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Hua Hui Tan

Norwegian Geotechnical Institute, Oslo, Norway

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Hua Hui Tan
Senior Engineer, Norwegian Geotechnical Institute, Oslo, Norway

SYNOPSIS: A finite-element method for the propagation of Rayleigh-type waves in layered solid-fluid media is presented. The method uses displacement as the only parameter to describe the solid and the fluid motions. A penalty function method and a selective integration technique are employed to change the elastic coefficient matrix of fluid from singular to nonsingular. Results obtained from the proposed method agree well with those obtained from available analytical solutions. The method can be used to study the propagation of Rayleigh-type waves in a two-dimensional, irregular solid-fluid system.

INTRODUCTION

The propagation of Rayleigh-type waves in a system consisting of a fluid layer over an elastic solid half-space has previously been solved analytically (Stoneley, 1926; Biot, 1952). These solutions are valid only when both the solid and the fluid are homogeneous materials. For the situation where the solid and/or the fluid are multilayered materials, numerical methods based on Thomson-Haskell matrix formulation are available (Haskell, 1953; Dorman, 1962). Dorman's method can only compute the vertical, but not the horizontal, distribution of particle motion and stress in the normal modes. Therefore, it cannot be used to study the propagation of Rayleigh waves in a two-dimensional, irregular, heterogeneous system located between two horizontally layered solid-fluid systems. It should also be noted that all methods mentioned above use mixed parameters to describe the equations of motion, i.e., they use pressure or displacement potentials in the fluid, and use displacement for the solid. Obviously, extra effort is required to couple the equations in the solid and fluid domains.

A finite-element approach to solve the problem of Rayleigh-type wave propagation in layered solid-fluid media is presented herein; it uses displacement as the only parameter in both the solid and the fluid. A penalty function and a selective integration technique are employed to convert the elastic coefficient matrix in the fluid from singular to nonsingular. The proposed method can compute both the horizontal and vertical distributions of particle motion and stress in the normal modes. More importantly, the method can further be used to form the dynamic stiffness matrix for a layered solid-fluid medium (Tan, 1990).

EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

Let us consider harmonic plane wave propagating in a solid-fluid system. Using cartesian coordinates, the displacements for Rayleigh-type waves in the solid and fluid can be expressed as

$$[\delta]^s = [XY]^T e^{i(\omega t - Kx)} \quad \text{in the solid domain } \Omega^s \qquad (1)$$

$$[\delta]^f = [QP]^T e^{i(\omega t - Kx)} \quad \text{in the fluid domain } \Omega^f \qquad (2)$$

in which the superscripts ^{s}, ^{f}, and ^{T} represent solid, fluid and the transposition, $[\delta]$ denotes the displacement vector, i is $\sqrt{-1}$, ω is the angular frequency, t is the time, K is the wave number, x is the distance, X and Y (or P and Q) are the amplitudes of solid (or fluid) displacements in the x and y directions at t = 0 and x = 0, and Ω is the domain. Neglecting the body forces and initial stresses, the equations of motion in the solid domain are

$$[K^2(\lambda+2\mu) - \omega^2 \rho^s]X - \mu X_{,yy} + iK(\lambda+\mu)Y_{,y} = 0 \qquad (3)$$

$$[K^2\mu - \omega^2 \rho^s]Y - (\lambda-2\mu)Y_{,yy} + iK(\lambda+\mu)X_{,y} = 0 \qquad (4)$$

in which the subscripts $_{,y}$ and $_{,yy}$ represent the first and the second order of differential operation with respect to y, λ and μ are the Lame's constant and shear modulus for the solid, and ρ is the density.

In the fluid domain, the displacement approach is employed and the fluid is treated as an elastic solid with a negligible shear modulus. Thus, for a fluid layer, introducing a zero shear modulus into equations (3) and (4) and replacing μ, X, Y, and ρ^s with β, P, Q, and ρ^f produce

$$(K^2\beta - \omega^2 \rho^f)P + iK\beta Q_{,y} = 0 \qquad (5)$$

$$-\omega^2 \rho^f Q - \beta Q_{,yy} + iK\beta P_{,y} = 0 \qquad (6)$$

where β is the bulk modulus of the fluid.

For Rayleigh-type waves traveling in a solid-fluid medium, it is assumed that the fluid motion remains irrotational (Biot, 1952). This assumption leads to

$$\omega_Z = \text{curl}[\delta]^f = 0 \qquad (7)$$

where ω_Z is the vorticity of the fluid. The assumption of zero vorticity is very important since it makes the displacement approach applicable in the fluid domain.

By use of equation (2), the vorticity in equation (7) can be written in the form of

$$\omega_z = (dP/dy + iKQ)e^{i(\omega t - Kx)} = 0 \quad (8)$$

The boundary conditions of the semi-infinite solid-fluid system are:

1) At the free surface, the normal stress in the fluid is zero

$$\sigma^f_{yy} = 0 \quad (9)$$

where σ_{yy} represents normal stress in the vertical y direction.

2) At the solid-fluid interface, the boundary conditions to be satisfied are continuity of normal stresses and continuity of vertical displacements between solid and fluid. In addition, since the shear stress in the fluid is zero, the shear stress in the solid at the solid-fluid interface must also be zero. These boundary conditions can be expressed as

$$\sigma^s_{yy} = \sigma^f_{yy} \quad (10)$$

$$Y = Q \quad (11)$$

$$\tau^s_{yy} = 0 \quad (12)$$

where τ_{xy} is the shear stress.

3) On the rigid base, the solid displacements are forced to be zero

$$X = Y = 0 \quad (13)$$

PENALTY FUNCTION METHOD AND THE WEAK FORM FORMULATION

The weak form which satisfies the basic equations of motion in the solid (equations 3 and 4) and in the fluid (equations 5 and 6) and all of the boundary conditions (equations 9 to 13) is

$$W(X,Y,P,Q) = \int_{\Omega^s} (\text{equation}(3)\cdot\bar{X} + \text{equation}(4)\cdot\bar{Y})d\Omega + \int_{\Omega^f} (\text{equation}(5)\cdot\bar{P} + \text{equation}(6)\cdot\bar{Q})d\Omega = 0 \quad (14)$$

where W is the weak form. The parameters under the bar sign represent virtual displacements. Note that in equation (14), the constraint requirement of equation (7) has not been introduced; upon assembly, equation (14) will give a singular system of equations because the shear modulus of the fluid is zero.

The singularity problem can be avoided if the constraint in equation (7) is included in the original problem. The irrotational constraint in the fluid can be introduced innto the system by a penalty function method (Zienkiewicz, 1986). The imposed penalty function will change the weak from in equation (14) to

$$W_\alpha(X,Y,P,Q,\alpha) = W(X,Y,P,Q) + \int_{\Omega^f}(\alpha\omega_z)\bar{\omega}_z d\Omega = 0 \quad (15)$$

where α is a large penalty number, $\bar{\omega}_z$ is the virtual vorticity in the fluid and W_α is the modified weak form which contains the penalty term. On assembly, equation (15) will produce a nonsingular system of equations in which solutions can be found. Results obtained from equation (15) are essentially independent of the chosen value of α, provided α is in the range of

$$10^{-1}\cdot\beta \le \alpha \le 10^{\bullet}\cdot\beta \quad (16)$$

Substituting equations (3), (4), (5), (6), and (7) into equation (15), integrating the high derivative terms by parts, and reorganizing the results yield

$$W_\alpha(X,Y,P,Q,\alpha) = \int_{\Omega^s} (a^sK^2 + ib^sK + e^s - c^s\omega^2)d\Omega + \int_{\Omega^f} (a^fK^2 + ib^fK + e^f - c^f\omega^2)d\Omega + \text{the boundary conditions in equations (9) through (13)} = 0 \quad (17)$$

where a^s, b^s, e^s, c^s, a^f, b^f, e^f, and c^f are defined by (Tan, 1988)

$$a^s = (\lambda + 2\mu)X\bar{X} + \mu Y\bar{Y} \quad (18)$$

$$b^s = \lambda(Y_{,y}\bar{X} - X\bar{Y}_{,y}) + \mu(X_{,y}\bar{Y} - Y\bar{X}_{,y}) \quad (19)$$

$$e^s = \mu X_{,y}\bar{X}_{,y} + (\lambda + 2\mu)Y_{,y}\bar{Y}_{,y} \quad (20)$$

$$c^s = \rho^s(X\bar{X} + Y\bar{Y}) \quad (21)$$

$$a^f = \beta P\bar{P} + \alpha Q\bar{Q} \quad (22)$$

$$b^f = \beta(Q_{,y}\bar{P} - P\bar{Q}_{,y}) + \alpha(-P_{,y}\bar{Q} + Q\bar{P}_{,y}) \quad (23)$$

$$e^f = \beta Q_{,y}\bar{Q}_{,y} + \alpha P_{,y}\bar{P}_{,y} \quad (24)$$

$$c^f = \rho^f(P\bar{P} + Q\bar{Q}) \quad (25)$$

FINITE-ELEMENT SPATIAL OISCRETIZATION

A semi-discretization technique is used to discretize the layered solid-fluid system. In the horizontal direction, the system is treated as a continuum. In the vertical direction, one-dimensional, two-node, linear, solid or fluid elements are used to model the system according to standard finite-element techniques.

Usually, one horizontal displacement and one vertical displacement are used as the degrees of freedom for each node. However, the node at the solid-fluid interface has three degrees of freedom. One of these corresponds to the vertical solid and vertical fluid displacement, and the other two correspond to a horizontal displacement for the solid and a perhaps different, horizontal displacement for the fluid. Therefore, the three degrees of freedom at this node are able to model a slip at the solid-fluid interface.

For N fluid layers over M solid layers, as shown in Figure 1, the system's nodal displacement vector, $[\delta]$, can be written as

$$[\delta] = [V]e^{i(\omega t - kx)} \quad (26)$$

$[V]$ is defined by

$$[V]^T = [V_1V_2\cdots V_{2N+1}V_{2N+2}\cdots V_{2(M+N)}V_{2(M+N)+1}] \quad (27)$$

where $V_i(i=1, \ldots 2(M + N) + 1)$ is solid of fluid nodal displacements as shown in Figure 1. Each element consists of four nodal displacements, two on the top of the element and two on the bottom.

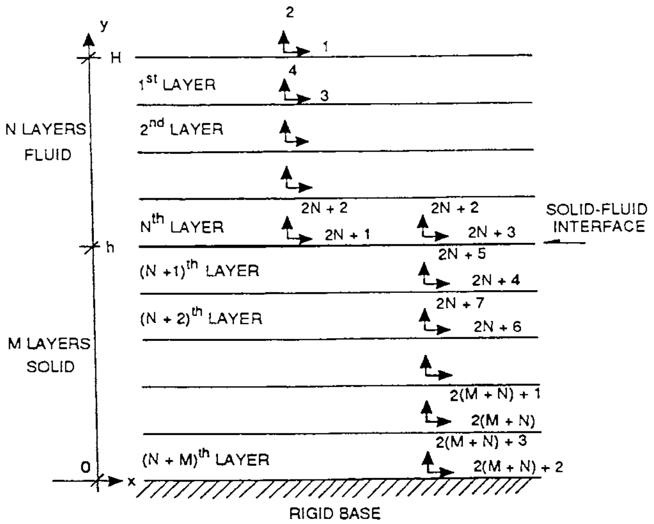

Fig. 1 Layered solid-fluid system, direction of coordinates and nubmering of degrees of freedom and layers. Note that at the solid-fluid interface, there are three degrees of freedom.

For a typical j^{th} fluid element ($1 \le j \le N$)

$$[V]_j^T = [V_{2j-1}\ V_{2j}\ V_{2j+1}\ V_{2j+2}] \quad (28)$$

where $[V]_j$ represents the nodal displacement vector of element j.

For the solid element at the solid-fluid interface, $j = N + 1$ only

$$[V]_{N+1}^T = [V_{2N+3}\ V_{2N+2}\ V_{2N+4}\ V_{2N+5}] \quad (29)$$

For a typical j^{th} solid element ($N + 2 \le j \le N + M$)

$$[V]_j^T = [V_{2j}\ V_{2j+1}\ V_{2j+2}\ V_{2j+3}] \quad (30)$$

On the right hand side of equations (28) through (30), the first and the third terms are horizontal displacements, the second and the fourth are vertical displacements. The motions of a point within any solid or fluid element consist of both vertical and horizontal displacements, and these displacements can be related to the elementy nodal displacements by

$$[v]_j = [N]_j[V]_j \quad (31)$$

in which $[v]_j$ is the displacement vector of a point, $[V]_j$ is the element nodal displacement vector as defined in equations (28) through (30), and $[N]_j$ is the matrix of the interpolation function for element j and is defined by

$$[N]_j = \begin{bmatrix} N_1^j & 0 & N_2^j & 0 \\ 0 & N_1^j & 0 & N_2^j \end{bmatrix} \quad j = 1, \ldots, (M + N) \quad (32)$$

where $N_1{}^j$ and $N_2{}^j$ are one-dimensional interpolation functions and are given by

$$N_1^j = (y_2^j - y)/h_j \quad (33)$$

$$N_2^j = (y - y_1^j)/h_j \quad (34)$$

where h_j is the thickness of layer j, $y_1{}^j$ and $y_2{}^j$ are the y coordinates of the top and bottom nodes of layer j.

Using standard finite-element discretization technique, the weak form can be changed from a continuous function to a discrete set of $2(N + M) + 1$ homogeneous complex linear equations

$$([A]K^2 + i[B]K + [E] - [C]\omega^2)[V] = 0 \quad (35)$$

where [A], [E], and [C] are $[2(N + M) + 1]\cdot[2(M + N)+ 1]$ symmetric matrices and [B] is a skew symmetric matrix with the same dimensions.

The four characteristic matrices in equation (35) consist of the contributions from the individual solid and fluid layer. A typical j^{th} layer of one-dimensional, four-node, linear solid or fluid element contributes a set of 4·4 submatrices $[A]_j{}^i$, $[E]_j{}^i$, $[C]_j{}^i$, and $[B]_j{}^i$, where i represents solid or fluid and j represents the layer number. Formulas for computing these submatrices will be given in the following section. These submatrices are then assembled according to the regular finite-element assembly technique to form the total [A], [E], [C], and [B] matrices (Waas, 1973).

For any given frequency, ω, equation (35) constitutes an eigenvalue problem in K and [V]. The eigenvalue $[K]_s$, $s = 1$ to $4(N + M) + 2$, can be determined as the roots of the polynomial

$$|[A]K^2 + i[B]K + [E] - [C]\omega^2| = 0 \quad (36)$$

and for each K_s, the corresponding eigenvectors $[V]_s$ can then be determined as a normalized solution to the homogeneous linear equations in equation (35). The physical meaning of the eigenvalues, K_s, is that they are wave numbers for generalized Rayleigh waves in the layered solid-fluid media.

SELECTIVE INTEGRATION METHOD

As mentioned in the previous section, in the process of forming the four characteristic submatrices, $[A]_j{}^i$, $[E]_j{}^i$ $[C]_j{}^i$, and $[B]_j{}^i$, integration of polynomial functions is required. For a system containing both a solid and a fluid, the selective integration method should be used for the fluid elements. In this method, all terms being integrated are divided into two groups. The first group contains no penalty number, α, and the second group does. Any term with no α will be integrated with a normal order integration rule, and terms with α will be integrated with a lower order integration rule.

The four characteristic submatrices for the solid and fluid are obtained by integrating the corresponding functions of equations (18) to (25). The solid submatrices obtained using this method are identical to those obtained by Waas (1973).

For fluid elements, the matrix $[E]_j{}^f$ is generated by integrating the e^f in equation (24). The highest order of the polynomial terms in e^f is zero. Integrating them by one-point Gaussian quadrature rule yields

$$[E]_j^f = 1/h_j \cdot \begin{bmatrix} \alpha & 0 & -\alpha & 0 \\ 0 & \beta_j & 0 & -\beta_j \\ -\alpha & 0 & \alpha & 0 \\ 0 & -\beta_j & 0 & \beta_j \end{bmatrix} \qquad (37)$$

where β_j is the bulk modulus of fluid at layer j. By comparison, the corresponding submatrix $[E]_j{}^s$ for solid is

$$[E]_j^s = 1/h_j \cdot \begin{bmatrix} \mu_j & 0 & -\mu_j & 0 \\ 0 & (2\mu_j+\lambda_j) & 0 & -(2\mu_j+\lambda_j) \\ -\mu_j & 0 & \mu_j & 0 \\ 0 & -(2\mu_j+\lambda_j) & 0 & (2\mu_j+\lambda_j) \end{bmatrix} \qquad (38)$$

where μ_j and λ_j are the shear modulus and Lame's elastic constant of solid at layer.

The submatrix $[B]_j{}^f$ for fluid is obtained by integrating the b^f term in equation (23). The highest order or the polynomial terms in b^f is one. Integrating them by one-point Gaussian quadrature rule yields

$$[B]_j^f = ½ \cdot \begin{bmatrix} 0 & (-\beta_j-\alpha) & 0 & (\beta_j-\alpha) \\ -(-\beta_j-\alpha) & 0 & (\beta_j-\alpha) & 0 \\ 0 & -(\beta_j-\alpha) & 0 & -(-\beta_j-\alpha) \\ -(\beta_j-\alpha) & 0 & (-\beta_j-\alpha) & 0 \end{bmatrix} \qquad (39)$$

For solid elements, the corresponding submatrix $[B]_j{}^s$ is

$$[B]_j^s = ½ \cdot \begin{bmatrix} 0 & (-\lambda_j+\mu_j) & 0 & (\lambda_j+\mu_j) \\ -(-\lambda_j+\mu_j) & 0 & (\lambda_j+\mu_j) & 0 \\ 0 & -(\lambda_j+\mu_j) & 0 & -(-\lambda_j+\mu_j) \\ -(\lambda_j+\mu_j) & 0 & (-\lambda_j+\mu_j) & 0 \end{bmatrix} \qquad (40)$$

The submatrix $[C]_j{}^i$ (i = solid of fluid) is derived by integrating c^s or c^f term in equation (21) or (25). $(C)_j{}^i$ is a function of material density, ρ_j, and the thickness of a layer. It represents the mass matrix of the layer. The highest polynomial terms in c^s or c^f is two and they do not contain the penalty value α; therefore, they sould be integrated by two-point Gaussian quadrature rule. For solid and fluid, the submatrix $[C]_j{}^i$ takes the form of

$$[C]_j^i = \begin{bmatrix} m_1 & 0 & m_2 & 0 \\ 0 & m_1 & 0 & m_2 \\ m_2 & 0 & m_1 & 0 \\ 0 & m_2 & 0 & m_1 \end{bmatrix} \qquad (41)$$

in which m_1 and m_2 are

$$m_1 = \rho_j \cdot h_j/3 \qquad (42)$$

$$m_2 = \rho_j \cdot h_j/6 \qquad (43)$$

The values of m_1 and m_2 above are derived from the consistent mass formulation. If lumped mass formulation is used, then m_2 is zero and m_1 is

$$m_1 = \rho_j\ h_j/2 \qquad (44)$$

In this study, the mass matrix actually used for solid or fluid elements is the average of the lumped and the consistent mass matrices.

Integrating a^f in equation (22) yields the submatrix $[A]_j{}^f$ in fluid. The highest order of polynomial terms in a^f is 2. In fluid, $[A]_j{}^f$ has the form of

$$[A]_j^f = h_j/6 \cdot \begin{bmatrix} b_1\beta_j & 0 & b_2\beta_j & 0 \\ 0 & a_1\alpha & 0 & a_2\alpha \\ b_2\beta_j & 0 & b_1\beta_j & 0 \\ 0 & a_2\alpha & 0 & a_1\alpha \end{bmatrix} \qquad (45)$$

The values of b_1, b_2, a_1 and a_2 depend on the integration rule being used. When two-point Gaussian rule was applied

$$a_1 = b_1 = 1/3 \qquad (46)$$

$$a_2 = b_2 = 1/6 \qquad (47)$$

However, when one-point integration rule was used

$$a_1 = a_2 = b_1 = b_2 = 1/4 \qquad (48)$$

The best values of a_1, a_2, b_1, and b_2, using the mentioned selective integration rule, will be $a_1 = a_2 = 1/4$, $b_1 = 1/3$ and $b_2 = 1/6$.

Using these values, the correct form of submatrix $[A]_j{}^f$ reads

$$[A]_j^f = h_j/6 \cdot \begin{bmatrix} 2\beta_j & 0 & \beta_j & 0 \\ 0 & 1.5\alpha & 0 & 1.5\alpha \\ \beta_j & 0 & 2\beta_j & 0 \\ 0 & 1.5\alpha & 0 & 1.5\alpha \end{bmatrix} \qquad (49)$$

By comparison, the submatrix $[A]_j{}^s$ for solid is

$$[A]_j^s = h_j/6 \cdot \begin{bmatrix} 2(2\mu_j+\lambda_j) & 0 & (2\mu_j+\lambda_j) & 0 \\ 0 & 2\mu_j & 0 & \mu_j \\ (2\mu_j+\lambda_j) & 0 & 2(2\mu_j+\lambda_j) & 0 \\ 0 & \mu_j & 0 & 2\mu_j \end{bmatrix} \qquad (50)$$

NUMERICAL EXAMPLE

The example given below concerns the propagation of fundamental Rayleigh waves in a system consisting of a finite depth of compressible fluid over an elastic solid half space. The analytical method presented by Biot (1952) will be used to check the approximate method presented herein.

The computational model used in the example is similar to Figure 1. The infinite depth of the elastic solid is represented by 10, 15, or 20 one-dimensional solid elements; the total solid depth (2100 meters) is about the length of a single fundamental Rayleigh wave. The density, S-wave velocity, and Poisson's ratio of the solid are 1000 kg/m³, 2250 m/sec, and 0.25, respectively. The total fluid depth (375.5 meters) is modeled by 10 one-dimensional fluid elements. The density and P-wave velocity of the fluid are 1000 kg/m³ and 1500 m/sec. These parameters are chosen to match Biot's data. A frequency of 1.0 Hz and a penalty value of 10 times fluid modulus are also used.

When solved using Biot's method, the wave number, K, of the first Rayleigh mode has a value of 4.01 [1/km]. The values obtained by the proposed numerical method are 3.94, 3.97, and 3.98 [1/km] when 10, 15, and 20 solid elements, respectively, were used in the computation. Thus, the numerical results can be improved and will converge towards the analytical solution as a finer finite-element mesh is used.

The fundamental Rayleigh wave mode shapes for the model using 20 solid elements are plotted in Figure 2. The mode shapes are normalized so that the horizontal motion of the solid at the solid-fluid interface has an amplitude of unity.

The example demonstrates that the finite-element method yields results that agree closely with Biot's solution.

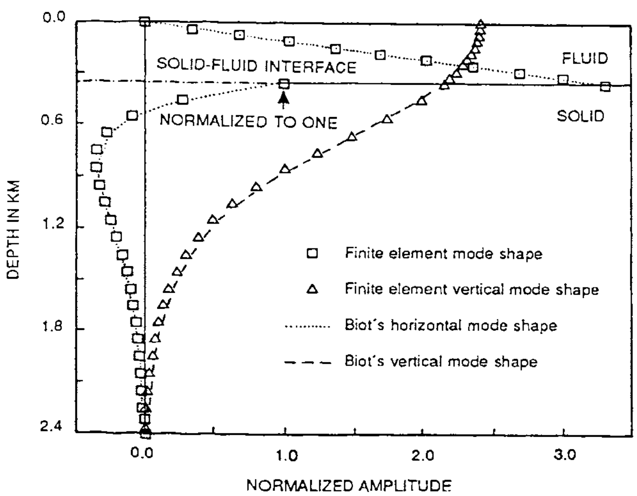

Fig. 2 Fundamental Rayleigh wave mode shape in a solid fluid medium, normalized by the horizontal displacement at the top of the solid surface.

CONCLUSIONS

A displacement approach for the propagation of Rayleigh wave in layered solid-fluid media has been developed. The method leads to motion equations in terms of displacements for both the solid and the fluid domains. The main advantages of this approach are that displacement compatibility and force equilibrium along the solid-fluid interface are easily satisfied and thus no special coupling equation is required. With this method, the solid and the fluid domains can be treated as a single system. A general purpose finite-element computer program for solids can be easily modified to include a fluid region since the same assembly process can be used for both the solid and fluid elements. Numerical example shows the proposed method agrees with the analytical solution. The method can further be used to study the propagation of Rayleigh-type waves in a two-dimensional, irregular solid-fluid system and to derive the dynamic stiffness matrix of a layered solid-fluid medium.

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