Lateral Impedance of Single Piles in Inhomogeneous Soil

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ABSTRACT

The flexural stiffness and damping (dynamic impedance) of a single vertical pile in an inhomogeneous soil deposit with continuously increasing stiffness with depth, is studied. An analytical formulation based on the Beam-on-Dynamic-Winkler-Foundation (BDWF) model is implemented. The model is used in conjunction with a virtual work approximation and pertinent shape functions for the deflected shape of a long flexible pile, which are analogous to those used in finite-element formulations. Explicit closed-form solutions are derived for: (1) the dynamic pile stiffness; and (2) the damping coefficient at the pile head. Both swaying and rocking vibrations are considered and all associated impedance coefficients (swaying, rocking, and cross swaying-rocking) are determined. Results from the method are found to be in good agreement with earlier solutions, while new results are developed. The errors resulting from the use of an "equivalent" homogeneous profile with average properties are discussed.

KEYWORDS

Pile, Stiffness, Damping, Impedance, Inhomogeneity, Winkler, Analysis

INTRODUCTION

The assumption of constant soil stiffness with depth is hardly the case in actual soil profiles. The increase in overburden stresses with depth combined with stress-induced non-linearities close to the foundation usually result to a profile whose stiffness increases with depth. Unfortunately, there is a general lack of solutions for embedded foundations in inhomogeneous media. With reference to laterally-loaded piles, solutions to the problem have been published by Banerjee & Davis (1978), Poulos & Davis (1980), Randolph (1981), and El-Marsafawi et al (1992), using numerical finite- and boundary-element formulations. Analytical solutions using Winkler models have also been derived by Hetenyi (1946), Barber (1953), and Franklin & Scott (1979). All the latter solutions refer to the case of a soil modulus that increases proportionally with depth. The scope of this paper is to extend the Winkler formulations to more general classes of inhomogeneous media for which no exact analytical solutions are presently available.

The problem studied is shown in Fig. (1a): a long laterally-loaded pile embedded in a soil profile whose stiffness increases monotonically with depth. The pile is considered a linearly visco-elastic solid cylindrical beam of diameter d, Young's modulus E_p, and linear hysteretic damping β_p. The pile is assumed to be long and flexible, therefore not deforming over its entire length, but only up to a certain depth L_a which is known as the "active" pile length (Randolph 1981). The soil is modeled as a linear viscoelastic material of Young's modulus E_s, Poisson's ratio ν_s, mass density ρ_s, and linear hysteretic damping β_s. Pile-soil interaction is modeled

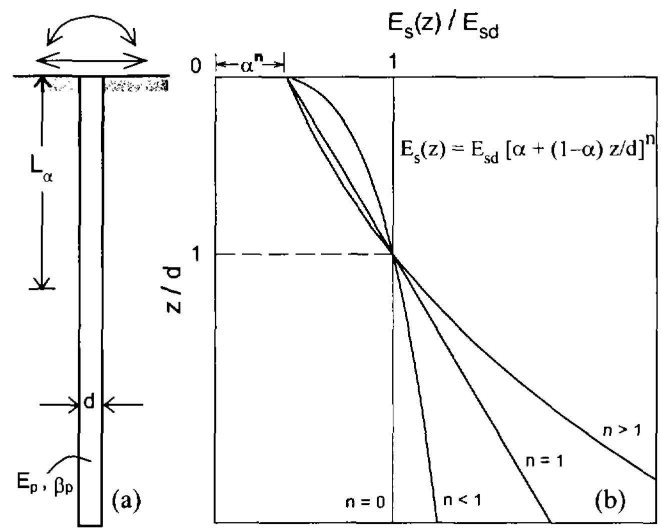

Fig. 1 (a) Problem considered. (b) Variation of soil stiffness with depth.

through a bed of independent Winkler springs and dashpots, uniformly distributed along the pile.

SOIL DESCRIPTION

The type of soil inhomogeneity considered in this paper is (see Fig 1b):

$$E_s(z) = E_{sd}\left[\alpha+(1-\alpha)\frac{z}{d}\right]^n \qquad (1)$$

where E_{sd} denotes the soil Young's modulus at the depth of one pile diameter from the surface (i.e., at z = d), while α, n are dimensionless parameters. As evident from Eqn (1), α stands for the ratio

$$\alpha = \left(\frac{E_{so}}{E_{sd}}\right)^{1/n} < 1 \qquad (2)$$

in which E_{so} denotes the Young's modulus at the soil surface (z = 0). Equation (1) can be used to describe several different types of soil inhomogeneity. For instance, with α = 0, Eqn (1) describes a profile with zero modulus at the surface. Also, with n = 1, a linear variation in E_s is obtained. Furthermore, setting either α = 1 or n = 0, the dependence of E_s on depth is eliminated, which corresponds to a homogeneous profile. Each case will be examined in the sequel. It is noted that because of their small variation with depth as compared to that in Young's modulus, the mass density, material damping, and Poisson's ratio of the soil are assumed to be constant with depth.

With reference to the modulus of the Winkler springs, given that k is approximately proportional to E_s (Roesset 1980; Dobry et al 1982; Gazetas et al 1992), it is postulated that k(z) follows the same variation with depth as $E_s(z)$, i.e.,

$$k(z) \approx k_d\left[\alpha+(1-\alpha)\frac{z}{d}\right]^n \qquad (3a)$$

where k_d denotes the value of k(z) at the depth of one pile diameter. k_d and E_{sd} can be related as

$$k_d \approx \delta\, E_{sd} \qquad (3b)$$

where δ is a dimensionless coefficient which ranges between approximately 1 and 1.5 (Novak et al 1978; Roesset 1980; Scott 1981; Dobry et al 1982).

Regarding the damping coefficient, it is known from earlier studies (Novak et al 1978; Gazetas & Dobry 1984) that c is approximately proportional to the shear wave velocity of the material (square root of E_s). Accordingly, one may write

$$c(z) \approx c_d\left[\alpha + (1-\alpha)\frac{z}{d}\right]^{n/2} \qquad (4)$$

which reveals a weaker dependence of c on depth. c_d stands for the dashpot modulus at depth z = d, which can be obtained from pertinent expressions available in the above references.

MODEL DEVELOPMENT

In harmonic flexural oscillations, the equation of motion of a uniform pile on a Winkler foundation is

$$E_p\, I_p \frac{\partial^4 Y}{\partial z^4}+m\frac{\partial^2 Y}{\partial t^2}+\left[k(z)+i\,\omega\, c(z)\right]Y=0 \qquad (5)$$

where Y(z) = Y(z) exp[i ωt] denotes the harmonic pile deflection, (E_p I_p) the pile flexural stiffness, m the pile mass per unit pile length, k(z) and c(z) the distributed springs and dashpots, ω the cyclic vibrational frequency, t the time, and i the imaginary unity.

In the case of a homogeneous soil (k, c = constant), the solution to Eqn (5) is elementary and can be found in textbooks (Scott 1981). The corresponding impedance coefficients in swaying, rocking, and cross-swaying rocking are, respectively, (Scott 1981; Pender 1993; Mylonakis & Gazetas 1999)

$$K_{hh} = 4\, E_p\, I_p\, \lambda^3 \qquad (6a)$$

$$K_{rr} = 2\, E_p\, I_p\, \lambda \qquad (6b)$$

$$K_{hr} = 2\, E_p\, I_p\, \lambda^2 \qquad (6c)$$

where

$$\lambda = \left[\frac{k + i\omega c - m\omega^2}{4E_p I_p}\right]^{1/4} \qquad (7)$$

is a Winkler parameter (units = 1/Length), which can be interpreted as a "wavenumber" controlling the attenuation of pile response with depth.

Basis of the proposed approximate solution is that the unknown deflection function Y(z) in Eqn (5) can be replaced by a pair of approximate functions χ(z) and ϕ(z). Of these two functions, χ(z) represents the deflected shape of the pile caused by a unit imposed head displacement (under zero rotation), whereas ϕ(z) is the deflected shape caused by a unit head rotation under zero displacement. For long piles, these functions can be approximated by the deflected shape of a long pile in homogeneous soil (Mylonakis & Gazetas 1999)

$$\chi(z) \approx e^{-\mu z}(\sin \mu z + \cos \mu z) \qquad (8a)$$

and

$$\phi(z) \approx \frac{e^{-\mu z}}{\mu} \sin \mu z \tag{8b}$$

In the above equations, μ is a shape parameter which, in homogeneous soil, is equal to the wavenumber λ of Eqn (7). In non-homogeneous soil, μ can be approximated by the mean value of λ within the active length, L_a, of the pile:

$$\mu = \frac{1}{L_a} \int_0^{L_a} \lambda(z, \omega)\, dz \tag{9}$$

It is reminded here that L_a is defined as the length beyond which the pile behaves as a semi-infinite beam that is, an increase in pile length does not lead to an increase in lateral stiffness at the pile head. Pertinent expressions for L_a in various types of soil profiles are reported by Randolph (1981) and Gazetas (1991).

Replacing Y(z) in Eqn (1) with $\chi(z)$, multiplying by $\phi(z)$ and integrating over the pile length, it can be shown (Roumbas 2000) that the stiffness and damping coefficients atop the pile can be obtained through the virtual-work equations

$$K_{ij} = E_p I_p \int_0^L \chi_i''(z) \chi_j''(z) dz + \int_0^L k(z)\, \chi_i(z) \chi_j(z) dz \tag{10a}$$

$$C_{ij} = \frac{2\beta_p I_p}{\omega} \int_0^L \chi_i''(z) \chi_j''(z) dz + \frac{2\beta_s}{\omega} \int_0^L k(z) \chi_i(z) \chi_j(z) dz + \int_0^L c(z) \chi_i(z) \chi_j(z) dz \tag{10b}$$

which are analogous to energy approximations used in finite-element formulations. The two terms in the right-hand side of Eqn (10a) stand for the contributions to the overall stiffness of the pile flexural stiffness and the soil stiffness, respectively. The contribution of pile inertia (i.e., m in Eqn 5) to the overall stiffness was found to be small and has been omitted from Eqn (10a). In the second equation, the first two terms stand for the contributions of material damping in the pile and the soil, respectively; the last term corresponds to the contribution of radiation damping in the soil.

The subscripts i and j refer to the two vibrational modes (i.e., swaying and rocking). For instance, using $\chi_i(z) = \chi_j(z) = \chi(z)$, the swaying impedance coefficients K_{hh} and C_{hh} are obtained. Similarly, with $\chi_i(z) = \chi_j(z) = \phi(z)$ the rocking impedance is obtained. With $\chi_i(z) = \chi(z)$ and $\chi_j(z) = \phi(z)$ generates the cross-swaying-rocking impedance. It is noted that, to derive the above equations it was implicitly assumed that $\chi(z)$ and $\phi(z)$ are real-valued functions. The validity of this approximation has been demonstrated in earlier studies by Gazetas & Dobry (1984) and Mylonakis & Gazetas (1999).

To demonstrate the use of Eqns (10), the pile stiffness coefficients in a soil profile whose Young's modulus increases linearly with depth (n =1) are obtained in closed form:

$$K_{hh} = E_p I_p \mu^3 + \frac{3 k_{sd} \left[(1-\alpha) + 2\alpha \mu d\right]}{8 \mu^2 d} \tag{11a}$$

$$K_{rr} = \frac{3}{2} E_p I_p \mu + \frac{3 k_{sd} \left[(1-\alpha) + 2\alpha \mu d\right]}{8 \mu^4 d} \tag{11b}$$

$$K_{hh} = E_p I_p \mu^2 + \frac{k_{sd} \left[3(1-\alpha) + 4\alpha \mu d\right]}{16 \mu^3 d} \tag{11c}$$

in which

$$\mu = \left(\frac{k_{sd}}{4 E_p I_p}\right)^{1/4} \frac{4}{5} \left(\frac{L_a}{d}\right)^{-1} \frac{\left[(1-\alpha)\frac{L_a}{d} + \alpha\right]^{5/4} - \alpha^{5/4}}{1-\alpha} \tag{11d}$$

Note that with $\alpha \to 1$, the above expressions duly reduce to those in Eqns (7).

Equivalent Homogeneous Profile

To determine pile impedances in an inhomogeneous soil, engineers often replace the actual profile with an "equivalent" homogeneous soil with average properties. In the realm of the present analysis, this can be done by replacing the wavenumber λ in Eqns (6) with the average wavenumber μ of Eqn (9), i.e.,

$$K_{hh}^{(h)} = 4 E_p I_p \mu^3 \tag{12a}$$

$$K_{rr}^{(h)} = 2 E_p I_p \mu \tag{12b}$$

$$K_{hr}^{(h)} = 2 E_p I_p \mu^2 \tag{12c}$$

where the superscript (h) stands for "homogeneous".

To develop insight into the solution, it appears useful to express Eqns (10) in terms of the solution for the equivalent homogeneous profile. Accordingly, the pile stiffness K_{ij} can be written as

$$K_{ij} = \chi_{ij} K_{ij}^{(h)} \tag{13}$$

as where χ_{ij} is a dimensionless coefficient defined as the ratio of Eqns (10) and (12). Apart form its obvious usefulness in normalizing pile stiffness, Eqn (13) has some additional advantages over Eqns (10): *First*, the performance of the equivalent homogeneous soil approximation can be readily assessed (i.e., by examining how close χ_{ij} is to 1). *Second*, potential errors arising from the use of the simplified Winkler model could be reduced using the above normalization. In other words, since Eqns (10) and (12) may involve systematic modeling errors arising from the use of the Winkler assumption, their ratio (χ_{ij}) would, in principle, be less sensitive to these errors. Support to this argument comes from the fact that χ_{ij} is much less dependent to the value of the Winkler coefficient δ than Eqns (10) and (12) (Roumbas 2000).

With reference to the damping coefficient, it is instructive to replace C_{ij} in Eqn (10b) with the dimensionless factor

$$D_{ij} = \frac{\omega C_{ij}}{2 K_{ij}} \qquad (14)$$

which expresses the ratio of the imaginary and real part of the impedance, and is analogous to the damping coefficient of a single-degree-of-freedom oscillator. From this expression, a second dimensionless coefficient, ξ_{ij}, can be defined

$$D_{ij} = \xi_{ij} \, D_{ij}^{(h)} \qquad (15)$$

which relates pile damping in the (actual) inhomogeneous profile and in the (substitute) homogeneous profile.

RESULTS

With reference to a linear variation in soil Young's modulus (n = 1), Fig 2 presents the three dimensionless stiffness coefficients χ_{ij} as functions of the inhomogeneity factor α. The analytical solutions in Eqns (11) are compared against results from an exact numerical Winkler solution using the computer program DAP (Mylonakis 1996). It is seen that the rocking coefficient χ_{rr} is very close to 1 for all values of α and E_p/E_{sd}, which confirms the validity of the "equivalent homogeneous soil" approximation in that response mode. In contrast, in swaying vibrations χ_{hh} is very sensitive to the value of α and decreases quickly with increasing inhomogeneity. This indicates that swaying stiffness is not controlled by the average soil properties within the active pile length (about 10 pile diameters), but instead by the soil stiffness within the first few pile diameters from the surface. An intermediate behavior is observed with the cross-swaying-rocking coefficient. The dashed line in the second graph of Fig 2 represents the approximation:

$$\chi_{hr} \approx 0.35\,\chi_{hh} + 0.65\,\chi_{rr} \qquad (16)$$

where

$$\chi_{rr} \approx 1 \qquad (17)$$

which reveals a stronger dependence of χ_{hr} on the rocking stiffness. The above expressions were found to fit the results reasonably well in all cases examined, and are recommended for quick approximate estimations of pile stiffness.

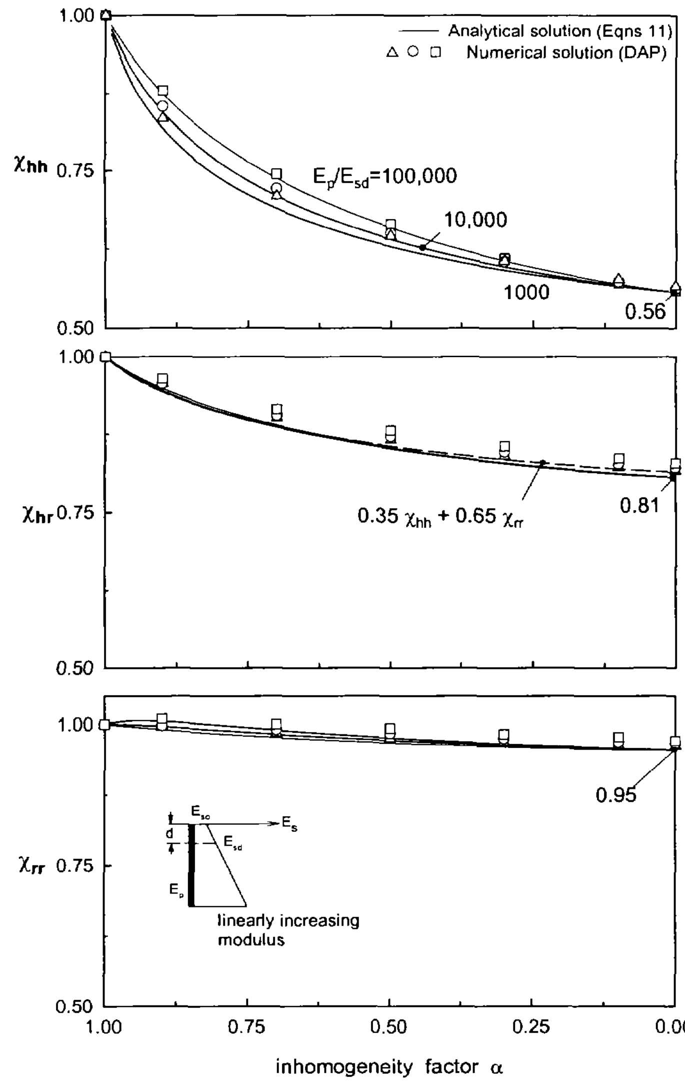

Fig 2. Normalized pile stiffness coefficients for a "linear" soil profile (n = 1, δ = 1.2).

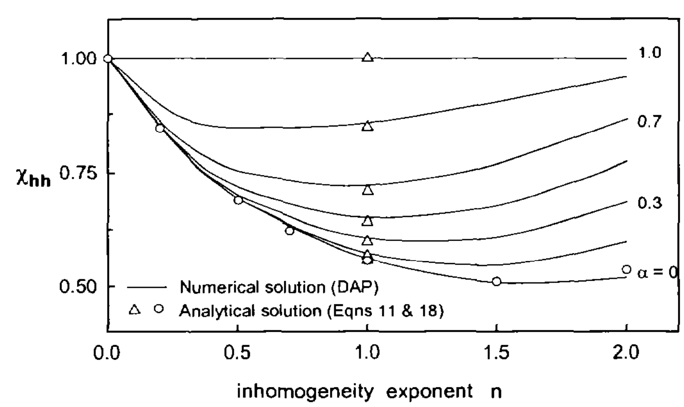

Fig 3. Normalized pile stiffness coefficients as function of n and a; (E_p/E_{sd} = 10,000; δ = 1.2)

Figure 3 presents results for the normalized stiffness coefficient K_{hh} as functions of both n and α. For α = 0, the corresponding analytical solution is (Roumbas 2000)

$$K_{hh} = E_p\, I_p\, \mu^3 + k_{sd}\, d \left(\frac{1}{\mu d} \right)^{n+1} \times$$
$$\times \Gamma(n+1) \left\{ 2^{-(1+n)} + 2^{-3(1+n)/2} \sin\left[\frac{\pi}{4}(1+n) \right] \right\} \quad (18a)$$

where Γ( ) denotes the Gamma function; μ is given by

$$\mu = \frac{4}{4+n} \left(\frac{L_a}{d} \right)^{n/4} \left(\frac{k_{sd}}{4\, E_p\, I_p} \right)^{1/4} \quad (18b)$$

It is seen that with either α = 1 or n = 0 (a homogeneous profile) χ_{hh} equals 1. With increasing n and decreasing α, however, χ_{hh} drops quickly and may reach values as low as 0.5. The increasing trend observed beyond approximately n = 1 can be explained by the fast increase, with increasing soil inhomogeneity, in Young's modulus with depth below z = d. Comparison between analytical and numerical results shows good agreement. The plotted values correspond to E_p/E_{sd} = 10000. However, as seen in Fig 2, pile-soil-stiffness contrast does not affect the results significantly, so the graph can be used, even if approximately, for other E_p/E_{sd} values. It is also mentioned that the effect of frequency on pile stiffness is small and has been neglected in these graphs. This, however, is strictly applicable to single piles, since frequency effects can be important in pile groups (Kaynia & Kausel 1982).

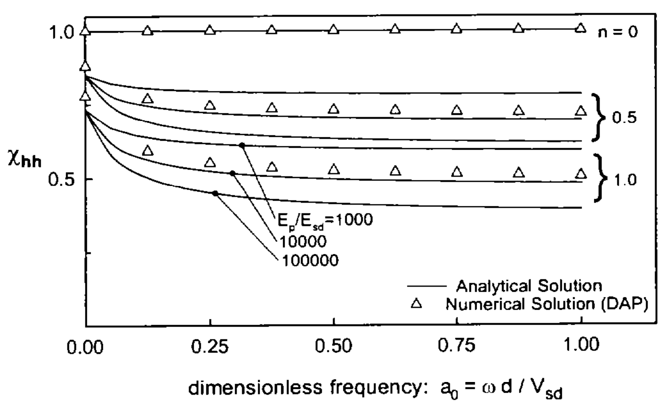

Fig 4. Normalized pile damping coefficients versus frequency; (a = 0; δ = 1.2)

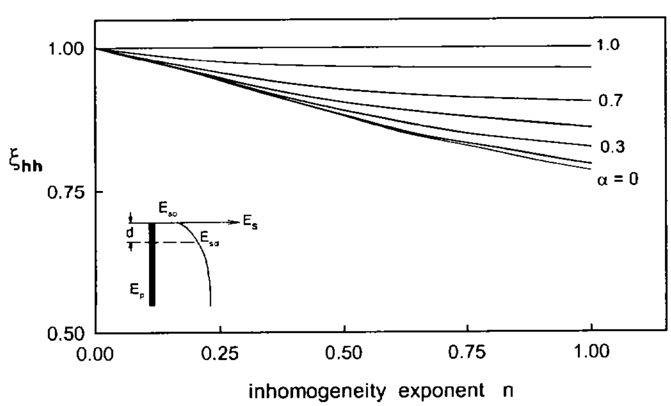

Fig 5. Normalized pile damping coefficient under static conditions; (E_p/E_{sd} = 10,000; δ = 1.2)

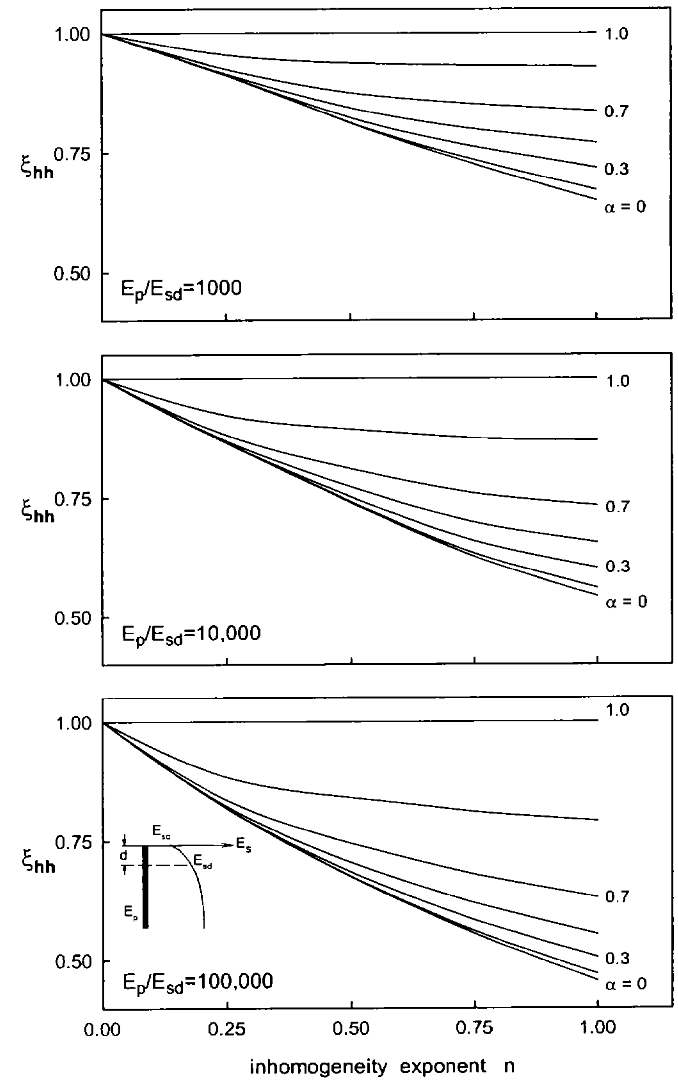

Fig 6. Normalized pile damping coefficient (a_0 =0.3; δ =1.2).

Results for the normalized damping coefficient ξ_{ij} are presented in Fig 4, referring to a soil profile with zero stiffness at the surface (α = 0). The results are plotted as function of the well-known dimensionless frequency $a_0 = \omega\, d / V_{sd}$, V_{sd} being the propagation velocity of shear waves at depth z = d. It can be seen that for a_0 greater than about 0.25, ξ_{ij} is practically frequency independent. It is also noted that ξ_{ij} are almost identical in all vibrational modes (not shown), despite the large differences in the corresponding stiffnesses. As a first approximation, two different frequencies will be examined here: (i) a_0 = 0 (static conditions); (ii) a_0 = 0.3.

Figure 5 shows results for the swaying damping coefficient ξ_{hh} under static conditions ($a_0 = 0$). The other two coefficients are almost identical and are not shown. It is seen that ξ_{hh} is always smaller that 1 and tends to decrease with increasing n or decreasing α. This indicates that a pile in an inhomogeneous profile possesses smaller damping than the same pile in a homogeneous profile with average properties. The reduction, however, is relatively small, being less than 25 percent even for strong soil inhomogeneity ($n = 1$; $\alpha = 0$). Since pile-soil stiffness contrast is not important at static frequencies (see Fig 4), only results for $E_p/E_{sd} = 10{,}000$ are provided here.

Corresponding results for $a_0 = 0.3$ are shown in Figure 6. In this dynamic case, a decrease in ξ_{hh} with increasing pile-soil stiffness contrast is observed. The general trends are analogous to those in Fig 5. The two graphs can be used for approximate computations of pile damping in inhomogeneous soil profiles of the type described by Eqn (1).

CONCLUSIONS

An approximate analytical solution for estimating the lateral stiffness and damping of a dynamically-loaded pile in an inhomogeneous soil deposit was presented. The method is based on a dynamic Winkler model and a virtual-work scheme combined with a pair of approximate shape functions for pile deflections under imposed head displacements and rotations. The main conclusions glean from the study are:

(1) The proposed analytical technique allows for closed-form solutions to be derived which provide useful insight on the physics of the problem.

(2) Results obtained with the proposed method are in good agreement with results from numerical solutions.

(3) The assumption of an equivalent homogeneous profile with average properties is realistic for the rocking stiffness, but may severely overestimate the swaying and cross-swaying-rocking stiffness. All three stiffness coefficients can be estimated with the help of Figure 3 and Eqns (16) and (17).

(4) Damping in an inhomogeneous profile is always smaller than that in a homogeneous profile with average properties. The reduction in damping is practically the same in all vibrational modes

(5) At low "static" frequencies, the reduction in damping is relatively small and practically independent of the pile-soil stiffness contrast E_p/E_{sd}. In the high frequency range, however, higher reductions in damping and some dependence on E_p/E_{sd} are observed

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