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K. B. Agarwal
University Sains Malaysia, Perak Branch Campus, Malaysia

B. Siva Ram
Consultants Firm, Australia

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A Numerical Solution of Wave Equation for Dynamic Compaction of Soil

K.B. Agarwal
Professor, School of Civil Engineering, University Sains Malaysia, Perak Branch Campus, Seri Iskandar, 31750 Tronoh, Perak Darul Ridzuan, Malaysia

B. Siva Ram
Geotechnical Engineer, Consultants firm - Australia

SYNOPSIS: It is a known fact that any disturbance at the ground surface, like the one created by vibratory, compactors or by application of blast pressure on detonation of a foamed propellant, is transmitted into ground until it is weak enough to travel deeper and farther. The ground acceleration at various points, induces compaction. The transmission of vibrations due to such surface dynamic loads are governed by the equation of motion based on Newton's second law. The equation of motion is presented in Euiler's Coordinates using tensor notation and is solved for surface displacements due to surface dynamic loads. These loads are likely to be experienced over a half space due to movement of vehicles, compactors etc. The paper presents a finite difference iterative method for solving the above equation which permits the simultaneous solution of two partial differential equations in plane strain condition. Results of the present analysis have been compared with those available from theory of elasticity.

INTRODUCTION

The explosives like dynamite, giletinite and T.N.T. have been successfully applied for compaction of deep loose deposits of sand and silt. The above method has been put forth by Lyman (1942) and consists of lowering capsuled charges inside deep boreholes located at two thirds of the depth of compaction required from the ground surface. The boreholes are plugged i.e. stemmed and the leads for detonation are laid up to the control point. The boreholes are spaced at predetermined distances and the sequence of detonation which are designed according to the requirement, serve to give optimum compaction.

The present method differs from above in respect of method of applying the charge. An explosive can be applied on surface to compact the surficial layers of 1.5 to 2 meters depth. This has specific advantage of reducing the cost and time involved in elaborately making the boreholes. A sensitive mat of propellant (expanded by foaming agent) has been used over a loose stretch of ground and detonated to achieve compaction in this study.

The evaluation of exact value of compacted density is considered difficult from the present knowledge of dynamic compaction. An attempt however, has been made to theoretically predict the depth of compaction. The analysis is based on solving numerically the wave equation for disturbance on the surface caused by a known profile of pressure versus time. The time history of vibration due to travel of pressure pulse through the medium as determined by solving wave equation has been put to use for evaluating the compaction. The criteria used in this study is the fact that compaction would not take place if the highest vertical acceleration obtained from acceleration time history falls below unity (Appolonia 1967).

GROUND VIBRATION DUE TO DYNAMIC LOADS

Transmission of vibration due to external dynamic loads is governed by the equation of motion based on Newton's law of motion. The accurate equation is presented below in the Euiler's coordinates using tensor notations by Grigorian (1960):

$$\rho \frac{D\dot{v}_i}{Dt} - \frac{\delta \sigma_m}{\delta x_i} - \frac{\delta s_{ij}}{\delta x_j} = \rho I_i \qquad ..(1)$$

where ρ = mass density;

$\dot{v}_i$ = velocity vector in arbitrarily chosen cartesian coordinate direction i = 1,2,3, corresponding to say x_1, x_2, x_3 and also

$$\dot{v}_i = \frac{v_i}{t};$$

v_i = displacement vector

σ_m = mean normal pressure

$$\frac{1}{3}\sigma_{ii} \text{ or } (\sigma_{11} + \sigma_{22} + \sigma_{33})/3$$

δ_{ij} = Kronecker delta;

S_{ij} = deviatoric stress tensor

$$\sigma_{ij} - \frac{1}{3}\sigma_{ij} \cdot \delta_{ij} \quad ;$$

ρI_i = external body force, like gravity per unit volume,

σ_{ii} = terms with repeated subscripts to be added for i = 1,2,3.

$$\frac{D}{Dt} = \frac{\delta}{\delta t} + \dot{v}_i \frac{\delta}{\delta x_i}$$

The above equation has been solved numerically for disturbance on the surface caused by the known profile of dynamic stresses. Hence the time history of vibration due to travel of pressure pulse in the sand medium and the surface displacements are determined.

A simpler but less accurate form of equation of motion is also given below:

$$\rho \frac{\delta \dot{v}_i}{\delta t} - \frac{\delta \sigma_m}{\delta x_i} - \frac{\delta S_{ij}}{\delta x_j} = \rho I_i \qquad ..(2)$$

The notations have already been defined in Eq. 1.

It may be noted that deviatoric stress tensor as well as mean normal stress can be represented in terms of displacement by following relationships.

$$S_{ij} = 2\, G(\varepsilon_{ij} - \frac{1}{3} \varepsilon_{kk}\, S_{ij}) \qquad ..(3)$$

$$\sigma_{ii} = K\, (\varepsilon_{ii}) \qquad ..(4)$$

where

G = shear modulus

K = bulk modulus

ε_{ij} = strain vector

A more generalised equation of strain in three directions for i and j each equal to 1,2,3 may be written in the following form

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\delta V_i}{\delta X_j} + \frac{\delta V_j}{\delta X_i} \right)$$

where j is the repeated index.

In present analysis two dimensional plane strain problem (Fig. 1) is considered in plane xz which allows equating strain and displacement in y direction to a constant.

The analysis presented here in considers the elastic properties of sand by developing a finite difference iterative algorithm for solving the wave equation which permits the simultaneous solution of two partial differential equations in plane strain condition.

STATEMENT OF PROBLEM

When the dynamic stress e.g. the blast pressure is applied on the surface of sand, the pulse is transmitted into the medium according to the equation of motion. Let equation 1 be represented in cartesian coordinates x,y,z; in which z represents the vertical direction across the strip Fig. 1. The stresses σ_x, σ_y and σ_z represent normal stresses and τ_{xy}, τ_{yz} and τ_{zx} the shear stresses in x,y,z directions respectively on any arbitrary element. Let u and v be the displacements in x and z directions respectively. The equations of motion in the horizontal and vertical, viz x and z, directions respectively can be written on the basis of Eq. 1 as follows:

$$\rho \left[\frac{D}{Dt} \frac{\delta u}{\delta t} \right] - \frac{\delta \sigma_m}{\delta x} - \frac{\delta}{\delta x} \left[\sigma_x - \frac{1}{3} (\sigma_x + \sigma_y + \sigma_z) \right] - \frac{\delta \tau_{xz}}{\delta z} - \rho I_x = 0 \qquad ..(5)$$

$$\rho \left[\frac{D}{Dt} \frac{\delta v}{\delta t} \right] - \frac{\delta \sigma_m}{\delta z} - \frac{\delta}{\delta z} \left[\sigma_z - \frac{1}{3} (\sigma_x + \sigma_y + \sigma_z) \right] - \frac{\delta \tau_{xz}}{\delta x} - \rho I_z = 0 \qquad ..(6)$$

The following assumptions have been made w.r.t. above problem:

(a) The application of dynamic stresses over sand surface is a plane strain problem

(b) Due to stress propagation in the media, in general, the small strain (ε_{ij}) will be induced and hence small strain theory is applicable.

where by:

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\delta u_i}{\delta x_j} + \frac{\delta u_j}{\delta x_i} \right) \qquad ..(7)$$

SOLUTION OF EQUATION OF MOTION

The terms containing stresses in Eqs. 5 and 6 ca be converted into strains which in turn lead to particle displacements. A uniform equation in terms of displacements can be obtained by the us of appropriate deformation modulii. The terms $[\frac{D}{Dt} \frac{\delta u}{\delta t}]$ and $[\frac{D}{Dt} \frac{\delta v}{\delta t}]$ in Eqns. 5 and 6 may be replaced by $\emptyset_x$ and $\emptyset_z$ respectively. These terms resresent the force generated by particle momentum. The rest of the terms in Eqns. 5 and 6 refer to stress terms i.e. internal reaction and body forces and let them be represented by w_x and w_z in respective equations. The equations o motion now become:

$$\emptyset_x + w_x = 0 \qquad ..(8)$$

$$\emptyset_z + w_z = 0 \qquad ..(9)$$

The equation of motion has been solved using finite difference technique using central difference scheme. Finite difference method is a wel known powerful mathematical tool for analysis of differential equations in which all the differential forms are approximated with the dependent parameters in terms of their values in the neighbouring domain with respect to all independent variables.

SOLUTION TECHNIQUE

Let a specified extent of half space be defined for the computational purpose. Let this area be divided into fine 3-D mesh with spacing Δx in X direction, Δz in vertical and Δt in the time space. Let the dependent parameters u and v, the particle displacements be suffixed according to Fig. 2. A common set of coordinates X, Z are defined from which the nodal distance of particle be designated U and V respectively. It may be noted that the coordinate axes X, Z are parallel to that of x, z system respectively.

On either side of a node at a distance Δx in X direction two nodes are marked and corresponding dependent parameters are suffixed with L and R respectively, meaning left or right. At distance Δz in vertical direction along Z axis, the dependent parameters are suffixed T and B meaning top and bottom. At time units Δt on time axis, the dependent parameters are suffixed p, f for the preceeding and subsequent time steps. Accordingly, designation of each of the dependent parameters may be suffixed with L or R, T or B, p or

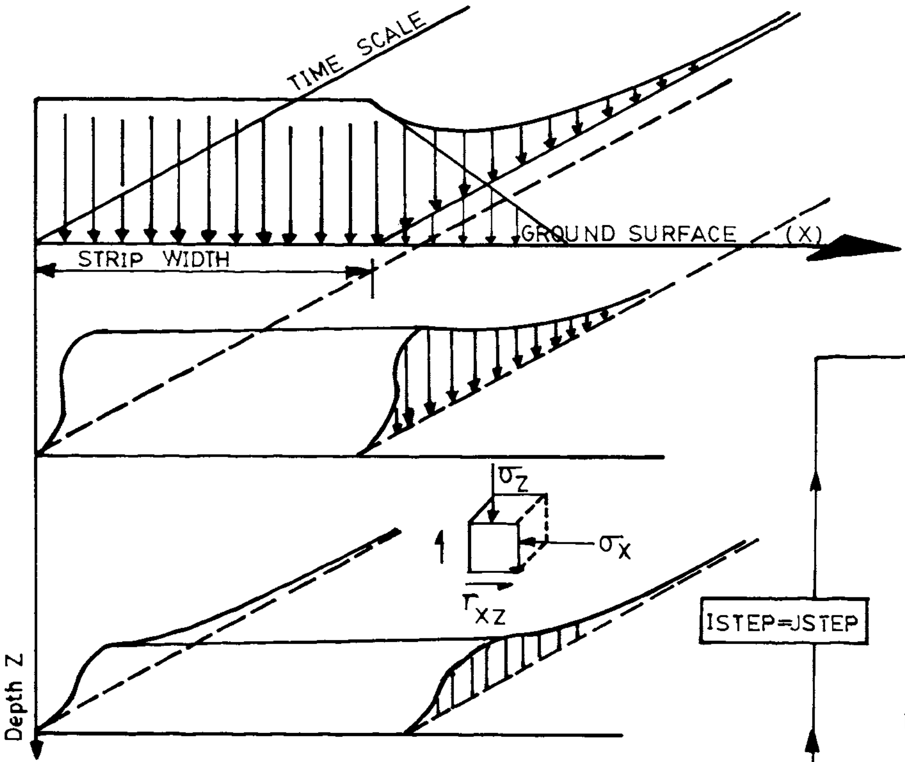


Fig. 1. Transmission of Dynamic Stress

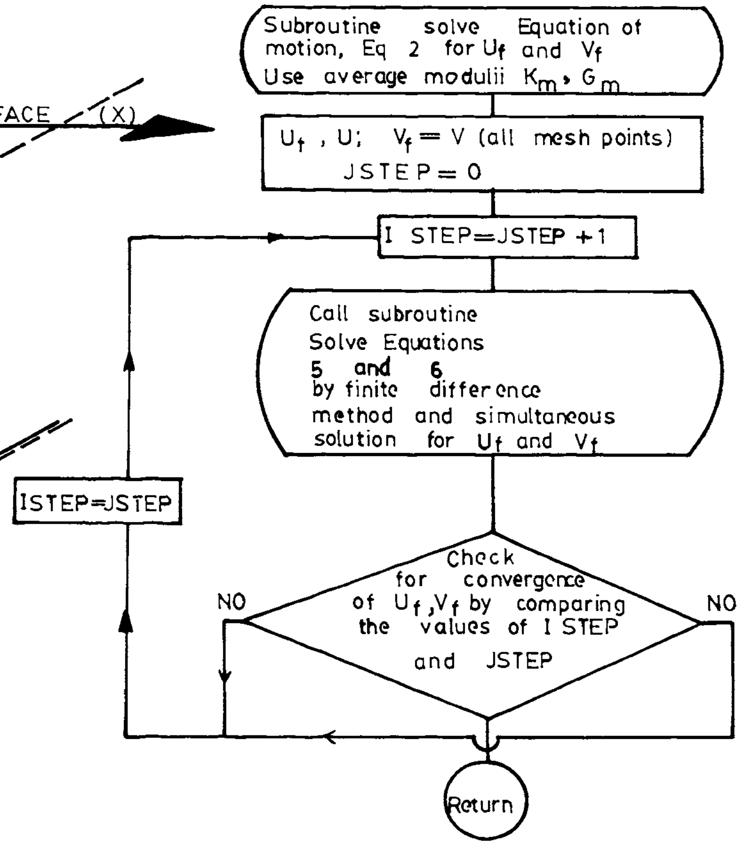
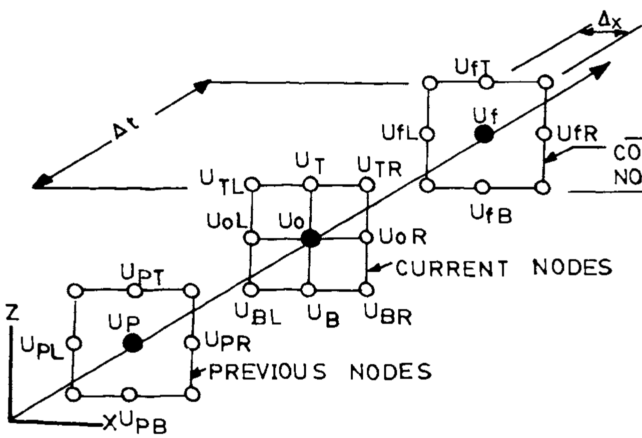


Fig. 3. Flow Chart for Solution of the Accurate Wave Equation



(a) Values of Independent Parameter U in the Neighbouring Domain

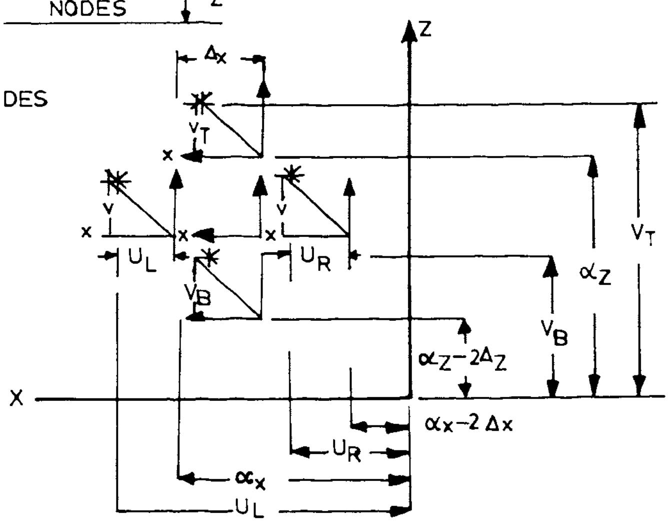


Fig. 2. Coordinate System

(b) Displacement w.r.t. Local Coordinate Axes(x,z) and Common Coordinate(x z) Axes f respectively. The equation of motion is then replaced by suitable difference equations. To evaluate dependent variables, the difference equations are solved algebraically.

Final form of equation of motion i.e. Eq. 1 in x and z direction using the nodal coordinates and finite difference scheme can be written in the following form:

for x direction $\emptyset_1 U_f + \emptyset_2 V_f + \emptyset_5 = 0$ ..(10)

for z direction $\emptyset_6 U_f + \emptyset_7 V_f + \emptyset_{10} = 0$ ..(11)

where $\emptyset_1, \emptyset_2, \emptyset_5, \emptyset_6, \emptyset_7$ and $\emptyset_{10}$ are themselves the nodal coordinates U_f and V_f on adjacent nodes of the computed one (viz. $U_{fL}, U_{fR}, V_{fL}, V_{fR}, U_{fT}$ etc.) Solving Eqs. 10 and 11 the value of U_f and V_f can be obtained and are given by Eqs. 12 and 13.

$$V_f = \frac{\emptyset_{10}\,\emptyset_1 - \emptyset_5\,\emptyset_6}{\emptyset_2\,\emptyset_6 - \emptyset_7\,\emptyset_8} \quad ..(12)$$

$$U_f = \frac{(V_f\,\emptyset_2 + \emptyset_5)}{\emptyset_1} \quad ..(13)$$

Since the values of U_{fL}, U_{fR}, V_{fL}, V_{fR} etc. the nodal coordinates of the computed time space are not known in advance, an iterative approach is adopted. The difficulty was surmounted by assuming the particle displacement in computed time space at all nodes for the initializing the iteration. Thereby:

$$U_{fL} = U_{pL} \quad ..(14)$$

$$U_{fR} = U_{pR} \quad ..(15)$$

etc.

It is seen also that by use of Eqs. 14 and 15 the accurate form of equation of motion in terms of finite differences assumes the corresponding finite differences of less accurate form i.e. the Eq. 2. Therefore the initialization is consistant with equation of motion i.e. Eq. 2. The algorithm of the computational scheme has been shown in flow chart, Fig. 3. The computations have been mainly carried out on I.B.M. 360 (with a core memory of 128 K-bites).

The solution of the hyperbolic partial differential equation (like wave equation) by difference method, requires certain conditions to be satisfied for stability and convergence. The stability refers to the situation where the time step interval is chosen such that the errors due to rounding off as well as due to neglecting of higher orders difference form may not accumulate leading to erroneous solution. Once the stability of the difference equation is obtained, the convergence is assured. The form of the equation of motion being comparable to Eq. 16, the stability criteria given by Eq. 17 has been adapted.

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial z^2} = 0 \quad ..(16)$$

$$\theta = 2\,\frac{\Delta z^2}{\Delta t^2} \quad (\text{for } \Delta x = \Delta z) \quad ..(17)$$

It may be noticed that Δz and Δt have different dimension i.e., of length and time respectively The value of $\Delta z/\Delta t$ would have dimensions of vel city. Therefore, the Eq. 17 should be used aft dividing it with square of characteristic wave velocity of the medium. The characteristic vel cities of the Rayleigh shear and the Compressio waves are not equal in magnitude. The first tw are close to each other. Their velocities are less than the compression wave velocity and ran between 10 to 50% of the later. It is therefor desirable that suitable time and length interva are chosen such that they satisfy the stability condition for all characteristic velocities. This can be achieved by using the lowest value, i.e., the Rayleigh wave velocity.

BOUNDARY CONDITIONS AND EFFECT OF GRAVITY

The half space is bounded by four boundaries. These are the top surface, the lateral boundarie on left and right side and bottoms boundary. Th bottom and the two lateral boundaries have been chosen very far off from the source of disturbar at surface. On these three boundaries other tha the surface, the condition of stresses and the particle positions are fixed once for all. In contrast the stresses in top surface are inducec by surface disturbance. At any time, the surfac vertical stresses are in balance with the applie pressure. Therefore the value of stress (mean stress + deviatoric stress) is in balance with vertical component of applied surface dynamic stress (the action of roller on surface is considered only the vertical direction).

ELASTIC HALF SPACE SOLUTION

A plane strain problem has been considered in elastic half space which is represented by size 2.125 m length 1.75 m height and of unit width for numerical computation by the present algorit This half space is represented by square grid 12 cm centre to centre. Time step is considered at intervals of 1/2 ms. The stress pulse is applie on a single central node at surface of this half space. It's magnitude at any instant, t is comp ted using the following equation and also depict in Fig. 4.

$$P = A\,t\,\mathrm{Exp}\,(-\zeta t) \quad ..(18$$

where, P is the applied stress in kg/cm^2, t is time in milisecond after start of the blast load ζ is a parameter defined by the ratio of shear wave velocity of the half space to half width of strip i.e. C_2/a, A a constant chosen such that the peak ground stress corresponding to $t = 1/\zeta$ is equal to the desired value. The point load i considered to represent strip width of 12.5 cm, therefore, a = 6.25 cm, is known.

Let the elastic property of the half space be defined indirectly by choosing shear wave velocity, C_2, Poisson's ratio μ, and density γ, viz. C_2 = 6.6 cm/ms, μ = 0.33 and γ = 1.6 gm/cc. The value of shear modulus G is obtained from the shear wave velocity using following equation

$$C_2 = \sqrt{\frac{G\,g}{\gamma}} \quad ..(19)$$

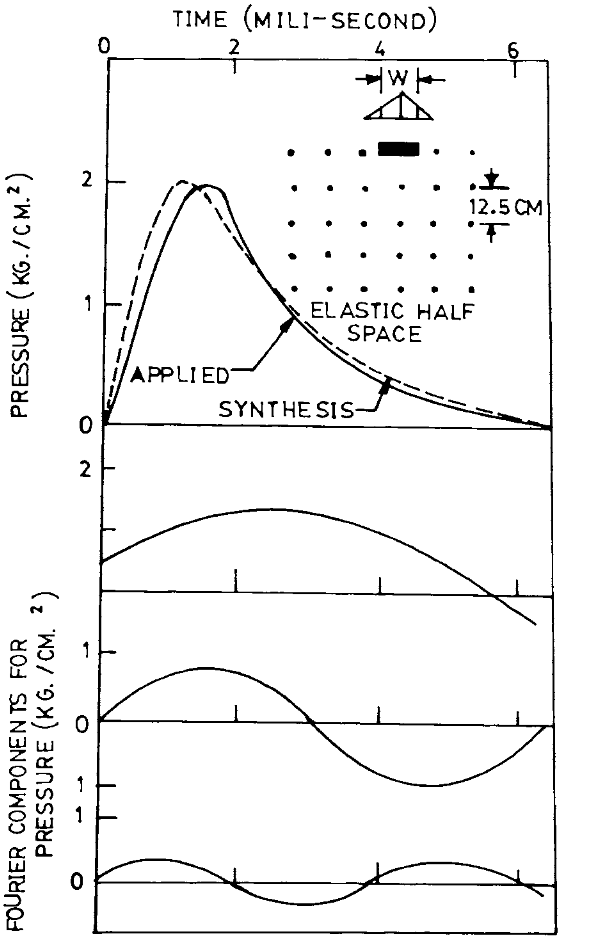


Fig. 4. Applied Dynamic Surface Stress Versus Time and Fourier Components

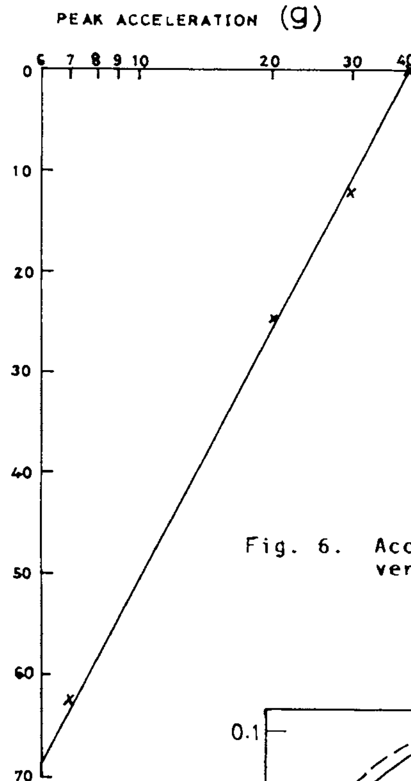


Fig. 6. Acceleration Peak versus Depth

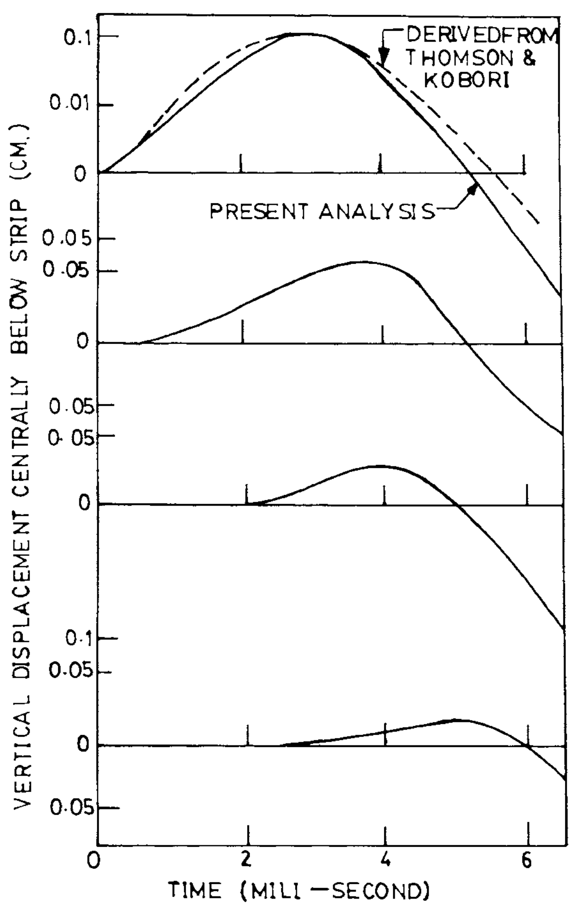


Fig. 5. Computed Displacement versus Time

where, g is the acceleration due to gravity. The above equation is used satisfying the units of C_2, G, g and γ. The computed value of G is, therefore, 70.9 kg/cm^2. The value of the bulk modulus K is found out using the following equation,

$$K = \frac{2G\ (1 + \mu)}{3\ (1 - 2\mu)} \qquad ..(20)$$

This gives K = 189 kg/cm^2. The applied blast pressure follows Eq.18. It is proposed to obtain peak applied stress P_m of 2 Kg/cm^2 peak ground stress. For this peak, the corresponding time can be evaluated by calculus which works out to be $t_m = 1/\zeta$. The value of ζ which is C_2/a, works out to be 1.056 per millisecond. In order to obtain the peak ground stress of 2 kg/cm^2 from Eq. 18 the constant A should have a value of 5.74 kg/cm^2.

COMPARISON WITH AVAILABLE SOLUTION

The vertical displacement versus time for the elastic half space has been computed using the algorithm. These displacements are depicted in Fig. 5 for points below the centre of blast strip. The surface vertical displacement at the centre of the loaded strip versus time from the present analysis has been compared with those derived using close form solutions. The results of Thomson and Kobori (1963) have been used to get alternative solution of the present problem. The computed values by the two methods show fairly good agreement upto 6.28 ms period of computations.

The elastic solutions for cyclic cosine loading over a strip provided by Thomson and Kobori have been extended to present case using Fourier cosine components of the ground stress pulse at any time which is represented for 6.28 ms of computations by the following equation

$$P = \sum_{i=1}^{3} A_i \cos\left(2\frac{\pi n t}{T} + \theta_i\right) \qquad ..(21)$$

where,

P = Applied blast pressure at any time (kg/cm^2)

A_i = amplitude of the Fourier cosine component (kg/cm^2)

θ_i = phase leg in cosine component (radians)

T = time period = 6.28 millisecond (2π)

n = frequency (cycles/millisecond)

t = time (millisecond)

The deflection amplitude B_1, B_2, B_3 (say B_i) can be computed using elastic solutions of Thomson and Kobori corresponding to each of the Fourier's component of the applied blast pressure, A_i,

$$\Delta = \sum_{1}^{3} B_i \cos\left(\frac{2\pi n t}{T} + \theta_i\right) \qquad ..(22)$$

where,

Δ = deflection in (mm)

B_i = the deflection amplitude (mm)

n, t, T and θ_i have already been defined.

The sum of the cosine components Eq. 21 together closely follow the applied ground stress, Fig. 4. The displacement amplitudes corresponding to abo Fourier components were worked out. The computec displacement versus time are shown with those of the present analysis in Fig. 5.

PREDICTION OF DEPTH OF COMPACTION

The solution of equation of motion includes evaluation of acceleration time history for various depths below the centre of dynamically loaded strip. These have also been obtained in the present analysis. The peak accelerations either positive or negative have been determined for various nodes below the loaded strip from the computed acceleration time history. Since the computations were terminated after half cycle of displacement, the peak acceleration was obtained for first half cycle of displacement assuming that the peak acceleration in subsequent displacement cycles shall be lesser.

The peak acceleration is then plotted on logrithmic scale on abscissa and depth along ordinate or simple scale. The peak acceleration versus depth is seen to be neerly linear in the present case as depicted in Fig. 6. Considering the fact that the depth of compaction would be limited to a maximum depth below which the peak acceleration seldom assumes 1g value. Using Fig. 6 this depth can be read off by extrapolating the straight line upto 1g acceleration. The same is found to be 170 cm.

CONCLUSIONS

The equation of motion has been solved taking acceleration time history for an arbitrary profile of surface vertical stresses generated due to application of a strip of foamed propellant. A method for determining depth of compaction has been demonstrated.

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