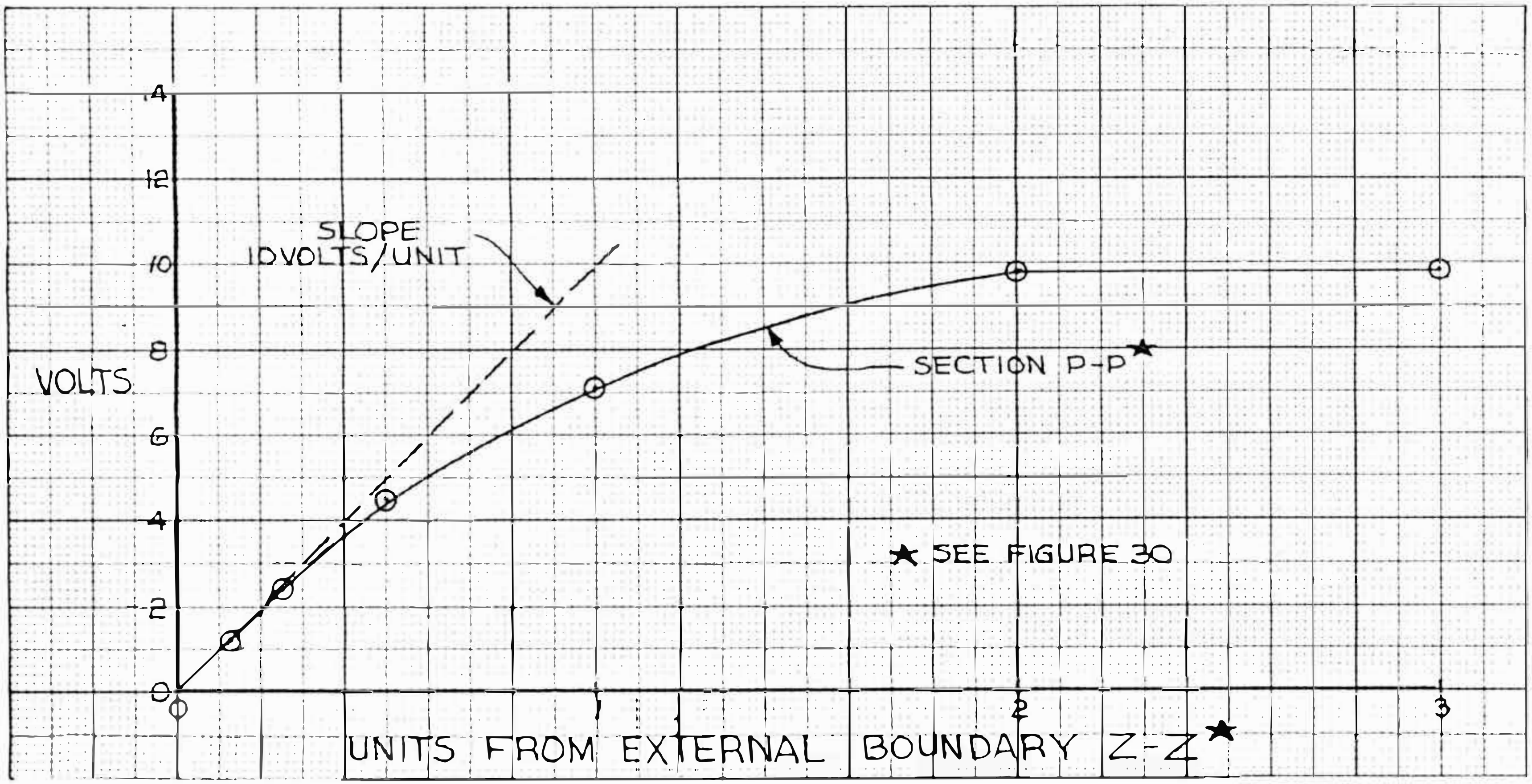

FIGURE 29
VOLTAGE FUNCTION NEAR POINT OF LARGE RELATIVE STRESS
FOR TORSION OF A RECTANGULAR BAR
WITH A RECTANGULAR HOLE

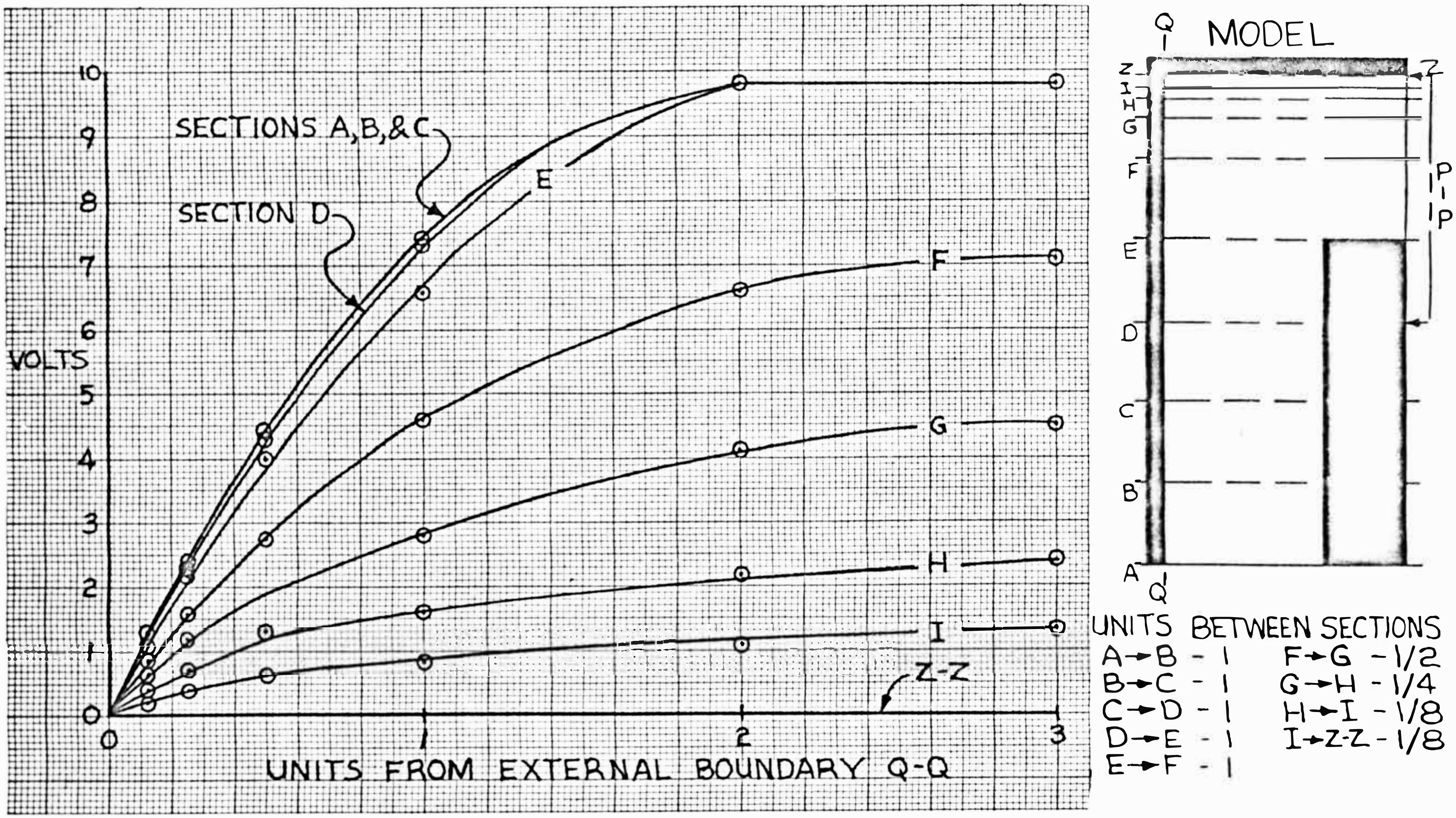

FIGURE 30
CROSS SECTIONS THROUGH VOLTAGE FUNCTION FOR TORSION OF A RECTANGULAR BAR WITH A RECTANGULAR HOLE

IV. CONCLUSIONS

The conducting sheet analogy is a tool of proven accuracy, and there is no doubt about its application to certain types of problems. The console arrangement allows the analogy to be used in a form similar in concept to the analog computer. This technique provides acceptably consistent results of sufficient accuracy for all but the most critical applications.

The analogy itself has several advantages that are enhanced by the use of the console arrangement. The experimental equipment involved is relatively inexpensive and easy to assemble. The analogy is particularly suited to the solution of problems which have unusually shaped physical boundaries since the boundaries may be easily constructed when cutting the paper model. This ease of construction may be contrasted to the extreme difficulty encountered in attempting to approximate these same boundaries to employ a digital computer for an iterative solution. Another advantage the analogy has over iterative methods is the relative ease with which critical points in a properly constructed model may be investigated by slight manipulations of the measuring probe. To obtain the equivalent data from a digital computer requires the use of a very fine iteration net. When choosing a fine net the number of points that must be iterated becomes impractically large, and excessive hours of machine time

are required. An additional feature of the analogy is that it allows the experimenter to move the measuring probe rapidly over the model and obtain a "physical" concept of the three dimensional surface that represents the function being studied.

There are some very definite limitations and disadvantages to the analogy-with-console arrangement. Obviously the set-up is valid only for problems whose mathematical description is similar in form to the describing equation for the conducting sheet. The analogy is most easily adapted to problems in which the variable function is either constant along the boundaries or independent of the boundary dimensions. These conditions may be simulated exactly in the analogy while other types of boundary values must be approximated. Another limitation is the accuracy of the voltmeter and ammeter employed in the console. The least accurate readings of these instruments provide quantities with only two significant figures, and the accuracy of these numbers is subject to the accuracy of the measuring equipment preceding the meters. However, the sample problems seem to indicate that the total deviations that may be expected are usually less than 10%.

Although the possibility was not investigated in this work, the use of the Power-Instrument and Current Regulator consoles should allow the approximation of the general

two dimensional Poisson equation

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = F(x,y) .$$

The analogous electrical equation is

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} = -iR ,$$

where i is a function of x and y and E and R are unchanged from their previous definitions. Thus, any physical problem which can be mathematically described by this Poisson equation could possibly be solved with the aid of the conducting sheet analogy.

APPENDIX

Operation of Power-Instrument Console

The operation of the Power-Instrument console is relatively simple. By switching the main power switch into the on position the tube filaments and voltmeter are energized. Moving the high voltage switch into the on position produces from zero to 300 volts D.C. at the power output terminals. The exact regulated voltage is determined by the position of the voltage control potentiometer. To determine the output of the power supply the voltmeter range and function switches should be set to measure up to 500 volts D.C. before the voltmeter and ammeter switches on the front panel are moved to the output position. The readings that appear on the voltmeter and ammeter are the output voltage and current of the supply. The voltmeter is read in the same manner as any standard V.T.V.M. and the ammeter indicates 5 milliamps per division for a total of 250 milliamps full-scale. When the voltmeter and ammeter function switches are in the off position the two instruments may be adjusted for zero drift while the console is in operation. In the probe position the voltmeter function switch places the probe jack on the front panel at the input of the voltmeter. The input position of the ammeter function switch places the meter movement across the ammeter input terminals on the front panel. In this position the meter reads one-half milliamp per division for 25 ma. full scale.

From time to time the voltage limits of the power supply must be reset due to drift caused by the aging of tubes and components. Two controls on the back of the console are provided for this purpose. (Figure 31 shows their location.) The adjusting procedure is as follows:

1. Switch on main power and high voltage.
2. Set voltmeter for D.C., 500 volt full scale.
3. Set voltage control in the full clockwise position.
4. With no load on the power output adjust the 250 K, 300 volt set potentiometer, shown in figure 31, until the voltmeter indicates 300 volts.
5. Set the voltage control in the full counter clockwise position.
6. Adjust the voltmeter for D.C., 1.5 volts full scale.
7. Adjust the 500 K, zero set potentiometer until the voltmeter indicates zero.
8. Check the 300 volt limit again and make adjustments if necessary.
9. Repeat the entire sequence until no further adjustment is required.

For problems requiring the use of the Current Regulator console the following connections between the Power-Instrument console and the Current Regulator console must be made.

1. Connect the power output of the Power-Instrument console to the power input of the Current Regulator

being sure to observe proper polarity. (red - positive, black - ground).

2. Connect the terminals labeled "To Ammeter Input" on the Current Regulator console to the terminals labeled "Ammeter Input" on the Power-Instrument console, again being sure to observe proper polarity.
3. Connect the input wires from the model to the spring clips on the Current Regulator console. (The upper row of spring clips is controlled by the upper row of potentiometers and switches.)
4. Connect the return current path from the model to the ground (black) side of the Power-Instrument console.
5. Regulate the current flow in each path by moving the slide switch to the up position and adjusting the potentiometer directly above it until the ammeter indicates the desired current flow. The ammeter function switch must be in the input position.
6. Do not exceed the specifications shown on the rear of the Current Regulator console.

When measuring resistances with the ohmmeter of the Power-Instrument console be sure that the selector switch on the probe is in the ohms position.

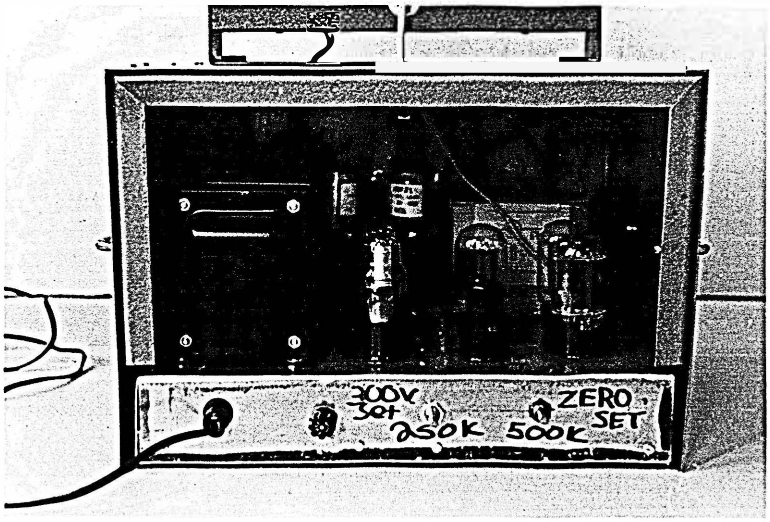

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Figure 31

Rear View of Power Instrument Console

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VITA

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He was appointed Instructor of Engineering Mechanics at the Missouri School of Mines and Metallurgy where he commenced work toward the Master of Science Degree in Mechanical Engineering.

June 30, 1961, he was married to Carol M. Dietz of St. Louis, Missouri.