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Dynamic Timoshenko Beam-Columns on Elastic Media

By Franklin Y. Cheng,[1] Member, ASCE, and Chris P. Pantelides,[2] Associate Member, ASCE

Abstract: Differential equations, stiffness coefficients, and fixed-end forces are formulated for a Timoshenko beam-column on an elastic foundation subjected to lateral time-dependent excitations and static axial loads. The theoretical formulation includes shear and bending deformations as well as rotatory inertia, with emphasis on two approaches. The two approaches differ in terms of the assumed shear component of the static axial load on the cross section. The first approach is based on the assumption that the shear component of the axial load is calculated from the total slope; in the second approach, the shear component of the axial load is calculated only from the bending slope. The dynamic stiffness coefficients and fixed-end forces are expressed in terms of nondimensional parameters associated with the effects of transverse and rotatory inertia, axial force, elastic media, and shear and bending deformations. When the individual effect is not considered, then the associated parameter can be dropped. The significance of the individual parameters on natural frequencies and dynamic response of typical beams is then extensively examined. The two approaches are also studied by comparing the response behavior and it is found that they differ appreciably with increasing axial loads and decreasing slenderness ratios. Comparison of the natural frequencies shows that the second approach gives higher values than the first.

Introduction

The theoretical analysis of the flexural vibration of beams was generally based on the Bernoulli-Euler theory, with consideration of lateral inertia forces and bending deformations. Ever since Timoshenko pointed out that the effects of cross-sectional dimensions on the frequencies of beams could be significant (Timoshenko 1921), a considerable amount of research work based on Timoshenko's beam theory has been published. Early researchers studied the vibrations of Timoshenko beams with various boundary conditions (Aalami and Atzori 1974; Abbas 1984; Anderson 1953; Huang 1961). Later investigators employed the Timoshenko theory in developing numerical techniques for electronic computation, such as consistent mass matrices and dynamic stiffness matrices by Archer (1963) and Cheng (1970), respectively. Cheng then extended his formulations for the frequency analysis of thin wall member grid systems (Cheng 1969), and for the response analysis of continuous beams and frames with various types of externally applied forces and foundation movements (Cheng et al. 1970).

It has been well-recognized that the axial force acting on a member can

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significantly affect the natural frequencies of that member. Djodjo studied the combined effects of axial loads and Timoshenko theory on the natural frequencies of beams (Djodjo 1969). Howson and Williams investigated the combined effect, based on both the analytical and experimental results (Howson and Williams 1973). Cheng and his associates further extended his dynamic stiffness approach for plane frames, the constituent members of which may have axial forces, transverse and rotatory inertia, and bending and shear deformations (Cheng et al. 1970; Cheng and Tseng 1973). He also studied the combined effects on the frequencies and responses of space structures based on the finite element approach (Cheng 1972).

The effects of elastic media on the flexural vibrations were examined by a number of researchers (Date and Tuma 1985; Djodjo 1969; Lunden and Akesson 1983; Pestel and Leckie 1963; Pilkey and Chang 1978; Wang and Stephens 1977). Cheng employed the Bernoulli-Euler theory and the transfer matrix technique (Tuma and Cheng 1983) to derive the dynamic stiffness and flexibility matrices for transverse and longitudinal vibrations (Cheng 1977).

The present paper is an extension of Cheng's work of dynamic stiffness formulation to include elastic media of Winkler type, axial forces, lateral and rotatory inertia, and shear and bending deformations. Furthermore, the formulations presented herein consist of two new approaches to how the shear component of the axial force is acting on a cross section. In the first approach, the shear component of the axial load is assumed to act perpendicularly to the tangent line of the total slope, which consists of the bending and shear slope. In the second approach, the shear component of the axial load is acting perpendicularly to the tangent line of the bending slope. The resulting stiffness coefficients and fixed-end forces are general and can be applied to large structural systems with the numerical procedures previously presented by the first writer. In addition, the stiffness coefficients and the fixed-end force formulations are distinguished in the sense that the complex roots are not treated separately, as solved previously in Cheng's work (1970; Cheng and Tseng 1973). These resulting formulations are expressed in terms of nondimensional parameters associated with the effects of lateral and rotatory inertia, axial force, elastic media, and shear and bending deformations. When the individual effect is not considered, then the associated parameter can be dropped. Numerical results are provided to show the significant effects of the individual parameters and those of the two approaches on natural frequencies, and dynamic response behavior.

Basic Motion Equations

Consider the beam element shown in Fig. 1. The total slope $\partial y/\partial x$ is a combination of the bending slope ψ and the shearing slope $\bar{\beta}$.

$$\frac{\partial y}{\partial x} = \psi + \bar{\beta} \quad (1)$$

The equilibrium equations for the free-body diagram shown in Fig. 1 yield

$$\frac{\partial V}{\partial x} = \rho A \frac{\partial^2 y}{\partial t^2} - w + qy \quad (2)$$

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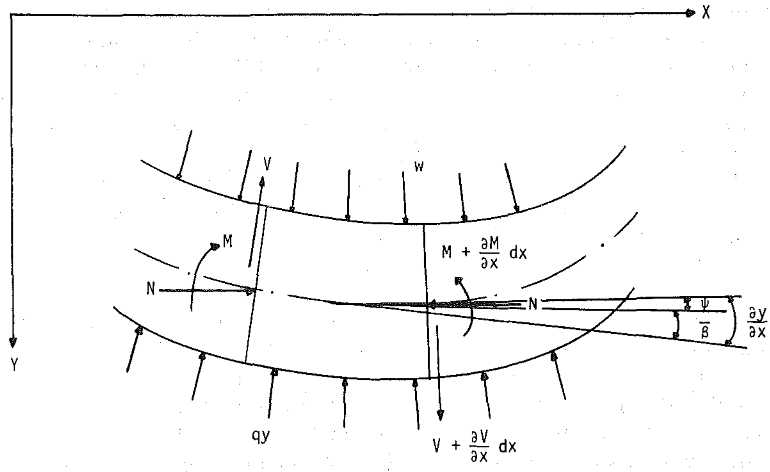

FIG. 1. Beam Element

$$V = \frac{\partial M}{\partial x} - N\frac{\partial y}{\partial x} + \rho I \frac{\partial^2 \psi}{\partial t^2} \quad (3)$$

where $\rho = \bar{\gamma}/g$; N = axial force; V = shear; M = bending moment; w = lateral load; q = foundation constant; A = cross-sectional area; g = acceleration of gravity; $\bar{\gamma}$ = weight per unit volume. From structural mechanics the bending moment, M, of a Timoshenko-beam is

$$M = -EI\frac{\partial \psi}{\partial x} \quad (4)$$

in which E = modulus of elasticity; and I = moment of inertia of cross section.

First Approach Formulation

Differential Equations

The effects of how the shear component of axial force acts on a cross section is shown in Figs. 2 and 3. From Fig. 2, the shear component of the axial load N, on the cross section is $N \sin(\partial y/\partial x) \cong N\,\partial y/\partial x$. Considering shear deformations

$$V + N\frac{\partial y}{\partial x} = \left(\frac{\partial y}{\partial x} - \psi\right)\phi \quad (5)$$

$$\phi = GAk \quad (6)$$

in which G = modulus of rigidity; and k = shear factor for cross section.

By eliminating ψ and y from Eqs. 2–5, the following complete equations in y and ψ are obtained

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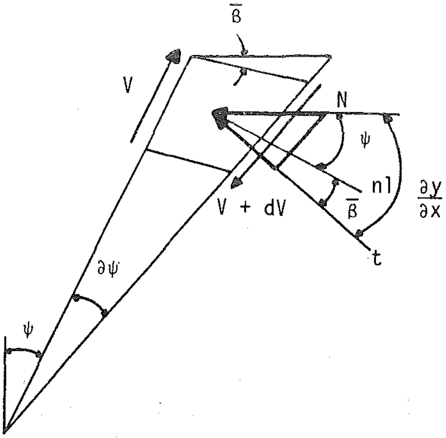

FIG. 2. First Approach

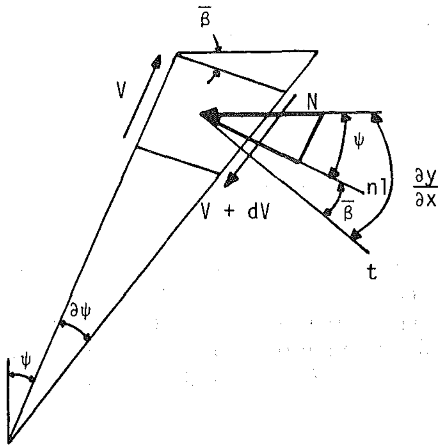

FIG. 3. Second Approach

$$EI\zeta \frac{\partial^4 y}{\partial x^4} + \rho\left(A + \frac{qI}{\phi}\right) \frac{\partial^2 y}{\partial t^2} + \left(N - \frac{EIq}{\phi}\right) \frac{\partial^2 y}{\partial x^2} - I\rho\left(\zeta + \frac{EA}{\phi}\right) \frac{\partial^4 y}{\partial x^2 \partial t^2}$$

$$+ \frac{\rho^2 IA}{\phi} \frac{\partial^4 y}{\partial t^4} + qy - \frac{I}{\phi}\left(\rho \frac{\partial^2 w}{\partial t^2} - E \frac{\partial^2 w}{\partial x^2}\right) - w = 0 \quad \text{(7)}$$

$$EI\zeta \frac{\partial^4 \psi}{\partial x^4} + \rho\left(A + \frac{qI}{\phi}\right) \frac{\partial^2 \psi}{\partial t^2} + \left(N - \frac{EIq}{\phi}\right) \frac{\partial^2 \psi}{\partial x^2} - I\rho\left(\zeta + \frac{EA}{\phi}\right) \frac{\partial^4 \psi}{\partial x^2 \partial t^2}$$

$$+ \frac{\rho^2 IA}{\phi} \frac{\partial^4 \psi}{\partial t^4} + q\psi \frac{\partial W}{\partial x} = 0 \quad \text{(8)}$$

where

$$\zeta = 1 - \frac{N}{\phi} \quad (9)$$

The solutions of Eqs. 7 and 8 can be found by assuming

$$y = Ye^{ipt} \quad (10a)$$

$$\psi = \Psi e^{ipt} \quad (10b)$$

$$w = We^{ipt} \quad (10c)$$

$$\xi = \frac{x}{L} \quad (10d)$$

where W = magnitude of forcing function; Y = normal function of y; Ψ = normal function of ψ; $i = \sqrt{-1}$; p = angular frequency; and L = length of beam. The functions Y and Ψ are then given by

$$Y(\xi) = C_1 \cosh b\alpha\xi + C_2 \sinh b\alpha\xi + C_3 \cos b\beta\xi + C_4 \sin b\beta\xi$$

$$+ \frac{W}{(q - \rho A p^2)} \quad (11)$$

$$\Psi(\xi) = C_1' \sinh b\alpha\xi + C_2' \cosh b\alpha\xi + C_3' \sin b\beta\xi + C_4' \cos b\beta\xi \quad (12)$$

where

$$\begin{matrix}\alpha \\ \beta\end{matrix} = \frac{1}{\sqrt{2}}\left[\begin{matrix}- \\ +\end{matrix}\; r^2 T + \left(r^4 T^2 - \frac{4U}{b^2}\right)^{1/2}\right]^{1/2} \quad (13)$$

Depending on the values of r^4T^2, $4U/b^2$ and r^2T, there are three distinct cases for α and β. These cases are α and β are complex, α is imaginary and β is real, and α and β are real. However, the solutions derived for stiffness coefficients and fixed-end forces are valid for all cases, as shown in Appendix I. Here, T, U, b^2, s^2, and r^2 are dimensionless parameters defined as

$$b^2 = \rho \frac{AL^4}{EI} p^2 \quad (14)$$

$$r^2 = \frac{I}{AL^2} \quad (15)$$

$$s^2 = \frac{EI}{\phi L^2} \quad (16)$$

$$T = \frac{\left(N - \dfrac{qEI}{\phi}\right)}{\zeta \rho p^2 I} + 1 + \frac{EA}{\zeta\phi} \quad (17)$$

$$U = \left(\frac{\rho I p^2}{\zeta\phi} - \frac{1}{\zeta}\right) + \frac{q}{A\zeta}\left(\frac{1}{\rho p^2} - \frac{I}{\phi}\right) \quad (18)$$

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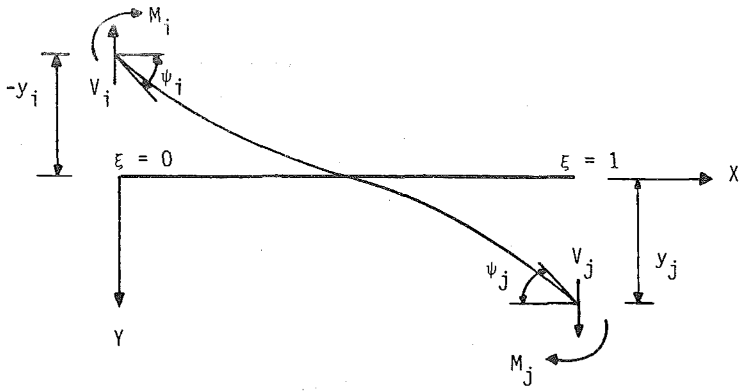

FIG. 4. Member End Forces and Deformations

where r = the variable associated with the rotatory inertia; s = the effect of shear deformations; and b = a function of natural frequencies. The constants in Eqs. 11 and 12 may be expressed as

$$C_1' = \frac{b}{L\alpha}\left(\zeta\alpha^2 + s^2 - \frac{qL^2}{\phi b^2}\right)C_1 \qquad (19)$$

$$C_2' = \frac{b}{L\alpha}\left(\zeta\alpha^2 + s^2 - \frac{qL^2}{\phi b^2}\right)C_2 \qquad (20)$$

$$C_3' = \frac{-b}{L\beta}\left(\zeta\beta^2 - s^2 + \frac{qL^2}{\phi b^2}\right)C_3 \qquad (21)$$

$$C_4' = \frac{b}{L\beta}\left(\zeta\beta^2 - s^2 + \frac{qL^2}{\phi b^2}\right)C_4 \qquad (22)$$

Stiffness Coefficients

The force-deformation relationships of a typical member are shown in Fig. 4 for which the boundary conditions are

$$\xi = 0, \quad M_i = \frac{-EI}{L}\frac{d\Psi}{d\xi}, \quad \Psi = \Psi_i \qquad (23a)$$

$$\xi = 1, \quad M_j = \frac{EI}{L}\frac{d\Psi}{d\xi}, \quad \Psi = \Psi_j \qquad (23b)$$

$$\xi = 0, \quad V_i = \phi\left(\frac{\zeta}{L}\frac{dY}{d\xi} - \Psi\right), \quad Y = -Y_i \qquad (23c)$$

$$\xi = 1, \quad V_j = \phi\left(\frac{\zeta}{L}\frac{dY}{d\xi} - \Psi\right), \quad Y = Y_j \qquad (23d)$$

Using Eqs. 11, 12, and 23 we can derive the dynamic stiffness matrix $[S]$. Let the end-displacements $\{\delta\}$ be expressed in terms of the constants C_1, C_2, C_3, C_4, through matrix $[R]$

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$$\{\delta\} = [R]\{C\} \quad (24)$$

then the end-forces $\{F\}$ can be expressed in terms of the same constants $\{C\}$ through matrix $[X]$

$$\{F\} = [X]\{C\} \quad (25)$$

From Eq. 24 we obtain

$$\{C\} = [R]^{-1}\{\delta\} \quad (26)$$

$$\{F\} = [X][R]^{-1}\{\delta\} = [S]\{\delta\} \quad (27)$$

$$\begin{Bmatrix} M_i \\ M_j \\ V_i \\ V_j \end{Bmatrix} = \begin{bmatrix} S11 & S12 & S13 & S14 \\ & S11 & S14 & S13 \\ & & S33 & S34 \\ SYMM. & & & S33 \end{bmatrix} \begin{Bmatrix} \Psi_i \\ \Psi_j \\ -Y_i \\ Y_j \end{Bmatrix} \quad (28a)$$

or in a condensed form

$$\left\{\frac{M}{V}\right\} = \left[\begin{array}{c|c} SM\Psi & SMY \\ \hline SV\Psi & SVY \end{array}\right]\left\{\frac{\Psi}{Y}\right\} \quad (28b)$$

The stiffness coefficients using the first approach are

$$S11 = \frac{EIb(\alpha\Lambda' + \beta B')(n'c\Lambda' - nc'B')}{|R'|L} \quad (29a)$$

$$S12 = \frac{EIb(\alpha\Lambda' + \beta B')(B'n - \Lambda'n')}{|R'|L} \quad (29b)$$

$$S13 = \frac{EIbB'\Lambda'[(1 - cc')(\alpha\Lambda' - \beta B') - nn'(\alpha B' + \beta\Lambda')]}{|R'|L} \quad (29c)$$

$$S14 = \frac{EIbB'\Lambda'(c' - c)(\alpha\Lambda' + \beta B')}{|R'|L} \quad (29d)$$

$$S33 = \frac{\zeta b\phi(n'cB' + nc'\Lambda')(\beta\Lambda' - \alpha B')}{|R'|L} \quad (29e)$$

$$S34 = \frac{\zeta b\phi(n\Lambda' + n'B')(\beta\Lambda' - \alpha B')}{|R'|L} \quad (29f)$$

in which $|R'| = nn'(\Lambda'^2 - B'^2) + 2\,(1 - cc')\Lambda'B'$; $\Lambda' = b/L\alpha\ (\zeta\alpha^2 + s^2 - qL^2/\phi b^2)$; $B' = b/L\beta\ (\zeta\beta^2 - s^2 + qL^2/\phi b^2)$; $n = \sinh b\alpha$, $n' = \sin b\beta$; $c = \cosh b\alpha$, and $c' = \cos b\beta$.

Fixed-End Forces

The fixed-end forces are determined for the following loading cases shown in Figs. 5 and 6. Considering the uniform forcing function shown in Fig. 5, one may derive the fixed-end forces as

$$Moi = \frac{EIbW\Lambda'B'[(1 - cc')(\alpha\Lambda' - \beta B') - (\alpha B' + \beta\Lambda')nn' + (c - c')(\alpha\Lambda' + \beta B')]}{|R'|(q - \rho Ap^2)L} \quad (30)$$

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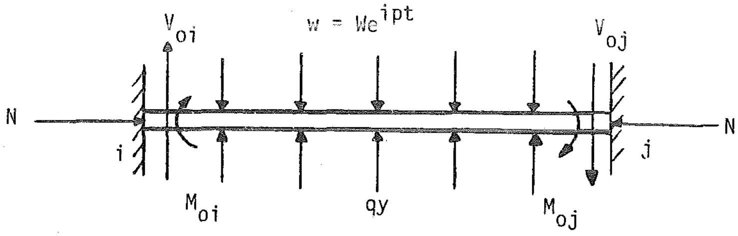

FIG. 5. Fixed End Forces, Uniform Forcing Function

$$Voi = \frac{\zeta b \phi W(\alpha B' - \beta \Lambda')[n(1 - c')\Lambda' + n'(1 - c)B']}{|R'|(q - \rho A p^2)L} \quad (31)$$

$$Moj = -Moi \quad (32)$$

$$Voj = -Voi \quad (33)$$

For the triangular forcing function shown in Fig. 6 the fixed-end forces may also be derived as

$$M_{oi} = \frac{EIbW(\alpha\Lambda' + \beta B')}{|R'|L}\left\{\frac{[n'\Lambda'(1 - c) + nB'(c' - 1)]}{LU\rho A p^2} - \frac{(c' - c)\Lambda' B'}{(q - \rho A p^2)}\right\} \quad (34)$$

$$M_{oj} = \frac{-EIbW}{|R'|L}\left\{\frac{(\alpha\Lambda' + \beta B')[n'\Lambda'(c - 1) + nB'(1 - c')]}{LU\rho A p^2} + \frac{\Lambda' B'[(\alpha\Lambda' - \beta B')(1 - cc') - nn'(\alpha B' + \beta\Lambda')]}{(q - \rho A p^2)}\right\} \quad (35)$$

$$V_{oi} = \frac{-\phi W}{|R'|L} \cdot \left\{\frac{[\zeta b(\alpha\Lambda' - \beta B') + L(B'^2 - \Lambda'^2)]nn' + [\zeta b(\alpha B' + \beta\Lambda') - 2LB'\Lambda'](1 - cc')}{LU\rho A p^2} + \frac{\zeta b(\alpha B' - \beta\Lambda')(c - c')}{LU\rho A p^2} + \frac{\zeta b(\beta\Lambda' - \alpha B')(n\Lambda' + n'B')}{(q - \rho A p^2)}\right\} - \frac{\phi W}{L}\left(\frac{1}{U\rho A p^2} - \frac{\zeta}{(q - \rho A p^2)}\right) \quad (36)$$

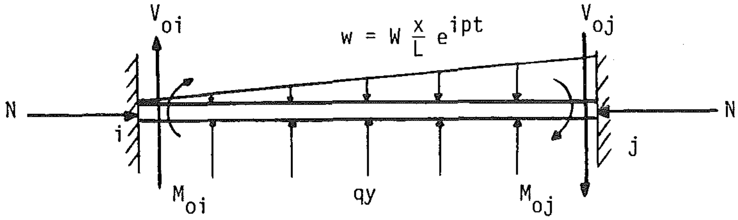

FIG. 6. Fixed End Forces, Triangular Forcing Function

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$$V_{oj} = \frac{-\phi W}{|R'|L}$$

$$\cdot \left\{ \frac{[\zeta b(\alpha\Lambda' - \beta B') + L(B'^2 - \Lambda'^2)]nn' + [\zeta b(\alpha B' + \beta\Lambda') - 2LB'\Lambda'](1 - cc')}{LU\rho Ap^2} \right.$$

$$\left. + \frac{\zeta b(\alpha B' - \beta\Lambda')(c - c')}{LU\rho Ap^2} + \frac{\zeta b(\beta\Lambda' - \alpha B')(nc'\Lambda' + cn'B')}{(q - \rho Ap^2)} \right\}$$

$$- \frac{\phi W}{L}\left(\frac{1}{U\rho Ap^2} - \frac{\zeta}{(q - \rho Ap^2)}\right) \qquad (37)$$

SECOND APPROACH FORMULATION

Differential Equations

From Fig. 3 the shear component of the axial load N on the cross section is $N\psi$, which can be expressed as

$$V + N\psi = \left(\frac{\partial y}{\partial x} - \psi\right)\phi \qquad (38)$$

By eliminating ψ and y from Eqs. 2, 3, 4, and 38, the following differential equations in y and ψ are obtained

$$EI\frac{\partial^4 y}{\partial x^4} + \rho\left(A\upsilon + \frac{qI}{\phi}\right)\frac{\partial^2 y}{\partial t^2} + \left(N\upsilon - \frac{EIq}{\phi}\right)\frac{\partial^2 y}{\partial x^2} - I\rho\left(1 + \frac{EA}{\phi}\right)\frac{\partial^4 y}{\partial x^2 \partial t^2}$$

$$+ \frac{\rho^2 IA}{\phi}\frac{\partial^4 y}{\partial t^4} + q\upsilon y - \frac{I}{\phi}\left(\frac{\rho\partial^2 w}{\partial t^2} - \frac{E\partial^2 w}{\partial x^2}\right) - w\upsilon = 0 \qquad (39)$$

$$EI\frac{\partial^4 \psi}{\partial x^4} + \rho\left(A\upsilon + \frac{qI}{\phi}\right)\frac{\partial^2 \psi}{\partial t^2} + \left(N\upsilon - \frac{EIq}{\phi}\right)\frac{\partial^2 \psi}{\partial x^2} - I\rho\left(1 + \frac{EA}{\phi}\right)\frac{\partial^4 \psi}{\partial x^2 \partial t^2}$$

$$+ \frac{\rho^2 IA}{\phi}\frac{\partial^4 \psi}{\partial t^4} + q\upsilon\psi - \frac{\upsilon\partial w}{\partial x} = 0 \qquad (40)$$

in which

$$\upsilon = 1 + \frac{N}{\phi} \qquad (41)$$

The solutions of Eqs. 39 and 40 can be found by using Eqs. 10*a*–*d* as

$$Y(\xi) = C_1 \cosh\lambda\alpha\xi + C_2 \sinh\lambda\alpha\xi + C_3 \cos\lambda\beta\xi + C_4 \sin\lambda\beta\xi$$

$$+ \frac{W}{(q - \rho Ap^2)} \qquad (42)$$

$$\Psi(\xi) = C_i' \sinh\lambda\alpha\xi + C_2' \cosh\lambda\alpha\xi + C_3' \sin\lambda\beta\xi + C_4' \cos\lambda\beta\xi \qquad (43)$$

where

$$\begin{matrix}\alpha \\ \beta\end{matrix} = \frac{1}{\sqrt{2}}\left[\begin{matrix}- \\ +\end{matrix}\; r^2 T' + \left(r^4 T'^2 - \frac{4U'}{\lambda^2}\right)^{1/2}\right]^{1/2} \qquad (44)$$

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Again, three cases exist for α and β and the solutions derived for stiffness coefficients and fixed-end forces are always real quantities. Here T', U' and λ^2 are dimensionless parameters defined as

$$T' = \frac{N\upsilon - qEI/\phi}{\rho p^2 I} + 1 + \frac{EA}{\phi} \quad (45)$$

$$U' = \left(\frac{\rho I p^2}{\phi} - \upsilon\right) + \frac{q}{A}\left(\frac{\upsilon}{\rho p^2} - \frac{I}{\phi}\right) \quad (46)$$

$$\lambda^2 = \frac{\rho A L^4}{EI} p^2 \quad (47)$$

where λ = a function of natural frequencies. The constants in Eqs. 42 and 43 may be expressed as

$$C_1' = \frac{\lambda}{\upsilon L \alpha}\left(\alpha^2 + s^2 - \frac{qL^2}{\lambda^2 \phi}\right) C_1 \quad (48)$$

$$C_2' = \frac{\lambda}{\upsilon L \alpha}\left(\alpha^2 + s^2 - \frac{qL^2}{\lambda^2 \phi}\right) C_2 \quad (49)$$

$$C_3' = \frac{-\lambda}{\upsilon L \beta}\left(\beta^2 - s^2 + \frac{qL^2}{\lambda^2 \phi}\right) C_3 \quad (50)$$

$$C_4' = \frac{\lambda}{\upsilon L \beta}\left(\beta^2 - s^2 + \frac{qL^2}{\lambda^2 \phi}\right) C_4 \quad (51)$$

Stiffness Coefficients

Based on Figs. 3 and 4, the force-deformation relationships can be expressed as

$$\xi = 0, \quad M_i = \frac{-EI}{L}\frac{d\Psi}{d\xi}, \quad \Psi = \Psi_i \quad (52a)$$

$$\xi = 1, \quad M_j = \frac{EI}{L}\frac{d\Psi}{d\xi}, \quad \Psi = \Psi_j \quad (52b)$$

$$\xi = 0, \quad V_i = \frac{\phi dY}{L d\xi} - \upsilon\phi\Psi, \quad Y = -Y_i \quad (52c)$$

$$\xi = 1, \quad V_j = \frac{\phi dY}{L d\xi} - \upsilon\phi\Psi, \quad Y = Y_j \quad (52d)$$

Using Eqs. 42, 43, and 52 and the procedures described in Eqs. 24–28, the dynamic stiffness coefficients are obtained as

$$S11 = \frac{EI\lambda(\alpha Z' + \beta H')(n'cZ' - nc'H')}{|RO|L} \quad (53a)$$

$$S12 = \frac{EI\lambda(\alpha Z' + \beta H')(nH' - n'Z')}{|RO|L} \quad (53b)$$

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$$S13 = \frac{EI\lambda Z'H'[(\alpha Z' - \beta H')(1 - cc') - (\alpha H' + \beta Z')nn']}{|RO|L} \quad \text{(53c)}$$

$$S14 = \frac{EI\lambda Z'H'(\alpha Z' + \beta H')(c' - c)}{|RO|L} \quad \text{(53d)}$$

$$S33 = \frac{\lambda\phi(\beta Z' - \alpha H')(n'cH' + nc'Z')}{|RO|L} \quad \text{(53e)}$$

$$S34 = \frac{\lambda\phi(\beta Z' - \alpha H')(nZ' + n'H')}{|RO|L} \quad \text{(53f)}$$

in which $Z' = \lambda/\upsilon L\alpha\ (\alpha^2 + s^2 - qL^2/\lambda^2\phi)$; $H' = \lambda/\upsilon L\beta\ (\beta^2 - s^2 + qL^2/\lambda^2\phi)$; $|RO| = nn'\ (Z'^2 - H'^2) + 2\ (1 - cc')Z'H'$; $n = \sinh \lambda\alpha$; $c = \cosh \lambda\alpha$; $c' = \cos \lambda\beta$; and $n' = \sin \lambda\beta$.

Fixed-End Forces

The fixed-end forces are determined for the forcing functions given in Figs. 5 and 6. For the uniform forcing function

$$M_{oi} = \frac{EI\lambda WZ'H'[(1 - cc')(\alpha Z' - \beta H') - nn'(\alpha H' + \beta Z') + (c - c')(\alpha Z' + \beta H')]}{|RO|(q - \rho Ap^2)L} \quad \text{(54)}$$

$$V_{oi} = \frac{\phi W\lambda(\alpha H' - \beta Z')[n(1 - c')Z' + n'(1 - c)H']}{|RO|(q - \rho Ap^2)L} \quad \text{(55)}$$

$$M_{oj} = -M_{oi}, \quad V_{oj} = -V_{oi} \quad \text{(56)}$$

For the triangular forcing function shown in Fig. 6

$$M_{oi} = \frac{EI\lambda W(\alpha Z' + \beta H')}{|RO|L}\left\{\frac{[n'Z'(1 - c) + nH'(c' - 1)]\upsilon}{LU\rho Ap^2} - \frac{Z'H'(c' - c)}{(q - \rho Ap^2)}\right\} \quad \text{(57)}$$

$$M_{oj} = \frac{-EI\lambda W}{|RO|L}\left\{\frac{\upsilon(\alpha Z' + \beta H')[n(1 - c')H' + n'(c - 1)Z']}{LU\rho Ap^2} + \frac{Z'H'[(1 - cc')(\alpha Z' - \beta H') - nn'(\alpha H' + \beta Z')]}{(q - \rho Ap^2)}\right\} \quad \text{(58)}$$

$$V_{oi} = \frac{-\phi W}{|RO|L} \cdot \left\{\frac{\upsilon nn'[\lambda(\alpha Z' - \beta H') + L\upsilon(H'^2 - Z'^2)] + \upsilon(1 - cc')[\lambda(\alpha H' + \beta Z') - 2L\upsilon Z'H']}{LU\rho Ap^2} + \frac{\upsilon\lambda(c - c')(\alpha H' - \beta Z')}{LU\rho Ap^2} + \frac{\lambda(nZ' + n'H')(\beta Z' - \alpha H')}{(q - \rho Ap^2)}\right\} - \frac{\phi W}{L}\left(\frac{\upsilon^2}{U\rho Ap^2} - \frac{1}{(q - \rho Ap^2)}\right) \quad \text{(59)}$$

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$$V_{oj} = \frac{-\phi W}{|RO|L}$$

$$\cdot \left\{ \frac{\upsilon n n'[\lambda(\alpha Z' - \beta H') + L\upsilon(H'^2 - Z'^2) + \upsilon(1 - cc')[\lambda(\alpha H' + \beta Z') - 2L\upsilon Z'H']}{LU\rho A p^2} \right.$$

$$\left. + \frac{\upsilon\lambda(c - c')(\alpha H' - \beta Z')}{LU\rho A p^2} + \frac{\lambda(n'cH' + nc'Z')(\beta Z' - \alpha H')}{(q - \rho A p^2)} \right\}$$

$$- \frac{\phi W}{L}\left[\frac{\upsilon^2}{U\rho A p^2} - \frac{1}{(q - \rho A p^2)}\right] \quad \text{(60)}$$

Sample Examples and Observations

Typical System Formulation and Response Analysis

Since the sign conventions adopted herein are exactly the same as those used in Cheng (1970) and Cheng et al. (1970), the basic formulation of equilibrium, compatibility, and inertia force matrices depicted in those references may be directly used in the present problem. For illustration, consider a general rigid frame shown in Fig. 7(a) subjected to dynamic forces $w_1(x)e^{ipt}$, $w_2(x)e^{ipt}$, and We^{ipt} as well as static axial loads N_1 and N_2. Member DE is also supported by an elastic media q. This structure has five degrees of freedom, which are shown in Fig. 7(b) with the nodal displacements signified by $X_{\psi 1}e^{ipt}$, $X_{\psi 2}e^{ipt}$, $X_{\psi 3}e^{ipt}$, $X_{y1}e^{ipt}$, $X_{y2}e^{ipt}$, and the nodal forces denoted by $P_{\psi 1}e^{ipt}$, $P_{\psi 2}e^{ipt}$, $P_{\psi 3}e^{ipt}$, $P_{y1}e^{ipt}$, and $P_{y2}e^{ipt}$. Terms Q_1, Q_2, Q_3, and Q_4 are inertia forces due to axial rigid-body motions U_1, U_2, U_3, and U_4. The internal moments and shears and their associated deformations of each member are numbered in i and j of which the positive direction is coincided with the typical sign shown in Fig. 4. The system matrix of Fig. 7 may be established as

$$\left\{\begin{array}{c} M_{oj}^1 + M_{oi}^2 \\ M_{oj}^2 \\ 0 \\ \hline V_{oj}^1 - V_{oi}^2 \sin\tau \\ W \end{array}\right\} = \left[\begin{array}{c|c} A_m\left(\sum_{l=1}^{NM} SM\Psi_l\right)A_m^T & A_m\left(\sum_{l=1}^{NM} SMY_l\right)A_y^T \\ \hline A_y\left(\sum_{l=1}^{NM} SV\Psi_l\right)A_m^T & A_y\left(\sum_{l=1}^{NM} SVY_l\right)A_y^T \end{array}\right]$$

$$- \left[\begin{array}{c} 0 \\ \hline p^2 A_q A_p A_q^T \end{array}\right] \left\{\begin{array}{c} X_{\psi_1} \\ X_{\psi_2} \\ X_{\psi_3} \\ \hline X_{y_1} \\ X_{y_2} \end{array}\right\} e^{ipt} \quad \text{(61)}$$

in which $NM = 4$, number of the constituent members; A_m and A_y = the dynamic equilibrium matrices relating external nodal forces to the internal forces of constituent members; A_p = the mass matrix expressing Q's in U's; and A_q is the transfer matrix to express Q's in global coordinates.

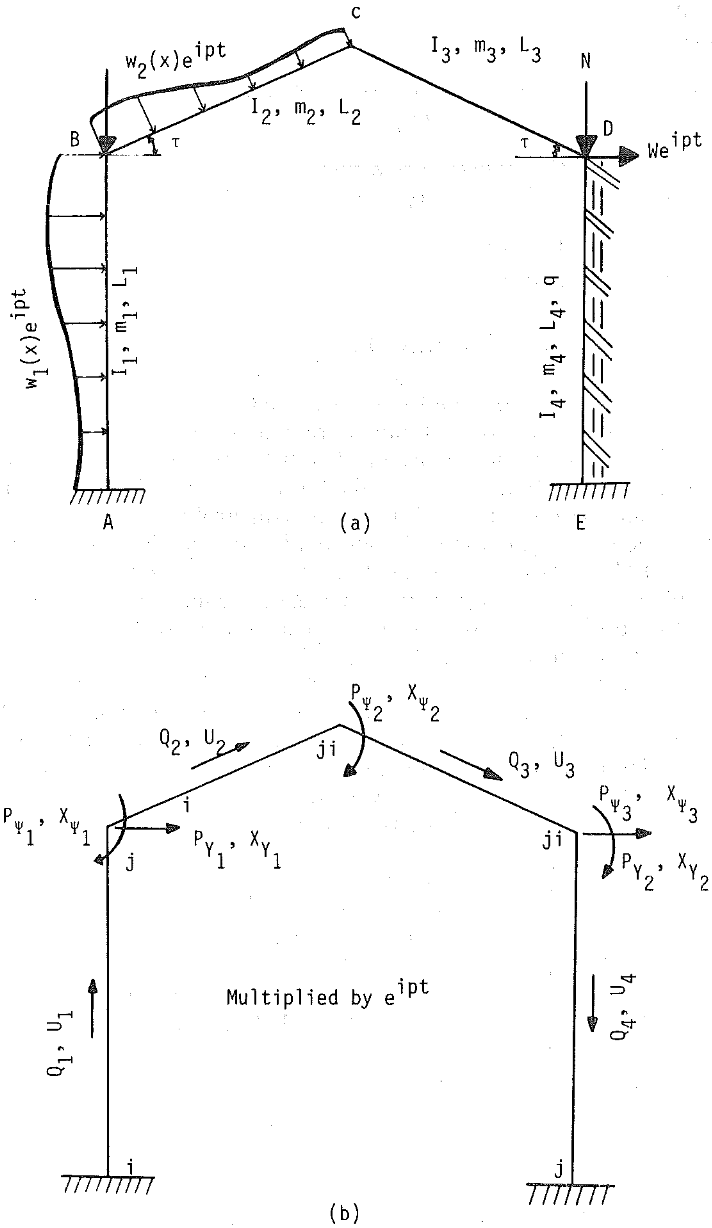

FIG. 7. General Scheme for Typical Frame Analysis: (*a*) Given Structure; (*b*) Typical Diagram

Note that the general formulation of Eq. 61 may be applied to various types of rigid frames and can be much simplified for a structure without sidesway or continuous beams. If the structure shown in Fig. 7 is prevented from sidesway, then Eq. 61 may be modified as

$$\begin{Bmatrix} M_{oj}^{1} + M_{oi}^{2} \\ M_{oj}^{2} \\ 0 \end{Bmatrix} = \left[A_m \left(\sum_{l=1}^{NM} SM\Psi_l \right) A_m^T \right] \begin{Bmatrix} X_{\psi_1} \\ X_{\psi_2} \\ X_{\psi_3} \end{Bmatrix} \quad (62)$$

As shown in Cheng (1970) and Cheng et al. (1970), for a given loading magnitude and forcing frequency, the displacement response can be first obtained from Eqs. 61 or 62 and then the internal forces can be calculated by using dynamic equilibrium matrices, stiffness coefficients, and the external displacements. For eigenvalue problems, the natural frequencies can be obtained by evaluating zero determinant of the structural stiffness and mass matrix shown in the brackets [ ] of the right hand side of Eqs. 61 or 62.

For convenient calculations, the stiffness coefficients and fixed-end forces are expressed in terms of physical parameters such as N/NE, q/qe, L/R, and b, for which some notations may be modified as

$$CC_1 = \frac{N}{\left(\dfrac{\pi^2 EI}{L^2}\right)} = \frac{N}{NE} = \frac{N\left(\dfrac{L}{R}\right)^2}{\pi^2 EA} \quad (63)$$

$$CC_2 = \frac{q}{\left(\dfrac{\pi^4 EI}{L^4}\right)} = \frac{q}{qe} = \frac{q\left(\dfrac{L}{R}\right)^4 I}{\pi^4 EA^2} \quad (64)$$

$$\zeta = 1 - \frac{N}{\phi} = 1 - \frac{2CC_1\pi^2(1+\nu)}{k\left(\dfrac{L}{R}\right)^2} \quad (65)$$

$$r^2 = \frac{I}{AL^2} = \frac{1}{\left(\dfrac{L}{R}\right)^2} \quad (66)$$

$$s^2 = \frac{EI}{\phi L^2} = \frac{2(1+\nu)}{K\left(\dfrac{L}{R}\right)^2} \quad (67)$$

$$T = \frac{N - \dfrac{qEI}{\phi}}{\zeta\rho p^2 I} + 1 + \frac{EA}{\zeta\phi} = \frac{\pi^2}{\zeta b^2 r^2}(CC_1 - CC_2\pi^2 s^2) + 1 + \frac{2(1+\nu)}{\zeta k} \quad (68)$$

$$U = \left(\frac{\rho I p^2}{\zeta\phi} - \frac{1}{\zeta}\right) + \frac{q}{A\zeta}\left(\frac{1}{\rho p^2} - \frac{I}{\phi}\right) = \frac{1}{\zeta}\left[r^2 s^2(b^2 - CC_2\pi^4) - 1 + \frac{CC_2\pi^4}{b^2}\right] \quad (69)$$

$$\Lambda' = \frac{b}{L\alpha}\left(\zeta\alpha^2 + s^2 - \frac{qL^2}{\phi b^2}\right) = \frac{b}{\alpha\left(\dfrac{L}{R}\right)\sqrt{\dfrac{I}{A}}}\left[\zeta\alpha^2 + s^2\left(1 - \frac{CC_2\pi^4}{b^2}\right)\right] \quad (70)$$

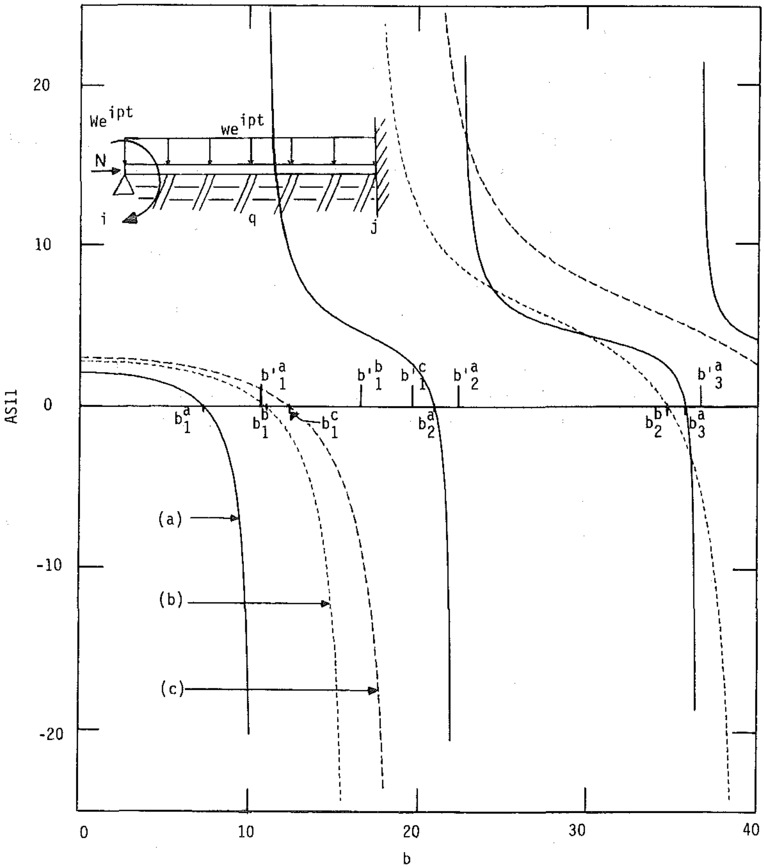

FIG. 8. Natural Frequencies Based on First Approach for Various Slenderness Ratios (*N/NE* = 0.6, *q* = 0): (*a*) *L/R* = 10; (*b*) *L/R* = 20; (*c*) *L/R* = 40

$$B' = \frac{b}{L\beta}\left(\zeta\beta^2 - s^2 + \frac{qL^2}{\phi b^2}\right) = \frac{b}{\beta\left(\frac{L}{R}\right)\sqrt{\frac{I}{A}}}\left[\zeta\beta^2 + s^2\left(\frac{CC_2\pi^4}{b^2} - 1\right)\right] \quad (71)$$

Example

A simple illustration may be shown from Fig. 8 for which the dynamic equilibrium matrix, stiffness coefficients, and system matrix may be established as

$$[A_m] = [1 \quad 0] \quad (72a)$$

$$[SM\Psi] = \begin{bmatrix} S11 & S12 \\ S21 & S22 \end{bmatrix} \quad (72b)$$

$$(W - M_{oi})e^{eipt} = [A_m][SM\Psi][A_m]^T\{X_\Psi\}e^{ipt} = S11\ X_{\Psi_1}e^{ipt} \quad (72c)$$

TABLE 1. Frequencies of Fig. 8

Frequencies (1)	Case a $L/R = 10$ (2)	Case b $L/R = 20$ (3)	Case c $L/R = 40$ (4)
(*a*) Hinged-fixed			
b_1^a	7.32	—	—
b_2^a	20.93	—	—
b_3^a	35.70	—	—
b_1^b	—	11.02	—
b_2^b	—	34.74	—
b_1^c	—	—	12.44
(*b*) Fixed-fixed			
$b_1'^a$	10.55	—	—
$b_2'^a$	22.20	—	—
$b_3'^a$	—	36.48	—
$b_1'^b$	—	16.45	—
$b_1'^c$	—	—	19.41

Note: $q = 0$, $N = 0.6\ NE$.

knowing the forcing frequency, p, and the loading magnitudes of W and w, one may calculate X_{ψ_t} from Eq. 72*c*. When the external loads are not applied, one may calculate the natural frequencies by evaluating the zero determinant of the righthand side of Eq. 72*c*, that, in fact, yields the stiffness-coefficient variations for different values of frequency parameter, b, as shown in Fig. 8. The graphs of $AS11$ [$AS11 = S11/(EI/L)$] are plotted for three cases of slenderness ratios, $L/R = 10, 20, 40$, with $q = 0$. and $N = 0.6NE$, where $NE = \pi^2EI/L^2$, Euler buckling load associated with these three slenderness ratios. Other parameters used are $\nu = 0.25$, $k = 2/3$, $\rho = 7829\ \text{kgm}^{-3}$, and $E = 212.95\ \text{kNm}^{-2}$. Fig. 8 shows two distinct characteristics: zero and infinite points. The former is associated with natural frequencies of the member shown in Fig. 8; the latter, however, is corresponding to the natural frequencies of the member with both ends fixed. Because the end-rotations are restrained, the end-moments must approach to infinity when the natural frequency reaches resonance. The frequency parameters corresponding to zero and infinite points are summarized in Table 1. Note that when L/R is reduced, the frequency parameter, b, decreases. For bending deformation only without axial force, the frequency parameter b is 15.394 and 22.368 of hinged-fixed and fixed-fixed conditions, respectively, for all slenderness ratios. Also, note that as $b \rightarrow 0$, the static stiffness are obtained as 2.095, 2.840, and 3.061, corresponding to $L/R = 10$, 20, and 40, respectively.

The effects of axial loads and elastic media on the natural frequencies and stiffness coefficients are shown in Fig. 9, the most interesting results from which are summarized in Table 2. In Table 2, $qe = \pi^4EI/L^4$; $NE = \pi^2EI/L^2$ (Euler buckling); and b_1 and b_1' = the frequency parameters corresponding to zero points (hinged-fixed) and infinite points (fixed-fixed), respectively. It is apparent that the axial force reduces the stiffnesses and natural frequencies, whereas the elastic media increases the stiffness and natural frequencies.

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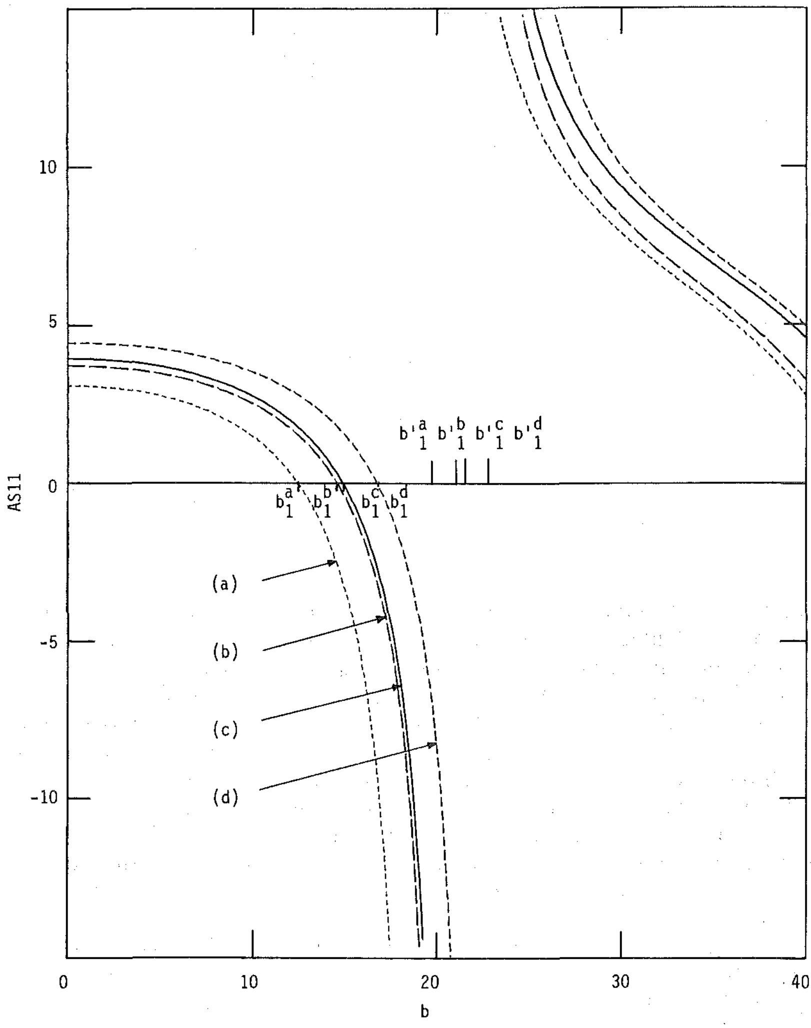

FIG. 9. Natural Frequencies Based on First Approach for Various Elastic Media and Axial Load Conditions (L/R = 40): (*a*) N/NE = 0.6, q = 0; (*b*) N/NE = 0.6, q/qe = 0.6; (*c*) N = 0, q = 0; (*d*) N = 0, q/qe = 0.6.

The comparison of the natural frequencies obtained by the first and second approaches is summarized in Table 3, in which the parameters b, with and without primes, correspond to the natural frequencies of fixed-fixed and hinged-fixed, respectively. It may be observed that the second approach yields higher frequencies than the first. Note that the axial load N used in this example is based on the fundamental buckling load of a simply supported beam (Euler buckling). Therefore, the influence of N on b is more than on b' for the same mode.

TABLE 2. Frequencies of Fig. 9

Cases (1)	q/qe (2)	Frequencies b_1 (3)	Frequencies b_1' (4)	Stiffness at $b = 0$ (5)
		(*a*) $N/NE = 0.6$		
a	0.0	12.45	19.41	3.061
b	0.6	14.60	20.81	3.687
		(*b*) $N/NE = 0.0$		
c	0.0	14.90	21.11	3.879
d	0.6	16.74	22.42	4.416

TABLE 3. Frequencies Using Two Approaches

Hinged-fixed or fixed-fixed (1)	Frequencies (2)
(*a*) First approach	
b_1^1	10.46
b_2^1	22.20
b_3^1	36.50
$b_1^{1\prime}$	12.95
$b_2^{1\prime}$	23.40
Stiffness $b = 0$	2.922
(*b*) Second approach	
b_1^2	10.95
b_2^2	24.11
$b_1^{2\prime}$	13.75
$b_2^{2\prime}$	25.67
Stiffness $b = 0$	3.035

Note: $L/R = 10$, $N/NE = 0.6$, $q/qe = 0.6$.

Observation of the Frequencies Based on the Two Approaches

Presented herein is the effect of two approaches on the fundamental and higher frequencies of a typical simply supported beam for which the boundary conditions are

$$Y(0) = 0, \quad M(0) = EI\Psi'(0) = 0 \quad \text{(73)}$$

$$Y(1) = 0, \quad M(0) = EI\Psi'(1) = 0 \quad \text{(74)}$$

Using the first approach yields the following frequency equation:

$$P1_n^2 = \frac{\phi}{2\rho I}\left\{\left[n^2\pi^2(\zeta r^2 + s^2) + 1 + \frac{qL^2r^2}{\phi}\right.\right. - \sqrt{\left[n^2\pi^2(\zeta r^2 + s^2) + 1 + \frac{qL^2r^2}{\phi}\right]^2 - 4\left[n^2\pi^2s^2r^2\left(\zeta n^2\pi^2 - \frac{N}{s^2\phi}\right) + \frac{qL^2r^2}{\phi}(n^2\pi^2s^2 + 1)\right]}\Bigg\} \quad \text{(75)}$$

where n = mode number. When the shear deformations and rotatory inertia effects are ignored

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$$PE_n^2 = \frac{1}{L^2}\left[n^4\pi^4\left(\frac{EI}{L^2\rho A} - \frac{N}{n^2\pi^2\rho A}\right) + \frac{qL^2}{\rho A}\right] \quad (76)$$

For the second approach, the frequency equation may be expressed as

$$P2_n^2 = \frac{\phi}{2\rho I}\left\{\left[n^2\pi^2(r^2 + s^2) + \upsilon + \frac{qL^2r^2}{\phi}\right]\right.$$

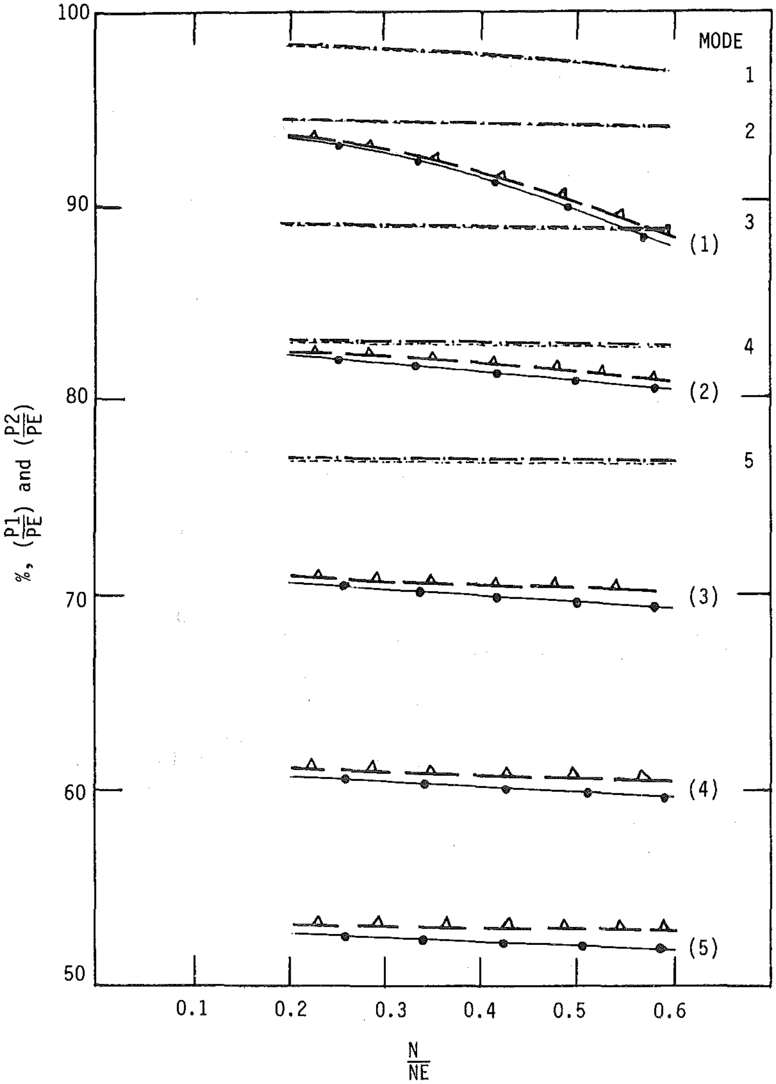

FIG. 10. Natural Frequencies Using First and Second Approach: First Approach L/R = 20: —●—●—●—, L/R = 40: -----; Second Approach L/R = 20:△△△, L/R = 40: — . — . — . —

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$$- \sqrt{\left[\left(n^2\pi^2(r^2+s^2)+\upsilon+\frac{qL^2r^2}{\phi}\right]^2 - 4\left[n^2\pi^2s^2r^2\left(n^2\pi^2-\frac{N\upsilon}{s^2\phi}\right)+\frac{qL^2r^2}{\phi}(n^2\pi^2s^2+\upsilon)\right]\right\}} \quad \ldots\ldots \quad (77)$$

When the shear deformations and rotatory inertia are neglected, Eq. 77 becomes identical to Eq. 76.

Using the numerical data given previously, a plot of $P1/PE$ and $P2/PE$

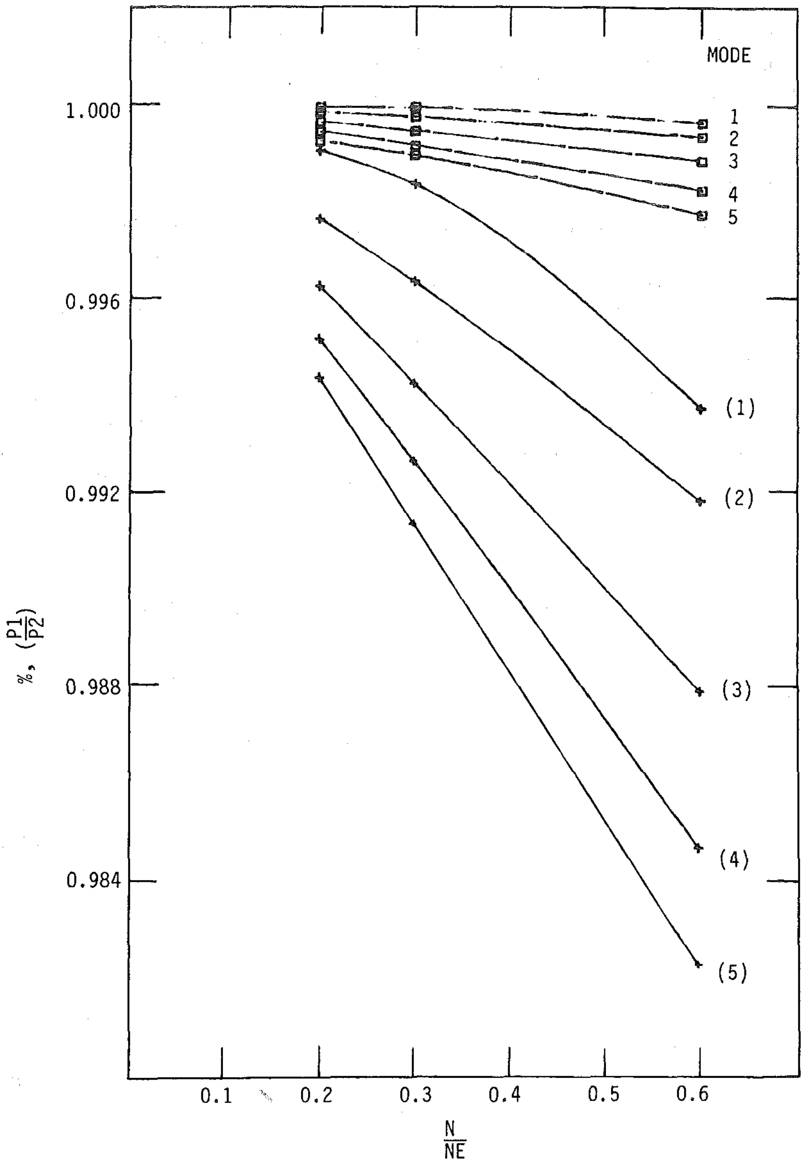

FIG. 11. Comparison of First and Second Approach Frequencies: L/R **= 20:** —+—+—, L/R **= 40:** —□—□—

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versus N/NE is given in Fig. 10, for $q/qe = 0$, and two slenderness ratios, $L/R = 40$ and $L/R = 20$. $P1$, PE and $P2$ are the natural frequencies of the first five modes from Eqs. 75, 76, and 77 respectively. NE is the fundamental Euler buckling load. It may be observed that the Timoshenko theory has significant effect on a smaller L/R and a higher frequency and that the differences between $P1/PE$ and $P2/PE$ become significant for higher modes and for larger axial loads.

The ratios of $P1$ and $P2$ are given in Fig. 11 for the first five modes associated with $q/qe = 0$. The figure shows that the first approach yields lower frequency than the second; the difference becomes more pronounced for higher axial loads and lower slenderness ratios.

Conclusions

Differential equations, stiffness coefficients, and fixed-end forces have been derived for general analysis of structural systems, the constituent members of which may have transverse and rotatory inertia, shear and bending deformations, axial force, and elastic foundation. The numerical procedures of determining natural frequency and dynamic response have been illustrated and compared for various typical beams. Two approaches were employed for considering the direction of the shear force component of the axial load. The first approach is based on the model that the shear component is calculated from the total slope due to bending and shear, and the second approach assumes that the shear component is calculated only from the bending slope.

Acknowledgments

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Appendix I. Complex Solution

Three distinct cases can arise from the solution of Eq. 13. Case 1: α and β are complex when

$$r^4T^2 < \frac{4U}{b^2} \quad (78)$$

then, from Eq. 13

$$\begin{matrix}\alpha\\ \beta\end{matrix} = \frac{1}{\sqrt{2}}\sqrt{\mp r^2T + i\sqrt{\frac{4U}{b^2} - r^4T^2}} \quad (79)$$

For simplicity, let

$$B = r^2T \quad (80)$$

$$D = \sqrt{\frac{4U}{b^2} - r^4T^2} \quad (81)$$

where b and D are both real and positive quantities. Eq. 79 can be written as

$$\frac{\alpha}{\beta} = \frac{1}{\sqrt{2}} \sqrt{\frac{-}{+} B + iD} \qquad (82)$$

Expressing Eq. 82 in polar form

$$\beta = \frac{1}{\sqrt{2}} \sqrt{B + iD} = \frac{1}{\sqrt{2}} [\sqrt{B^2 + D^2}\, e^{i \arctan\left(\frac{D}{B}\right)}]^{1/2}$$

Two cases exist

$$\beta_0 = \frac{1}{\sqrt{2}} (B^2 + D^2)^{1/4} \left\{ \cos\left[\frac{1}{2} \arctan\left(\frac{D}{B}\right)\right] + i \sin\left[\frac{1}{2} \arctan\left(\frac{D}{B}\right)\right]\right\} \qquad (83)$$

and

$$\beta_1 = -\frac{1}{\sqrt{2}} (B^2 + D^2)^{1/4} \left\{ \cos\left[\frac{1}{2} \arctan\left(\frac{D}{B}\right)\right] + i \sin\left[\frac{1}{2} \arctan\left(\frac{D}{B}\right)\right]\right\} \qquad (84)$$

Similarly

$$\alpha = \frac{i}{\sqrt{2}} (\sqrt{B^2 + D^2}\, e^{i \arctan\left(\frac{-D}{B}\right)})^{1/2} \qquad (85)$$

and the two cases are

$$\alpha_0 = \frac{1}{\sqrt{2}} (B^2 + D^2)^{1/4} \left\{ \sin\left[\frac{1}{2} \arctan\left(\frac{D}{B}\right)\right] + i \cos\left[\frac{1}{2} \arctan\left(\frac{D}{B}\right)\right]\right\} \qquad (86)$$

and

$$\alpha_1 = \frac{-1}{\sqrt{2}} (B^2 + D^2)^{1/4} \left\{ \sin\left[\frac{1}{2} \arctan\left(\frac{D}{B}\right)\right] + i \cos\left[\frac{1}{2} \arctan\left(\frac{D}{B}\right)\right]\right\} \qquad (87)$$

Comparing α_0 and β_0, the real part of α_0 is equal to the imaginary part of β_0, and the imaginary part of α_0 is equal to the real part of β_0. Similar arguments hold true if we compare α_1 and β_1.

Simplifying the two cases into one; with symbols F and Q that are real

$$\text{Re}(\alpha) = \text{Im}(\beta) = F \qquad (88)$$

$$\text{Im}(\alpha) = \text{Re}(\beta) = Q \qquad (89)$$

the α and β can be written as

$$\alpha = F + iQ \qquad (90a)$$

$$\beta = Q + iF \qquad (90b)$$

To prove whether the stiffness coefficients and fixed-forces are real, the writers investigate a typical stiffness coefficient $S11$ given in Eq. 29*a*. Let us first study the quantities Λ', B', n, n', c, and c', which are used in $|R'|$. Then $|R'|$ and other quantities used in $S11$ are studied. For

$$\Lambda' = \frac{b}{L\alpha}\left(\zeta\alpha^2 + s^2 - \frac{qL^2}{\phi b^2}\right)$$

Let

$\frac{b}{L} = R1$, $s^2 - \frac{qL^2}{\phi b^2} = R2$, then

$$\Lambda' = \frac{R1}{(F^2 + Q^2)} \{[F\zeta(F^2 + Q^2) + FR^2] + [Q\zeta(F^2 + Q^2) - QR2]i\} \quad (91)$$

Similarly

$$B' = \frac{R1}{(F^2 + Q^2)} \{[Q\zeta(F^2 + Q^2) - QR2] + [F\zeta(F^2 + Q^2) + FR2]i\} \quad (92)$$

Comparing Eqs. 91 and 92, we can write

$$\Lambda' = F_1 + iQ_1 \quad (93a)$$

$$B' = Q_1 + iF_1 \quad (93b)$$

where F_1 and Q_1 = real quantities. n and n' may be expressed as

$$n = \sinh b\alpha = \sinh b(F + iQ) = \sinh bF \cos bQ + i \cosh bF \sin bQ \quad (94)$$

$$n' = \sin b\beta = \sin b(Q + iF) = \sin bQ \cos bFi + \sin bFi \cos bQ \quad (95)$$

Using the identities

$$\cos bFi = \cosh bF \quad (96)$$

$$\sin bFi = i \sinh bF \quad (97)$$

Eq. 95 becomes

$$n' = \sin b\beta = \sin bQ \cosh bF + i \sinh bF \cos bQ \quad (98)$$

Comparing Eqs. 94 and 98, we have

$$n = F_2 + iQ_2 \quad (99a)$$

$$n' = Q_2 + iF_2 \quad (99b)$$

where F_2 and Q_2 = real quantities. Similarly, c and c' can be expressed as

$$C = \cosh b\alpha = \cosh b(F + iQ) = \cosh bF \cos bQ + i \sinh bF \sin bQ \quad (100)$$

$$C' = \cos b\beta = \cos b(Q + iF) = \cos bQ \cos bFi - \sin bQ \sin bFi \quad (101)$$

Using Eqs. 96–97

$$C' = \cos bQ \cosh bF = i \sin bQ \sinh bF \quad (102)$$

Comparing Eqs. 100 and 102 yields

$$C = F_3 + iQ_3 \qquad (103a)$$

$$C' = F_3 - iQ_3 \qquad (103b)$$

where F_3 and Q_3 = real quantities.

The expression of $|R'|$ is

$$|R'| = nn'(\Lambda'^2 - B'^2) + 2(1 - cc')\Lambda'B' \qquad (104)$$

Using Eqs. 93*a*–93*b*, 99*a*–99*b*, 103*a*–103*b*, we have

$$nn' = (F_2 + iQ_2)(Q_2 + iF_2) = i(Q_2^2 + F_2^2) = iQ_4 \qquad (105)$$

$$\Lambda'^2 - B'^2 = (F_1 + iQ_1)^2 - (Q_1 + iF_1)^2 = 2F_1^2 - 2Q_1^2 = F4 \qquad (106)$$

$$cc' = (F_3 + iQ_3)(F_3 - iQ_3) = (F_3)^2 + (Q_3)^2 = F_5 \qquad (107)$$

$$\Lambda'B' = (F_1 + iQ_1)(Q_1 + iF_1) = iQ_1^2 + iF_1^2 = iQ_5 \qquad (108)$$

Using Eqs. 105–108, Eq. 104 becomes

$$|R'| = (iQ_4)(F_4) + 2(1 - F_5)(iQ_5) = iQ_6 \qquad (109)$$

where Q_4, Q_5, Q_6, F_4, F_5 = real and $|R'|$ is therefore a purely imaginary quantity. Recall Eq. 29*a* of $S11$.

$$S11 = \left(\frac{EIb}{L}\right)\frac{(\alpha\Lambda' + \beta B')(n'c\Lambda' - nc'B')}{|R'|} \qquad (110)$$

Using Eqs. 90*a*–*b* and 93*a*–*b*

$$\alpha\Lambda' + \beta B' = (F + iQ)(F_1 + iQ_1) + (Q + iF)(Q_1 + iF_1)$$

$$\alpha\Lambda' + \beta B' = 2i(QF_1 + FQ_1) = iQ_7 \qquad (111)$$

Based on Eqs. 93*a*–*b*, 99*a*–*b*, and 103*a*–*b*

$$n'c\Lambda' - nc'B' = (Q_2 + iF_2)(F_3 + iQ_3)(F_1 + iQ_1)$$

$$- (F_2 + iQ_2)(F_3 - iQ_3)(Q_1 + iF_1) = F_6 \qquad (112)$$

where F_6 and Q_7 are real. Substituting Eqs. 109, 111, and 112 in Eq. 110 yields

$$S11 = \left(\frac{EIb}{L}\right)\frac{(iQ_7)(F_6)}{iQ_6}$$

Since EI, b, L are real

$$S11 = \frac{EIb}{L}\frac{(Q_7)(F_6)}{Q_6} \qquad (113)$$

Hence $S11$ = a real quantity. Similar results follow for the rest of the stiffness coefficients and fixed-end forces.

Case 2: α is purely imaginary and β is real when

$$r^4T^2 > \frac{4U}{b^2} \text{ and } \sqrt{r^4T^2 - \frac{4U}{b^2}} < r^2T \qquad (114)$$

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For simplicity let

$$B = r^2 T$$

$$\Gamma = \sqrt{r^4 T^2 - \frac{4U}{b^2}}$$

where B and Γ are both real and positive quantities Eq. 13 can be written as

$$\begin{matrix}\alpha \\ \beta\end{matrix} = \frac{1}{\sqrt{2}} \sqrt{\mp B + \Gamma} \quad \text{(115)}$$

Since from Eq. 114, $\Gamma < B$ is true, then

$$\alpha = \frac{1}{\sqrt{2}} i\sqrt{B - \Gamma} = i\Delta \quad \text{(116)}$$

where Δ = a real and positive quantity, and hence α = a purely imaginary quantity. Simplifying

$$\beta = \frac{1}{\sqrt{2}} \sqrt{B + \Gamma} = \Theta \quad \text{(117)}$$

where Θ = a real and positive quantity; and β = a real quantity. Consequently

$$\Lambda' = \frac{b}{Li\Delta}\left[\zeta(i\Delta)^2 + s^2 - \frac{qL^2}{\phi b^2}\right] = -i\left(\frac{b}{L\Delta}\right)\left[-\zeta\Delta^2 + s^2 - \frac{qL^2}{\phi b^2}\right] = i\Delta_1 \quad \text{(118)}$$

where Δ_1 is real, and Λ' = a purely imaginary quantity. Similarly, B' can also be proved to be a real quantity as

$$B' = \Theta_1 \quad \text{(119)}$$

For n, n', c, and c', one may write

$$n = \sinh b\alpha = \sinh ib\Delta = i \sin b\Delta = i\Delta_2 \quad \text{(120)}$$

$$n' = \sin b\beta = \Theta_2 \quad \text{(121)}$$

$$c = \cosh b\alpha = \cosh ib\Delta = \cos b\Delta = \Theta_3 \quad \text{(122)}$$

$$c' = \cos b\beta = \Theta_4 \quad \text{(123)}$$

where Δ_2, Θ_2, Θ_3, and Θ_4 are real quantities. For the quantities in $|R'|$, we have

$$nn' = (i\Delta_2)(\Theta_2) = i\Delta_3 \quad \text{(124)}$$

$$\Lambda'^2 - B'^2 = (i\Delta_1)^2 - (\Theta_1)^2 = -\Delta_1^2 - \Theta_1^2 = \Theta_5 \quad \text{(125)}$$

$$cc' = \Theta_3\Theta_4 = \Theta_6 \quad \text{(126)}$$

$$\Lambda' B' = (i\Delta_1)(\Theta_1) = i\Delta_4 \quad \text{(127)}$$

The final expression of $|R'|$ is then

$$|R'| = (i\Delta_3)(\Theta_5) + 2(1 - \Theta_6)(i\Delta_4) = \Delta_5 i \quad (128)$$

The other quantities in $S11$ are

$$\alpha A' + \beta B' = (i\Delta)(i\Delta_1) + \Theta\Theta_1 = -\Delta\Delta_1 + \Theta\Theta_1 = \Theta_7 \quad (129)$$

$$n'cA' - nc'B' = (\Theta_2\Theta_3\Delta_1 - \Delta_2\Theta_4\Theta_1)i = \Delta_6 i \quad (130)$$

Substituting in Eqs. 128–130 into Eq. 29*a*,

$$S11 = \left(\frac{EIb}{L}\right)\frac{(\Theta_7)(\Delta_6 i)}{(\Delta_5 i)} = \frac{EI\Theta_7\Delta_6}{L\Delta_5} \quad (131)$$

Hence $S11$ = a real quantity. Similar results follow for the rest of the stiffness coefficients and fixed-end forces.

Case 3: α and β are real when

$$r^4T^2 > \frac{4U}{b^2} \quad \text{and} \quad r^2T < \sqrt{r^4T^2 - \frac{4U}{b^2}} \quad (132)$$

Similar reasoning can be shown to hold true for the second approach formulation.

Appendix II. References

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Appendix III. Notation

The following symbols are used in this paper:

A = cross-sectional area;
C, C' = integration constants;
E = modulus of elasticity;
G = modulus of rigidity;
I = moment of inertia of cross section;
i = $\sqrt{-1}$;
k = shear factor for cross section;
L = length of beam element;
L/R = slenderness ratio;
M = internal moments;
N = static axial load;
$P1, P2$ = natural frequency of Timoshenko theory;
PE = natural frequency of Bernoulli-Euler theory;
$[S]$ = notation for stiffness matrix;
$[S_{ij}]$ = notation for stiffness coefficients;
$T, U, b, r, s, \alpha, \beta, \zeta, \xi$ = dimensionless notation;
V = internal shears;
W = magnitude of forcing function;
q = elastic foundation constant;
t = time;
w = forcing function;
x = rectangular coordinate abscissa;
y = rectangular coordinate ordinate;
$\underline{\beta}$ = shearing slope;
$\bar{\gamma}$ = weight per unit volume;
ν = Poisson's ratio; and
ψ, ψ_i, and ψ_j = rotational deformations.

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