Summer 2016

Pointwise and uniform convergence of fourier series on SU(2)

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POINTWISE AND UNIFORM CONVERGENCE OF FOURIER SERIES ON $SU(2)$

by

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A DISSERTATION

Presented to the Faculty of the Graduate School of the

MISSOURI UNIVERSITY OF SCIENCE AND TECHNOLOGY

In Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

2016

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ABSTRACT

Let f be a Lipschitz function on the special unitary group $SU(2)$. We prove that the Fourier partial sums of f converge to f uniformly on $SU(2)$, thereby extending theorems of Caccioppoli, Mayer, and a special case of Ragozin. Pointwise convergence theorems for the Fourier series of functions on $SU(2)$, due to Liu and Qian, were obtained by Clifford algebra techniques. We obtain similar versions of these theorems using simpler proof techniques: classical harmonic analysis and group theory.

ACKNOWLEDGMENTS

First and Foremost I would like to express my sincere gratitude to my thesis advisor Dr. David Grow for introducing me to the world of non commutative harmonic analysis. Without his guidance, patience, and constant encouragement, completing this dissertation would have been impossible. I would also like to thank my committee members, especially Dr. Stephen Clark, Dr. Roman Dwilewicz, and Dr. Leon Hall for providing excellent mathematics courses that prepared me for research.

I would like to thank Dr.Vy Le. His enthusiasm for mathematics inspired me to continue studying mathematics. My master's advisor Dr. Peter Kuchment, and Dr. Christopher Pope gave me their time and encouragement while I was studying mathematics at Texas A&M university I am eternally grateful for their support.

I would like to thank my classmates Thomas Matthews, Roger Bunn, and Paul Runnion. Their friendship made getting through graduate school at Missouri S&T much easier. Dr. G. (Dan) Waddill was also there when I needed to vent. I would like to thank him for his patience.

Last but not least, I would like to thank my wife Heidi Myers for her love and support.

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1 INTRODUCTION

In this section we give a brief history of pointwise and uniform convergence of Fourier series on $SU(2)$ and spheres and show how the results of this thesis fit into the previous body of knowledge. There are two reasons to restrict our attention to $SU(2)$. First, $SU(2)$ is the most elementary compact, connected, simply connected, simple, nonabelian matrix Lie group. Second, there are many open questions regarding convergence theory for Fourier series in $SU(2)$, some of which will be examined in section 4. Consequently, more general settings such as $SU(N)$, or a compact, connected, nonabelian group G, are not considered in this thesis.

The classical Fourier series of a Lebesgue-integrable complex function f on the group $\mathrm{T} = [-\pi, \pi)$, with addition modulo 2π, is

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$$

where

$$\hat{f}(n) = \frac{1}{2\pi}\int_{\mathrm{T}} f(x)e^{-inx}dx$$

for $n = 0, \pm 1, \pm 2, \ldots$. This thesis will explore some features of the analogous representation for functions on the compact group $SU(2)$ of complex 2×2 unitary matrices with determinant one.

The natural replacements on $SU(2)$ for the exponential functions $e_n(x) = e^{inx}$ $(n = 0, \pm 1, \pm 2, \ldots)$ are the continuous irreducible unitary representations $\pi^m : SU(2) \to U(\mathcal{H}_m)$ $(m = 0, 1, 2, \ldots)$ of the elements of $SU(2)$ as unitary operators on an $(m+1)-$dimensional Hilbert space $\mathcal{H}_m$. Whereas the exponential functions on T satisfy the identity $e_n(x + y) = e_n(x)e_n(y)$, the representations of $SU(2)$ satisfy $\pi^m(xy) = \pi^m(x)\pi^m(y)$ for all matrices x and y in $SU(2)$. That is, each π^m is a (continuous) homomorphism from the group $SU(2)$ into the group $U(\mathcal{H}_m)$ of unitary operators on $\mathcal{H}_m$.

If f is a complex function on $SU(2)$, integrable with respect to normalized Haar measure $\mu(dx)$ on $SU(2)$, then the mth Fourier coefficient of f is the operator

$$\hat{f}(\pi^m) = \int_{\mathrm{SU(2)}} f(x)\pi^m(x^{-1})\mu(dx) \quad (m = 0, 1, 2, \ldots.)$$

acting on the space $\mathcal{H}_m$, and the Nth partial sum of the Fourier series of f is

$$S_N f(x) = \sum_{m=0}^{N} (m+1)\mathrm{tr}\left(\hat{f}(\pi^m)\pi^m(x)\right) \quad (N = 0, 1, 2, \ldots; x \in SU(2)).$$

Note two new features for the Fourier partial sums of a complex function f on the non-abelian group $SU(2)$ which did not appear on the abelian group T : (1) the trace of the mth operator function $\hat{f}(\pi^m)\pi^m(x)$ is used in order to obtain a complex function; (2) the dimension $\dim(\pi^m) = m + 1$ of the mth representation appears as a factor on the mth term in the Fourier partial sum. The necessity of these features for accurately representing functions on $SU(2)$ is emphasized by the following special case of a general 1927 theorem due to F. Peter and H. Weyl.

Theorem: [F], pp.108-110. If $f \in L^2(SU(2))$ then $\|S_N f - f\|_2 \to 0$ as $N \to \infty$.

This theorem implies mean convergence of the Fourier series of f on $SU(2)$ to f. We now ask what smoothness assumptions on f guarantee pointwise, uniform, and absolute convergence of its Fourier series to f. Recall a theorem of Dirichlet and Jordan which says that if f is a continuous function of bounded variation on T then the Fourier series of f converges uniformly to f on T([Z], p. 57). It follows that if f is continuously differentiable on T, i.e. $f \in C^1(\mathrm{T})$, then the Fourier series of f converges to f uniformly. For smooth functions on $SU(2)$ we have the following theorems.

Theorem: [F], p.168. Let $x \in SU(2)$ and $f \in C^2(SU(2))$. Then

$$f(x) = \sum_{m=0}^{\infty} (m+1)\mathrm{tr}(\hat{f}(\pi^m)\pi^m(x)),$$

and the series converges uniformly and absolutely.

Theorem: [Ma1]. Let $f \in C^1(SU(2))$. Then for all $x \in SU(2)$,

$$f(x) = \sum_{m=0}^{\infty} (m+1)\mathrm{tr}(\hat{f}(\pi^m)\pi^m(x)),$$

and the series converges uniformly. There exists a function in $C^1(SU(2))$ whose Fourier series does not converge absolutely.

A matrix x belongs to $SU(2)$ if and only if there exist complex numbers x_{11} and x_{12} satisfying $|x_{11}|^2 + |x_{12}|^2 = 1$ such that

$$x = \begin{pmatrix} x_{11} & x_{12} \\ -\overline{x_{12}} & \overline{x_{11}} \end{pmatrix}.$$

Therefore

$$d(x,y) = \sqrt{|x_{11} - y_{11}|^2 + |x_{12} - y_{12}|^2}$$

defines a natural Euclidean metric on $SU(2)$ and hence shows $SU(2)$ is isometrically homeomorphic to the unit sphere

$$S^3 = \left\{\xi \in \mathbb{R}^4 : |\xi| = 1\right\}$$

in $\mathbb{R}^4$. Consequently, the uniform convergence portion of Mayer's theorem above was actually obtained in 1932 by Caccioppoli in [Ca] using classical harmonic analysis techniques on the unit sphere S^3. In fact, in this same paper, Caccioppoli showed that the Fourier series of any function in $\mathrm{Lip}_1(\mathrm{S}^3)$ converges pointwise to the function on S^3. Let us pause to contrast these theorems on $SU(2)$ with absolute convergence or uniform convergence results for smooth functions on T.

Theorem: [Z], p.240. (Bernstein) If $f \in \text{Lip}_\alpha(\text{T})$ for some $\alpha > \frac{1}{2}$, then $S_N f \to f$ absolutely as $N \to \infty$.

This theorem is sharp; i.e. there exists $f \in \text{Lip}_{\frac{1}{2}}(\text{T})$ for which $S_N f$ does not converge absolutely to f as $N \to \infty$. However we do have the following uniform convergence result.

Theorem: [Z], p.63. If $f \in \text{Lip}_\alpha(\text{T})$ for some $\alpha \in (0, 1]$, then $S_N f \to f$ uniformly as $N \to \infty$.

Comparing these two theorems with Mayer's C^1 counterexample on $SU(2)$ suggests that a function on $SU(2)$ must satisfy more stringent smoothness requirements in order to be guaranteed absolute convergence of its Fourier series. To make this precise, we introduce the following notion.

Definition: [AH], p.68. Let k be a non-negative integer, and $\gamma \in [0, 1]$. The space $C^{k,\gamma}(S^{d-1})$ consists of functions on S^{d-1} that are k times continuously differentiable and such that the kth order partial derivatives are Hölder continuous, of exponent γ.

Recalling the homeomorphism between $SU(2)$ and S^3, we see that if $\alpha \in (1, 2]$, then $\text{Lip}_\alpha(SU(2)) \cong C^{1,\gamma}(S^3)$ for $1 + \gamma = \alpha$. The following result in Pini's 1985 paper improves on the $C^2(SU(2))$ absolute convergence theorem that appears in Faraut's book.

Theorem: [P] Let $f \in \text{Lip}_\alpha(SU(2))$ for some $\alpha > \frac{3}{2}$. Then for all $x \in SU(2)$,

$$f(x) = \sum_{m=0}^{\infty} (m+1)\text{tr}(\hat{f}(\pi^m)\pi^m(x))$$

and the series converges absolutely. There exists a function in $\text{Lip}_{\frac{3}{2}}(SU(2))$ whose Fourier series does not converge absolutely.

Actually, this convergence theorem is a special case of an absolute convergence theorem proved by Shapiro [Sh] in 1961 for unit spheres S^{n-1} in $\mathbb{R}^n$ using classical harmonic analysis techniques. However, the paper by Pini gives an explicit example of a function in $\mathrm{Lip}_{\frac{3}{2}}(SU(2))$ whose Fourier series does not converge absolutely and this was not present in [Sh].

As a consequence of the above theorems, the question of how much smoothness is required of a function f on $SU(2)$ in order to be guaranteed an absolutely convergent Fourier series on $SU(2)$ is essentially closed. It is natural to ask the question: "To which $C^{k,\gamma}(S^3)$ space must a function f on $SU(2)$ belong in order to be guaranteed a uniformly convergent Fourier series?" The Fourier series of a function on a unit sphere S^{d-1} in $\mathbb{R}^d$ is sometimes called its Fourier-Laplace series, or just the Laplace series of f. The partial sums of the Fourier-Laplace series of f on S^{d-1} are given by

$$S_n f = \sum_{k=0}^{n} \mathcal{P}_{k,d} f$$

where $\mathcal{P}_{k,d}$ is the projection of f into the space $\mathcal{Y}_k^d$ of spherical harmonics of order k in d dimensions defined by

$$(\mathcal{P}_{k,d} f)(\xi) = \frac{N_{n,d}}{|S^{d-1}|} \int\limits_{S^{d-1}} f(\eta) P_{n,d}(\eta \cdot \xi) dS^{d-1}(\eta),$$

where $P_{n,d}$ is a Legendre polynomial of degree n with dimension d , $\frac{1}{|S^{d-1}|} dS^{d-1}(\eta)$ denotes the normalized surface measure on S^{d-1}, and $N_{n,d}$ is the dimension of the space $\mathcal{Y}_k^d$ [AH], p.26. In the special case when $d = 2$, the partial sums of the Fourier-Laplace series of f on S^2, denoted by $Q_n f$, are given by

$$(Q_n f)(\eta) = \frac{n+1}{4\pi} \int\limits_{S^2} f(\xi) P_n^{(1,0)}(\eta \cdot \xi) dS^2(\xi),$$

where $P_n^{(1,0)}$ is a Jacobi polynomial, and $\frac{1}{4\pi}dS^2(\xi)$ denotes the normalized surface measure on S^2 [AH], p.151.

Theorem: [AH], p.152. Assume that $f \in C^{k,\gamma}(S^2)$ for some $k \geq 0$ and some $\gamma \in (0,1]$, and further assume $k+\gamma > \frac{1}{2}$. Then

$$\|f - Q_n f\|_\infty \leq \frac{c}{n^{k+\gamma-\frac{1}{2}}}$$

for a suitable constant $c > 0$. In particular, the Laplace partial sums $Q_n f$ of f are uniformly convergent to f on S^2.

Ragozin proved the following uniform convergence theorem in 1972.

Theorem: [R], [AH], p.68. Let $d \geq 3$ and $f \in C^{k,\gamma}(S^{d-1})$ for some $k \geq 0$ and some $\gamma \in (0,1]$, and further assume $k+\gamma > \frac{d}{2}-1$. Then $S_N f$ converges uniformly to f on S^{d-1}.

A uniform convergence theorem for functions in $C^{0,1}(S^2)$ would show $\|f - S_n f\|_\infty \to 0$ with rate of decay $\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$ as $n \to \infty$. However, Ragozin's uniform convergence theorem is not applicable when $f \in C^{0,1}(S^3)$ due to the constraint $k+\gamma > \frac{d}{2}-1$. This is the first gap in the uniform convergence theory for Fourier series on non-abelian groups. The main result of this thesis closes this gap.

Theorem: Let $f \in \mathrm{Lip}_1(SU(2))$. Then for all $x \in SU(2)$,

$$f(x) = \sum_{m=0}^{\infty}(m+1)\mathrm{tr}(\hat{f}(\pi^m)\pi^m(x)),$$

and the series converges uniformly. Moreover, to each $\alpha \in (0,1)$ there corresponds $f \in \mathrm{Lip}_\alpha(SU(2))$ such that the Fourier series of f does not converge pointwise at the identity matrix of $SU(2)$.

This theorem strengthens the uniform convergence result in Caccioppolli [Ca] and Mayer [Ma1] and gives a sharp result for the uniform convergence of Fourier series on $SU(2)$. A key ingredient in the proof of this result was the discovery of an identity for the Nth partial sum of the Fourier series of an integrable function f on $SU(2)$:

$$S_N f(x) = -\frac{1}{\pi}\int_0^{\pi} [Q_x f](\theta) D'_{N+1}(\theta)\sin(\theta)d\theta. \tag{1}$$

Here D'_{N+1} denotes the derivative of the Dirichlet kernel on T :

$$D_{N+1}(\theta) = \frac{1}{2} + \sum_{m=1}^{N} \cos((m+1)\theta)\ (-\pi \leq \theta \leq \pi)$$

and

$$[Q_x f](\theta) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} f(xy(\phi,\theta,\psi))\sin(\phi)d\psi d\phi$$

where

$$y(\phi,\theta,\psi) = \begin{pmatrix} \cos(\theta) + i\sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi)e^{i\psi} \\ -\sin(\theta)\sin(\phi)e^{-i\psi} & \cos(\theta) - i\sin(\theta)\cos(\phi) \end{pmatrix}$$

is the spherical coordinate parametrization of a general element y in $SU(2)$. Note that (1) reduces the question of convergence of the Fourier partial sums of f at x on the three dimensional manifold $SU(2)$ to an analysis of the behavior of the function $[Q_x f]$ on the one-dimensional interval $[0,\pi]$.

The identity (1) also allowed us to obtain several pointwise convergence theorems for the Fourier series of functions on $SU(2)$. We later learned that these pointwise convergence results had been anticipated by Liu and Qian in 2004 [QHMS] using Clifford algebra techniques. We provide a simpler proof of these theorems using classical harmonic analysis and group theory.

2 FUNDAMENTALS

2.1 GEOMETRY AND TOPOLOGY OF $SU(2)$ AND $su(2)$

In this section we will introduce some elementary geometric and topological properties of $SU(2)$ and $su(2)$. The elementary properties derived in this section help lay the groundwork for the construction of Haar measure and representation theory needed to develop the notion of Fourier series on $SU(2)$. We begin with preliminary definitions.

2.1.1 The Topology of $M_n(\mathbb{C})$ and GL$(n;\mathbb{C})$. We will need the notion of the exponential function defined on a matrix Lie group. The following definitions will be our starting point.

Definition 2.1.1: The set $M_n(\mathbb{C})$ denotes the space of all $n \times n$ matrices with complex entries.

Definition 2.1.2: The Hilbert-Schmidt norm on $M_n(\mathbb{C})$ is defined as

$$|||X||| = \left(\sum_{k,l=1}^{n} |X_{kl}|^2 \right)^{\frac{1}{2}}.$$

The Hilbert-Schmidt norm on $M_n(\mathbb{C})$ satisfies the property

$$|||XY||| \leq |||X|||\,|||Y|||$$

for every $X, Y \in M_n(\mathbb{C})$.

Definition 2.1.3: The general linear group over the complex numbers, denoted GL(n;$\mathbb{C}$), is the group of all $n \times n$ invertible matrices with complex entries.

Notation: We frequently denote the $n \times n$ identity matrix by e especially when the dimension n is clear from context.

Remark: The Hilbert-Schmidt norm on $M_n(\mathbb{C})$ induces a topology on $\mathrm{GL}(n;\mathbb{C})$. Let $\{x_m\}_{m=1}^{\infty}$ be a sequence of complex matrices in GL(n;ℂ). We say that x_m converges to a matrix x if each entry of x_m converges (as $m\to\infty$) to the corresponding entry of x; i.e. if $(x_m)_{kl}$ converges to x_{kl} for all $1 \le k,\ l \le n$.

Definition 2.1.4: A matrix Lie group is any subgroup G of GL(n;ℂ) with the following property: If $\{x_m\}_{m=1}^{\infty}$ is any sequence of matrices in G and x_m converges to some matrix x then either $x \in G$, or x is not invertible. (I.e., a matrix Lie group is a closed subgroup of GL(n;ℂ) for some n.)

Notation: In this thesis we will denote elements of a matrix Lie group with lowercase letters such as $x, y, z, \ldots$ etc.

Definition 2.1.5: Let X be an $n \times n$ real or complex matrix. We define the matrix exponential of X, e^X or $\exp(X)$, by the usual power series:

$$e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!}.$$

We pause to list some properties of the matrix exponential.

Proposition 2.1.6: Let X and Y be matrices in $M_n(\mathbb{C})$, then the following properties hold

1. $e^0 = I$.
2. $(e^X)^* = e^{X^*}$.
3. e^X is invertible and $\left(e^X\right)^{-1} = e^{-X}$.
4. $e^{(a+b)X} = e^{aX}e^{bX}$ for all $a, b \in \mathbb{C}$.
5. If $XY = YX$, then $e^{X+Y} = e^X e^Y = e^Y e^X$
6. If C is invertible, then $e^{CXC^{-1}} = Ce^X C^{-1}$.
7. $|||e^X||| \le e^{|||X|||}$.

The proofs of these properties are straightforward. The matrix exponential is well-defined and continuous due to property 7 and the Weierstrass-M test.

Definition 2.1.7: Let G be a matrix Lie group. The matrix Lie algebra of G, denoted by $\mathfrak{g}$, is the set of all matrices X such that e^{tX} is in G for every real number t.

2.1.2 The Geometry of $SU(2)$ of S^3. In this section we will introduce the primary matrix Lie group that will be used in this dissertation. Spherical geometry in four dimensions will also be examined in detail.

Definition 2.1.8: The two-dimensional special unitary group is defined as

$$SU(2) = \{x \in \mathrm{GL}(2, \mathbb{C}) \,|\, \det(x) = 1 \,\text{and}\, x^* = x^{-1}\}.$$

The asterisk denotes the complex conjugate transpose operator, and x^* is called the adjoint of x. A matrix x meeting the first condition on elements in the set $SU(2)$ is called a unitary matrix, and the second condition on the determinant is the source of the term special. The special unitary group is nonempty because the identity matrix, $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, is in the set, but note the zero matrix is not in $SU(2)$. We claim that $SU(2)$ is a group under matrix multiplication. To prove closure, note that if $x, y \in SU(2)$ then $(xy^{-1})^* = (y^{-1})^* x^* = (y^*)^* x^{-1} = yx^{-1} = (xy^{-1})^{-1}$. The multiplication property of determinants yields $\det(xy^{-1}) = \det(x)\det(y^{-1}) = \frac{\det(x)}{\det(y)} = 1$. So $SU(2)$ is a subgroup of $\mathrm{GL}(n, \mathbb{C})$ with respect to matrix multiplication, and hence $SU(2)$ is a group under the operation of matrix multiplication. Since matrix multiplication is not in general commutative, the group $SU(2)$ is non-abelian. The special unitary group is a matrix Lie group because if we take a sequence of matrices $\{x_j\} \subseteq SU(2)$ such that x_j converges to x, then $e = x_j x_j^*$ and $\det(x_j) = 1$ for every $j \in \mathbb{N}$. So as $j \to \infty$, we conclude $x^* = x^{-1}$ holds and, since the determinant is an analytic function on $\mathrm{GL}(n, \mathbb{C})$, we have $\det(x) = 1$. Consequently both conditions of a matrix Lie group are satisfied.

Let $x \in SU(2)$ be given by

$$x = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

From the conditions $\det(x) = 1$ and $x^* = x^{-1}$, we have

$$\begin{pmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta} \end{pmatrix} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}.$$

This gives us the relations $\gamma = -\overline{\beta}$ and $\delta = \overline{\alpha}$, so

$$x = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}.$$

Thus a general matrix in $SU(2)$ is completely determined by its first row (or first column if you use the transpose of x). The equation $\det(x) = 1$ then implies $|\alpha|^2 + |\beta|^2 = 1$, so each element of $SU(2)$ corresponds to a unique point on the unit ball in $\mathbb{C}^2$. If $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$, then $|\alpha|^2 + |\beta|^2 = \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$, so $SU(2)$ also can be identified with the unit sphere S^3 in $\mathbb{R}^4$. The relationship $\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$ implies there are only three independent real variables, and hence we say the real dimension of $SU(2)$ is three. Due to symmetry, we conclude S^3 can be covered by eight hemispheres that come from solving $\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$ for any one variable in terms of the other three variables. Without loss of generality, we will take the α_1-axis as the vertical axis of S^3. We can obtain a three-dimensional section of the four-dimensional sphere by fixing α_1 and writing $\alpha_2^2 + \beta_1^2 + \beta_2^2 = 1 - \alpha_1^2$. If $-1 \leq \alpha_1 < 0$ then we are on the lower hemisphere of S^3 and if $0 < \alpha_1 \leq 1$ then we are on the upper hemisphere of S^3. If $\alpha_1 = 0$, then we are on the equator of S^3. The mapping $f : SU(2) \to S^3$, given by

$$f\begin{pmatrix} \alpha_1 + i\alpha_2 & \beta_1 + i\beta_2 \\ -(\beta_1 - i\beta_2) & \alpha_1 - i\alpha_2 \end{pmatrix} = (\alpha_1, \alpha_2, \beta_1, \beta_2)$$

is the natural diffeomorphism from $SU(2)$ onto S^3. Hence, $SU(2) \cong S^3$, and we may describe a point in $SU(2)$ in three equivalent ways: as a matrix in $SU(2)$, as a unit vector on S^3 in $\mathbb{R}^4$, or as a pair of complex numbers in $\mathbb{C}^2$ whose squares of respective moduli sum to unity. If $\alpha_1 = \pm 1$, then these points correspond to the north and south poles of S^3, respectively, because $f(e) = (1, 0, 0, 0)$, and $f(-e) = (-1, 0, 0, 0)$.

Unitary matrices are diagonalizable, and the characteristic polynomial of $x \in SU(2)$ is

$$p(\lambda) = \lambda^2 - 2\mathrm{Re}(\alpha)\lambda + 1. \tag{2}$$

The coefficients of the characteristic polynomial imply the product of the eigenvalues of x must be unity, and their sum must be $2\mathrm{Re}(\alpha)$. It follows that both eigenvalues have modulus one. This means the eigenvalues of x lie on the unit circle in the complex plane, so for some $\theta \in \mathbb{R}$, let $\lambda = e^{i\theta}$ be an eigenvalue of x with normalized eigenvector $\begin{pmatrix} u \\ -\overline{v} \end{pmatrix}$. Then the other eigenvalue is $\overline{\lambda} = e^{-i\theta}$ with eigenvector $\begin{pmatrix} v \\ \overline{u} \end{pmatrix}$. This eigenvector is unique up to sign for $0 < \theta < \pi$. Hence every matrix $x \in SU(2)$ can be written as

$$x = y \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} y^{-1}$$

for some $y \in SU(2)$, where the columns of y are normalized eigenvectors of x corresponding to the eigenvalues λ and $\overline{\lambda}$ respectively. Since $\mathrm{Re}(\alpha) = \alpha_1$, and the fact that the trace of a matrix is the sum of the eigenvalues we get $\mathrm{tr}(x) = 2\cos(\theta)$. We restrict $\theta \in [0, \pi]$ because $\alpha_1 \in [-1, 1]$, and interpret the angle theta as the geodesic distance between e and the matrix $x \in SU(2)$. If $\theta \in [0, \frac{\pi}{2})$ then x corresponds to a point on the upper hemisphere of S^3 and if $\theta \in (\frac{\pi}{2}, \pi]$, then x corresponds to a point on the lower hemisphere of S^3. If $\theta = \frac{\pi}{2}$, then x corresponds to a point on the equator of S^3. The following two definitions are adapted from [A], pp. 274-276.

Definition 2.1.9: A latitude is a horizontal slice through the unit sphere S^3 in $\mathbb{R}^4$, a locus of the form $\{(\alpha_1, \alpha_2, \beta_1, \beta_2) \in S^3 \mid \alpha_1 = c\}$ where $c \in [-1, 1]$, or equivalently, as a subset of the form

$$\left\{ x = \begin{pmatrix} \alpha_1 + i\alpha_2 & \beta_1 + i\beta_2 \\ -(\beta_1 - i\beta_2) & \alpha_1 - i\alpha_2 \end{pmatrix} \in SU(2) \mid \mathrm{tr}(x) = 2c \right\}$$

in the special unitary group.

The equation $\alpha_2^2 + \beta_1^2 + \beta_2^2 = 1 - \alpha_1^2$ implies every group element of $SU(2)$ is contained in a latitude and the matrix diagonalization of $x \in SU(2)$ implies the latitudes are the conjugacy classes of $SU(2)$:

$$cl(x) = \{y^{-1}xy \mid y \in SU(2)\} = \{z \in SU(2) \mid \mathrm{tr}(z) = 2\cos(\theta)\}$$

where $e^{i\theta}$ and $e^{-i\theta}$ are the eigenvalues of x.

Definition 2.1.10: Let W be any two-dimensional subspace of $\mathbb{R}^4$ which contains the north pole $(1, 0, 0, 0)$. The intersection L of W with the unit sphere S^3, which is the set of unit vectors in W, is a longitude of S^3. We denote $f^{-1}[L]$ as a longitude of $SU(2)$ where f is the natural diffeomorphism from $SU(2)$ onto S^3. L is a unit circle in the plane W, and a great circle in S^3, meaning a circle in S^3 of maximal radius one.

Example 2.1.11: We now list some properties of longitudes in S^3 and $SU(2)$.

(a) L meets the equator of S^3 in two points $\pm p = \pm(0, \alpha_2, \beta_1, \beta_2)$ where $\alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$.

(b) The north pole and p form an orthonormal basis of W.

(c) If $A = f^{-1}(p)$ then the longitude $f^{-1}[L]$ in $SU(2)$ has parametrization $H(t) = \cos(t)e + \sin(t)A$ where $t \in \mathbb{R}$.

(d) $f^{-1}[L]$ is a subgroup of $SU(2)$.

(e) Any two longitudes in $SU(2)$ are conjugate subgroups.

(f) Every element $H \in SU(2) \setminus \{\pm e\}$ lies on a unique longitude.

Proof: (a),(b), and (c) are clear.

(d) Since A belongs to the equator of $SU(2)$, the eigenvalues of A are $\pm i$ and there exists $y \in SU(2)$ such that $A = y^{-1}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} y$. Hence $A^2 = -e$. The addition formulas for sine and cosine yield $H(s+t) = H(s)H(t)$ for every $s, t \in \mathbb{R}$.

(e) For $j = 1, 2$ let

$$f^{-1}[L_j] = \{\cos(t)e + \sin(t)A_j | t \in \mathbb{R}\}$$

be any two longitudes in $SU(2)$; here A_j are two matrices on the equatorial latitude of $SU(2)$. Therefore the eigenvalues of A_j are $\pm i$ and there exist $y_j \in SU(2)$ such that $y_j^{-1}A_j y_j = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, and thus $A_2 = w^{-1}A_1 w$ where $w = y_1 y_2^{-1} \in SU(2)$. Then

$$\begin{aligned} w^{-1}f^{-1}[L_1]w &= \{w^{-1}(\cos(t)e + \sin(t)A_1)w \,| t \in \mathbb{R}\} \\ &= \{\cos(t)e + w^{-1}\sin(t)A_1 w \,| t \in \mathbb{R}\} \\ &= f^{-1}[L_2]. \end{aligned}$$

(f) Let $H \in SU(2) \setminus \{e, -e\}$. Then

$$H = \begin{pmatrix} \alpha_1 + i\alpha_2 & \beta_1 + i\beta_2 \\ -(\beta_1 - i\beta_2) & \alpha_1 - i\alpha_2 \end{pmatrix}$$

where $(\alpha_1, \alpha_2, \beta_1, \beta_2) \in S^3$ and $-1 < \alpha_1 < 1$. Set

$$A = \begin{pmatrix} \frac{i\alpha_2}{\sqrt{1-\alpha_1^2}} & \frac{\beta_1+i\beta_2}{\sqrt{1-\alpha_1^2}} \\ \frac{-(\beta_1-i\beta_2)}{\sqrt{1-\alpha_1^2}} & \frac{-i\alpha_2}{\sqrt{1-\alpha_1^2}} \end{pmatrix}$$

and observe that A belongs to the equator of $SU(2)$. Then

$$\hat{L} = \{\cos(t)e + \sin(t)A \,|t \in \mathbb{R}\}$$

is a longitude in $SU(2)$. Choose $t_0 \in (0, \pi)$ such that $\sin(t_0) = \sqrt{1 - \alpha_1^2}$ and $\cos(t_0) = \alpha_1$. Then

$$\begin{aligned} \cos(t_0)e + \sin(t_0)A &= \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{pmatrix} + \begin{pmatrix} i\alpha_2 & \beta_1 + i\beta_2 \\ -(\beta_1 - i\beta_2) & -i\alpha_2 \end{pmatrix} \\ &= H \end{aligned}$$

so $\hat{L}$ contains H. If $\hat{L}_1 = \{\cos(t)e + \sin(t)A_1 \,|t \in \mathbb{R}\}$ is another longitude in $SU(2)$ which contains H then $p_1 = f(A_1)$ and $p = f(A)$ are points on the equator of S^3 where the two-dimensional subspace W of $\mathbb{R}^4$ spanned by $(1, 0, 0, 0)$ and $f(H)$ meets the equator of S^3. Hence $p_1 = \pm p$ by (a). Thus $A_1 = \pm A$ and consequently $\hat{L}_1 = \hat{L}$.

Since $SU(2) \cong S^3$, we may express $x = \begin{pmatrix} \alpha_1 + i\alpha_2 & \beta_1 + i\beta_2 \\ -(\beta_1 - i\beta_2) & \alpha_1 - i\alpha_2 \end{pmatrix} \in SU(2)$ in spherical coordinates given by $\alpha_1 = \cos(\theta)$, $\alpha_2 = \sin(\theta)\cos(\phi)$, $\beta_1 = \sin(\theta)\sin(\phi)\cos(\psi)$, $\beta_2 = \sin(\theta)\sin(\phi)\sin(\psi)$, where $\theta \in [0, \pi]$, $\phi \in [0, \pi]$, and $\psi \in [0, 2\pi]$. There are two useful ways to express x using spherical coordinates. Diagonalizing x we obtain

$$x(\phi,\theta,\psi) = y \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} y^{-1}$$

where,

$$\begin{aligned} y(\phi,\psi) &= \begin{pmatrix} e^{i\frac{\psi}{2}} \cos\left(\frac{\phi}{2}\right) & ie^{i\frac{\psi}{2}} \sin\left(\frac{\phi}{2}\right) \\ ie^{-i\frac{\psi}{2}} \sin\left(\frac{\phi}{2}\right) & e^{-i\frac{\psi}{2}} \cos\left(\frac{\phi}{2}\right) \end{pmatrix} \\ &= \begin{pmatrix} e^{i\frac{\psi}{2}} & 0 \\ 0 & e^{-i\frac{\psi}{2}} \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\phi}{2}\right) & i\sin\left(\frac{\phi}{2}\right) \\ i\sin\left(\frac{\phi}{2}\right) & \cos\left(\frac{\phi}{2}\right) \end{pmatrix}. \end{aligned}$$

Also, note

$$\begin{aligned} x(\phi,\theta,\psi) &= \begin{pmatrix} \cos(\theta) + i\sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi)e^{i\psi} \\ -\sin(\theta)\sin(\phi)e^{-i\psi} & \cos(\theta) - i\sin(\theta)\cos(\phi) \end{pmatrix} \\ &= \cos(\theta)e + \sin(\theta)S(\phi,\psi), \end{aligned}$$

where $S(\phi,\psi) = \begin{pmatrix} i\cos(\phi) & \sin(\phi)e^{i\psi} \\ -\sin(\phi)e^{-i\psi} & -i\cos(\phi) \end{pmatrix}$, and

$$\begin{aligned} x^{-1}(\phi,\theta,\psi) &= x^*(\phi,\theta,\psi) \\ &= -x(\phi,\pi-\theta,\psi) \\ &= \cos(\theta)e - \sin(\theta)S(\phi,\psi). \end{aligned}$$

The matrix $S(\phi,\psi)$ is skew-symmetric with $\mathrm{tr}(S(\phi,\psi)) = 0$. We will see below that $S(\phi,\psi)$ belongs to $su(2)$ and to the equatorial latitude of $SU(2)$. If $\mathbf{r} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \beta_1\mathbf{e}_3 + \beta_2\mathbf{e}_4$, then $\mathbf{r}$ is a position vector on S^3. In spherical coordinates,

$$\mathbf{r}(\phi,\theta,\psi) = \cos(\theta)\mathbf{e}_1 + \sin(\theta)\mathbf{s}(\phi,\psi),$$

where $\theta \in [0,\pi]$, $\phi \in [0,\pi]$, $\psi \in [0,2\pi]$, and

$$\mathbf{s}(\phi,\psi) = \cos(\phi)\mathbf{e}_2 + \sin(\phi)\cos(\psi)\mathbf{e}_3 + \sin(\phi)\sin(\psi)\mathbf{e}_4.$$

Hence, $x(\phi,\theta,\psi)$ and $\mathbf{r}(\phi,\theta,\psi)$ are equivalent ways of expressing points in $SU(2)$ or S^3.

Example 2.1.12: For $x, y \in SU(2)$, the product

$$\begin{aligned} x(\phi_0,\theta_0,\psi_0)y^{-1}(\phi,\theta,\psi) &= (\cos(\theta_0)e + \sin(\theta_0)S(\phi_0,\psi_0))\,(\cos(\theta)e - \sin(\theta)S(\phi,\psi)) \\ &= \cos(\theta_0)\cos(\theta)e + \sin(\theta_0)\cos(\theta)S(\phi_0,\psi_0) \\ &\quad - \cos(\theta_0)\sin(\theta)S(\phi,\psi) - \sin(\theta_0)\sin(\theta)S(\phi_0,\psi_0)S(\phi,\psi). \end{aligned}$$

On the other hand, $SU(2)$ is a group, so

$$x(\phi_0,\theta_0,\psi_0)y^{-1}(\phi,\theta,\psi) = \cos(\Theta)e + \sin(\Theta)S(\Phi,\Psi)$$

for some $\Theta \in [0,\pi]$,$\Phi \in [0,\pi]$, and $\Psi \in [0,2\pi]$. Separating out real and imaginary parts of the first row entries of the matrix $x(\phi_0,\theta_0,\psi_0)y^{-1}(\phi,\theta,\psi)$ yields the following system of equations,

$$\begin{aligned} \cos(\Theta) = \cos(\theta)\cos(\theta_0) + \sin(\theta)\sin(\theta_0)(\cos(\phi)\cos(\phi_0) \\ + \sin(\phi)\sin(\phi_0)\cos(\psi - \psi_0)), \end{aligned}$$

$$
\begin{aligned}
\sin(\Theta)\cos(\Phi) &= -\cos(\theta_0)\sin(\theta)\cos(\phi) + \sin(\theta_0)\cos(\phi_0)\cos(\theta) \\
&\quad + \sin(\theta)\sin(\theta_0)\sin(\phi)\sin(\phi_0)\sin(\psi_0 - \psi), \\
\sin(\Theta)\sin(\Phi)\cos(\Psi) &= -\sin(\theta)\sin(\phi)\cos(\theta_0)\cos(\psi) + \sin(\theta_0)\sin(\phi_0)\cos(\theta)\cos(\psi_0) \\
&\quad + \sin(\theta)\sin(\phi)\sin(\theta_0)\cos(\phi_0)\sin(\psi) \\
&\quad - \sin(\theta_0)\sin(\phi_0)\sin(\theta)\cos(\phi)\sin(\psi_0), \\
\sin(\Theta)\sin(\Phi)\sin(\Psi) &= -\sin(\theta)\sin(\phi)\cos(\theta_0)\sin(\psi) + \sin(\theta_0)\sin(\phi_0)\cos(\theta)\sin(\psi_0) \\
&\quad - \sin(\theta)\sin(\phi)\sin(\theta_0)\cos(\phi_0)\cos(\psi) \\
&\quad + \sin(\theta_0)\sin(\phi_0)\sin(\theta)\cos(\phi)\cos(\psi_0).
\end{aligned}
$$

The entries of $x(\phi_0, \theta_0, \psi_0)y^{-1}(\phi, \theta, \psi)$ are analytic functions of the coordinates ϕ, θ, and ψ and hence have bounded derivatives with respect to the coordinates.

The first equation has a geometrical interpretation. Consider two position vectors $\mathbf{r}_0$ and $\mathbf{r}_1$ on S^2 given in ordinary spherical coordinates in $\mathbb{R}^3$ with $\psi, \psi_0 \in [0, 2\pi]$ and $\phi, \phi_0 \in [0, \pi]$ by $\mathbf{r}_0 = \cos(\psi_0)\sin(\phi_0)\mathbf{i} + \sin(\psi_0)\sin(\phi_0)\mathbf{j} + \cos(\phi_0)\mathbf{k}$ and $\mathbf{r}_1 = \cos(\psi)\sin(\phi)\mathbf{i} + \sin(\psi)\sin(\phi)\mathbf{j} + \cos(\phi)\mathbf{k}$. The dot product of $\mathbf{r}_0$ and $\mathbf{r}_1$ is

$$
\begin{aligned}
\mathbf{r}_0 \cdot \mathbf{r}_1 &= \cos(\phi_0)\cos(\phi) + \sin(\phi_0)\sin(\psi_0)\sin(\phi)\sin(\psi) + \cos(\psi_0)\sin(\phi_0)\sin(\phi)\cos(\psi) \\
&= \cos(\phi_0)\cos(\phi) + \sin(\phi_0)\sin(\phi)\left(\cos(\psi)\cos(\psi_0) + \sin(\psi)\sin(\psi_0)\right) \\
&= \cos(\phi)\cos(\phi_0) + \sin(\phi)\sin(\phi_0)\cos(\psi - \psi_0).
\end{aligned}
$$

The right hand side is the parenthetical expression in the first equation of the system. Since $\mathbf{r}_0$ and $\mathbf{r}_1$ are unit vectors, the left hand side is the cosine of the angle between $\mathbf{r}_0$ and $\mathbf{r}_1$. Let

$$\cos(\tau) = \cos(\phi)\cos(\phi_0) + \sin(\phi)\sin(\phi_0)\cos(\psi - \psi_0),$$

and note $|\tau|$ measures the geodesic distance on S^2 between the tips of the vectors $\mathbf{r}_0$ and $\mathbf{r}_1$. This equation is a law of cosines for spherical triangles in S^2 with vertices at the tips of the vectors $\mathbf{k}, \mathbf{r_0}$, and $\mathbf{r}_1$ and side lengths ϕ, ϕ_0, and τ. The term $\psi - \psi_0$ can be taken without loss of generality to be non-negative and is the interior angle between the arcs ϕ and ϕ_0. The first equation in the system reduces to

$$\cos(\Theta) = \cos(\theta)\cos(\theta_0) + \sin(\theta)\sin(\theta_0)\cos(\tau),$$

and we get another law of cosines in S^2. By the same argument as for S^2, we conclude Θ is the angle between two position vectors defined on S^3 in spherical coordinates and $|\Theta|$ measures the geodesic distance between the two position vectors on S^3. The spherical triangle on S^3 has side lengths θ, θ_0 and Θ, and the angle τ measures the interior angle between the arcs θ and θ_0. We conclude the first equation in the system above is a law of cosines for S^3 and can be viewed as a composition of the laws of cosines on S^2 with itself. The pattern for the composition laws will persist for the higher $n-$dimensional spheres S^{n-1}. See [AH], p.21. For $t \in [0, 2\pi]$, the geodesic connecting $\mathbf{r}_0$ and $\mathbf{r}_1$ is given by

$$\mathbf{r}(t) = \cos(t)\mathbf{r}_0 + \sin(t)\mathbf{r}_1.$$

If $\mathbf{r}_0$ and $\mathbf{r}_1$ are not antipodal, then the component of $\mathbf{r}_1$ perpendicular to $\mathbf{r}_0$ is $\mathbf{r}_1 - \cos(\xi)\mathbf{r}_0$ and $\|\mathbf{r}_1 - \cos(\xi)\mathbf{r}_0\| = \sin(\xi)$, where ξ is the angle between $\mathbf{r}_0$ and $\mathbf{r}_1$. Hence we can define $\mathbf{r}$ as

$$\mathbf{r}(t) = \cos(t)\mathbf{r}_0 + \sin(t)\left(\frac{\mathbf{r}_1 - \cos(\xi)\mathbf{r}_0}{\sin(\xi)}\right).$$

2.1.3 Topological Properties of $SU(2)$. We now study some topological properties of $SU(2)$. The following classical theorem will be useful in subsequent sections.

Theorem 2.1.12: (Heine-Borel) The compact sets in a Euclidean space are the sets which are closed and bounded.

Example 2.1.13: The unit spheres $S^{n-1} \subset \mathbb{R}^n$ are compact sets in $\mathbb{R}^n$.

Proposition 2.1.14: The topological group $SU(2)$ is compact.

Proof: Since $SU(2) \cong S^3$ and, by the Heine-Borel theorem, S^3 is a compact subset of $\mathbb{R}^4$, it follows that $SU(2)$ is a compact matrix Lie group.

Definition 2.1.15: A path in $M_n(\mathbb{C})$ is a continuous function $t \to \mathrm{A}(t) \in M_n(\mathbb{C})$, where t belongs to some interval of real numbers, so the entries $a_{ij}(t)$ of $\mathrm{A}(t)$ are continuous functions of the real variable t. The path is called smooth, or differentiable, if the functions $a_{ij}(t)$ are differentiable.

Example 2.1.16: Three paths on $SU(2)$ that pass through $\pm e \in SU(2)$ are given by

$$\omega_1(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix},$$
$$\omega_2(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix},$$
$$\omega_3(t) = \begin{pmatrix} \cos(t) & i\sin(t) \\ i\sin(t) & \cos(t) \end{pmatrix}$$

where $0 \leq t \leq \pi$.

Definition 2.1.17: A matrix Lie group G is path connected if given any two matrices A and B in G, there exists a continuous path $A(t)$, $a \leq t \leq b$, lying in G with $A(a) = A$ and $A(b) = B$.

Remark: For matrix Lie groups the notions of path connected and connected are equivalent. See [H], p. 22 for a proof.

Proposition 2.1.18: $SU(2)$ is connected.

Proof: Let $x = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(2)$ where (α, β) is a unit vector in $\mathbb{C}^2$, and let $\alpha = u\cos(\theta)$ and $\beta = v\sin(\theta)$ for some $u, v \in \mathbb{C}$ with $|u| = |v| = 1$ and some $\theta \in [0, \pi/2]$. Set $u = e^{i(\phi+\psi)}$ and $v = e^{i(\phi-\psi)}$ for some $\phi, \psi \in \mathbb{R}$. For $t \in [0,1]$ the function

$$H(t) = (e^{i(\phi+\psi)}\cos(\theta t),\, e^{i(\phi-\psi)}\sin(\theta t))$$

defines a path from the identity matrix e to the matrix $x \in SU(2)$ defined above. We conclude $SU(2)$ is connected.

Remark: This coordinate system on $SU(2)$ is called the Euler coordinate system when the parameters θ, ϕ, and ψ are restricted to appropriate intervals. See $[V]$, p.98 for more details on the Euler coordinate system.

Definition 2.1.19: A matrix Lie group G is simply connected if it is connected and, in addition, every loop in G can be shrunk continuously to a point in G. More precisely, assume G is connected. Then G is simply connected if given any continuous path $A(t)$, $0 \leq t \leq 1$, lying in G with $A(0) = A(1)$, there exists a continuous function $A(s,t)$, $0 \leq s,\, t \leq 1$, taking values in G and having the following properties:

1. $A(s,0) = A(s,1)$ for all s,
2. $A(0,t) = A(t)$,
3. $A(1,t) = A(1,0)$ for all t.

One interpretation of the preceding definition is that $A(t)$ is a single loop and a family of loops $A(s,t)$ parameterized by s shrinks $A(t)$ to a point. Condition 1 guarantees we have a loop for all s. Condition 2 specifies that $A(t)$ is a loop and condition 3 says that when $s = 1$, the loop $A(t)$ is a point.

The following elementary result is needed to deduce that $SU(2)$ is simply connected.

Proposition 2.1.20: Let X be a topological space such that $X = U \cup V$, where U and V are open sets of X. Suppose $U \cap V$ is nonempty and path connected. If U and V are simply connected, then X is simply connected.

Proof : See [Mu] (Corollary 59.2, p. 385.)

Proposition 2.1.21: $SU(2)$ is simply connected.

Proof: See [Mu] (Theorem 59.3, pp. 385-386.)

Remarks:

1. Note that $S^1 \cong \mathrm{T}$ is not simply connected, but the argument given above can be used to show that S^n for natural numbers $n \geq 2$ are simply connected.

2. If we use stereographic projection using the south pole to construct a homeomorphism from S^3 to $\mathbb{R}^3$, we can show that S^3 has an atlas consisting of two coordinate charts, and this atlas is minimal. Stereographic projection is a key ingredient in the proof of Proposition 2.1.18.

2.1.4 Geometrical and Topological Properties of $su(2)$. We now examine the matrices on the equatorial latitude of $SU(2)$ in more detail. If $x \in SU(2)$ and belongs to the equatorial latitude, $\mathrm{tr}(x) = 0$ so

$$x = \begin{pmatrix} i\alpha_2 & \beta_1 + i\beta_2 \\ -(\beta_1 - i\beta_2) & -i\alpha_2 \end{pmatrix},$$

where $\alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$. The matrix x is skew-symmetric with trace equal to zero and determinant one.

Definition 2.1.22: The space $su(2)$ is defined as

$$\begin{aligned} su(2) &= \{X \in \mathrm{M}_2(\mathbb{C}) \,|\, X^* = -X \,\mathrm{and}\, \mathrm{tr}(X) = 0\} \\ &= \left\{ X = \begin{pmatrix} i\alpha_2 & \beta_1 + i\beta_2 \\ -(\beta_1 - i\beta_2) & -i\alpha_2 \end{pmatrix} \,|\, \alpha_2, \beta_1, \beta_2 \in \mathbb{R} \right\}. \end{aligned}$$

The map g on $su(2)$ given by

$$g\begin{pmatrix} i\alpha_2 & \beta_1 + i\beta_2 \\ -(\beta_1 - i\beta_2) & -i\alpha_2 \end{pmatrix} = (0, \alpha_2, \beta_1, \beta_2)$$

transforms the subset of $su(2)$ whose elements have determinant one, i.e. the equator of $SU(2)$, onto a copy of S^2, the unit sphere in three dimensions. In particular, g is a homeomorphism from $su(2)$ onto S^2. The set $su(2)$ is not a multiplicative group because the matrix $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}^2 = -e$, which is not in $su(2)$. Note the 2×2 zero matrix is an element of $su(2)$, but not an element of $SU(2)$, so $su(2)$ is not a subset of $SU(2)$. It follows $SU(2)$ cannot be a vector space due to the absence of the zero matrix, but $su(2)$ is a real vector space. It is straight-forward to show real linear combinations of elements in $su(2)$ are elements in $su(2)$, so $su(2)$ is a closed real subspace of $M_2(\mathbb{C})$. Consider the following three matrices in the intersection of $SU(2)$ and $su(2)$:

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

From the homeomorphism from $SU(2)$ onto S^3 we have $f(X_1) = \mathbf{e}_2$, $f(X_2) = \mathbf{e}_3$ and $f(X_3) = \mathbf{e}_4$, where $\mathbf{e}_2 = (0,1,0,0)^\top$, $\mathbf{e}_3 = (0,0,1,0)^\top$, and $\mathbf{e}_4 = (0,0,0,1)^\top$. Thus the elements of $\{e, X_1, X_2, X_3\} \subseteq SU(2)$ are identified with the standard basis vectors in $\mathbb{R}^4$, and similarly the elements of $\{X_1, X_2, X_3\} \subseteq su(2)$ are identified with the standard basis vectors in $\mathbb{R}^3$ under the map g and serve as a basis for $su(2)$. Moreover, $\{X_1, X_2, X_3\}$ form an orthonormal basis with respect to the inner product on $su(2)$ defined by

$$\big(X, Y\big) = \frac{1}{2}\mathrm{tr}\big(XY^*\big),$$

which induces the norm

$$\|X\| = \sqrt{\frac{1}{2}\operatorname{tr}\big(XX^*\big)}.$$

Recall the commutator of two square matrices A and B is $[A,B] = AB - BA$. The commutator is sometimes abbreviated as $\operatorname{ad}_A B$, and a straight-forward computation shows if X and Y are elements of $su(2)$ then $[X,Y]$ is also an element of $su(2)$, but XY and YX are not in general elements of $su(2)$. For $\{i,j,k\} \in \{1,2,3\}$, the commutator relations satisfy $[X_i, X_j] = \varepsilon_{ijk}X_k$ where ε_{ijk} takes the values $1, -1$, or 0 when $\{i,j,k\}$ is an even, odd, or no permutation of $\{1,2,3\}$ respectively. Hence, the vector space $su(2)$ is isomorphic to $\mathbb{R}^3$ with the commutator on $su(2)$ corresponding to twice the cross product of the standard basis vectors defined on $\mathbb{R}^3$.

Notice $X_1 = \omega_1'(0)$, $X_2 = \omega_2'(0)$, and $X_3 = \omega_3'(0)$, where ω_1, ω_2, and ω_3 are the paths on $SU(2)$ through e and $-e$ in Example 2.1.13, and where $'$ denotes differentiation with respect to t. The matrices X_1, X_2, and X_3 can be interpreted as tangential directions of the paths ω_j for $j = 1,2,3$ at the identity matrix. Elementary computations, show $\exp(tX_j) = \omega_j(t)$ for $j = 1,2,3$ and all real t. These observations suggest that $su(2)$ is the matrix Lie algebra of $SU(2)$. In fact the function exp is an onto mapping from $su(2)$ to $SU(2)$. This is clear because every matrix $x \in SU(2)$ is conjugate to a matrix of the form $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ for some $\theta \in [0,\pi]$, so given $x \in SU(2)$ there exists a $y = \begin{pmatrix} \gamma & \delta \\ -\overline{\delta} & \overline{\gamma} \end{pmatrix} \in SU(2)$ such that $x = \exp\left\{\theta y X_1 y^{-1}\right\}$. The matrix

$$y\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}y^{-1} = \begin{pmatrix} i\left(|\gamma|^2 - |\delta|^2\right) & -2i\gamma\delta \\ -2i\overline{\gamma}\overline{\delta} & i\left(|\delta|^2 - |\gamma|^2\right) \end{pmatrix}$$

is skew-symmetric and has trace equal to zero so belongs to $su(2)$. Hence, exp is onto. The matrix exponential is not a one-to-one map from $su(2)$ to $SU(2)$ because

$$\exp\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

for all real θ. To prove $su(2)$ is the Lie algebra of $SU(2)$, note $\exp(X)$ is unitary if and only if $(\exp X)^* = \exp(-X)$. Proposition 2.1.7 implies $(\exp X)^* = \exp(X^*)$, and for every $t \in \mathbb{R}$, $\exp(tX)\exp(tX^*) = e$ on $SU(2)$. Differentiating with respect to t yields,

$$\begin{aligned} 0 &= \frac{d}{dt}\exp(tX)\exp(tX^*) \\ &= X\exp(tX)\exp(tX^*) + \exp(tX)X^*\exp(tX^*). \end{aligned}$$

Setting $t = 0$ we obtain $X^* = -X$. We also will use the following identity from linear algebra relating the trace and determinant of a matrix $X : \det(\exp X) = e^{\operatorname{tr}(X)}$. One way to see this identity is to note every square matrix is similar to an upper triangular matrix, i.e. $X = YUY^{-1}$ for some invertible matrix Y and some upper triangular matrix U with eigenvalues λ_i for $i = 1, \ldots, n$. Next,

$$\begin{aligned} \det(\exp X) &= \det\big(\exp(YUY^{-1})\big) \\ &= \det\big(Y\exp(U)Y^{-1})\big) \\ &= \det\exp(U) \\ &= \prod_{i=1}^{n} e^{\lambda_i} \\ &= e^{\sum_{i=1}^{n}\lambda_i} \\ &= e^{\operatorname{tr}(X)}. \end{aligned}$$

If $X \in su(2)$ then $\exp X \in SU(2)$, which implies $\det(\exp X) = 1$. For any real number t, we have

$$1 = \det(\exp(tX)) = e^{\mathrm{tr}(X)t}.$$

Differentiating both sides with respect to t yields the equation

$$0 = \mathrm{tr}(X)e^{\mathrm{tr}(X)t},$$

so $\mathrm{tr}(X) = 0$. Therefore the conditions $X^* = -X$ and $\mathrm{tr}(X) = 0$ imply $su(2)$ contains the Lie algebra of $SU(2)$. To prove the reverse inclusion, assume $X^* = -X$ and $\mathrm{tr}(X) = 0$. Note that

$$\begin{aligned} \det(\exp X) &= e^{\mathrm{tr}(X)} \\ &= e^0 \\ &= 1, \end{aligned}$$

and Proposition 2.1.7 implies

$$\begin{aligned} \exp X\,(\exp X)^* &= \exp X \exp X^* \\ &= \exp X \exp(-X) \\ &= \exp(X - X) \\ &= \exp(0) \\ &= e. \end{aligned}$$

Therefore, $\exp X$ is unitary and we conclude $\exp X \in SU(2)$. Hence X is in the Lie algebra of $SU(2)$ if X traceless and skew-Hermitian. Consequently, $su(2)$ is the Lie algebra of $SU(2)$.

Remark: If $X \in su(2)$ and $g \in SU(2)$, then the conjugation of X with g, i.e. the matrix gXg^{-1}, is in $su(2)$ because $\exp(tgXg^{-1}) = g(\exp tX)g^{-1} \in SU(2)$ for all $t \in \mathbb{R}$. The matrix gXg^{-1} is sometimes abbreviated by $\mathrm{Ad}_g X$ and is related to ad_X by $e^{\mathrm{ad}_X Y} = \mathrm{Ad}(\exp X)Y$ or $\mathrm{ad}_X Y = \frac{d}{dt}\mathrm{Ad}(\exp(tX))Y\Big|_{t=0}$ for $X, Y \in su(2)$. The Ad operation has geometric properties which correspond to rotations. If $g \in SU(2)$, then

$$\begin{aligned}\big(\mathrm{Ad}_g X, \mathrm{Ad}_g Y\big) &= \frac{1}{2}\mathrm{tr}\big(gXg^{-1}(gYg^{-1})^*\big)\\ &= \frac{1}{2}\mathrm{tr}\big(gXg^{-1}gY^*g^*\big)\\ &= \frac{1}{2}\mathrm{tr}\big(XY^*\big)\\ &= \big(X, Y\big).\end{aligned}$$

Hence, the inner product on $su(2)$ is Ad invariant. For more on this topic see [A], p. 279 and [Fo], p. 145.

From the characteristic polynomial (2) of x in $SU(2)$ we have $\mathrm{Re}(\alpha) = 0$, for any x on the equator of $SU(2)$, so the eigenvalues of such a matrix are $\pm i$. Hence x is conjugate to X_1 and $x^2 = -e$. The matrices X_1, X_2, X_3 in $SU(2)$ satisfy the relations $X_1^2 = X_2^2 = X_3^2 = -e$, $X_1X_2 = X_3$, $X_2X_3 = X_1$, and $X_3X_1 = X_2$. The real linear span of the set $\{e, X_1, X_2, X_3\}$ make up the quaternion algebra. The set $\{\frac{-i\hbar}{2}X_1, \frac{-i\hbar}{2}X_2, \frac{-i\hbar}{2}X_3\}$ are the Pauli spin matrices from quantum mechanics [L], p.825 where $\hbar$ is Planck's constant. Every matrix $x \in SU(2)$ may be written uniquely as

$$x = \alpha_1 e + \alpha_2 X_1 + \beta_1 X_2 + \beta_2 X_3$$

where $(\alpha_1, \alpha_2, \beta_1, \beta_2) \in S^3$, so $SU(2)$ can be identified with the real norm one quaternions.

2.1.5 Central Functions on $SU(2)$. We will now introduce an important set of functions defined on $SU(2)$. Their analytic and geometric properties will be used throughout the thesis.

Definition 2.1.23: A function f on $SU(2)$ is called central (or a class function) if, for every $x, y \in SU(2)$, $f(x) = f(yxy^{-1})$ or equivalently $f(xy) = f(yx)$.

Since every matrix in $SU(2)$ is diagonalizable, for a central function

$$\begin{aligned} f(x) &= f\left(y\omega_1(\theta)y^{-1}\right) \\ &= f(\omega_1(\theta)) \end{aligned}$$

Thus, for every central function f on $SU(2)$, we can find a corresponding function F on $[-1, 1]$ such that

$$\begin{aligned} f(x) &= F\left(\frac{1}{2}\mathrm{tr}(x)\right) \\ &= F(\cos(\theta)) \end{aligned}$$

where $e^{\pm i\theta}$ are the eigenvalues of x.

Remarks on central functions:

1. Since $\mathrm{tr}(\omega_1(\theta)) = \mathrm{tr}(\omega_1^{-1}(\theta)) = \mathrm{tr}(\omega_1(-\theta))$ for every $\theta \in [0, \pi]$, we conclude $f(\omega_1(-\theta)) = f(\omega_1(\theta))$.

2. As a consequence of 1 we obtain $f(x) = f(x^{-1})$ for every $x \in SU(2)$.

3. Central functions depend only on the geodesic distance θ measured from e to $x \in SU(2)$ and hence central functions are constant on conjugacy classes, which are latitudes in $SU(2)$.

The following metric on $SU(2)$ will be used in several computations throughout the thesis.

Definition 2.1.24: For each $x, y \in SU(2)$, define $d : SU(2) \times SU(2) \to \mathbb{R}^+$ by

$$d(x,y) = \sqrt{\frac{1}{2}\mathrm{tr}((x-y)(x-y)^*)}.$$

This function is a metric on $SU(2)$. To see this, if $x = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, y = \begin{pmatrix} \gamma & \delta \\ -\overline{\delta} & \overline{\gamma} \end{pmatrix} \in SU(2)$, then $\frac{1}{2}\mathrm{tr}((x-y)(x-y)^*) = |\alpha-\gamma|^2 + |\beta-\delta|^2$, so our distance function is just the Euclidean metric in $\mathbb{C}^2$ applied to the first rows of x and y, and agrees with the distance function on $SU(2)$ defined in the introduction (cf. p.8). This metric also induces the Hilbert-Schmidt norm on $SU(2)$ in Definition 2.1.2. Since $\mathrm{tr}(uv) = \mathrm{tr}(vu)$, the distance function is left and right translation invariant. That is, for every $x, y, z \in SU(2)$,

$$d(zx, zy) = d(xz, yz) = d(x,y).$$

Example 2.1.25:

(a) For all $x \in SU(2)$, $d(x,-x) = d(xx^{-1}, -xx^{-1}) = d(e,-e) = 2$.

(b) For all $x, y \in SU(2)$,

$$\begin{aligned}\mathrm{tr}((x-y)(x-y)^*) &= \mathrm{tr}(xx^*) + \mathrm{tr}(yy^*) - 2\mathrm{tr}(xy^*) \\ &= 2\mathrm{tr}(e) - 2\mathrm{tr}(xy^*) \\ &= 4 - 2\mathrm{tr}(xy^*).\end{aligned}$$

In particular, $d^2(x,x^{-1}) = d^2(x,x^*) = 2 - \mathrm{tr}(x^2)$. The eigenvalues of x are given by $e^{\pm i\theta}$ for $\theta \in [0,\pi]$, so the eigenvalues of x^2 are $e^{\pm i2\theta}$. Consequently, $d^2(x,x^{-1}) = 2 - 2\cos(2\theta) = 4\sin^2(\theta)$. Hence $d(x,x^{-1}) = 2\sin(\theta)$ and, by translation invariance, $d(x^2,e) = 2\sin(\theta)$.

(c) Since $x = w\omega_1(\theta)w^{-1}$ for some $w \in SU(2)$, and some $\theta \in [0, \pi]$, consider the matrix $u = w\omega_1\left(\frac{\theta}{2}\right)w^{-1}$. Then $u^2 = x$ and from translation invariance and part (b), $d(x, e) = d(x^{-1}, e) = d(u^2, e) = 2\sin\left(\frac{\theta}{2}\right)$.

There appears to be no simple way to express the distance between two general points in $SU(2)$. If $x \sim \omega_1(\theta_0)$ and $y \sim \omega_1(\theta)$ where $\sim$ denotes similarity of matrices, then

$$\begin{aligned} \mathrm{tr}(xy^*) &= \left(\alpha\overline{\gamma} + \beta\overline{\delta}\right) + \left(\overline{\alpha}\gamma + \overline{\beta}\delta\right) \\ &= 2\mathrm{Re}\left(\alpha\overline{\gamma} + \beta\overline{\delta}\right) \\ &= 2\cos(\Theta) \end{aligned}$$

for some $\Theta \in [0, \pi]$. Hence

$$\begin{aligned} d(x, y) &= \sqrt{2 - 2\cos(\Theta)} \\ &= 2\sin\left(\frac{\Theta}{2}\right). \end{aligned}$$

If $\alpha \in (0, 1]$ and there exists a real number $M > 0$ such that

$$d(f(x), f(y)) \leq M d^{\alpha}(x, y)$$

for all $x, y \in SU(2)$, then we write $f \in \mathrm{Lip}_{\alpha}(SU(2))$. In particular if f is a central function on $SU(2)$, then for $\theta_0, \theta \in [0, \pi]$,

$$\begin{aligned} d(f(x), f(y)) &= d(f(\omega_1(\theta)), f(\omega_1(\theta_0))) \\ &\leq M\left(2 - 2\cos(\theta - \theta_0)\right)^{\frac{\alpha}{2}} \\ &= 2^{\alpha}\left|\sin\left(\frac{\theta - \theta_0}{2}\right)\right|^{\alpha}. \end{aligned}$$

This final lemma for central functions will be used in the next section.

Lemma 2.1.26: The following statements are equivalent for a function $f : SU(2) \to \mathbb{C}$.

1. For each $y \in SU(2)$, the value of $f(y)$ depends only on the $\theta-$coordinate of y.
2. For each $y \in SU(2)$, the value of $f(y)$ depends only on the trace of y.
3. For each $y, z \in SU(2)$, $f(zyz^{-1}) = f(y)$.

Proof: (1) $\Rightarrow$ (2). Suppose (1) holds. Since $\theta \mapsto \cos(\theta)$ is an injection on $[0, \pi]$, for each $y \in SU(2)$ the value of $f(y)$ depends only on $2\cos(\theta) = e^{i\theta} + e^{-i\theta} = \text{tr}(y)$ where $e^{\pm i\theta}$ are the eigenvalues of y. (2) $\Rightarrow$ (3). Suppose (2) holds and let $y, z \in SU(2)$. Then $\det(zyz^{-1} - \lambda e) = \det(z(y - \lambda e)z^{-1}) = \det(y - \lambda e)$. Hence y and zyz^{-1} have the same eigenvalues, so by (2) $f(zyz^{-1}) = f(y)$. (3) $\Rightarrow$ (1). Suppose (3) holds and let $y \in SU(2)$. There exists $z \in SU(2)$ which diagonalizes y; i.e. $zyz^{-1} = \omega_1(\theta_0)$, and $\theta_0 \in [0, \pi]$ is unique from Example 2.1.26. By property (3) we conclude (1) holds.

This completes our discussion of the geometry and topology of $SU(2)$, and in the next subsection we develop the Haar measure on $SU(2)$.

2.2 HAAR MEASURE AND FUNCTION SPACES ON SU(2)

In this section we will construct the Haar measure on $SU(2)$ and describe some of the function spaces on $SU(2)$ used in our main result.

2.2.1 The Haar Measure on $SU(2)$ and its Properties. We begin with some preliminary definitions. We have the following theorem due to Von Neumann.

Theorem 2.2.1: On every compact group G there exists a unique regular Borel probability measure μ which is left invariant, in the sense that for $g \in G$ and $f \in C(G)$,

$$\int_G f(gx)\mu(dx) = \int_G f(x)\mu(dx).$$

This μ is also right invariant:

$$\int_G f(xg^{-1})\mu(dx) = \int_G f(x)\mu(dx),$$

and satisfies the relation

$$\int_G f(x^{-1})\mu(dx) = \int_G f(x)\mu(dx).$$

This μ is called the Haar measure of G.

Proof: See [Ru2], p. 123.

We expect Haar measure on $SU(2)$ to coincide with Lebesgue measure on S^3 for the following reasons. First, $SU(2)$ is a compact group and homeomorphic to S^3. Second, Lebesgue measure on S^3 is rotation invariant. Finally, multiplication of matrices in $SU(2)$ correspond to orthogonal transformations, i.e. rotations and reflections, in Euclidean space. The following lemma justifies the expectation.

Lemma 2.2.2: [DE], p.152. The map $SU(2) \to S^3$, mapping the matrix $x \in SU(2)$ to its first row, is a homeomorphism. Via this homeomorphism and the natural identification of $\mathbb{C}^2$ with $\mathbb{R}^4$, the normalized Lebesgue measure on S^3 coincides with the normalized Haar measure on $SU(2)$.

Proof: See [HR], pp. 133-134.

2.2.2 Integration and Convolution on $L^1(SU(2))$. To construct the normalized Lebesgue measure on a sphere of radius r in $\mathbb{R}^4$ we need to compute the Jacobian matrix for the spherical coordinate transformation $\alpha_1 = r\cos(\theta)$, $\alpha_2 = r\sin(\theta)\cos(\phi)$, $\beta_1 = r\sin(\theta)\sin(\phi)\cos(\psi)$, $\beta_2 = r\sin(\theta)\sin(\phi)\sin(\psi)$, where $r \in [0,\infty)$, $\theta \in [0,\pi]$, $\phi \in [0,\pi]$, and $\psi \in [0,2\pi]$. The Jacobian matrix J of this transformation on $\mathbb{R}^4$ is given in block form as $J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$ where

$$J_{11} = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta)\cos(\phi) & r\cos(\theta)\cos(\phi) \end{pmatrix},$$

$$J_{12} = \begin{pmatrix} 0 & 0 \\ -r\sin(\theta)\sin(\phi) & 0 \end{pmatrix},$$

$$J_{21} = \begin{pmatrix} \sin(\theta)\sin(\phi)\cos(\psi) & r\cos(\theta)\sin(\phi)\cos(\psi) \\ \sin(\theta)\sin(\phi)\sin(\psi) & r\cos(\theta)\sin(\phi)\sin(\psi) \end{pmatrix},$$

$$J_{22} = \begin{pmatrix} r\sin(\theta)\cos(\phi)\cos(\psi) & -r\sin(\theta)\sin(\phi)\sin(\psi) \\ r\sin(\theta)\cos(\phi)\sin(\psi) & r\sin(\theta)\sin(\phi)\cos(\psi) \end{pmatrix},$$

and the metric tensor on $\mathbb{R}^4$ is given by the matrix

$$J^\top J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\sin(\theta) & 0 \\ 0 & 0 & 0 & r^2\sin(\theta)\sin(\phi) \end{pmatrix}.$$

The volume element on $\mathbb{R}^4$ is given by

$$\sqrt{\det(J^\top J)}drd\psi d\phi d\theta = r^3\sin^2(\theta)\sin(\phi)drd\psi d\phi d\theta.$$

The volume element shows the spherical coordinate parametrization is degenerate when $\phi \in \{0, \pi\}$ or $\theta \in \{0, \pi\}$. To find the normalization we take $r = 1$ to restrict to S^3 and note

$$\int_0^\pi \int_0^\pi \int_0^{2\pi} \sin^2(\theta)\sin(\phi)d\psi d\phi d\theta = 2\pi^2.$$

The quantity $\frac{1}{2\pi^2}\sin^2(\theta)\sin(\phi)d\psi d\phi d\theta$ is normalized Lebesgue measure on S^3 . Lebesgue measure is rotation invariant on S^3 in the following sense. If R is a 3×3 orthogonal matrix, f is an integrable function on S^3, and $\mathbf{x} = \mathbf{x}(\phi,\theta,\psi)\in S^3$ then

$$\frac{1}{2\pi^2}\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi} f(R\mathbf{x})\sin^2(\theta)\sin(\phi)d\psi d\phi d\theta = \frac{1}{2\pi^2}\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi} f(\mathbf{x})\sin^2(\theta)\sin(\phi)d\psi d\phi d\theta.$$

Normalized Lebesgue measure is, up to a positive constant multiple, the only rotation invariant probability measure on S^3 and directly corresponds to left, right and inverse invariance of the Haar measure on $SU(2)$.

Definition 2.2.3: For $1\le p<\infty$, the L^p norm of a measurable function f on $SU(2)$ is denoted by $\|f\|_{L^p(SU(2))}$ and is defined by

$$\|f\|_{L^p(SU(2))} = \left(\int_{SU(2)} |f(x)|^p\mu(dx)\right)^{\frac{1}{p}}.$$

Let Φ denote the map from $[0,\pi]\times[0,\pi]\times[0,2\pi]$ onto $SU(2)$ given by

$$(\phi,\theta,\psi)\xrightarrow{\Phi} y(\phi,\theta,\psi) = \begin{pmatrix} \cos(\theta)+i\sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi)e^{i\psi} \\ -\sin(\theta)\sin(\phi)e^{-i\psi} & \cos(\theta)-i\sin(\theta)\cos(\phi) \end{pmatrix}.$$

The following result reduces integrals on $SU(2)$ with respect to normalized Haar measure to a three-dimensional Lebesgue integral.

Proposition 2.2.4: [F], p. 135. If f is an integrable function on $SU(2)$, then

$$\int_{SU(2)} f(x)\mu(dx) = \frac{1}{2\pi^2}\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi} f\circ\Phi(\phi,\theta,\psi)\sin^2(\theta)\sin(\phi)d\psi d\phi d\theta.$$

In particular, if f is a central function then the above proposition reduces to

$$\int_{SU(2)} f(x)\mu(dx) = \frac{2}{\pi}\int_0^{\pi} f(\omega_1(\theta))\sin^2(\theta)d\theta$$

where $\omega_1(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ for all $0 \le \theta \le \pi$.

Definition 2.2.5: Let μ be the normalized Haar measure on $SU(2)$. The convolution product of two integrable functions f_1 and f_2 is defined for all $x \in SU(2)$ by

$$(f_1 \star f_2)(x) = \int_{SU(2)} f_1(xy^{-1})f_2(y)\mu(dy).$$

If $f_1 \in L^p(SU(2))$ and $f_2 \in L^q(SU(2))$, then $f_1 \star f_2 \in C(SU(2))$ if $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f_1 \in L^r(SU(2))$ and $f_2 \in L^s(SU(2))$, then $f_1 \star f_2 \in L^t(SU(2))$ if $1 < r, s < \infty$ and $\frac{1}{t} = \frac{1}{r} + \frac{1}{s} - 1 > 0$. These facts are standard and their proofs can be found in [HR], Vol. 1, pp. 295-296.

The change of variables $z = xy^{-1}$ yields

$$\begin{aligned}(f_2 \star f_1)(x) &= \int_{SU(2)} f_2(xy^{-1})f_1(y)\mu(dy) \\ &= \int_{SU(2)} f_1(z^{-1}x)f_2(z)\mu(dz)\end{aligned}$$

due to the inverse invariance of Haar measure. As a consequence, $f_1 \star f_2 \neq f_2 \star f_1$ because $SU(2)$ is non-abelian. The convolution is commutative on abelian groups.

Example 2.2.7: Suppose f_1 is a central function on $SU(2)$. Then

$$(f_1 \star f_2)(x) = \int_{SU(2)} f_1(xy^{-1})f_2(y)\mu(dy)$$

$$= \int_{SU(2)} f_1(y^{-1}x)f_2(y)\mu(dy).$$

Let $z = y^{-1}x$. Then due to the inverse invariance of Haar measure,

$$\int_{SU(2)} f_1(y^{-1}x)f_2(y)\mu(dy) = \int_{SU(2)} f_1(z)f_2(xz^{-1})\mu(dz)$$
$$= (f_2 \star f_1)(x).$$

In this case convolution is commutative. Recall the matrix $\omega_1(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ from Example 2.1.13. Since f_1 is central, $f_1(z) = f_1(\omega_1(\theta))$ for some $\theta \in [0, \pi]$, so by inverse invariance of Haar measure and the fact $\mathrm{tr}(\omega_1(\theta)) = \mathrm{tr}(\omega_1^{-1}(\theta))$,

$$\int_{SU(2)} f_1(z)f_2(xz^{-1})\mu(dz) = \int_{SU(2)} f_1(z^{-1})f_2(xz)\mu(dz^{-1})$$
$$= \int_{SU(2)} f_1(z)f_2(xz)\mu(dz).$$

That is, Definition 2.2.5 is equivalent to

$$(f_1 \star f_2)(x) = \int_{SU(2)} f_1(y)f_2(xy^{-1})\mu(dy) = \int_{SU(2)} f_1(y)f_2(xy)\mu(dy).$$

Fix $x(\phi_0, \theta_0, \psi_0) \in SU(2)$ in spherical coordinates, and let $f_1, f_2 \in L^1(SU(2))$ where f_1 is a central function. Then

$$(f_1 \star f_2)(x(\phi_0, \theta_0, \psi_0)) = \int_{SU(2)} f_1(y)f_2(xy)\mu(dy)$$
$$= \frac{2}{\pi}\int_0^{\pi} f_1(\omega_1(\theta))(Q_{x(\phi_0,\theta_0,\psi_0)}f_2)(\omega_1(\theta))\sin^2(\theta)d\theta,$$

where the central function

$$(Q_{x(\phi_0,\theta_0,\psi_0)}f)((\omega_1(\theta)) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} f(x(\phi_0,\theta_0,\psi_0)y^{-1}(\phi,\theta,\psi))\sin(\phi)d\psi d\phi$$

will be studied below.

2.2.3 The Q_x Operator on $L^1(SU(2))$ and its Properties. The following definition will be useful in developing the integral form for the Nth partial sum of the Fourier series on $SU(2)$.

Definition 2.2.8: For a fixed $x \in SU(2)$ and $y \in SU(2)$ in spherical coordinates and $f \in L^1(SU(2))$ define $y \longmapsto (Q_x f)(y)$ by

$$(Q_x f)(y) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} f(x(\phi_0,\theta_0,\psi_0)y^{-1}(\phi,\theta,\psi))\sin(\phi)d\psi d\phi,$$

and define $\theta \longmapsto [Q_x f](\theta)$ on $[0,\pi]$ by

$$[Q_x f](\theta) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} f(x(\phi_0,\theta_0,\psi_0)y^{-1}(\phi,\theta,\psi))\sin(\phi)d\psi d\phi.$$

Using this definition, we can express the preceding convolution more succinctly as

$$(f_1 \star f_2)(x(\phi_0,\theta_0,\psi_0)) = \frac{2}{\pi}\int_0^{\pi} f_1(\omega_1(\theta))(Q_x f_2)(y)\sin^2(\theta)d\theta,$$

and this formula will be useful in our disscussion of the Fourier partial sum operator on $SU(2)$ in spherical coordinates below.

Remark: By Lemma 2.1.26, the function $y \longmapsto (Q_x f)(y)$ in Definition 2.2.8 is a central function on $SU(2)$ whether or not f is a central function on $SU(2)$.

We will now record some properties of the mapping $\theta \longmapsto [Q_x f](\theta)$ on $[0,\pi]$. The following inequality will be needed in this endeavor.

Theorem 2.2.9: (Jensen Inequality) Let φ be a convex function on $(-\infty,\infty)$ and f an integrable function on $[a,b]$. Then

$$\varphi\left(\frac{1}{b-a}\int_a^b f(t)dt\right) \leq \frac{1}{b-a}\int_a^b \varphi(f(t))dt.$$

Proof: [RF], p. 133.

Example 2.2.10: The function $\varphi(x)=x^p$ for $p\geq 1$ is convex on $[0,\infty)$, so by the Jensen inequality

$$\left(\int_0^1 |f(t)|dt\right)^p \leq \int_0^1 |f(t)|^p dt$$

for every $f\in L^p[0,1]$.

Proposition 2.2.11: The function $\theta \longmapsto [Q_x f](\theta)$ defined on $[0,\pi]$ satisfies the following properties.

(a) $[Q_x f](\theta) = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} f(x(\phi_0,\theta_0,\psi_0)y(\phi,\theta,\psi))\sin(\phi)d\psi d\phi$.

(b) If $f\in L^p(SU(2))$ for some $p\geq 1$ then the function $\theta\longmapsto [Q_x f](\theta)$ belongs to $\left(L^p(0,\pi), \frac{2}{\pi}\sin^2(\theta)d\theta\right)$ for $p\geq 1$.

(c) If $f\in C(SU(2))$, then the function $\theta\longmapsto [Q_x f](\theta)$ is continuous on $[0,\pi]$, and

1. $\lim_{\theta\to 0^+}[Q_x f](\theta) = f(x)$;
2. $\lim_{\theta\to \pi^-}[Q_x f](\theta) = f(-x)$.

(d) If $f\in \mathrm{Lip}_1(SU(2))$, then the function $\theta\longmapsto [Q_x f](\theta)$ belongs to $\mathrm{Lip}_1[0,\pi]$. Moreover, $\theta\longmapsto [Q_x f](\theta)$ has finite total variation independent of $x\in SU(2)$.

Proof: Let $f\in L^p(SU(2))$ for some $p\geq 1$, and denote the measures $d\mu = \frac{2}{\pi}\sin^2(\theta))d\theta$, $d\nu = \frac{1}{2\pi^2}\sin^2(\theta)\sin(\phi)d\psi d\phi d\theta$. Let $x=x(\phi_0,\theta_0,\psi_0)\in SU(2)$ be parameterized in spherical coordinates defined previously. Part (a) follows from Example 2.2.7. For (b), the Jensen Inequality and translation invariance of Haar measure on

$SU(2)$ yield

$$\begin{aligned}
\|[Q_x f](\theta)\|_{\left(L^p(0,\pi),\frac{2}{\pi}\sin^2(\theta)d\theta\right)} &= \frac{2}{\pi}\int_0^\pi |(Q_x f)(\theta)|^p \sin^2(\theta))d\theta \\
&= \int_0^\pi \left| \frac{1}{4\pi}\int_0^\pi \int_0^{2\pi} f(xy(\phi,\theta,\psi))\sin(\phi)d\psi d\phi \right|^p d\mu \\
&\leq \int_0^\pi \int_0^\pi \int_0^{2\pi} |f(xy(\phi,\theta,\psi))|^p d\nu \\
&= \int_0^\pi \int_0^\pi \int_0^{2\pi} |f(y(\phi,\theta,\psi))|^p d\nu \\
&= \|f\|^p_{L^p(SU(2))},
\end{aligned}$$

Part (c) is a direct consequence of the Lebesgue Dominated Convergence theorem. For $x = x(\phi_0, \theta_0, \psi_0), y = y(\phi, \theta, \psi) \in SU(2)$ parameterized in spherical coordinates, as $\theta \to 0^+$, $xy^{-1} \to xe = x$, so 1 of (c) holds and by a similar argument 2 of (c) holds.

For (d), suppose $|f(u) - f(v)| \leq Kd(u, v)$ for some constant $K \geq 0$ and all $u, v \in SU(2)$; here d is the translation invariant metric of Definition 2.1.24. Let $x, y \in SU(2)$, $\theta_1, \theta_2 \in [0, \pi]$, and $d\nu = \frac{1}{4\pi}\sin(\phi)d\psi d\phi$. Then using the note following Example 2.1.25,

$$\begin{aligned}
|[Q_x f](\theta_2) - [Q_x f](\theta_1)| &\leq \int_0^\pi \int_0^{2\pi} |f(xy(\phi,\theta_2,\psi)) - f(xy(\phi,\theta_1,\psi))|d\nu \\
&\leq K\int_0^\pi \int_0^{2\pi} d(xy(\phi,\theta_2,\psi)\,, xy(\phi,\theta_1,\psi))d\nu \\
&= K\int_0^\pi \int_0^{2\pi} d(y(\phi,\theta_2,\psi), y(\phi,\theta_1,\psi))d\nu \\
&= K\int_0^\pi \int_0^{2\pi} d(\omega_1(\theta_2), \omega_1(\theta_1))d\nu
\end{aligned}$$

$$= 2K\left|\sin\left(\frac{\theta_2-\theta_1}{2}\right)\right|\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\sin(\phi)d\psi d\phi$$

$$\leq K|\theta_2-\theta_1|.$$

Let $P := \theta_0 = 0 < \theta_1 < \ldots < \theta_n = \pi$ be a partition of $[0,\pi]$. Then,

$$\begin{aligned}
V([Q_xf],P) &= \sum_{k=1}^{n}|[Q_xf](\theta_k)-[Q_xf](\theta_{k-1})| \\
&\leq \int_0^{\pi}\int_0^{2\pi}\sum_{k=1}^{n}|f(x(\phi_0,\theta_0,\psi_0)y(\phi,\theta_k,\psi))-f(x(\phi_0,\theta_0,\psi_0)y(\phi,\theta_{k-1},\psi))|d\nu \\
&\leq K\int_0^{\pi}\int_0^{2\pi}\sum_{k=1}^{n}d(x(\phi_0,\theta_0,\psi_0)y(\phi,\theta_k,\psi)\,,x\,(\phi_0,\theta_0,\psi_0)y(\phi,\theta_{k-1},\psi))d\nu \\
&\leq K\int_0^{\pi}\int_0^{2\pi}\sum_{k=1}^{n}d(y(\phi,\theta_k,\psi)\,,y(\phi,\theta_{k-1},\psi))d\nu \\
&= K\int_0^{\pi}\int_0^{2\pi}\sum_{k=1}^{n}d(\omega_1(\theta_k),\omega_1(\theta_{k-1}))d\nu \\
&= \frac{K}{4\pi}\int_0^{\pi}\int_0^{2\pi}\sum_{k=1}^{n}2\left|\sin\left(\frac{\theta_k-\theta_{k-1}}{2}\right)\right|\sin(\phi)d\psi d\phi \\
&= K\sum_{k=1}^{n}2\left|\sin\left(\frac{\theta_k-\theta_{k-1}}{2}\right)\right| \\
&\leq K\sum_{k=1}^{n}|\theta_k-\theta_{k-1}| \\
&= K\pi.
\end{aligned}$$

Remark: The central function Q_xf on $SU(2)$ is a special case of the quotient integral formula found in [DE], p. 21, and we also note the similarity of Q_x with the spherical mean of a function on $\mathbb{R}^3$ as defined in [M], p. 84.

When $x = e$, Q_xf has a geometric interpretation. Let x, y, and g be matrices in $SU(2)$ parameterized in spherical coordinates.

Definition 2.2.12: The orthogonal projection of a function f in $L^2(SU(2))$ onto the space of square integrable central functions on $SU(2)$ is defined as

$$(Qf)(x) = \int_{SU(2)} f(gxg^{-1})\mu(dg).$$

We pause to verify that the operator Q has the properties asserted in this definition. It is clear Q is a linear operator on $L^2(SU(2))$. For $x, z \in SU(2)$

$$\begin{aligned}(Qf)(zxz^{-1}) &= \int_{SU(2)} f(gzxz^{-1}g^{-1})\mu(dg) \\ &= \int_{SU(2)} f((gz)x(gz)^{-1})\mu(dg) \\ &= \int_{SU(2)} f(wxw^{-1})\mu(dwz^{-1}) \\ &= \int_{SU(2)} f(wxw^{-1})\mu(dw) \\ &= (Qf)(x)\end{aligned}$$

by translation invariance of Haar measure. So Qf is a central function on $SU(2)$ and for convenience we write $[Qf](\theta) = (Qf)(x)$ where x is unitarily equivalent to $\omega_1(\theta)$ for $\theta \in [0, \pi]$. Next, Q is a projection because

$$\begin{aligned}Q^2f(x) &= \int_{SU(2)} (Qf)(yxy^{-1})\mu(dy) \\ &= \int_{SU(2)} (Qf)(x)\mu(dy) \\ &= (Qf)(x) \int_{SU(2)} \mu(dy) \\ &= Qf(x).\end{aligned}$$

Let $f_1, f_2 \in L^2(SU(2))$ and

$$< f_1, f_2 >= \int\limits_{SU(2)} f_1(x)\overline{f_2(x)}\mu(dx)$$

denote the usual inner product on $L^2(SU(2))$. We now verify Q is a self-adjoint operator on $L^2(SU(2))$. An application of Fubini's theorem and a change of variables yields

$$\begin{aligned}< Qf_1, f_2 > &= \int\limits_{SU(2)} (Qf_1)(x)\overline{f_2(x)}\mu(dx) \\ &= \int\limits_{SU(2)} \int\limits_{SU(2)} f_1(yxy^{-1})\overline{f_2(x)}\mu(dy)\mu(dx) \\ &= \int\limits_{SU(2)} f_1(w)\overline{\int\limits_{SU(2)} f_2(y^{-1}wy)\mu(dy)}\mu(dw) \\ &=< f_1, Qf_2 > .\end{aligned}$$

Hence Q is self-adjoint on $L^2(SU(2))$, and so an orthogonal projection on $L^2(SU(2))$. By the Cauchy-Schwarz inequality and the Fubini theorem,

$$\begin{aligned}\|Qf\|^2_{L^2(SU(2))} &= \int\limits_{SU(2)} |(Qf)(x)|^2\mu(dx) \\ &= \int\limits_{SU(2)} \left| \int\limits_{SU(2)} f(yxy^{-1})\mu(dy)\right|^2 \mu(dx) \\ &\le \int\limits_{SU(2)} \int\limits_{SU(2)} |f(yxy^{-1})|^2\mu(dy)\mu(dx) \\ &= \int\limits_{SU(2)} \int\limits_{SU(2)} |f(yxy^{-1})|^2\mu(dx)\mu(dy) \\ &= \int\limits_{SU(2)} \int\limits_{SU(2)} |f(z)|^2\mu(dz)\mu(dy)\end{aligned}$$

$$= \|f\|^2_{L^2(SU(2))} \int\limits_{SU(2)} \mu(dy)$$

$$= \|f\|^2_{L^2(SU(2))}.$$

Hence, $\|Q\|_{L^2(SU(2))} \leq 1$. Clearly $Q(1) = 1$, so $\|Q\|_{L^2(SU(2))} = 1$.

Theorem 2.2.13: Let $f \in L^1(SU(2))$ and $0 \leq \theta \leq \pi$, then

$$[Qf](\theta) = \frac{1}{4\pi} \int\limits_0^{\pi} \int\limits_0^{2\pi} f(y(\phi, \theta, \psi)) \sin(\phi) d\psi d\phi.$$

Proof: This is an easy consequence of the previous definitions.

Remark: If we fix $x \in SU(2) \setminus \{e\}$, then $f \longmapsto Q_x f$ is neither a projection nor self-adjoint operator on $L^2(SU(2))$.

If $f \in L^1(SU(2))$ and $x \in SU(2)$ then

$$\frac{2}{\pi} \int\limits_0^{\pi} |[Q_x f](\theta)| \sin^2(\theta) d\theta \leq \frac{1}{2\pi^2} \int\limits_0^{\pi} \int\limits_0^{\pi} \int\limits_0^{2\pi} |f(xy(\phi, \theta, \psi))| \sin^2(\theta) \sin(\phi) d\psi d\phi d\theta$$

$$= \int\limits_{SU(2)} |f(xy)| \mu(dy)$$

$$= \int\limits_{SU(2)} |f(y)| \mu(dy)$$

$$= \|f\|_{L^1(SU(2))}.$$

Therefore $f \mapsto [Q_x f]$ is a bounded linear transformation from $L^1(SU(2))$ into the function space $L^1([0, \pi], d\mu)$ with norm at most one. If $f \in L^\infty(SU(2))$ then for almost every $\theta \in [0, \pi]$

$$|[Q_x f](\theta)| \leq \frac{1}{4\pi} \int\limits_0^{\pi} \int\limits_0^{2\pi} |f(x(\phi_0, \theta_0, \psi_0) y(\phi, \theta, \psi))| \sin(\phi) d\psi d\phi$$

$$\leq \|f\|_{L^\infty(SU(2))}.$$

Therefore $f \mapsto [Q_x f]$ is a bounded linear transformation from $L^\infty(SU(2))$ into $L^\infty\left([0,\pi], \frac{2}{\pi}\sin^2(\theta)d\theta\right)$ with norm at most one. By the Riesz-Thorin interpolation theorem, [K], p. 97, $f \mapsto [Q_x f]$ is of strong type (p,p) for all $1 < p < \infty$ as well, with norm at most one; i.e., for all $f \in L^p(SU(2))$,

$$\|[Q_x f]\|_{L^p\left(0,\pi], \frac{2}{\pi}\sin^2(\theta)d\theta\right)} \leq \|f\|_{L^p(SU(2))}.$$

Moreover, if $f \in C(SU(2))$ then $[Q_x f] \in C([0,\pi])$ and $\|[Q_x f]\|_\infty \leq 1$ with equality if $f = 1$.

2.2.4 Differential Operators on $SU(2)$. We will now begin our discussion of differentiation on $SU(2)$. The main objective is to obtain the Laplace operator $\triangle$ in spherical coordinates on $SU(2)$ which will useful in deriving the Fourier series of $f \in L^2(SU(2))$ in the next section.

Definition 2.2.14: Let U be an open subset of $SU(2)$. A complex function f is of class C^1 on U provided:

i) for every $g \in U$, and $X \in su(2)$, the function

$$t \to f(g\exp(tX))$$

is differentiable at $t = 0$, and then one puts

$$(\rho(X)f)(g) = \frac{d}{dt} f(g\exp(tX))|_{t=0};$$

ii) the map

$$su(2) \times U \to \mathbb{C}, \quad (X,g) \mapsto (\rho(X)f)(g),$$

is continuous.

iii) For $k > 1$ a natural number, a function f is C^k on U if f is C^1, and if, for every $X \in su(2)$, the function $\rho(X)f$ is C^{k-1}.

Remark: The operator $\rho(X)$ is also called an infinitesimal generator and ρ is a representation on $su(2)$.

Example 2.2.15: Express $g \in SU(2)$ in spherical coordinates, and for $t \in \mathbb{R}$, let

$$g(t) = \begin{pmatrix} \cos(\theta(t)) + i\sin(\theta(t))\cos(\phi(t)) & \sin(\theta(t))\sin(\phi(t))e^{i\psi(t)} \\ -\sin(\theta(t))\sin(\phi(t))e^{-i\psi(t)} & \cos(\theta(t)) - i\sin(\theta(t))\cos(\phi(t)) \end{pmatrix}$$

be a smooth path in $SU(2)$.

The ordered triple $(\phi(0),\, \theta(0),\, \psi(0))$ will be denoted by $(\phi,\, \theta,\, \psi)$. By the chain rule,

$$\frac{d}{dt} f(g\exp(tX))|_{t=0} = \frac{\partial f}{\partial \theta}\theta'(0) + \frac{\partial f}{\partial \phi}\phi'(0) + \frac{\partial f}{\partial \psi}\psi'(0).$$

If $X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, then

$$\begin{aligned} g\exp(tX_1) &= \begin{pmatrix} \cos(\theta) + i\sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi)e^{i\psi} \\ -\sin(\theta)\sin(\phi)e^{-i\psi} & \cos(\theta) - i\sin(\theta)\cos(\phi) \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \\ &= \begin{pmatrix} e^{it}(\cos(\theta) + i\sin(\theta)\cos(\phi)) & \sin(\theta)\sin(\phi)e^{i(\psi - t)} \\ -\sin(\theta)\sin(\phi)e^{-i(\psi - t)} & e^{-it}(\cos(\theta) - i\sin(\theta)\cos(\phi)) \end{pmatrix}. \end{aligned}$$

Since $SU(2)$ is a group under matrix multiplication we must also have for $t \in \mathbb{R}$,

$$g\exp(tX_1) = \begin{pmatrix} \cos(\theta(t)) + i\sin(\theta(t))\cos(\phi(t)) & \sin(\theta(t))\sin(\phi(t))e^{i\psi(t)} \\ -\sin(\theta(t))\sin(\phi(t))e^{-i\psi(t)} & \cos(\theta(t)) - i\sin(\theta(t))\cos(\phi(t)) \end{pmatrix}.$$

We have the following relationships:

$$\begin{aligned}\cos(\theta(t)) &= \cos(\theta)\cos(t) - \sin(\theta)\cos(\phi)\sin(t),\\ \sin(\theta(t))\cos(\phi(t)) &= \cos(\theta)\sin(t) + \sin(\theta)\cos(\phi)\cos(t),\\ \sin(\theta(t))\sin(\phi(t))\cos(\psi(t)) &= \sin(\theta)\sin(\phi)\cos(\psi - t).\end{aligned}$$

We differentiate each equation with respect to t and evaluate at $t = 0$. For the first equation,

$$-\sin(\theta)\theta'(0) = -\sin(\theta)\cos(\phi).$$

So, $\theta'(0) = \cos(\phi)$. For the second equation,

$$\cos(\theta)\cos^2(\phi) - \sin(\theta)\sin(\phi)\phi'(0) = \cos(\theta),$$

so $\phi'(0) = \frac{-\sin(\phi)\cos(\theta)}{\sin(\theta)}$. The angle ψ maps to $\psi - t$ when g is multiplied on the right by $\exp(tX_1))$, so $\psi'(0) = -1$. Hence,

$$\rho(X_1) = \cos(\phi)\frac{\partial}{\partial\theta} - \frac{\sin(\phi)\cos(\theta)}{\sin(\theta)}\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\psi}.$$

Similar computations lead to

$$\begin{aligned}\rho(X_2) = \sin(\phi)\cos(\psi)&\frac{\partial}{\partial\theta}\\ &+ \frac{\cos(\theta)\cos(\phi)\cos(\psi) + \sin(\theta)\sin(\psi)}{\sin(\theta)}\frac{\partial}{\partial\phi}\\ &+ \frac{\sin(\theta)\cos(\phi)\cos(\psi) - \cos(\theta)\sin(\psi)}{\sin(\theta)\sin(\phi)}\frac{\partial}{\partial\psi},\end{aligned}$$

and

$$\rho(X_3) = \sin(\phi)\sin(\psi)\frac{\partial}{\partial\theta}$$

$$+\frac{\cos(\theta)\cos(\phi)\sin(\psi)-\sin(\theta)\cos(\psi)}{\sin(\theta)}\frac{\partial}{\partial\phi}$$
$$+\frac{\cos(\theta)\cos(\psi)+\sin(\theta)\cos(\phi)\sin(\psi)}{\sin(\theta)\sin(\phi)}\frac{\partial}{\partial\psi},$$

where $X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

In matrix notation,

$$\begin{pmatrix} \rho(X_1) \\ \rho(X_2) \\ \rho(X_3) \end{pmatrix} = M \begin{pmatrix} \frac{\partial}{\partial\theta} \\ \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi} \\ \frac{1}{\sin(\theta)\sin(\phi)}\frac{\partial}{\partial\psi} \end{pmatrix},$$

where the columns of M are denoted as follows. The first column of M will be denoted by the vector $\mathbf{f}$ and is defined as

$$\mathbf{f} = \begin{pmatrix} \cos(\phi) \\ \sin(\phi)\cos(\psi) \\ \sin(\phi)\sin(\psi) \end{pmatrix}.$$

The second column of M will be denoted by the vector $\mathbf{g}$ and is defined as

$$\mathbf{g} = \begin{pmatrix} -\cos(\theta)\sin(\phi) \\ \cos(\theta)\cos(\phi)\cos(\psi)+\sin(\theta)\sin(\psi) \\ \cos(\theta)\cos(\phi)\sin(\psi)-\sin(\theta)\cos(\psi) \end{pmatrix}.$$

The third column of M will be denoted by the vector $\mathbf{h}$ and is defined as

$$\mathbf{h} = \begin{pmatrix} -\sin(\theta)\sin(\phi) \\ \sin(\theta)\cos(\phi)\cos(\psi)-\cos(\theta)\sin(\psi) \\ \sin(\theta)\cos(\phi)\sin(\psi)+\cos(\theta)\cos(\psi) \end{pmatrix}.$$

A straightforward computation with the standard euclidean dot product and norm on $\mathbb{R}^3$ shows the matrix M is orthogonal and $\det(M) = 1$. Consequently,

$$\begin{pmatrix} \frac{\partial}{\partial \theta} \\ \frac{1}{\sin(\theta)} \frac{\partial}{\partial \phi} \\ \frac{1}{\sin(\theta)\sin(\phi)} \frac{\partial}{\partial \psi} \end{pmatrix} = M^{\top} \begin{pmatrix} \rho(X_1) \\ \rho(X_2) \\ \rho(X_3) \end{pmatrix}.$$

Remark: The set

$$\left\{ \frac{\partial}{\partial \theta}, \frac{1}{\sin(\theta)} \frac{\partial}{\partial \phi}, \frac{1}{\sin(\theta)\sin(\phi)} \frac{\partial}{\partial \psi} \right\}$$

forms an orthonormal basis for the space of tangent vectors on $SU(2)$ and is called an orthonormal frame. From this orthonormal frame we deduce that an orthonormal basis for the dual space to the space of tangent vectors, called the cotangent space, is

$$\{d\theta, \sin(\theta) d\phi, \sin(\theta)\sin(\phi) d\psi\}.$$

Remark: The $\rho(X_j)$ operators are analogous to angular momentum operators in spherical coordinates up to scaling factors as seen in [L], p. 380, but in one extra dimension.

The following computation will be used in the next subsection. Write

$$\rho(X_j) = f_j(\phi, \theta, \psi) \frac{\partial}{\partial \theta} + g_j(\phi, \theta, \psi) \frac{\partial}{\partial \phi} + h_j(\phi, \theta, \psi) \frac{\partial}{\partial \psi},$$

where for $j = 1, 2, 3$, f_j denotes component j of the first column of M, g_j denotes component j of the second column of M scaled by $\frac{1}{\sin(\theta)}$ and h_j denotes component j of the third column of M scaled by $\frac{1}{\sin(\theta)\sin(\phi)}$.

Definition 2.2.16: The Casimir operator of ρ is denoted by Ω_ρ and is defined by $\Omega_\rho = \sum_{j=1}^{3} \rho(X_j)^2$.

Remark: This definition is independent of the orthonormal basis for $su(2)$ as long as $\{X_1, X_2, X_3\}$ forms a basis for $su(2)$. The Casimir operator also commutes with the ρ operators defined above. See [F], pp. 120-122, for details. The Casimir operator is analogous to the total angular momentum operator in physics [L], p. 365.

Next, we will show that

$$\Omega_\rho = \frac{1}{\sin^2(\theta)} \left[\frac{\partial}{\partial\theta} \left(\sin^2(\theta) \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin(\phi)} \frac{\partial}{\partial\phi} \left(\sin(\phi) \frac{\partial}{\partial\phi} \right) + \frac{1}{\sin^2(\phi)} \frac{\partial^2}{\partial\psi^2} \right].$$

To begin we compute

$$\begin{aligned}
\rho(X_j)^2 &= \left(f_j(\phi,\theta,\psi) \frac{\partial}{\partial\theta} + g_j(\phi,\theta,\psi) \frac{\partial}{\partial\phi} + h_j(\phi,\theta,\psi) \frac{\partial}{\partial\psi} \right)^2 \\
&= \left(f_j^2 \frac{\partial^2}{\partial\theta^2} + g_j^2 \frac{\partial^2}{\partial\phi^2} + h_j^2 \frac{\partial^2}{\partial\psi^2} \right) \\
&+ \left(2 f_j g_j \frac{\partial^2}{\partial\theta\partial\phi} + 2 g_j h_j \frac{\partial^2}{\partial\phi\partial\psi} + 2 h_j f_j \frac{\partial^2}{\partial\psi\partial\theta} \right) \\
&+ \left(f_j \frac{\partial f_j}{\partial\theta} + g_j \frac{\partial f_j}{\partial\phi} + h_j \frac{\partial f_j}{\partial\psi} \right) \frac{\partial}{\partial\theta} \\
&+ \left(f_j \frac{\partial g_j}{\partial\theta} + g_j \frac{\partial g_j}{\partial\phi} + h_j \frac{\partial g_j}{\partial\psi} \right) \frac{\partial}{\partial\phi} \\
&+ \left(f_j \frac{\partial h_j}{\partial\theta} + g_j \frac{\partial h_j}{\partial\phi} + h_j \frac{\partial h_j}{\partial\psi} \right) \frac{\partial}{\partial\psi}.
\end{aligned}$$

Let $\mathbf{f} = \mathbf{f}_1$ be the first column of M, and let $\mathbf{g}_1$ be the second column of M scaled by $\frac{1}{\sin(\theta)}$, and let $\mathbf{h}_1$ be the third column of M scaled by $\frac{1}{\sin(\theta)\sin(\phi)}$, and for $j = 1, 2, 3$, define the column vector $\zeta_j = \begin{pmatrix} f_j, & g_j, & h_j \end{pmatrix}^\top$. Let $\mathbf{x}$ and $\mathbf{y}$ be column vectors in $\mathbb{R}^3$, and set $< \mathbf{x},\mathbf{y} >= \mathbf{x}^\top \mathbf{y}$, $\|\mathbf{x}\|^2 = \mathbf{x}^\top \mathbf{x}$, and let D denote the operator

$$D = \left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}, \frac{\partial}{\partial\psi} \right\rangle.$$

Then

$$\Omega_\rho = \left(\|\mathbf{f}_1\|^2\frac{\partial^2}{\partial\theta^2} + \|\mathbf{g}_1\|^2\frac{\partial^2}{\partial\phi^2} + \|\mathbf{h}_1\|^2\frac{\partial^2}{\partial\psi^2}\right)$$
$$+ \left(2 < \mathbf{f}_1, \mathbf{g}_1 > \frac{\partial^2}{\partial\theta\partial\phi} + 2 < \mathbf{g}_1, \mathbf{h}_1 > \frac{\partial^2}{\partial\phi\partial\psi} + 2 < \mathbf{h}_1, \mathbf{f}_1 > \frac{\partial^2}{\partial\psi\partial\theta}\right)$$
$$+ \sum_{j=1}^{3}\left(< \zeta_j, Df_j > \frac{\partial}{\partial\theta} + < \zeta_j, Dg_j > \frac{\partial}{\partial\phi} + < \zeta_j, Dh_j > \frac{\partial}{\partial\psi}\right).$$

Since M is orthogonal, $\|\mathbf{f}_1\|^2 = 1$, $\|\mathbf{g}_1\|^2 = \frac{1}{\sin^2(\theta)}$, $\|\mathbf{h}_1\|^2 = \frac{1}{\sin^2(\theta)\sin^2(\phi)}$, and the second term in parentheses is equal to zero. Elementary computations yield

$$\sum_{j=1}^{3} < \zeta_j, Df_j > = 2\frac{\cos(\theta)}{\sin(\theta)},$$
$$\sum_{j=1}^{3} < \zeta_j, Dg_j > = \frac{\cos(\phi)}{\sin^2(\theta)\sin(\phi)},$$
$$\sum_{j=1}^{3} < \zeta_j, Dh_j > = 0.$$

Hence

$$\Omega_\rho = \sum_{j=1}^{3}\rho(X_j)^2$$
$$= \frac{\partial^2}{\partial\theta^2} + 2\frac{\cos(\theta)}{\sin(\theta)}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2} + \frac{\cos(\phi)}{\sin^2(\theta)\sin(\phi)}\frac{\partial}{\partial\phi} + \frac{1}{\sin^2(\theta)\sin^2(\phi)}\frac{\partial^2}{\partial\psi^2}$$
$$= \frac{1}{\sin^2(\theta)}\frac{\partial}{\partial\theta}\left(\sin^2(\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2(\theta)}\left(\frac{\partial^2}{\partial\phi^2} + \frac{\cos(\phi)}{\sin(\phi)}\frac{\partial}{\partial\phi} + \frac{1}{\sin^2(\phi)}\frac{\partial^2}{\partial\psi^2}\right)$$
$$= \frac{1}{\sin^2(\theta)}\left[\frac{\partial}{\partial\theta}\left(\sin^2(\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin(\phi)}\frac{\partial}{\partial\phi}\left(\sin(\phi)\frac{\partial}{\partial\phi}\right) + \frac{1}{\sin^2(\phi)}\frac{\partial^2}{\partial\psi^2}\right].$$

Remark: The operator Ω_ρ is the same as the angular part of the Laplace operator in $\mathbb{R}^4$ in spherical coordinates, denoted by $\triangle_{S^3}$. Thus we will call Ω_ρ the Laplace operator on $SU(2)$ in spherical coordinates, and also frequently denote Ω_ρ by $\triangle$. The operator $\triangle_{S^2} = \frac{1}{\sin(\phi)}\frac{\partial}{\partial\phi}\left(\sin(\phi)\frac{\partial}{\partial\phi}\right) + \frac{1}{\sin^2(\phi)}\frac{\partial^2}{\partial\psi^2}$ is the Laplace-Beltrami operator on $\mathbb{R}^3$ in spherical coordinates.

Example 2.2.17: If Ω_ρ operates on an $f \in C^2(SU(2))$ which is a class function, then

$$\begin{aligned}\Omega_\rho f &= \frac{d^2 f}{d\theta^2} + 2\cot(\theta)\frac{df}{d\theta} \\ &= \frac{1}{\sin^2(\theta)}\frac{d}{d\theta}\left(\sin^2(\theta)\frac{df}{d\theta}\right).\end{aligned}$$

The solutions to the eigenvalue problem $\Omega_\rho f = \lambda f$ subject to $\lim\limits_{\theta\to 0}\sin(\theta)f(\theta) = \lim\limits_{\theta\to\pi}\sin(\theta)f(\theta) = 0$, are $\lambda_n = -n(n+2)$ and $f_n(\cos(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)}$ for n a non-negative integer [BR], p. 309. The functions $U_n(\cos(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)}$ are the Chebychev polynomials of the second kind [Sz], p.60, and we use these polynomials and their generalizations to construct the Fourier series of a function $f \in L^2(SU(2))$ in spherical coordinates in the next subsection.

Example 2.2.18: Let $f, g \in C^1(SU(2))$ and let $x \in SU(2)$ and $X \in su(2)$. With respect to the usual inner product defined on $L^2(SU(2))$,

$$< \rho(X)f, g >= - < f, \rho(X)g > .$$

To see this we compute

$$\begin{aligned}< \rho(X)f, g > &= \int\limits_{SU(2)} \frac{d}{dt} f(x\exp(tX))\Big|_{t=0} \overline{g(x)}\mu(dx) \\ &= \frac{d}{dt}\int\limits_{SU(2)} f(x\exp(tX))\overline{g(x)}\mu(dx)\Big|_{t=0} \\ &= \frac{d}{dt}\int\limits_{SU(2)} f(x)\overline{g(x\exp(-tX))}\mu(dx)\Big|_{t=0}\end{aligned}$$

$$= \int\limits_{SU(2)} f(x)\overline{\frac{d}{dt}g(x\exp(-tX))}\Big|_{t=0}\mu(dx)$$

$$= - < f, \rho(X)g > .$$

If we assume $f, g \in C^2(SU(2))$, then we conclude the Laplace operator is symmetric because

$$< \triangle f, g > = < g, \triangle f >,$$

and the negative of the Laplace operator is positive since

$$< -\triangle f, f > = \int\limits_{SU(2)} \sum_{j=1}^{3} |\rho(X_j)f(x)|^2 \mu(dx).$$

For completeness we list the expressions for the gradient, divergence, and curl on $SU(2)$ in spherical coordinates. Let $\widehat{\theta} = \frac{\partial}{\partial\theta}$, $\widehat{\phi} = \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}$, $\widehat{\psi} = \frac{1}{\sin(\theta)\sin(\phi)}\frac{\partial}{\partial\psi}$, and $F \in C^1(SU(2))$. Define a vector field $\mathbf{V}(\theta, \phi, \psi) = f\widehat{\theta} + g\widehat{\phi} + h\widehat{\psi}$, where the functions f, g, and h are sufficiently smooth functions on $SU(2)$ in the parameters (θ, ϕ, ψ). Elementary computations from [Pa], p. 29 yield the gradient, divergence and curl on $SU(2)$, respectively:

$$\begin{aligned}
\nabla F &= \frac{\partial F}{\partial\theta}\widehat{\theta} + \frac{1}{\sin(\theta)}\frac{\partial F}{\partial\phi}\widehat{\phi} + \frac{1}{\sin(\theta)\sin(\phi)}\frac{\partial F}{\partial\psi}\widehat{\psi}, \\
\nabla \cdot \mathbf{V} &= \frac{\partial f}{\partial\theta} + 2\frac{\cos(\theta)}{\sin(\theta)}f + \frac{1}{\sin(\theta)}\frac{\partial g}{\partial\phi} + \frac{\cos(\phi)}{\sin^2(\theta)\sin(\phi)}g + \frac{1}{\sin(\theta)\sin(\phi)}\frac{\partial h}{\partial\psi}, \\
\nabla \times \mathbf{V} &= \frac{1}{\sin(\theta)\sin(\phi)}\left(\frac{\partial}{\partial\phi}(\sin(\phi)h) - \frac{\partial g}{\partial\psi}\right)\widehat{\theta} \\
&\quad + \frac{1}{\sin(\theta)\sin(\phi)}\left(\frac{\partial f}{\partial\psi} - \sin(\phi)\frac{\partial}{\partial\theta}(\sin(\theta)h)\right)\widehat{\phi} \\
&\quad + \frac{1}{\sin(\theta)}\left(\frac{\partial}{\partial\theta}(\sin(\theta)g) - \frac{\partial f}{\partial\phi}\right)\widehat{\psi}.
\end{aligned}$$

This ends our discussion on Haar measure and function spaces on $SU(2)$, and in the next subsection we will discuss the representation theory of $SU(2)$.

2.3 REPRESENTATION THEORY ON SU(2)

In this subsection we will define the Fourier partial sums for a square integrable function on $SU(2)$ and obtain integral representations for them in spherical coordinates. A key ingredient is to identify all continuous irreducible unitary representations on $SU(2)$ up to equivalence. To construct representations of a Lie group, one usually obtains representations of the Lie algebra. Once this is done, the matrix exponential is used to transfer the Lie algebra representations to the Lie group. However $SU(2)$ has such an elementary structure that we can skip working with $su(2)$ and use a direct approach.

2.3.1 The Subspaces M_m and $\tilde{M}_m$. The following subspaces are fundamental in developing the representation theory on $SU(2)$. We begin with some preliminary definitions.

Definition 2.3.1: A function $F : \mathbb{C}^2 \to \mathbb{C}$ is homogeneous of degree m if for $a > 0$ and $(u, v) \in \mathbb{C}^2$, $F(au, av) = a^m F(u, v)$.

Let M be the vector space of all polynomials $P(u, v) = \sum c_{jk} u^j v^k$ in two complex variables, and let $M_m \subset M$ be the subspace of homogeneous polynomials of degree m:

$$M_m = \left\{ P : P(u, v) = \sum_{j=0}^{m} c_j u^j v^{m-j}, c_0, c_1 \ldots c_m \in \mathbb{C} \right\}.$$

The set M_m is a finite dimensional complex vector space with $\dim(M_m) = m + 1$.

Definition 2.3.2 To each $P \in M_m$ we associate a unique continuous function $p : SU(2) \to \mathbb{C}$ by the formula

$$p(y) = P(\alpha, \beta) \text{ where } y = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(2).$$

We shall call such a function p, obtained by lifting P from M_m to $SU(2)$, a homogeneous polynomial on $SU(2)$ of degree m and denote by $\tilde{M}_m$ the vector space of all homogeneous polynomials on $SU(2)$ of degree m.

Remark: Since $\tilde{M}_m \subset C(SU(2)) \subset L^2(SU(2))$, there is a natural inner product on $\bigcup_{m=0}^{\infty} \tilde{M}_m$:

$$(p,q) = \int_{SU(2)} p(y)\overline{q(y)}\mu(dy).$$

Proposition 2.2.4 implies

$$(p,q) = \frac{1}{2\pi^2} \int_0^{\pi} \int_0^{\pi} \int_0^{2\pi} p \circ \Phi(\phi,\theta,\psi)\overline{q \circ \Phi(\phi,\theta,\psi)} \sin^2(\theta)\sin(\phi) d\psi d\phi d\theta$$

where

$$y(\phi,\theta,\psi) \equiv \Phi(\phi,\theta,\psi) = \begin{pmatrix} \cos(\theta) + i\sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi)e^{i\psi} \\ -\sin(\theta)\sin(\phi)e^{-i\psi} & \cos(\theta) - i\sin(\theta)\cos(\phi) \end{pmatrix}.$$

But, normalized Haar measure μ on $SU(2)$ corresponds to normalized surface measure σ on the unit sphere S^3 by Lemma 2.2.2, so if

$$\mathbf{x}(\phi,\theta,\psi) = (\cos(\theta), \sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi)\cos(\psi), \sin(\theta)\sin(\phi)\sin(\psi))$$

for $\theta \in [0,\pi]$,$\phi \in [0,\pi]$, and $\psi \in [0,2\pi]$ then

$$\begin{aligned}(p,q) &= \frac{1}{2\pi^2} \int_0^{\pi} \int_0^{\pi} \int_0^{2\pi} p(y(\phi,\theta,\psi))\overline{q(y(\phi,\theta,\psi))} \sin^2(\theta)\sin(\phi) d\psi d\phi d\theta \\ &= \frac{1}{2\pi^2} \int_0^{\pi} \int_0^{\pi} \int_0^{2\pi} P(\mathbf{x}(\phi,\theta,\psi))\overline{Q(\mathbf{x}(\phi,\theta,\psi))} \sin^2(\theta)\sin(\phi) d\psi d\phi d\theta\end{aligned}$$

$$= \int_{S^3} P(\mathbf{x})\overline{Q(\mathbf{x})}d\sigma(\mathbf{x})$$

$$\equiv < P, Q > .$$

The inner product $< \cdot, \cdot >$ is not convenient for computations, so we will define another inner product on M_m via

$$(F_1|F_2) = \frac{1}{\pi^2} \int_{\mathbb{C}^2} F_1(u,v)\overline{F_2(u,v)}e^{-(|u|^2+|v|^2)}\lambda(du)\lambda(dv),$$

where F_1 and F_2 are homogeneous polynomials on $\mathbb{C}^2$ and λ denotes Lebesgue measure on $\mathbb{C} \simeq \mathbb{R}^2$. We will show the inner products $< \cdot, \cdot >$ and $(\cdot \mid \cdot)$ are proportional on M_m. To do this, we need some preliminary facts.

Definition 2.3.3: The gamma function is defined for $\text{Re}(x) > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

Using the substitution $t = r^2$, we obtain

$$\Gamma(x) = 2\int_0^\infty r^{2x-1}e^{-r^2}dr. \tag{3}$$

The following facts about the gamma function are standard [Sz], p.14:

1. $\Gamma(x+1) = x\Gamma(x)$;
2. $\Gamma(n+1) = n!$ for every $n \in \mathbb{N}$.

Definition 2.3.4: For $\text{Re}(p) > 0$ and $\text{Re}(q) > 0$, the beta function is defined by

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt.$$

Using the substitution $t = \sin^2(\theta)$ yields

$$B(p,q) = 2\int_0^{\frac{\pi}{2}} \sin^{2p-1}(\theta)\cos^{2q-1}(\theta)d\theta.$$

Applying previous identites and a polar coordinate change-of-variables gives

$$\begin{aligned}
\Gamma(p)\Gamma(q) &= \left(2\int_0^\infty y^{2p-1}e^{-y^2}dy\right)\left(2\int_0^\infty x^{2q-1}e^{-x^2}dx\right) \\
&= 4\int_0^\infty\int_0^\infty y^{2p-1}x^{2q-1}e^{-(x^2+y^2)}dxdy \\
&= 4\int_0^\infty\int_0^{\frac{\pi}{2}} r^{2p+2q-1}\sin^{2p-1}(\theta)\cos^{2q-1}(\theta)e^{-r^2}drd\theta \\
&= \left(2\int_0^\infty r^{2p+2q-1}e^{-r^2}dr\right)\left(2\int_0^{\frac{\pi}{2}}\sin^{2p-1}(\theta)\cos^{2q-1}(\theta)d\theta\right) \\
&= \Gamma(p+q)B(p,q).
\end{aligned}$$

In particular, when p and q are positive integers, we have

$$B(p,q) = \frac{(p-1)!(q-1)!}{(p+q-1)!}.$$

Lemma 2.3.5: If $F : \mathbb{C}^2 \to \mathbb{C}$ is homogeneous of degree $m > -4$, then

$$\int_{S^3} F(\mathbf{x})d\sigma(\mathbf{x}) = \frac{1}{\pi^2\Gamma\left(\frac{m}{2}+2\right)}\int_{\mathbb{C}^2} F(u,v)e^{-(|u|^2+|v|^2)}\lambda(du)\lambda(dv).$$

Proof: Using polar coordinates,

$$\int_{\mathbb{C}^2} F(u,v)e^{-(|u|^2+|v|^2)}\lambda(du)\lambda(dv) = 2\pi^2\int_0^\infty\int_{S^3} F(r\mathbf{x})e^{-r^2}r^3d\sigma(\mathbf{x})dr$$

$$= 2\pi^2 \left(\int_0^\infty r^{m+3} e^{-r^2} dr \right) \left(\int_{S^3} F(\mathbf{x}) d\sigma(\mathbf{x}) \right)$$

$$= \pi^2 \Gamma\left(\frac{m}{2} + 2\right) \int_{S^3} F(\mathbf{x}) d\sigma(\mathbf{x}).$$

Lemma 2.3.6: If p, q, r and s are nonnegative integers then,

$$\frac{1}{\pi^2 p! r!} \int_{\mathbb{C}^2} z^p \overline{z}^q w^r \overline{w}^s e^{-(|z|^2+|w|^2)} \lambda(dz)\lambda(dw) = \delta_{pq}\delta_{rs}.$$

Proof: A direct computation using Fubini's theorem and polar coordinates yields

$$\begin{aligned}
\int_{\mathbb{C}^2} z^p \overline{z}^q w^r \overline{w}^s e^{-(|z|^2+|w|^2)} \lambda(dz)\lambda(dw) &= \left(\int_{\mathbb{C}} z^p \overline{z}^q e^{-|z|^2} \lambda(dz) \right) \left(\int_{\mathbb{C}} w^r \overline{w}^s e^{-|w|^2} \lambda(dw) \right) \\
&= \left(\int_0^\infty \int_0^{2\pi} e^{i(p-q)\theta} e^{-l^2} l^{p+q+1} dl d\theta \right) \\
&\times \left(\int_0^\infty \int_0^{2\pi} e^{i(r-s)\theta} e^{-l^2} l^{r+s+1} dl d\theta \right) \\
&= 4\pi^2 \left(\int_0^\infty e^{-l^2} l^{p+q+1} dl \right) \left(\int_0^\infty e^{-l^2} l^{r+s+1} dl \right) \delta_{pq}\delta_{rs} \\
&= \pi^2 \Gamma\left(\frac{1}{2}(p+q+2)\right) \Gamma\left(\frac{1}{2}(r+s+2)\right) \delta_{pq}\delta_{rs} \\
&= \begin{cases} \pi^2 p! r! & \text{if } p = q \text{ and } r = s, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Proposition 2.3.7: The spaces M_m $(m = 0, 1, 2, \ldots)$ are mutually orthogonal in $L^2(S^3)$, and

$$\left\{ E_j(u, v) = \sqrt{\frac{(m+1)!}{j!(m-j)!}} u^j v^{m-j} : 0 \leq j \leq m \right\}$$

is an orthonormal basis for M_m.

Proof: Let $F : \mathbb{C}^2 \to \mathbb{C}$ and $G : \mathbb{C}^2 \to \mathbb{C}$ be homogeneous polynomials of degrees m and n respectively. Then $F\overline{G}$ is a homogenous polynomial of degree $m+n \geq 0$, so Lemma 2.3.5 shows that

$$\begin{aligned} < F, G > &= \int_{S^3} F(\mathbf{x})\overline{G(\mathbf{x})} d\sigma(\mathbf{x}) \\ &= \frac{1}{\pi^2 \Gamma\left(\frac{m+n}{2} + 2\right)} \int_{\mathbb{C}^2} F(u,v)\overline{G(u,v)} e^{-(|u|^2+|v|^2)} \lambda(du)\lambda(dv) \\ &= \frac{1}{\pi^2 \Gamma\left(\frac{m+n}{2} + 2\right)} (F|G). \end{aligned}$$

In particular, when $m = n$ then $< F, G >= \frac{1}{\pi^2(m+1)!}(F|G)$. Hence $< \cdot, \cdot >$ and $(\cdot|\cdot)$ are proportional on M_m. The orthogonality of monomials in Lemma 2.3.6 gives

$$\begin{aligned} (u^j v^{m-j} | u^k v^{n-k}) &= \frac{1}{\pi^2} \int_{\mathbb{C}^2} u^j v^{m-j} \overline{u}^k \overline{v}^{n-k} e^{-(|u|^2+|v|^2)} \lambda(du)\lambda(dv) \\ &= j!(m-j)! \delta_{jk} \delta_{m-j,n-k} \\ &= j!(m-j)! \delta_{jk} \delta_{mn}. \end{aligned}$$

This implies $(F|G) = 0 =< f, g >$ if $m \neq n$. If $m = n$ then

$$\begin{aligned} < u^j v^{m-j}, u^k v^{m-k} > &= \frac{1}{\pi^2 \Gamma(m+2)} (u^j v^{m-j} | u^k v^{m-k}) \\ &= \frac{j!(m-j)!}{(m+1)!} \delta_{jk}, \end{aligned}$$

and hence $< E_j, E_k >= \delta_{jk}$ if $0 \leq j, k \leq m$.

By Proposition 2.3.7, Definition 2.3.2, and its subsequent remark, the spaces $\tilde{M}_m$ are mutually orthogonal in $L^2(SU(2))$ with respect to the inner product $(\cdot, \cdot)$ and

$$\left\{ e_j \left(\begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} \right) = \sqrt{\frac{(m+1)!}{j!(m-j)!}} \alpha^j \beta^{m-j} : 0 \leq j \leq m \right\}$$

is an orthonormal basis for $\tilde{M}_m$.

2.3.2 Continuous, Irreducible, Unitary, Representations on $SU(2)$. Let V be a nonzero normed vector space over $\mathbb{R}$ or $\mathbb{C}$, and let $L(V)$ denote the algebra of bounded operators on V.

Definition 2.3.9: [Ba], p. 37. An action ν of a group G on a set X is a function $\nu : G \times X \to X$, for which we write $\nu(g, x) = gx$, satisfying the following conditions for every $g, h \in G$, $x \in X$ and $e \in G$, the identity element:

1. $\nu(gh, x) = \nu(g, \nu(h, x))$;
2. $ex = x$.

Definition 2.3.10: [Ba], p. 38. Let G be a topological group and X be a topological space. A group action $\nu : G \times X \to X$ is a continuous group action if the function ν is continuous.

Example 2.3.11: [Ba], p. 39. We use the natural correspondence between functions f on $\tilde{M}_m$ and F on M_m given in Definition 2.3.2 to define a continuous group action of $SU(2)$ on the set $\tilde{M}_m$ by

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} f\left(\begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix}\right) = F\left(\alpha u - \overline{\beta} v, \beta u + \overline{\alpha} v\right)$$

On a monomial from $\tilde{M}_m$

$$\begin{aligned}
\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} u^r v^{m-r} &= \left(\alpha u - \overline{\beta} v\right)^r \left(\beta u + \overline{\alpha} v\right)^{m-r} \\
&= \sum_{i=0}^{r} \sum_{j=0}^{m-r} \binom{r}{i} \binom{m-r}{j} (\alpha u)^i \left(-\overline{\beta} v\right)^{r-i} (\beta u)^j \left(\overline{\alpha} v\right)^{m-r-j} \\
&= \sum_{k=0}^{m} \left(\sum_{i=0}^{k} \binom{r}{i} \binom{m-r}{k-i} \alpha^i \left(-\overline{\beta}\right)^{r-i} \beta^{k-i} \left(\overline{\alpha}\right)^{m-r-k+i} \right) u^k v^{m-k},
\end{aligned}$$

where the substitution $j = k - i$ was used to interchange the order of summation. From linearity, $SU(2)$ acts on the vector space $\tilde{M}_m$ and produces another element of $\tilde{M}_m$, so the group action is invariant.

Definition 2.3.12: A continuous representation of $SU(2)$ acting on a Hilbert space V is a map π from $SU(2)$ into $L(V)$, the space of bounded linear operators on V, such that:

1. $\pi(x_1 x_2) = \pi(x_1)\pi(x_2)$ for all $x_1, x_2 \in SU(2)$;
2. $\pi(e) = id_V$;
3. for every $v \in V$, the map from $SU(2)$ into V given by $x \mapsto \pi(x)v$ is continuous.

Definition 2.3.13: Let $\mathcal{H}$ be a Hilbert space. A representation π of $SU(2)$ on $\mathcal{H}$ is said to be unitary if, for every $g \in SU(2)$, $\pi(g)$ is a unitary operator; i.e. for every $g \in SU(2)$ and for every $v \in \mathcal{H}$, $\|\pi(g)v\| = \|v\|$.

Definition 2.3.14: Let π_1 and π_2 be two representations of $SU(2)$ on V_1 and V_2, respectively. If a continuous linear map $A : V_1 \to V_2$ satisfies $A\pi_1(g) = \pi_2(g)A$ for every $g \in SU(2)$, then we call A an intertwining operator or say A intertwines π_1 and π_2.

Definition 2.3.15: Let π_1 and π_2 be two representations of $SU(2)$ on V_1 and V_2, respectively. We say π_1 is equivalent to π_2 if there exists a linear isometry $A : V_1 \to V_2$ which is onto and intertwines π_1 and π_2.

Definition 2.3.16: A subspace $W \subset V$ is said to be invariant for the representation π of $SU(2)$ on V if, for every $g \in SU(2)$, $\pi(g)W \subseteq W$.

Definition 2.3.17: The left regular representation of $SU(2)$ on $L^2(SU(2))$ is given by

$$(L(g)f)(x) = f(g^{-1}x), \quad (x \in SU(2)),$$

and the right regular representation of $SU(2)$ on $L^2(SU(2))$, given by

$$(R(g)f)(x) = f(xg), \quad (x \in SU(2)).$$

The following proposition shows how the left and right regular representations of $SU(2)$ on $L^2(SU(2))$ relate to the differential operators ρ and $\triangle$ discussed in Section 2.2. We will use this proposition in the proof of our main result.

Proposition 2.3.18: Let $g \in SU(2)$, $X \in su(2)$ and $L(g)$ and $R(g)$ denote the left and right translation operators, respectively, on $L^2(SU(2))$.

(a) If $f \in C^1(SU(2))$, then $\rho(X)L(g)f = L(g)\rho(X)f$, and $\rho(X)R(g)f = R(g)\rho(\mathrm{Ad}(g^{-1})X)f$.

(b) If $f \in C^2(SU(2))$, then $\triangle(f \circ L(g)) = (\triangle f) \circ L(g)$ and $\triangle(f \circ R(g)) = (\triangle f) \circ R(g)$.

Proof: See [F], pp. 160-162, and Definition 2.2.11 and the remark on p. 26.

The following elementary example is a demonstration of the preceding six definitions.

Example 2.3.19: Let g, g_1, g_2 and $x \in SU(2)$, and f_1 and $f_2 \in L^2(SU(2))$. We will show the left and right regular representations of $SU(2)$ on $L^2(SU(2))$ are equivalent, unitary representations. The left and right regular represenations are linear maps on $L^2(SU(2))$, and from the translation invariance of Haar measure on $SU(2)$,

$$\begin{aligned}(L(g)f_1, L(g)f_2) &= \int_{SU(2)} f_1(g^{-1}y)\overline{f_2(g^{-1}y)}\mu(dy) \\ &= \int_{SU(2)} f_1(z)\overline{f_2(z)}\mu(dgz) \\ &= \int_{SU(2)} f_1(z)\overline{f_2(z)}\mu(dz) \\ &= (f_1, f_2),\end{aligned}$$

and similarly

$$(R(g)f_1, R(g)f_2) = (f_1, f_2).$$

Consequently, the left and right regular representations are unitary operators on $L^2(SU(2))$ and taking $f_1 = f_2$ implies the norms on $L^2(SU(2))$ of the left and right regular representations are equal to one. Next, for all $x \in SU(2)$,

$$\begin{aligned} L(g_1)(L(g_2)f)(x) &= (L(g_2)f)(g_1^{-1}x) \\ &= f(g_2^{-1}g_1^{-1}x) \\ &= f((g_1g_2)^{-1}x) \\ &= (L(g_1g_2)f)(x), \end{aligned}$$

and

$$\begin{aligned} R(g_1)(R(g_2)f)(x) &= (R(g_2)f)(xg_1) \\ &= f(xg_1g_2) \\ &= (R(g_1g_2)f)(x), \end{aligned}$$

so requirement 1 of Definition 2.3.10 holds.

Requirement 2 of Definition 2.3.10 clearly holds. We will now prove the right regular representation is continuous on $L^2(SU(2))$. Let $f \in L^2(SU(2))$, and $\epsilon > 0$ be given, and let $g_1, g_2 \in SU(2)$. Since R satisfies requirement 1 and is unitary

$$\begin{aligned} \|R(g_1)f - R(g_2)f\|_2 &= \|R(g_2^{-1})R(g_1)f - R(g_2^{-1})R(g_2)f\|_2 \\ &= \|R(g_2^{-1}g_1)f - R(g_2^{-1}g_2)f\|_2 \\ &= \|R(g_2^{-1}g_1)f - R(e)f\|_2 \\ &= \|R(g_2^{-1}g_1)f - f\|_2 \end{aligned}$$

$$= \|R(g)f - f\|_2$$

where $g = g_2^{-1}g_1$. Since $C(SU(2))$ is dense in $L^2(SU(2))$, we may select $\varphi \in C(SU(2))$, such that $\|\varphi - f\|_2 < \frac{\epsilon}{3}$ ([RF], p. 153). Since $SU(2)$ is compact, and hence φ is uniformly continuous on $SU(2)$, there exists a neighborhood N of e in $SU(2)$ such that $\|R(g)\varphi - \varphi\|_\infty < \frac{\epsilon}{3}$ for all $g \in N$. Consequently, for all $g_2^{-1}g_1 \in N$, we have

$$\begin{aligned}
\|R(g)f - f\|_2 &\leq \|R(g)f - R(g)\varphi\|_2 + \|R(g)\varphi - \varphi\|_2 + \|\varphi - f\|_2 \\
&\leq \|\varphi - f\|_2 + \|R(g)\varphi - \varphi\|_\infty + \|\varphi - f\|_2 \\
&< \frac{2\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}$$

This completes the verification of requirement 3 for $R(g)$ and the proof for $L(g)$ is similar. Consequently, both $R(g)$ and $L(g)$ are unitary representations on $L^2(SU(2))$. Next, the left and right regular representations of $SU(2)$ on $L^2(SU(2))$ are unitarily equivalent. To see this, for all $f \in L^2(SU(2))$ and $g, x \in SU(2)$, let $(Af)(x) = f(x^{-1})$, and note

$$\begin{aligned}
(A(L(g)f)(x)) &= (L(g)f)(x^{-1}) \\
&= f(g^{-1}x^{-1}) \\
&= f((xg)^{-1}) \\
&= (Af)(xg) \\
&= ((R(g)A)f)(x).
\end{aligned}$$

Hence, A intertwines the right and left regular representations, and the fact that A is a unitary operator on $L^2(SU(2))$ is clear from the inverse invariance of Haar measure on $SU(2)$. Finally, if $V = L^2(SU(2))$, $W = \tilde{M}_m$ for some integer $m \geq$

0, $x = \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix} \in SU(2)$, $g = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(2)$ then the right regular representation of $f \in L^2(SU(2))$ restricted to $\tilde{M}_m$ is defined by

$$\begin{aligned}(\pi^m(g)f)(x) &= f(xg) \\ &= F\left(\alpha u - \overline{\beta} v, \beta u + \overline{\alpha} v\right).\end{aligned}$$

We pause to check that this yields a representation. Let $g_1 = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$ and $g_2 = \begin{pmatrix} \gamma & \delta \\ -\overline{\delta} & \overline{\gamma} \end{pmatrix}$ belong to $SU(2)$. Note,

$$\begin{aligned}(\pi^m(g_1 g_2)f)(x) &= f(x g_1 g_2) \\ &= F\left(\left(\alpha\gamma - \beta\overline{\delta}\right) u - \left(\overline{\beta}\gamma + \overline{\alpha}\overline{\delta}\right) v,\, (\alpha\delta + \beta\overline{\gamma})\, u + \left(\overline{\alpha\gamma} - \overline{\beta}\delta\right) v\right),\end{aligned}$$

and

$$\begin{aligned}\pi^m(g_1)\left((\pi^m(g_2)f)(x)\right) &= (\pi^m(g_2)f)(x g_1) \\ &= f((x g_1) g_2) \\ &= F\left(\left(\alpha\gamma - \beta\overline{\delta}\right) u - \left(\overline{\beta}\gamma + \overline{\alpha}\overline{\delta}\right) v,\, (\alpha\delta + \beta\overline{\gamma})\, u + \left(\overline{\alpha\gamma} - \overline{\beta}\delta\right) v\right).\end{aligned}$$

The second requirement of Definition 2.3.12 is satisfied because of the identity

$$(\pi^m(e)f)(x) = f(x),$$

and the third requirement is satisfied because $x \mapsto f_x$ is continuous from $SU(2)$ to $L^p(SU(2)$ for all $1 \leq p < \infty$. Thus, the requirements of a continuous representation of $SU(2)$ on $\tilde{M}_m$ are satisfied. The computation in Example 2.3.11 shows $\tilde{M}_m$ is an invariant subspace of $L^2(SU(2))$.

Remark: Let $g \in SU(2)$ with $g \neq e$. Urysohn's Lemma (cf. [RF], p. 239) guarantees $\varphi \in C(SU(2))$ such that φ is positive and real with $\|\varphi\|_2 = 1$ and the supports of φ and $R(g)\varphi$ are disjoint. Then,

$$\begin{aligned}\|R(g)\varphi - \varphi\|_2^2 &= (R(g)\varphi - \varphi, R(g)\varphi - \varphi) \\ &= \|R(g)\varphi\|_2^2 - 2\int\limits_{SU(2)} \varphi(xg)\overline{\varphi(x)}\mu(dx) + \|\varphi\|_2^2 \\ &= 2.\end{aligned}$$

Hence $\|R(g) - R(e)\|_{\text{op}} \geq \sqrt{2}$, where

$$\|R(g) - R(e)\|_{\text{op}} = \sup\{\|(R(g) - R(e))f\|_2 : \ f \in L^2(SU(2), \|f\|_2 = 1\}.$$

Hence the right regular representation R is not a continuous mapping from $SU(2)$ into the bounded linear operators on $L^2(SU(2))$, equipped with the operator norm.

Definition 2.3.20: The representation π of $SU(2)$ on V is said to be irreducible if the only invariant closed subspaces for π are $\{0\}$ and V.

Remark: From the definition, a one dimensional representation is irreducible.

Theorem 2.3.21: The restriction π^m of the right regular representation R of $SU(2)$ to $\tilde{M}_m$ is irreducible.

Proof: See [F], pp. 137-140.

Theorem 2.3.22: Let π be an irreducible representation of $SU(2)$ on a finite dimensional complex vector space V. Then π is equivalent to one of the representations π^m.

Proof: See [F], pp. 140-141.

Remark: Example 2.3.18 and the previous theorem shows that to get all of the continuous, irreducible unitary representations on $SU(2)$, the choice of whether to use the right or left regular representation is arbitrary.

2.3.3 The Dual Object of $SU(2)$ and Schur's Orthogonality Relations. In this section we will find a natural ordering of the continuous, irreducible, unitary representations on $SU(2)$. We begin with the following definition.

Definition 2.3.23: The dual object of $SU(2)$, denoted by $\widehat{SU(2)}$, is defined to be the set of equivalence classes of continuous irreducible unitary representations of $SU(2)$.

Theorem 2.3.24: $\widehat{SU(2)} = \{\pi^0, \pi^1, \pi^2, \ldots\}$

Proof: See [Fo], p.143.

Remark: The dual object of $SU(2)$ may be parameterized by the nonnegative integers, and thus gives us a way to order the continuous irreducible unitary representations on $SU(2)$ according to increasing dimension; i.e. we order these matrices by their size. We will call this ordering the natural ordering on $\widehat{SU(2)}$.

Theorem 2.3.25: Every continuous irreducible unitary representation of $SU(2)$ is finite dimensional.

Proof: See [F], p. 105.

Remark: Since $SU(2)$ is compact, the continuous irreducible unitary representations on $SU(2)$ are $n \times n$ matricial functions on $SU(2)$. Therefore, equivalent continuous irreducible unitary representations on $SU(2)$ such as L and R in Definition 2.3.17 are similar of matrices of dimension $n \times n$. Since L and R are equivalent, we will only determine the matrix elements of R below.

Theorem 2.3.26: (Schur orthogonality relations) Let $g \in SU(2)$, and for each nonnegative integer n, let $\pi^n \in \widehat{SU(2)}$ and $\{\pi^n_{ij}(g) : i, j = 0, .., n\}$ be a set of coordinate functions for π^n on $SU(2)$ with respect to a fixed orthonormal basis $\{e_0, ..., e_n\}$

in the representation space $\tilde{M}_n$ of π^n :

$$\pi^n_{ij}(g) = (\pi^n(g)e_j, e_i).$$

Then the set of functions $\sqrt{n+1}\pi^n_{ij}$ form an orthonormal set in $L^2(SU(2))$:

$$\int\limits_{SU(2)} \pi^n_{ij}(g)\overline{\pi^m_{kl}(g)}\mu(dg) = \frac{1}{n+1}\delta_{nm}\delta_{ik}\delta_{jl}.$$

Proof: See [F], p. 105 or [HR], vol. 1, p. 344.

Example 2.3.27: Let $g = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(2)$. The matrix elements of $\pi^m(g)$ on $SU(2)$ are given by

$$\pi^m_{jk}(g) = (\pi^m(g)e_k, e_j),$$

where

$$e_j\left(\begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix}\right) = \sqrt{\frac{(m+1)!}{j!(m-j)!}}u^j v^{m-j} : 0 \leq j \leq m.$$

Note

$$\begin{aligned}(\pi^m(g)e_k)\left(\begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix}\right) &= e_k\left(\begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}\right) \\ &= \sqrt{\frac{(m+1)!}{k!(m-k)!}}(\alpha u - \overline{\beta}v)^k(\beta u + \overline{\alpha}v)^{m-k},\end{aligned}$$

and on the other hand, since $\{e_1, e_2, ..., e_m\}$ is an orthonormal basis for $\tilde{M}_m$,

$$\begin{aligned}(\pi^m(g)e_k)\left(\begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix}\right) &= \sum_{j=0}^{m} \pi^m_{jk}(g)e_j\left(\begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix}\right) \\ &= \sum_{j=0}^{m} \sqrt{\frac{(m+1)!}{j!(m-j)!}}\pi^m_{jk}(g)u^j v^{m-j}.\end{aligned}$$

Hence, for every $u, v \in \mathbb{C}$,

$$\sum_{j=0}^{m} \sqrt{\frac{(m+1)!}{j!(m-j)!}}\pi_{jk}^{m}(g)u^{j}v^{m-j} = \sqrt{\frac{(m+1)!}{k!(m-k)!}}(\alpha u - \overline{\beta}v)^{k}(\beta u + \overline{\alpha}v)^{m-k}.$$

In particular, if $u = e^{2\pi it}$ and $v = 1$, then the previous identity reads

$$\sum_{j=0}^{m} \sqrt{\frac{(m+1)!}{j!(m-j)!}}\pi_{jk}^{m}(g)e^{2\pi ijt} = \sqrt{\frac{(m+1)!}{k!(m-k)!}}(\alpha e^{2\pi it} - \overline{\beta})^{k}(\beta e^{2\pi it} + \overline{\alpha})^{m-k}.$$

The value of the coordinate function $\pi_{jk}^{m}(g)$ is obtained using standard Fourier analysis,

$$\pi_{jk}^{m}(g) = \sqrt{\frac{j!(m-j)!}{k!(m-k)!}}\int_{0}^{1}(\alpha e^{2\pi it} - \overline{\beta})^{k}(\beta e^{2\pi it} + \overline{\alpha})^{m-k}e^{-2\pi ijt}dt.$$

Remark: It is difficult to use this formula for a given matrix $g \in SU(2)$ in spherical coordinates to deduce the values of the individual matrix elements $\pi_{jk}^{m}(g)$. See [HR] , vol. 2, p. 130 for $m = 0, 1, 2, 3$. The following example is useful.

Example 2.3.28: Suppose $\omega_1(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$, then

$$\begin{aligned}\pi_{jk}^{m}(\omega_1(\theta)) &= \sqrt{\frac{j!(m-j)!}{k!(m-k)!}}\int_{0}^{1}\left(e^{ik\theta}e^{2\pi ikt}\right)\left(e^{-i(m-k)\theta}\right)e^{-2\pi ijt}dt \\ &= e^{i(2k-m)\theta}\sqrt{\frac{j!(m-j)!}{k!(m-k)!}}\int_{0}^{1}\left(e^{2\pi i(k-j)t}\right)dt \\ &= e^{i(2k-m)\theta}\sqrt{\frac{j!(m-j)!}{k!(m-k)!}}\delta_{jk}.\end{aligned}$$

Hence the functions $e^{i(2k-m)\theta}$, $0 \leq k \leq m$, are the eigenvalues of $\pi^{m}(\omega_1(\theta))$.

2.3.4 Characters and Dirichlet Kernel on $SU(2)$. In this section we will develop the irreducible characters on $SU(2)$. We begin with the following definition.

Definition 2.3.29: Let $x \in SU(2)$ and let π^m be a finite dimensional unitary representation of $SU(2)$. The character χ_m of π^m is the function on $SU(2)$ given by

$$\chi_m(x) = \mathrm{tr}(\pi^m(x)).$$

Since the characters depend only on the trace, they are central functions on $SU(2)$. In particular, if $x = e \in SU(2)$, then $\chi_m(e) = \mathrm{tr}(\pi^m(e)) = m + 1$.

Example 2.3.30: Since every $x \in SU(2)$ is unitarily equivalent to $\omega_1(\theta)$ for $0 \leq \theta \leq \pi$ and $\mathrm{tr}(\pi(\omega_1(\theta))) = \mathrm{tr}(\pi(\omega_1^{-1}(\theta))) = \mathrm{tr}(\pi(\omega_1(-\theta)))$, the characters on $SU(2)$ must be real, inverse invariant, and even functions. On $SU(2)$, if χ_m is the character of π^m, and $0 < \theta < \pi$, then

$$\begin{aligned}\chi_m(\omega_1(\theta)) &= \mathrm{tr}(\pi^m(\omega_1(\theta))) \\ &= \sum_{k=0}^{m} e^{i(2k-m)\theta} \\ &= \frac{\sin((m+1)\theta)}{\sin(\theta)}.\end{aligned}$$

These functions are the Chebychev polynomials of the second kind denoted by $U_m(\cos(\theta)) = \frac{\sin((m+1)\theta)}{\sin(\theta)}$ as in Example 2.2.13. As $\theta \to 0^+$, $\chi_m(e) = \chi_m(e^{-1}) = m+1$, and as $\theta \to \pi^-$, $\chi_m(-e) = \chi_m((-e)^{-1}) = (-1)^m(m+1)$.

Remark: The characters on T are $\chi_m(t) = e^{imt}$ where $m \in \mathbb{Z}$. This family of functions is uniformly bounded, whereas the family of characters on $SU(2)$ are Chebychev polynomials of the second kind and this family is unbounded. This causes a difference when comparing convergence of Fourier series on T and convergence of Fourier series on the space of central functions on $SU(2)$ which we will discuss below.

Example 2.3.31: From the definition of convolution and the Schur orthogonality relations,

$$(\chi_n \star \chi_m)(x) = \begin{cases} \frac{1}{n+1}\chi_n(x) & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

To see this, note that

$$\begin{aligned}
(\chi_n \star \chi_m)(x) &= \int\limits_{SU(2)} \chi_n(xy^{-1})\chi_m(y)\mu(dy) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{m} \pi_{ij}^n(x) \int\limits_{SU(2)} \overline{\pi_{ij}^n(y)}\pi_{kk}^m(y)\mu(dy) \\
&= \begin{cases} \frac{1}{n+1}\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{m} \pi_{ij}^n(x)\delta_{ik}\delta_{jk} & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases} \\
&= \begin{cases} \frac{1}{n+1}\sum_{i=0}^{n} \pi_{ii}^n(x) & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases} \\
&= \begin{cases} \frac{1}{n+1}\chi_n(x) & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}
\end{aligned}$$

Definition 2.3.32: The Dirichlet kernel on T is defined as

$$D_N(\theta) = \sum_{m=-N}^{N} e^{im\theta}$$

for $N = 0, 1, 2, ...$

Remarks:

1. A useful alternative expression for the Dirichlet kernel on T is given by $D_N(\theta) = 1 + 2\sum_{m=0}^{N-1}\cos((m+1)\theta) = \frac{\sin\left(\left(N+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)}$,

and as $\theta \to 0$, D_N tends to $2N+1$. This means D_N has linear growth at the origin.

2. Observe that $|D_N(\theta)| \leq \frac{1}{\sin(\theta/2)} \leq \frac{\pi}{|\theta|}$ if $|\theta| \leq \pi$. On the other hand, for $k = \pm 1, \pm 2, .., \pm N$, the points $\frac{k\pi}{N}$ are uniformly distributed in the set $\left[-\pi, -\frac{\pi}{N}\right] \cup \left[\frac{\pi}{N}, \pi\right]$ and

$$\begin{aligned} D_N\left(\frac{k\pi}{N}\right) &= \frac{\sin\left(\left(N+\frac{1}{2}\right)\frac{k\pi}{N}\right)}{\sin\left(\frac{k\pi}{2N}\right)} \\ &= \frac{\sin\left(k\pi + \frac{k\pi}{2N}\right)}{\sin\left(\frac{k\pi}{2N}\right)} \\ &= \frac{(-1)^k \sin\left(\frac{k\pi}{2N}\right)}{\sin\left(\frac{k\pi}{2N}\right)} \\ &= (-1)^k. \end{aligned}$$

Hence, outside every open neighborhood of the origin in T, D_N is uniformly bounded but does not tend to zero.

Definition 2.3.33: Let $x \in SU(2)$. The Dirichlet kernel on $SU(2)$ is defined by

$$\mathbf{D}_N(x) = \sum_{m=0}^{N} (m+1)\chi_m(x),$$

for $N = 0, 1, 2, ...$

Remarks:

1. In spherical coordinates

$$\mathbf{D}_N(x) = \sum_{m=0}^{N} (m+1)\frac{\sin((m+1)\theta)}{\sin(\theta)}$$

where $x \in SU(2)$ is unitarily equivalent to $\omega_1(\theta)$, $0 \leq \theta \leq \pi$. Since the characters χ_m are real, inverse invariant, and even, so is the Dirichlet kernel on $SU(2)$.

2. The growth of the Dirichlet kernel is much different on $SU(2)$ than T. For example, $\mathbf{D}_N(e) = \frac{N(N+1)(2N+1)}{6}$, $\mathbf{D}_N(X_1) = (-1)^N N$, and $\mathbf{D}_N(-e) = (-1)^{N+1}\frac{N(N+1)}{2}$.

The Dirichlet kernel on $SU(2)$ is oscillatory as $N \to \infty$ at X_1 and $-e$ and consequently isn't uniformly bounded as $N \to \infty$ outside every neighborhood of the identity. This divergent behavior is typical near almost every point in $SU(2)$; see Problem 6 in section 4 for more details.

Each term in the Dirichlet kernel is a class or central function on $SU(2)$, and if x is unitarily equivalent to $\omega_1(\theta)$, then Example 2.3.34 implies

$$\begin{aligned}\mathbf{D}_N(x) &= \frac{-1}{2\sin(\theta)}\frac{d}{d\theta}\left(1 + 2\sum_{m=0}^{N}\cos((m+1)\theta)\right)\\ &= \frac{-1}{2\sin(\theta)}D'_{N+1}(\theta)\end{aligned}$$

where $D_k(\theta)$, $k = 0, 1, 2, ...$ is the Dirichlet kernel on T.

The Dirichlet kernels on $SU(2)$ and T integrate to unity over their respective groups. The computation on $SU(2)$, using Proposition 2.2.4 and Example 2.3.35, is as follows:

$$\begin{aligned}\int\limits_{SU(2)} \mathbf{D}_N(x)\mu(dx) &= \frac{2}{\pi}\int\limits_0^{\pi} \mathbf{D}_N(\omega_1(\theta))\sin^2(\theta)d\theta\\ &= \frac{2}{\pi}\sum_{m=0}^{N}(m+1)\int\limits_0^{\pi}\frac{\sin((m+1)\theta)}{\sin(\theta)}\sin^2(\theta)d\theta\\ &= \frac{2}{\pi}\sum_{m=0}^{N}(m+1)\int\limits_0^{\pi}\sin((m+1)\theta)\sin(\theta)d\theta\\ &= \left(\frac{2}{\pi}\right)\left(\frac{\pi}{2}\right)\\ &= 1.\end{aligned}$$

2.3.5 Schur's Lemma and Fourier Coefficients on $SU(2)$. We will need the following lemma to compute the Fourier coefficients of a square integrable function on $SU(2)$ below.

Theorem 2.3.34 (Schur's Lemma):

i) Let π_1 and π_2 be two finite dimensional irreducible representations of $SU(2)$ on V_1 and V_2, respectively. Let $A : V_1 \to V_2$ be a linear map which intertwines the representations π_1 and π_2 :

$$A\pi_1(g) = \pi_2(g)A$$

for all $g \in SU(2)$. Then either $A = 0$, or A is an isomorphism.

ii) Let π be an irreducible $\mathbb{C}-$linear representation of $SU(2)$ on a finite dimensional complex vector space V. Let $A : V \to V$ be a $\mathbb{C}-$ linear map which commutes with the representation π :

$$A\pi(g) = \pi(g)A$$

for every $g \in SU(2)$. Then there exists $\lambda \in \mathbb{C}$ such that

$$A = \lambda I.$$

Proof: i) Suppose $A \neq 0$. Let $v_1 \in \ker(A)$, so $Av_1 = 0$. Then for all $g \in SU(2)$,

$$\begin{aligned} 0 &= \pi_2(g)Av_1 \\ &= A\pi_1(g)v_1. \end{aligned}$$

Hence, $\pi_1(g)v_1 \in \ker(A)$ for all $g \in SU(2)$, i.e. $\pi_1(g)\ker(A) \subset \ker(A)$. But π_1 is irreducible and $A \neq 0$ so $\ker(A) = \{0\}$. Let $v_2 \in \mathrm{Im}(A)$; then there exists $v \in V_1$ such that $Av = v_2$. For every $g \in SU(2)$,

$$\begin{aligned} A\pi_1(g)v &= \pi_2(g)Av \\ &= \pi_2(g)v_2 \end{aligned}$$

Hence, $\pi_2(g)\mathrm{Im}(A) \subset \mathrm{Im}(A)$. But π_2 is irreducible and $A \neq 0$ so $\mathrm{Im}(A) = V_2$. Hence, A is a linear isomorphism from V_1 to V_2.

ii) The fundamental theorem of algebra guarantees that the polynomial $\det(A - \lambda I) = 0$ has a root $\lambda_0 \in \mathbb{C}$, so there exists a nonzero vector v such that $(A - \lambda_0 I)v = 0$. But $\pi(g)A = A\pi(g)$ for all g in $SU(2)$ implies $(A - \lambda_0 I)\pi(g) = \pi(g)(A - \lambda_0 I)$. Because $0 \neq v \in \ker(A - \lambda_0 I)$, part i) implies $A - \lambda_0 I = 0$ on V.

Definition 2.3.35: Let $f \in L^1(SU(2))$ and n be a nonnegative integer, then we define the Fourier transform of f at π^n as an operator on $\tilde{M}_n$ in the following way

$$\hat{f}(\pi^n) = \int_{SU(2)} f(g)\pi^n(g^{-1})\mu(dg).$$

Suppose we choose an orthonormal basis for $\tilde{M}_n$, so that $\pi^n(g)$ is represented by the matrix $[\pi^n_{ij}(g)]$. Then

$$\begin{aligned} \hat{f}(\pi^n)_{ij} &= \int_{SU(2)} f(g)\pi^n_{ij}(g^{-1})\mu(dg) \\ &= \int_{SU(2)} f(g)\overline{\pi^n_{ji}(g)}\mu(dg). \end{aligned}$$

Remark: Since $\dim(\tilde{M}_n) = n + 1$, the Fourier transform $\hat{f}(\pi^n)$ may be regarded as an $(n+1) \times (n+1)$ matrix.

Proposition 2.3.36: Let $a, b \in \mathbb{C}$, $x \in SU(2)$ and $f, g \in L^1(SU(2))$. Then for any $n \geq 0$:

1. $(\widehat{af + bg})(\pi^n) = a\widehat{f}(\pi^n) + b\widehat{g}(\pi^n)$;
2. $(\widehat{f \star g})(\pi^n) = \widehat{f}(\pi^n)\widehat{g}(\pi^n)$;
3. $\widehat{\overline{f}}(\pi^n) = \overline{\widehat{f}}(\pi^n)$;
4. $\widehat{(L_x f)}(\pi^n) = \widehat{f}(\pi^n)\pi^n(x^{-1})$ and $\widehat{(R_x f)}(\pi^n) = \pi^n(x)\widehat{f}(\pi^n)$;

5. If $f \in C^2(SU(2))$, then $\widehat{\triangle f}(\pi^n) = -n(n+2)\widehat{f}(\pi^n)$;

6. If f is central, then $\hat{f}(\pi^n)\pi^n(x) = \pi^n(x)\hat{f}(\pi^n)$.

Proof: See [F], p.168, and [Fo], p. 135, for the proof. The proof of 6 is as follows. Note that if f is central, then

$$\begin{aligned}
\hat{f}(\pi^n)\pi^n(x) &= \int_{SU(2)} f(y)\pi^n(y^{-1}x)\mu(dy) \\
&= \int_{SU(2)} f(xy^{-1})\pi^n(y)\mu(dy) \\
&= \int_{SU(2)} f(y^{-1}x)\pi^n(y)\mu(dy) \\
&= \int_{SU(2)} f(y)\pi^n(xy^{-1})\mu(dy) \\
&= \pi^n(x)\hat{f}(\pi^n).
\end{aligned}$$

Example 2.3.37: If $f \in L^1(SU(2))$ is a central function, then by Schur's lemma and 6 of Proposition 2.3.36, to each integer $n \geq 0$ there corresponds $c_n \in \mathbb{C}$ such that $\hat{f}(\pi^n) = c_n I_{(n+1)\times(n+1)}$. Taking the trace of both sides of the last identity we find

$$\begin{aligned}
(n+1)c_n &= \mathrm{tr}(\hat{f}(\pi^n)) \\
&= \mathrm{tr}\left(\int_{SU(2)} f(y)\pi^n(y^{-1})\mu(dy)\right) \\
&= \int_{SU(2)} f(y)\mathrm{tr}(\pi^n(y^{-1}))\mu(dy) \\
&= \int_{SU(2)} f(y)\chi_n(y^{-1})\mu(dy) \\
&= \int_{SU(2)} f(y)\chi_n(y)\mu(dy).
\end{aligned}$$

Hence,

$$c_n = \frac{1}{n+1} \int_{SU(2)} f(y)\chi_n(y)\mu(dy)$$

are the Fourier coefficients for f.

Example 2.3.38: The following computation involving the Fourier transform will be useful. Let π^n be a continuous, irreducible, unitary representation of $SU(2)$ and $f \in L^1(SU(2))$. Note that

$$\begin{aligned}
\operatorname{tr}(\hat{f}(\pi^n)\pi^n(x)) &= \operatorname{tr}\left(\int_{SU(2)} f(y)\pi^n(y^{-1})\mu(dy)\pi^n(x) \right) \\
&= \operatorname{tr}\left(\int_{SU(2)} f(y)\pi^n(y^{-1}x)\mu(dy) \right) \\
&= \int_{SU(2)} f(y)\operatorname{tr}(\pi^n(y^{-1}x))\mu(dy) \\
&= \int_{SU(2)} f(y)\chi_n(y^{-1}x)\mu(dy) \\
&= (f \star \chi_n)(x).
\end{aligned}$$

Define $P_n f = (n+1)f \star \chi_n$ for each integer $n \geq 0$. Then P_n is a linear operator on $L^1(SU(2))$ and using Example 2.3.19,

$$\begin{aligned}
P_n^2 f &= P_n((n+1)f \star \chi_n) \\
&= (n+1)(P_n(f \star \chi_n)) \\
&= (n+1)^2(f \star (\chi_n \star \chi_n)) \\
&= (n+1)f \star \chi_n \\
&= P_n f.
\end{aligned}$$

Therefore P_n is an idempotent, and also an orthogonal projection onto $\tilde{M}_n$ if $f \in L^2(SU(2))$. In particular, if f is a central function, then for all $x \in SU(2)$, Example 2.3.37 implies

$$\begin{aligned}(P_n f)(x) &= (n+1)(f \star \chi_n)\\ &= (n+1)\mathrm{tr}(\hat{f}(\pi^n)\pi^n(x))\\ &= (n+1)\mathrm{tr}(c_n I_{(n+1)\times(n+1)}\pi^n(x))\\ &= (n+1)c_n \mathrm{tr}(\pi^n(x))\\ &= (n+1)c_n \chi_n(x).\end{aligned}$$

2.3.6 The Peter-Weyl Theorem. The following theorem is key to mean convergence of Fourier series on $SU(2)$.

Theorem 2.3.39: (Peter-Weyl) For all $\pi^n \in \widehat{SU(2)}$ and $i, j \in \{0, 1, .., n\}$, define the coordinate functions π^n_{ij} as in Theorem 2.3.26. The set of functions $\sqrt{n+1}\pi^n_{ij}$ is an orthonormal basis for $L^2(SU(2))$. For $f \in L^2(SU(2))$, we have

$$f = \sum_{m=0}^{\infty} \sum_{i,j=0}^{m} (m+1)(f, \pi^m_{ij})\pi^m_{ij},$$

where $c^m_{ij} = (m+1)(f, \pi^m_{ij}) = (m+1)\int_{SU(2)} f(x)\overline{\pi^m_{ij}(x)}\mu(dx)$ are the Fourier coefficients for f and the series converges in the metric of $L^2(SU(2))$.

Proof: (See [HR] pp. 26-28).

Remarks:

1. For $x \in SU(2)$ and N a nonnegative integer, the Nth partial sum of the Fourier series of $f \in L^2(SU(2))$ is denoted by

$$(S_N f)(x) = \sum_{m=0}^{N} \sum_{i,j=0}^{m} (m+1)(f, \pi^m_{ij})\pi^m_{ij}(x)$$

$$= \sum_{m=0}^{N} \sum_{i,j=0}^{m} c_{ij}^{m} \pi_{ij}^{m}(x)$$

and $S_N f$ is a trigonometric polynomial of degree N and belongs to $\bigoplus_{n=0}^{N} \tilde{M}_n$. The direct sum is orthogonal because the coordinate functions π_{ij}^m are orthonormal functions on $SU(2)$ by the Schur orthogonality relations. The terms in the partial sums $(S_N f)(x)$ are grouped in blocks, i.e.

$$(S_0 f)(x) = c_{11}^{0} \pi_{11}^{0}(x),$$

$$(S_1 f)(x) = [c_{11}^{0} \pi_{11}^{0}(x)] + [c_{11}^{1} \pi_{11}^{1}(x) + c_{12}^{1} \pi_{12}^{1}(x) + c_{21}^{1} \pi_{21}^{1}(x) + c_{22}^{1} \pi_{22}^{1}(x)],$$

and $(S_2 f)(x)$ will have fourteen terms. For convenience we will find an alternative expression for $(S_N f)(x)$ in terms of the linear operators P_n in Example 2.3.38. The definition of c_{ij}^m in the statement of the Peter-Weyl theorem yields

$$\begin{aligned}
\sum_{i,j=0}^{m} c_{ij}^{m} \pi_{ij}^{m}(x) &= (m+1) \sum_{i,j=0}^{m} \left(\int_{SU(2)} f(y) \overline{\pi_{ij}^{m}(y)} \mu(dy) \right) \pi_{ij}^{m}(x) \\
&= (m+1) \int_{SU(2)} f(y) \sum_{i,j=0}^{m} \left(\overline{\pi_{ij}^{m}(y)} \pi_{ij}^{m}(x) \mu(dy) \right) \\
&= (m+1) \int_{SU(2)} f(y) \left(\sum_{i,j=0}^{m} \pi_{ji}^{m}(y^{-1}) \pi_{ij}^{m}(x) \right) \mu(dy) \\
&= (m+1) \int_{SU(2)} f(y) \left(\sum_{j=0}^{m} \pi_{jj}^{m}(y^{-1}x) \right) \mu(dy) \\
&= (m+1) \int_{SU(2)} f(y) \operatorname{tr}(\pi^{m}(y^{-1}x)) \mu(dy) \\
&= (m+1) \int_{SU(2)} f(y) \chi_m(y^{-1}x)) \mu(dy) \\
&= (m+1)(f \star \chi_m)(x)
\end{aligned}$$

$$= (P_m f)(x).$$

Therefore, the Fourier partial sums $(S_N f)(x)$ for $f \in L^2(SU(2))$ can be expressed more succinctly as a sum of orthogonal projections:

$$(S_N f)(x) = \sum_{m=0}^{N} (P_m f)(x).$$

The purpose of this thesis is to examine under what conditions the above sequence of partial sums converges in the pointwise or uniform senses.

2. On T, the linear span of the characters e^{imt}, where m is an integer and $t \in [0, 2\pi)$, is dense in $L^2(\mathrm{T})$, and so all of the vector space analogues of $\tilde{M}_n$ on T are one dimensional. On $SU(2)$ the linear span of the characters χ_m for m a nonnegative integer cannot be dense in $L^2(SU(2))$ because the characters on $SU(2)$ are the Chebychev polynomials of the second kind given by $\chi_m(x) = \frac{\sin((m+1)\theta)}{\sin(\theta)}$ for $\theta \in [0, \pi]$ and these are functions of a single variable whereas a general function on $SU(2)$ is a function of three independent real variables. This is one reason to suspect that convergence theory on $SU(2)$ for central and non-central functions on $SU(2)$ will be different than on T.

3. The appearance of the $m + 1$ factor in the Peter-Weyl theorem on $SU(2)$ is also much different than on T because all of the vector spaces $\tilde{M}_n$ on T are one dimensional and so this factor is equal to one in Fourier partial sums on T. This is another reason the convergence theory for Fourier series on $SU(2)$ is different than convergence of Fourier series on T. In particular, some convergence problems on T can be resolved by requiring the Fourier coefficients to be sparse. The $m+1$ factor in the Fourier partial sums on $SU(2)$ prevents proving analogous convergence theorems by making the Fourier coefficents sparse.

Example 2.3.40 : Let $x, y, g \in SU(2)$ and $f \in \bigoplus_{m=0}^{N} \tilde{M}_m$. The Peter-Weyl theorem implies f can be represented as

$$f(x) = \sum_{m=0}^{N} \sum_{i,j=0}^{m} c_{ij}^m \pi_{ij}^m(x).$$

Recall from Example 2.2.7 that

$$(Q_x f)(y) = \int\limits_{SU(2)} f(xgyg^{-1})\mu(dg),$$

and using the Schur orthogonality relations we obtain

$$\begin{aligned}
\int\limits_{SU(2)} \pi_{ij}^m(xgyg^{-1})\mu(dg) &= \int\limits_{SU(2)} \sum_{k,l,r=0}^{m} \pi_{ik}^m(x)\pi_{kl}^m(g)\pi_{lr}^m(y)\overline{\pi_{jr}^m(g)}\mu(dg) \\
&= \sum_{k,l,r=0}^{m} \pi_{ik}^m(x)\pi_{lr}^m(y) \int\limits_{SU(2)} \pi_{kl}^m(g)\overline{\pi_{jr}^m(g)}\mu(dg) \\
&= \frac{1}{m+1} \sum_{k,l,r=0}^{m} \pi_{ik}^m(x)\pi_{lr}^m(y)\delta_{kj}\delta_{lr} \\
&= \sum_{m=0}^{N} \frac{1}{m+1} \sum_{i,j=0}^{m} c_{ij}^m \pi_{ij}^m(x) \sum_{l=0}^{m} \pi_{ll}^m(y) \\
&= \sum_{m=0}^{N} \frac{1}{m+1} \chi_m(y) \sum_{i,j=0}^{m} c_{ij}^m \pi_{ij}^m(x).
\end{aligned}$$

Consequently,

$$\begin{aligned}
(Q_x f)(y) &= \sum_{m=0}^{N} \sum_{i,j=0}^{m} c_{ij}^m \int\limits_{SU(2)} \pi_{ij}^m(xgyg^{-1})\mu(dg) \\
&= \sum_{m=0}^{N} \frac{1}{m+1} \chi_m(y)(P_m f)(x)
\end{aligned}$$

where $(P_m f)(x) = (m+1)(f \star \chi_m)(x)$. In particular, if $f \in \tilde{M}_n$ then

$$\begin{aligned}(Q_x f)(y) &= \int\limits_{SU(2)} f(xgyg^{-1})\mu(dg) \\ &= \frac{\chi_n(y)}{n+1} f(x)\end{aligned}$$

and since $\chi_n \in \tilde{M}_n$,

$$\begin{aligned}(Q_x \chi_n)(y) &= \int\limits_{SU(2)} \chi_n(xgyg^{-1})\mu(dg) \\ &= \frac{\chi_n(x)}{n+1} \chi_n(y).\end{aligned}$$

Therefore $\frac{\chi_n(x)}{n+1}$ are eigenvalues of Q_x with eigenfunctions χ_n. This result is a particular case of the Funk-Hecke Theorem for spheres in [F], p. 204.

Remark: For functions f which are not trigonometric polynomials on $SU(2)$, Theorem 2.2.9 yields

$$\begin{aligned}(Q_x f)(y) &= \int\limits_{SU(2)} f(xgyg^{-1})d\mu(g) \\ &= \frac{1}{4\pi} \int\limits_0^{\pi} \int\limits_0^{2\pi} f(x(\phi_0, \theta_0, \psi_0) y(\phi, \theta, \psi)) \sin(\phi) d\psi d\phi\end{aligned}$$

in spherical coordinates.

Example 2.3.41: We also can write a formula which will be useful in finding an integral representation for the Fourier partial sums of a central function on $SU(2)$. Suppose x and y are unitarily equivalent to $\omega_1(\theta_0)$ and $\omega_1(\theta)$, respectively. Multiplying both sides of the identity

$$\int\limits_{SU(2)} \chi_n(xgyg^{-1})\mu(dg) = \frac{\chi_n(x)}{n+1} \chi_n(y)$$

by $n+1$ and summing from zero to N yields, from Definition 2.3.33 and Example 2.3.30,

$$\begin{aligned}
\int_{SU(2)} \mathbf{D}_N(xgyg^{-1})\mu(dg) &= \sum_{n=0}^{N} \chi_n(x)\chi_n(y) \\
&= \frac{1}{\sin(\theta_0)\sin(\theta)} \sum_{n=0}^{N} \sin((n+1)\theta_0)\sin((n+1)\theta) \\
&= \frac{1}{2\sin(\theta_0)\sin(\theta)} \sum_{n=0}^{N} (\cos((n+1)(\theta_0-\theta)) - \cos((n+1)(\theta_0+\theta))) \\
&= \frac{1}{4\sin(\theta_0)\sin(\theta)} (D_{N+1}(\theta_0-\theta) - 1 - (D_{N+1}(\theta_0+\theta) - 1)) \\
&= \frac{1}{4\sin(\theta_0)\sin(\theta)} (D_{N+1}(\theta_0-\theta) - D_{N+1}(\theta_0+\theta)).
\end{aligned}$$

We would rather have an integral form for S_N involving a convolution between f and a kernel, analogous to the integral form of the Fourier partial sums on T. On T, we find an integral representation for the Nth partial sum of a square integrable function f as follows. Recall that

$$(S_N f)(x) = \sum_{j=-N}^{N} c_j e^{ijx}$$

where $c_j = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ijt}dt$. Substituting these expressions for the coefficients into the Fourier partial sum for f yields

$$\begin{aligned}
(S_N f)(x) &= \sum_{j=-N}^{N} \left(\frac{1}{2\pi} \int_0^{2\pi} f(t)e^{-ijt}dt \right) e^{ijx} \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(t) \sum_{j=-N}^{N} e^{ij(x-t)}dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} D_N(x-t)f(t)dt
\end{aligned}$$

where $D_N(t) = \sum_{j=-N}^{N} e^{ijt}$ is the Dirichlet kernel on T. Now we will find an analogous expression for the Nth Fourier partial sum of a square integrable function f on $SU(2)$ using a similar argument as on T. We will demonstrate that

$$(S_N f)(x) = \int_{SU(2)} \mathbf{D}_N(y) f(xy^{-1}) \mu(dy).$$

Substituting in $S_N f$ the expressions for the coefficients c_{ij}^n of f, the Peter-Weyl theorem yields

$$\begin{aligned}
(S_N f)(x) &= \sum_{n=0}^{N} (P_n f)(x) \\
&= \sum_{n=0}^{N} (n+1)(f \star \chi_n)(x) \\
&= \left(f \star \left(\sum_{n=0}^{N} (n+1) \chi_n \right) \right)(x) \\
&= \int_{SU(2)} f(y) \mathbf{D}_N(y^{-1}x) \mu(dy).
\end{aligned}$$

A change of variables then yields

$$(S_N f)(x) = \int_{SU(2)} \mathbf{D}_N(y) f(xy^{-1}) \mu(dy)$$

as desired.

Remarks:

1. In spherical coordinates, let $x = x(\theta_0, \phi_0, \psi_0)$ and $y = y(\theta, \phi, \psi)$ be matrices in $SU(2)$. Then $\mathbf{D}_N(y) = \frac{-1}{2\sin(\theta)} D'_{N+1}(\theta)$. Suppressing the arguments in x and y and using Proposition 2.2.4, the integral form for the Nth Fourier partial sum in spherical

coordinates for $f \in L^2(SU(2))$ is

$$\begin{aligned}
(S_N f)(x) &= \int\limits_{SU(2)} \mathbf{D}_N(y) f(xy^{-1})\mu(dy) \\
&= \frac{1}{2\pi^2} \int\limits_0^{\pi} \int\limits_0^{\pi} \int\limits_0^{2\pi} \frac{-1}{2\sin(\theta)} D'_{N+1}(\theta) f(xy^{-1}) \sin^2(\theta) \sin(\phi) d\psi d\phi d\theta \\
&= \frac{-1}{\pi} \int\limits_0^{\pi} D'_{N+1}(\theta) \sin(\theta) \left(\frac{1}{4\pi} \int\limits_0^{\pi} \int\limits_0^{2\pi} f(xy^{-1}) \sin(\phi) d\psi d\phi \right) d\theta \\
&= \frac{-1}{\pi} \int\limits_0^{\pi} D'_{N+1}(\theta) \sin(\theta) [Q_x f](\theta) d\theta.
\end{aligned}$$

2. From the Peter-Weyl theorem, the set $\{\chi_n : n \in \{0, 1, 2, ...\}\}$ forms an orthonormal basis for the subspace of central functions on $L^2(SU(2))$. In this case we get the following elementary expression for the Fourier partial sums:

$$(S_N f)(x) = \frac{1}{2\pi \sin(\theta_0)} \int_{-\pi}^{\pi} f(\omega_1(\theta)) \sin(\theta) D_{N+1}(\theta_0 - \theta) d\theta$$

To see this, Example 2.3.41 implies that the Nth Fourier partial sum of a square integrable, central function f on $SU(2)$ is

$$\begin{aligned}
(S_N f)(x) &= \sum_{n=0}^{N} P_n f(x) \\
&= \sum_{n=0}^{N} (n+1) c_n \chi_n(x) \\
&= \sum_{n=0}^{N} \left(\int\limits_{SU(2)} f(y) \chi_n(y) \mu(dy) \right) \chi_n(x) \\
&= \int\limits_{SU(2)} f(y) \sum_{n=0}^{N} \chi_n(y) \chi_n(x) \mu(dy)
\end{aligned}$$

$$= \frac{2}{\pi} \int_0^{\pi} f(\omega_1(\theta)) \frac{1}{4\sin(\theta_0)\sin(\theta)} \left(D_{N+1}(\theta_0 - \theta) - D_{N+1}(\theta_0 + \theta)\right) \sin^2(\theta) d\theta$$

$$= \frac{1}{2\pi \sin(\theta_0)} \int_{-\pi}^{\pi} f(\omega_1(\theta)) \sin(\theta) D_{N+1}(\theta_0 - \theta) d\theta$$

where x is unitarily equivalent to $\omega_1(\theta_0)$ and y is unitarily equivalent to $\omega_1(\theta)$. We will see this representation for the Nth Fourier partial sum agrees with the expansion of f in terms of zonal functions on the sphere in the next section. This completes our discussion of representation theory on $SU(2)$, and in the next subsection we will construct the Fourier series for a square integrable function on $SU(2)$ and study some applications to partial differential equations.

2.4 FOURIER SERIES ON SU(2)

In this section we will use elementary examples to illustrate some of the differences in the convergence of Fourier series on the abelian group T and the non-abelian group $SU(2)$. Applications of Fourier series on $SU(2)$ to partial differential equations will be given at the end of this section, as well as more sophisticated convergence results for multiple Fourier series on T and $SU(2)$, which will lead to the main result in the next section.

2.4.1 Spherical Harmonics on S^3. We first give an alternative derivation for the Fourier series on $SU(2)$ in spherical coordinates with the goal of determining an alternative formula for the matrix elements of the continuous irreducible unitary representations on $SU(2)$. The following theorem is useful in this endeavor.

Theorem 2.4.1: Every nonzero function f in $\tilde{M}_m$ is an eigenfunction of $\triangle$ with eigenvalue $k_m = -m(m+2)$.

Proof: See [F], pp. 163-164.

Example 2.4.2: Let π^m denote the $m+1$ dimensional continuous irreducible unitary representation of $SU(2)$ on $\tilde{M}_m$ and let $g \in SU(2)$. Since the matrix elements

$\pi^m_{jk}(g)$, of $\pi^m(g)$, belong to $\tilde{M}_m$,

$$\triangle \pi^m_{jk}(g) = -m(m+2)\pi^m_{jk}(g),$$

where k_m are constants. In particular,

$$\triangle \chi_m(g) = -m(m+2)\chi_m(g),$$

must also hold for all $g \in SU(2)$. This is consistent with the results in Example 2.2.13.

The equation $\triangle u = -\lambda u$ is called the Helmholtz equation and can be solved by the method of separation of variables. The primary references for the following calculation are [AH], pp. 74-81 and [St], pp. 270-277. We propose to solve the Helmholtz equation on S^3 which is equivalent to solving the Helmholtz equation on $SU(2)$. Using spherical coordinates in S^3 and the expression for $\triangle$ in these coordinates (cf. p. 59), we will solve

$$\frac{1}{\sin^2(\theta)}\left[\frac{\partial}{\partial\theta}\left(\sin^2(\theta)\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin(\phi)}\frac{\partial}{\partial\phi}\left(\sin(\phi)\frac{\partial u}{\partial\phi}\right) + \frac{1}{\sin^2(\phi)}\frac{\partial^2 u}{\partial\psi^2}\right] = -\lambda u \qquad (4)$$

in $0 < \theta < \pi$, $0 < \phi < \pi$, and $0 < \psi < 2\pi$, subject to the boundary conditions

$$u(\theta,\phi,0) = u(\theta,\phi,2\pi) \text{ and } \frac{\partial u}{\partial\psi}(\theta,\phi,0) = \frac{\partial u}{\partial\psi}(\theta,\phi,2\pi) \qquad (5)$$

if $0 \le \theta \le \pi$, $0 \le \phi \le \pi$, and the regularity conditions

$$u(\theta,\phi,\psi) \text{ is finite} \qquad (6)$$

if $\phi \in \{0,\pi\}$ and $\theta \in \{0,\pi\}$.

In the method of separation of variables we seek a nontrivial solution to (4), (5) and (6) of the form $u(\theta, \phi, \psi) = p(\theta)q(\phi)r(\psi)$. Substituting this functional form for u in (4) and multiplying through by $\frac{\sin^2(\theta)\sin^2(\phi)}{p(\theta)q(\phi)r(\psi)}$ and rearranging, we have

$$\frac{\sin^2(\phi)}{p(\theta)}\frac{d}{d\theta}\left(\sin^2(\theta)\frac{dp}{d\theta}\right) + \frac{\sin(\phi)}{q(\phi)}\frac{d}{d\phi}\left(\sin(\phi)\frac{dq}{d\phi}\right) + \lambda\sin^2(\theta)\sin^2(\phi) = -\frac{1}{r(\psi)}\frac{d^2r}{d\psi^2}. \tag{7}$$

Since the right hand side of (7) is a function of ψ while the left hand side of (7) is a function of θ and ϕ, it follows that there exists a constant μ such that

$$-\frac{1}{r(\psi)}\frac{d^2r}{d\psi^2} = \mu \tag{8}$$

for all $0 < \psi < 2\pi$ and

$$\frac{\sin^2(\phi)}{p(\theta)}\frac{d}{d\theta}\left(\sin^2(\theta)\frac{dp}{d\theta}\right) + \frac{\sin(\phi)}{q(\phi)}\frac{d}{d\phi}\left(\sin(\phi)\frac{dq}{d\phi}\right) + \lambda\sin^2(\theta)\sin^2(\phi) = \mu \tag{9}$$

for all $0 < \theta < \pi, 0 < \phi < \pi$. Dividing (9) by $\sin^2(\phi)$ and rearranging leads to

$$\frac{1}{p(\theta)}\frac{d}{d\theta}\left(\sin^2(\theta)\frac{dp}{d\theta}\right) + \lambda\sin^2(\theta) = \frac{\mu}{\sin^2(\phi)} - \frac{1}{q(\phi)\sin(\phi)}\frac{d}{d\phi}\left(\sin(\phi)\frac{dq}{d\phi}\right).$$

By a similar argument, there exists a constant ν such that

$$\frac{\mu}{\sin^2(\phi)} - \frac{1}{q(\phi)\sin(\phi)}\frac{d}{d\phi}\left(\sin(\phi)\frac{dq}{d\phi}\right) = \nu \tag{10}$$

for all $0 < \phi < \pi$ and

$$\frac{1}{p(\theta)}\frac{d}{d\theta}\left(\sin^2(\theta)\frac{dp}{d\theta}\right) + \lambda\sin^2(\theta) = \nu \tag{11}$$

for all $0 < \theta < \pi$.

Applying the first boundary condition of (2) with $u(\theta,\phi,\psi) = p(\theta)q(\phi)r(\psi)$ yields $0 = p(\theta)q(\phi)(r(2\pi) - r(0))$ for all $0 \le \theta \le \pi$, $0 \le \phi \le \pi$. If $r(2\pi) \ne r(0)$, then it follows that $0 = p(\theta)q(\phi)$ for all $0 \le \theta \le \pi$, $0 \le \phi \le \pi$, and hence $u(\theta,\phi,\psi) = 0$ on $0 \le \theta \le \pi$, $0 \le \phi \le \pi$, and $0 \le \psi \le 2\pi$. This contradicts the assumed nontrivality of u so we must have $r(0) = r(2\pi)$. Similar arguments using (5) and (6) and the nontrivality of $u(\theta,\phi,\psi) = p(\theta)q(\phi)r(\psi)$ leads to the conclusions $r'(0) = r'(2\pi)$ and $q(0), q(\pi), p(0)$, and $p(\pi)$ are finite. Consequently, taking into account (8), (10), and (11), we arrive at the following system of coupled boundary value problems satisfied by $u(\theta,\phi,\psi) = p(\theta)q(\phi)r(\psi)$:

$$\frac{d^2 r}{d\psi^2} + \mu r(\psi) = 0,\ r(0) = r(2\pi),\ r'(0) = r'(2\pi); \tag{12}$$

$$\frac{1}{\sin(\phi)}\frac{d}{d\phi}\left(\sin(\phi)\frac{dq}{d\phi}\right) + \left(\nu - \frac{\mu}{\sin^2(\phi)}\right) q(\phi) = 0,\ q(0) \text{ and } q(\pi) \text{ are finite}; \tag{13}$$

$$\frac{1}{\sin^2(\theta)}\frac{d}{d\theta}\left(\sin^2(\theta)\frac{dp}{d\theta}\right) + \left(\lambda - \frac{\nu}{\sin^2(\theta)}\right) p(\theta) = 0,\ p(0) \text{ and } p(\pi) \text{ are finite.} \tag{14}$$

The eigenvalues of (12) are $\mu_m = m^2$ and the eigenfunctions, up to a constant multiple, are $r_m(\psi) = e^{im\psi}$ where m is an integer. These eigenfunctions are orthonormal on the interval $[0, 2\pi]$ with respect to the weight function $w(\psi) = 1$ because

$$\frac{1}{2\pi}\int_0^{2\pi} r_n(\psi)\overline{r_m(\psi)}d\psi = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)\psi}d\psi = \delta_{nm}.$$

Substituting $\mu = m^2$ into (13) gives

$$\frac{1}{\sin(\phi)}\frac{d}{d\phi}\left(\sin(\phi)\frac{dq}{d\phi}\right) + \left(\nu - \frac{m^2}{\sin^2(\phi)}\right) q(\phi) = 0,\ q(0) \text{ and } q(\pi) \text{ are finite.} \tag{15}$$

The eigenvalues of (15) are $\nu_l = l(l+1)$, and the eigenfunctions up to a constant multiple are the associated Legendre functions given by $q_{lm}(\phi) = P_l^m(\cos(\phi))$ where

$$P_l^m(s) = \frac{(-1)^m}{2^l l!}(1-s^2)^{\frac{m}{2}} \frac{d^{l+m}}{ds^{l+m}}(s^2-1)^l,$$

$s = \cos(\phi)$, l is a nonnegative integer satisfying $l \geq |m|$. See [St], p. 273. The second linearly independent solution of (15) is discarded because it is singular at the origin, so it doesn't satisfy the condition $q(0)$ is finite. See [Sz], p. 65. The associated Legendre functions ([St], pp. 260-261) are orthogonal on $[0, \pi]$ with respect to the weight function $w(\phi) = \sin(\phi)$, i.e.

$$\int_0^\pi P_l^m(\cos(\phi)) P_{l'}^m(\cos(\phi)) \sin(\phi) d\phi = 0,$$

for nonnegative integers l and l' with $l \neq l'$. The normalizing constants are determined using

$$\int_0^\pi [P_l^m(\cos(\phi))]^2 \sin(\phi) d\phi = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!}.$$

See [St], pp. 275-276.

Remark: On S^2 the eigenfunctions of the Helmholtz equation are

$$Y_l^m(\phi, \psi) = (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{(l+m)!}} P_l^m(\cos(\phi)) e^{im\psi},$$

where m and l are integers satisfying $0 \leq |m| \leq l$ and are the normalized spherical harmonics on S^2 found in [St], p. 275, i.e:

$$\frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} Y_l^m(\phi, \psi) Y_{l'}^m(\phi, \psi)^* \sin(\phi) d\psi d\phi = \delta_{ll'}$$

where $*$ denotes complex conjugation. The spherical harmonics are polynomials in the variables $x = \cos(\psi)\sin(\phi)$, $y = \sin(\psi)\sin(\phi)$, and $z = \cos(\phi)$ of even degree if $l-m$ is even and of odd degree if $l-m$ is odd, and they form a complete orthonormal set in $L^2(S^3)$. See [AH], p. 90 for a proof. The index m is called the magnetic quantum number and the index l is called the orbital quantum number in quantum mechanics ([St], p. 279).

Substituting $\nu = l(l+1)$ into (14), it only remains to solve

$$\frac{1}{\sin^2(\theta)}\frac{d}{d\theta}\left(\sin^2(\theta)\frac{dp}{d\theta}\right) + \left(\lambda - \frac{l(l+1)}{\sin^2(\theta)}\right)p(\theta) = 0,\ p(0) \text{ and } p(\pi) \text{ are finite.} \quad (16)$$

The eigenvalues of (16) are $\lambda_n = n(n+2)$ and the eigenfunctions up to a constant multiple are $p_{n,l}(\theta) = P_{n,4,l}(\cos(\theta))$ where

$$P_{n,4,l}(t) = \frac{(-1)^{n-l} n! \Gamma\left(\frac{3}{2}\right)}{2^n (n-l)! \Gamma\left(n+\frac{3}{2}\right)}(1-t^2)^{-\left(\frac{l+1}{2}\right)}\left(\frac{d}{dt}\right)^{n-l}(1-t^2)^{n+\frac{1}{2}},$$

$t = \cos(\theta)$, n is a nonnegative integer, and $0 \le l \le n$. See [AH], p. 76. Similarly, the second linearly independent solution of (16) is discarded because it is singular at the origin. The eigenfunctions $P_{n,4,l}(\cos(\theta))$ are called associated Legendre functions of dimension four. The associated Legendre functions of dimension four are orthogonal on $[0, \pi]$ with respect to the weight function $w(\theta) = \sin^2(\theta)$, i.e.

$$\int_0^\pi P_{n,4,l}(\cos(\theta))P_{n',4,l}(\cos(\theta))\sin^2(\theta)d\theta = 0,$$

for nonnegative integers n and n' with $n \neq n'$. Also

$$\frac{2}{\pi}\int_0^\pi [P_{n,4,l}(\cos(\theta))]^2 \sin^2(\theta)d\theta = \frac{(n!)^2}{(n+1)(n-l)!(n+l+1)!}.$$

Hence,

$$\overline{P}_{n,4,l}(t) = \frac{\sqrt{(n+1)(n-l)!(n+l+1)!}}{n!} P_{n,4,l}(t)$$

are the normalized associated Legendre functions of dimension four on $[-1, 1]$. See [AH], p. 80-81.

Remark: Our definition of $\overline{P}_{n,4,l}(t)$ differs slightly from [AH], because the surface measure on S^3 is normalized whereas [AH] does not normalize the surface measure.

Therefore, the normalized eigenfunctions for $\triangle$ on S^3 in spherical coordinates are of the form

$$\begin{aligned} u(\theta, \phi, \psi) &= c_{nlm} \overline{P}_{n,4,l}(\cos(\theta)) P_l^m(\cos(\phi)) e^{i\psi} \\ &\equiv Y_{nlm}(\theta, \phi, \psi) \end{aligned}$$

where m, l, n are integers satisfying $0 \leq |m| \leq l \leq n$ $(n = 0, 1, 2...)$, and

$$c_{nlm} = \sqrt{\frac{(n+1)(n-l)!(n+l+1)!(2l+1)(l-m)!}{(l+m)!(n!)^2}}$$

are chosen so that the functions $Y_{nlm}(\theta, \phi, \psi)$ are orthonormal on S^3; i.e.

$$\frac{1}{2\pi^2} \int_0^{\pi} \int_0^{\pi} \int_0^{2\pi} Y_{nlm}(\theta, \phi, \psi) Y_{n'l'm'}(\theta, \phi, \psi) \sin^2(\theta) \sin(\phi) d\psi d\phi d\theta = \delta_{nn'} \delta_{ll'} \delta_{mm'}.$$

Consequently, the Fourier series of $u \in L^1(S^3)$ in spherical coordinates is formally

$$u(\theta, \phi, \psi) \sim \sum_{n=0}^{\infty} \sum_{l=0}^{n} \sum_{m=-l}^{l} \gamma_{nlm} Y_{nlm}(\theta, \phi, \psi),$$

where

$$\gamma_{nlm} = \frac{1}{2\pi^2} \int_0^{\pi} \int_0^{\pi} \int_0^{2\pi} u(\theta, \phi, \psi) Y^*_{nlm}(\theta, \phi, \psi) \sin^2(\theta) \sin(\phi) d\psi d\phi d\theta.$$

Remarks:

1. The functions $Y_{nlm}(\theta, \phi, \psi)$ are the three dimensional spherical harmonics and, when lifted to $SU(2)$, provide an alternative representation for the matrix elements for the continuous irreducible unitary representations of $SU(2)$ given in Example 2.3.27.

2. In quantum mechanics, the index n is sometimes labeled by integers and half-integers instead of integers and is used to describe the spin of particles. Particles with integer spin (or even integers for us) are called bosons and particles with half-integer spin (or odd integers for us) are called fermions. See [DD], pp. 90-91.

3. The separation of variables presented here can be extended to $\mathbb{R}^n$. It can be shown that the separated solutions of the $\triangle$ eigenvalue problem are higher dimensional spherical harmonics which are complete in $L^2(S^{d-1})$. These functions were mentioned in the introduction. See [AH] and [DX] for more information on these topics.

Example 2.4.3: When $l = 0$ the associated Legendre functions of dimension four reduce to $P_{n,4,0}(\cos(\theta)) = U_n(\cos(\theta))$ as in Example 2.3.26. Functions on S^3 which only depend on the variable θ are called zonal functions, and their Fourier series reduces to the formal expansion

$$u(\theta) \sim \sum_{n=0}^{\infty} (n+1) c_n U_n(\cos(\theta)),$$

where

$$(n+1)c_n = \frac{2}{\pi} \int_0^{\pi} \sin((n+1)\theta) u(\theta) \sin(\theta) d\theta,$$

in agreement with the results for central functions on $SU(2)$.

Theorem 2.4.4 (Addition Formula) If $\{Y_{nlm}\}$ is an orthonormal basis for M_n, where $-l \leq m \leq l$ and $l = 0, 1, 2, ...$ are integers, then

$$\sum_{n=0}^{N}(n+1)U_n(\cos(\Theta)) = \sum_{n=0}^{N}\sum_{l=0}^{n}\sum_{m=-l}^{l} Y^*_{nlm}(\theta,\phi,\psi)Y_{nlm}(\theta_0,\phi_0,\psi_0),$$

where $\cos(\Theta)$ is defined immediately preceding Example 2.1.26.

Proof: See [AH], p. 21.

The integral form for the Fourier partial sums in the general case is derived as follows. Let $u \in L^2(S^3)$ and

$$(S_N u)(\theta_0,\phi_0,\psi_0) = \sum_{n=0}^{N}\sum_{l=0}^{n}\sum_{m=-l}^{l} \gamma_{nlm}Y_{nlm}(\theta_0,\phi_0,\psi_0),$$

where

$$\gamma_{nlm} = \frac{1}{2\pi^2}\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi} u(\theta,\phi,\psi)Y^*_{nlm}(\theta,\phi,\psi)\sin^2(\theta)\sin(\phi)d\psi d\phi d\theta.$$

Substituting for the coefficients, interchanging the orders of summation and integration, and applying the addition formula yields

$$\begin{aligned}
(S_N u)(\theta_0,\phi_0,\psi_0) &= \sum_{n=0}^{N}\sum_{l=0}^{n}\sum_{m=-l}^{l} \gamma_{nlm}Y_{nlm}(\theta_0,\phi_0,\psi_0) \\
&= \frac{1}{2\pi^2}\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi} u(\theta,\phi,\psi)\sin^2(\theta)\sin(\phi) \\
&\times \left(\sum_{n=0}^{N}\sum_{l=0}^{n}\sum_{m=-l}^{l} Y^*_{nlm}(\theta,\phi,\psi)Y_{nlm}(\theta_0,\phi_0,\psi_0)\right) d\psi d\phi d\theta \\
&= \frac{1}{2\pi^2}\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi} u(\theta,\phi,\psi)\sin^2(\theta)\sin(\phi) \\
&\times \left(\sum_{n=0}^{N}(n+1)U_n(\cos(\Theta))\right) d\psi d\phi d\theta
\end{aligned}$$

$$= \frac{1}{2\pi^2} \int_0^{\pi} \int_0^{\pi} \int_0^{2\pi} \mathbf{D}_N(\omega_1(\Theta)) u(\theta, \phi, \psi) \sin^2(\theta) \sin(\phi) d\psi d\phi d\theta.$$

where $\cos(\Theta)$ is the angle between the two unit vectors with initial points at the origin and terminal points at $(\theta_0, \phi_0, \psi_0)$ and (θ, ϕ, ψ). This formula for the Nth partial sum agrees with the previous result for noncentral functions on $SU(2)$, i.e.

$$\frac{1}{2\pi^2} \int_0^{\pi} \int_0^{\pi} \int_0^{2\pi} \mathbf{D}_N(\omega_1(\Theta)) u(\theta, \phi, \psi) \sin^2(\theta) \sin(\phi) d\psi d\phi d\theta = (\mathbf{D}_N \star \tilde{u})(x)$$

where $x = x(\theta_0, \phi_0, \psi_0) \in SU(2)$, and $\tilde{u}$ is the lifting of u to $SU(2)$ in Definition 2.3.2.

Remark: In spherical coordinates on S^3 if $\cos(\Theta) = \mathbf{r}_1 \cdot \mathbf{r}_2$, then $\cos(\Theta) = R\mathbf{r}_1 \cdot R\mathbf{r}_2$ where R is a rotation matrix. The function $(\theta, \phi, \psi) \to \mathbf{D}_N(\omega_1(\theta))$ is an example of a reproducing kernel for the space of spherical harmonics of degree N on S^3. See [F], p. 202.

2.4.2 Elementary Convergence and Divergence of Fourier Series. We will now present two examples of computations to illustrate divergence of Fourier series on $SU(2)$.

Example 2.4.5: [Ma3]. Each $x \in SU(2)$ is unitarily equivalent to $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ for a unique $\theta \in [0, \pi]$. Define a central function f on $SU(2)$ by

$$f(x) = \begin{cases} 1 & \text{if} \quad \theta \in [0, \frac{\pi}{2}), \\ \frac{1}{2} & \text{if} \quad \theta = \frac{\pi}{2}, \\ 0 & \text{if} \quad \theta \in (\frac{\pi}{2}, \pi]. \end{cases}$$

It is clear that $f \in L^\infty(SU(2))$. By Proposition 2.2.4 and Examples 2.3.30 and 2.3.37, the Fourier coefficients of f are

$$(n+1)a_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin((n+1)\theta)\sin(\theta)d\theta.$$

If $n = 0$, then

$$a_0 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin^2(\theta)d\theta = \frac{1}{2}.$$

If $n \geq 1$, then

$$\begin{aligned}(n+1)a_n &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin((n+1)\theta)\sin(\theta)d\theta \\ &= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} (\cos(n\theta) - \cos((n+2)\theta))d\theta \\ &= \frac{1}{\pi}\left(\frac{\sin(\frac{n\pi}{2})}{n} - \frac{\sin((n+2)\frac{\pi}{2})}{n+2}\right).\end{aligned}$$

This quantity will be zero when n is even, so suppose $n = 2k+1$ where k is a nonnegative integer. Then

$$\begin{aligned}(2k+2)a_{2k+1} &= \frac{1}{\pi}\left(\frac{(-1)^k}{2k+1} - \frac{(-1)^{k+1}}{2k+3}\right) \\ &= \frac{4(-1)^k}{\pi}\left(\frac{k+1}{(2k+1)(2k+3)}\right).\end{aligned}$$

Using Examples 2.3.30 and 2.4.3, if x is unitarily equivalent to $\omega_1(\theta_0)$ then the Nth partial sum of the Fourier series of f is given by

$$(S_N f)(x) = \frac{1}{2} + \sum_{k=0}^{\left\lfloor \frac{N-1}{2} \right\rfloor} \frac{4(-1)^k}{\pi}\left(\frac{k+1}{(2k+1)(2k+3)}\right)\frac{\sin(2(k+1)\theta_0)}{\sin(\theta_0)}.$$

The continuity of $S_N f$ and the facts that $\lim\limits_{\theta_0\to 0}\frac{\sin(2(k+1)\theta_0)}{\sin(\theta_0)} = 2(k+1) = \lim\limits_{\theta_0\to\pi}\frac{\sin(2(k+1)\theta_0)}{\sin(\theta_0)}$ imply

$$(S_N f)(\pm e) = \frac{1}{2} + \sum_{k=0}^{\left\lfloor \frac{N-1}{2}\right\rfloor} \frac{8(-1)^k}{\pi}\left(\frac{(k+1)^2}{(2k+1)(2k+3)}\right).$$

Therefore, the Fourier series for f diverges at $\pm e$ since the N^{th} term doesn't go to zero as N tends to infinity.

Example 2.4.6: [CW]. Define a central function f on $SU(2)$ by $f(x) = \frac{2}{2-\mathrm{tr}(x)}$. Note that if x is unitarily equivalent to $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}$, where $\theta \in [0,\pi]$, then $f(x) = \frac{1}{1-\cos(\theta)}$. Proposition 2.2.4 implies

$$\begin{aligned}||f||^p_{L^p(SU(2)} &= \frac{2}{\pi}\int_0^\pi \left(\frac{1}{1-\cos(\theta)}\right)^p \sin^2(\theta)d\theta \\ &= \frac{2}{\pi}\int_0^\pi \left(\frac{1}{2\sin^2\left(\frac{\theta}{2}\right)}\right)^p \left(2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)^2 d\theta \\ &= \frac{2^{3-p}}{\pi}\int_0^\pi \sin^{2-2p}\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)d\theta.\end{aligned}$$

The integral will converge when $2-2p > -1$. Thus f belongs to $L^p(SU(2))$ for all $p < \frac{3}{2}$. Using the elementary identity

$$\frac{2\sin((m+1)\theta)\sin(\theta)}{1-\cos(\theta)} = D_{m+1}(\theta) + D_m(\theta)$$

where $D_m(\theta)$ is the mth Dirichlet kernel on $\top$, the Fourier coefficients of f can be computed using Proposition 2.2.4, Examples 2.3.30 and 2.4.3:

$$\begin{aligned}(m+1)a_m &= \int\limits_{SU(2)} f(y)\overline{\chi_m(y)}\mu(dy) \\ &= \frac{2}{\pi}\int_0^\pi \left(\frac{1}{1-\cos(\theta)}\right)\frac{\sin((m+1)\theta)}{\sin(\theta)}\sin^2(\theta)d\theta \\ &= \frac{1}{\pi}\int_0^\pi (D_{m+1}(\theta)+D_m(\theta))d\theta = 2.\end{aligned}$$

Thus the Fourier series for f at x, which is unitarily equivalent to $\omega_1(\theta_0)$, is

$$2\sum_{m=0}^{\infty}\frac{\sin((m+1)\theta_0)}{\sin(\theta_0)}.$$

We will show that this series diverges for all $\theta_0 \in [0,\pi]$ by slightly modifying the proof in [Ha], p. 125-126.

First, it is clear if $\theta_0 = 0$ or $\theta_0 = \pi$ the series must diverge because the argument of Example 2.4.5 shows that m-th term does not approach 0 as $m \to \infty$. Assume $0 < \theta_0 < \pi$. If $\theta_0 = \frac{p}{q}\pi$ where p and q are positive integers, then $0 < \theta_0 = \frac{p}{q}\pi < \pi$ implies $0 < p < q$. Without loss of generality, we may assume p and q are relatively prime so $s = q$ is the smallest positive integer s such that $s\left(\frac{p}{q}\right)$ is a positive integer. When $m = 0, 1, .., q-2$, the values of $\sin\left((m+1)\cdot\frac{p}{q}\pi\right)$ are

$$\left\{\sin\left(\frac{p}{q}\pi\right), \sin\left(\frac{2p}{q}\pi\right), ..., \sin\left(\frac{(q-1)p}{q}\pi\right)\right\},$$

when $m = q, q+1, ..., 2q-2$, the values of $\sin\left((m+1)\cdot\frac{p}{q}\pi\right)$ are

$$\left\{(-1)^p\sin\left(\frac{p}{q}\pi\right), (-1)^p\sin\left(\frac{2p}{q}\pi\right), ..., (-1)^p\sin\left(\frac{(q-1)p}{q}\pi\right)\right\},$$

and so forth, so the sequence is repeating and and not identically zero. Consequently the nth term doesn't tend to a finite limit as $m \to \infty$.

Now assume $\theta_0 = \alpha\pi$ where $0 < \alpha < 1$ is an irrational number. Since the sine function is bounded in absolute value by one, $\sin((m+1)\theta_0)$ cannot tend to $\pm\infty$ as $m \to \infty$. Assume that $\lim_{m\to\infty}\sin((m+1)\alpha\pi) = L$ where L is finite. Then

$$\begin{aligned} 0 &= \lim_{m\to\infty}\sin((m+1)\alpha\pi) - \sin(m\alpha\pi) \\ &= 2\lim_{m\to\infty}\cos\left(\left(m+\frac{1}{2}\right)\alpha\pi\right)\sin\left(\frac{\alpha\pi}{2}\right). \end{aligned}$$

Since $\sin\left(\frac{\alpha\pi}{2}\right) \neq 0$, $\lim\limits_{m\to\infty} \cos\left(\left(m+\frac{1}{2}\right)\alpha\pi\right) = 0$. Therefore $\left(m+\frac{1}{2}\right)\alpha\pi = \left(k_m+\frac{1}{2}\right)\pi + \epsilon_m$ where k_m is an integer depending on m and ϵ_m tends to zero as $m \to \infty$. Hence,

$$\begin{aligned}\alpha\pi &= \left(m+\frac{1}{2}\right)\alpha\pi - \left(m-\frac{1}{2}\right)\alpha\pi \\ &= \left(k_m+\frac{1}{2}\right)\pi + \epsilon_m - \left(k_{m-1}+\frac{1}{2}\right)\pi - \epsilon_{m-1} \\ &= (k_m - k_{m-1})\pi + (\epsilon_m - \epsilon_{m-1}).\end{aligned}$$

The left hand side is a fixed number in $(0,\pi)$ whereas the right hand side is an integer multiple of π added to a sequence that tends to zero as $m \to \infty$. Therefore, there exists an integer $m_0 > 0$ such that $k_m = k_{m-1}$ for all $m \geq m_0$. Thus

$$\begin{aligned}\alpha\pi &= \lim_{m\to\infty} \epsilon_m - \epsilon_{m-1} \\ &= 0,\end{aligned}$$

a contradiction, and we conclude the series $\sum_{m=0}^{\infty} \frac{\sin((m+1)\theta_0)}{\sin(\theta_0)}$ diverges for every $0 \leq \theta_0 \leq \pi$.

Remarks: 1. For $N \in \mathbb{N}$ and $0 < \theta_0 < \pi$ let $S_N = \sum_{m=0}^{N} \sin((m+1)\theta_0)$. To find a closed form for the partial sums note that

$$\begin{aligned}2\sin\left(\frac{\theta_0}{2}\right) S_N(\theta_0) &= \sum_{m=0}^{N} 2\sin\left(\frac{\theta_0}{2}\right)\sin((m+1)\theta_0) \\ &= \sum_{m=0}^{N} \cos\left(\left(m-\frac{1}{2}\right)\theta_0\right) - \cos\left(\left(m+\frac{3}{2}\right)\theta_0\right) \\ &= \cos\left(\frac{\theta_0}{2}\right) - \cos\left(\left(N+\frac{3}{2}\right)\theta_0\right).\end{aligned}$$

Hence,

$$S_N(\theta_0) = \frac{\cos\left(\frac{\theta_0}{2}\right) - \cos\left(\left(N+\frac{3}{2}\right)\theta_0\right)}{2\sin\left(\frac{\theta_0}{2}\right)}.$$

In classical harmonic analysis $S_N(\theta_0)$ is denoted by $\tilde{D}_{N+1}(\theta_0)$, which is the conjugate Dirichlet kernel on T. See $[Z]$, p. 2.

2. If $p = \frac{3}{2}$, then there is a central function whose Fourier series diverges almost everywhere. See [Me2] for details. If $p > \frac{3}{2}$, then $S_N f \to f$ a.e. whenever f is a central functon on $SU(2)$. See [Po] for details.

The following theorem is useful for proving convergence of Fourier series of smooth functions on $SU(2)$.

Theorem 2.4.7: [F], pp. 168-169. If $f \in C^2(SU(2))$, then

$$\sum_{m=0}^{\infty}(m+1)^{3/2}|||\widehat{f}(\pi^m)||| < \infty,$$

and

$$f(x) = \sum_{m=0}^{\infty}(m+1)\mathrm{tr}(\widehat{f}(\pi^m)\pi^m(x)),$$

where the series converges uniformly and absolutely on $SU(2)$.

2.4.3 Applications to the Poisson and Heat Equation on $SU(2)$. We will now present some applications to elliptic and parabolic partial differential equations on $SU(2)$. The computations in this section will be formal, and more rigorous arguments will be provided in the appendix.

Example 2.4.8: For $\alpha \in \mathbb{R}$, let q_α be the central function determined on $SU(2)\backslash\{\pm e\}$ by,

$$q_\alpha(\omega_1(\theta)) = \frac{\sin(\alpha(\pi-\theta))}{\sin(\theta)}$$

where

$$\omega_1(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \; 0 \leq \theta \leq \pi.$$

The function q_α is integrable because Proposition 2.2.4 implies

$$\begin{aligned}\|q_\alpha\|_{L^1(SU(2))} &= \frac{2}{\pi}\int_0^\pi |q_\alpha(\omega_1(\theta))|\sin^2(\theta)d\theta\\ &= \frac{2}{\pi}\int_0^\pi |\sin(\alpha(\pi-\theta))|\sin(\theta)d\theta\\ &\leq \frac{4}{\pi}.\end{aligned}$$

Using the identity

$$\sin(\alpha(\pi-\theta))\sin((m+1)\theta) = \frac{(m+1)\sin(\alpha\pi)}{(m+1)^2-\alpha^2},$$

the Fourier coefficients are computed using Proposition 2.2.4 and Examples 2.3.30 and 2.3.37 as follows:

$$\begin{aligned}(m+1)c_m &= \int_{SU(2)} q_\alpha(x)\chi_m(x)\mu(dx)\\ &= \frac{2}{\pi}\int_0^\pi \frac{\sin(\alpha(\pi-\theta))}{\sin(\theta)}\left(\frac{\sin((m+1)\theta)}{\sin(\theta)}\right)\sin^2(\theta)d\theta\\ &= \frac{2}{\pi}\int_0^\pi \sin(\alpha(\pi-\theta))\sin((m+1)\theta)d\theta\\ &= \frac{2(m+1)\sin(\alpha\pi)}{\pi((m+1)^2-\alpha^2)}.\end{aligned}$$

Example 2.4.9: Let $\lambda \in \mathbb{C}\setminus\{-m(m+2) : m = 0,1,2,...\}$, and let f be a continuous function on $SU(2)$. We shall solve the equation $-\triangle u + \lambda u = f$, where u is a C^2-function on $SU(2)$. Computing the Fourier transform of both sides of the partial differential equation at the representation π^m and using the properties of the Fourier transform in Proposition 2.3.36,

$$(m(m+2)+\lambda)\widehat{u}(\pi^m) = \widehat{f}(\pi^m).$$

Hence $\widehat{u}(m) = \frac{\widehat{f}(m)}{m(m+2)+\lambda}$ since $\lambda \neq -m(m+2)$. Multiplying both sides by the representation π^m evaluated at an arbitrary $x \in SU(2)$ yields

$$\widehat{u}(\pi^m)\pi^m(x) = \frac{\widehat{f}(\pi^m)\pi^m(x)}{m(m+2)+\lambda}.$$

Taking the trace of both sides gives

$$\operatorname{tr}(\widehat{u}(\pi^m)\pi^m(x)) = \operatorname{tr}\left(\frac{\widehat{f}(\pi^m)\pi_m(x)}{m(m+2)+\lambda}\right).$$

Lastly, we multiply both sides by $m+1$ and sum over all continuous irreducible unitary representations of $SU(2)$:

$$\begin{aligned}\sum_{m=0}^{\infty}(m+1)\operatorname{tr}(\widehat{u}(\pi^m)\pi^m(x)) &= \sum_{m=0}^{\infty}(m+1)\operatorname{tr}\left(\frac{\widehat{f}(\pi^m)\pi_m(x)}{m(m+2)+\lambda}\right)\\ &= \sum_{m=0}^{\infty}\left(\frac{m+1}{m(m+2)+\lambda}\right)\operatorname{tr}(\widehat{f}(\pi^m)\pi^m(x)).\end{aligned}$$

The left hand side of the above equation is $u(x)$ by Theorem 2.4.7 , so

$$u(x) = \sum_{m=0}^{\infty}\left(\frac{m+1}{m(m+2)+\lambda}\right)\operatorname{tr}(\widehat{f}(\pi^m)\pi^m(x)).$$

The series on the right converges uniformly on $SU(2)$. Setting $\lambda = 1-\alpha^2$ and assuming the Fourier series of q_α in Example 2.4.8 converges pointwise to q_α on $SU(2)$, we claim that

$$\sum_{m=0}^{\infty}\left(\frac{m+1}{m(m+2)+\lambda}\right)\operatorname{tr}(\widehat{f}(\pi^m)\pi^m(x)) = \frac{\pi}{2\sin(\alpha\pi)}\int_{SU(2)} q_\alpha(xy^{-1})f(y)\mu(dy).$$

To see this note that

$$u(x) = \sum_{m=0}^{\infty}(m+1)\left(\frac{1}{(m+1)^2-\alpha^2}\right)\operatorname{tr}(\widehat{f}(\pi^m)\pi^m(x))$$

$$
\begin{aligned}
&= \frac{\pi}{2\sin(\alpha\pi)} \sum_{m=0}^{\infty} (m+1)c_m \left(\int_{SU(2)} \chi_m(xy^{-1}) f(y) \mu(dy) \right) \\
&= \frac{\pi}{2\sin(\alpha\pi)} \int_{SU(2)} f(y) \left(\sum_{m=0}^{\infty} (m+1)c(m)\chi_m(xy^{-1}) \right) \mu(dy) \\
&= \frac{\pi}{2\sin(\alpha\pi)} \int\limits_{SU(2)} q_\alpha(xy^{-1}) f(y)\mu(dy).
\end{aligned}
$$

Example 2.4.10: Let f be a continuous function on $SU(2)$. We shall solve the equation $-\triangle u = f$, where u is a C^2- function on $SU(2)$. Let q be the central function determined on $SU(2)\backslash\{\pm e\}$ by,

$$
q(\omega_1(\theta)) = \frac{(\pi - \theta)\cos(\theta)}{2\sin(\theta)}, \ 0 < \theta < \pi.
$$

The function q is integrable because

$$
\begin{aligned}
\|q\|_{L^1(SU(2))} &= \frac{2}{\pi} \int_0^\pi |q(\omega_1(\theta))| \sin^2(\theta) d\theta \\
&= \frac{1}{\pi} \int_0^\pi |(\pi - \theta)\cos(\theta)| \sin(\theta) d\theta \\
&\leq 2.
\end{aligned}
$$

We will now show the Fourier coefficients are given by

$$
c_m = \begin{cases} \frac{1}{4} & m = 0 \\ \frac{2}{m(m+2)} & m \neq 0. \end{cases}
$$

If $m \neq 0$, then the Fourier coefficients of q are given by

$$
\begin{aligned}
(m+1)c_m &= \frac{2}{\pi} \int_0^\pi \frac{(\pi - \theta)\cos(\theta)}{2\sin(\theta)} \left(\frac{\sin((m+1)\theta)}{sin(\theta)} \right) \sin^2(\theta) d\theta \\
&= \frac{1}{\pi} \int_0^\pi (\pi - \theta)\cos(\theta)\sin((m+1)\theta) d\theta
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{\pi}(\pi-\theta)(\sin((m+2)\theta)+\sin(m\theta))d\theta \\
&= \frac{1}{\pi}\left(\frac{\pi}{m}+\frac{\pi}{m+2}\right) \\
&= \frac{2m+2}{m(m+2)} \\
&= \frac{2(m+1)}{m(m+2)}.
\end{aligned}
$$

If $m=0$, then the Fourier coefficient is

$$
\begin{aligned}
c_0 &= \frac{2}{\pi}\int_0^{\pi}\frac{(\pi-\theta)\cos(\theta)}{2\sin(\theta)}\sin^2(\theta)d\theta \\
&= \frac{1}{\pi}\int_0^{\pi}(\pi-\theta)\cos(\theta)\sin(\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{\pi}(\pi-\theta)\sin(2\theta)d\theta \\
&= \frac{1}{4}.
\end{aligned}
$$

If u is a C^2- solution of $-\triangle u = f$, then $\widehat{-\triangle u} = \hat{f}$ and using Proposition 2.3.36 implies $m(m+2)\widehat{u}(\pi^m) = \widehat{f}(\pi^m)$. When $m=0$ we get that $\int_{SU(2)} f(x)\mu(dx) = 0$ is a necessary condition for the PDE to have a solution. Next, $\widehat{u}(\pi^m) = \frac{\widehat{f}(\pi^m)}{m(m+2)}$. Multiplying both sides by the representation π^m evaluated at an arbitrary $x \in SU(2)$ yields

$$
\widehat{u}(\pi^m)\pi^m(x) = \frac{\widehat{f}(\pi^m)\pi_m(x)}{m(m+2)}.
$$

Taking the trace of both sides gives

$$
\mathrm{tr}(\widehat{u}(\pi^m)\pi^m(x)) = \mathrm{tr}\left(\frac{\widehat{f}(\pi^m)\pi_m(x)}{m(m+2)}\right).
$$

Lastly, we multiply both sides by $m+1$ and sum over all continuous irreducible unitary representations of $SU(2)$:

$$\sum_{m=0}^{\infty}(m+1)\operatorname{tr}(\widehat{u}(\pi^m)\pi^m(x)) = \sum_{m=1}^{\infty}(m+1)\operatorname{tr}\left(\frac{\widehat{f}(\pi^m)\pi^m(x)}{m(m+2)}\right)$$
$$= \sum_{m=1}^{\infty}\left(\frac{m+1}{m(m+2)}\right)\operatorname{tr}(\widehat{f}(\pi^m)\pi^m(x)).$$

The left hand side of the above equation is $u(x)$ by Theorem 2.4.7, so assuming the Fourier series of q converges pointwise to q on $SU(2)\setminus\{e\}$, we have

$$\begin{aligned}
u(x) &= \sum_{m=1}^{\infty}(m+1)\left(\frac{1}{m(m+2)}\right)\operatorname{tr}(\widehat{f}(\pi^m)\pi^m(x)) \\
&= \sum_{m=1}^{\infty}(m+1)c_m\left(\int_{SU(2)}\chi_m(xz^{-1})f(z)\mu(dz)\right) \\
&= \int_{SU(2)} f(z)\left(\sum_{m=1}^{\infty}(m+1)c_m\chi_m(xz^{-1})\right)\mu(dz) \\
&= \int_{SU(2)} q(xy^{-1})f(y)\mu(dy).
\end{aligned}$$

Example 2.4.11: We will now study the diffusion equation on $SU(2)$. Let f be a smooth function on $SU(2)$. We seek a function u continuous on $[0,\infty)\times SU(2)$, and C^2 on $(0,\infty)\times SU(2)$, such that

$$\frac{\partial u}{\partial t} = \triangle u \tag{17}$$

for all $t>0$, and $u(0,x)=f(x)$ for all $x\in SU(2)$.

To solve the problem we will use the Fourier method. Let $u_m(t,x)=e^{-m(m+2)t}v_m(x)$, where $v\in\tilde{M_m}$. Since $v_m\in\tilde{M_m}$, v_m satisfies $\triangle v_m=-m(m+2)v_m$ by Theorem 2.4.1. The Laplace operator is linear and time independent so the function u_m is a solution to the heat equation. To satisfy the initial condition we use an eigenfunction expansion:

$$u(t,x) = \sum_{m=0}^{\infty} e^{-m(m+2)t} v_m(x).$$

The initial condition implies

$$\sum_{m=0}^{\infty} v_m(x) = f(x).$$

If f is a C^2-function, then the convergence theorem 2.4.7 implies f is equal to its Fourier series on $SU(2)$:

$$f(x) = \sum_{m=0}^{\infty} (m+1)\mathrm{tr}(\widehat{f}(\pi^m)\pi^m(x));$$

here the series converges absolutely and uniformly on $SU(2)$. Choose $v_m(x) = (m+1)\mathrm{tr}(\widehat{f}(\pi^m)\pi^m(x))$. Then the solution to the Cauchy problem (17) is given by

$$u(t,x) = \sum_{m=0}^{\infty} (m+1)e^{-m(m+2)t}\mathrm{tr}(\widehat{f}(\pi^m)\pi^m(x)),$$

and the series converges absolutely and uniformly on $[0,\infty) \times SU(2)$. To prove uniqueness, assume u_1 and u_2 are solutions to the Cauchy problem. The linearity of the diffusion equation implies $w = u_1 - u_2$ also satisfies the diffusion equation with a homogeneous initial condition. Consider the energy function of w :

$$E(t) = \int_{SU(2)} w^2(t,x)\mu(dx), \ (t \geq 0).$$

It is clear $E(t) \geq 0$, and the initial condition implies $E(0) = 0$. Next,

$$\begin{aligned} E'(t) &= 2\int_{SU(2)} w(t,x)w_t(t,x)\mu(dx) \\ &= 2\int_{SU(2)} w(t,x)\triangle w(t,x)\mu(dx) \leq 0 \end{aligned}$$

since $-\triangle$ is a positive operator. Therefore E is decreasing and $E(0) = 0$, implies $E(t) \leq 0$. Hence, $E(t) = 0$ for all $t \geq 0$ which yields $w(t) = 0$ and implies the solution of the Cauchy problem is unique.

Remark: Another proof of the uniqueness of the Cauchy problem based on the maximum principle can be found in [F], pp. 176-177.

Definition 2.4.12: For $x \in SU(2)$ and $t > 0$, the heat kernel H on $SU(2)$ is given by

$$H(t,x) = \sum_{m=0}^{\infty}(m+1)e^{-m(m+2)t}\chi_m(x).$$

The series defining the heat kernel on $SU(2)$ is absolutely and uniformly convergent because $|\chi_m(x)| \leq m+1$, with equality when $x = e$. We will also need the following function.

Definition 2.4.13: [WW], p. 457. Let (z, τ) be a pair of complex numbers with $\mathrm{Im}(\tau) < 0$. The theta function, denoted by Θ_3, is defined as $\Theta_3(z,\tau) = 1 + 2\sum_{j=1}^{\infty} e^{-i\pi j^2\tau}\cos(2jz)$. If $(z,\tau) = \left(\frac{\theta}{2}, \frac{-it}{\pi}\right)$, then $\Theta_3\left(\frac{\theta}{2}, \frac{-it}{\pi}\right) = 1 + 2\sum_{m=1}^{\infty} e^{-m^2t}\cos(m\theta)$. From the Poisson summation formula ([F], pp. 174-175,180-181), we also have the identity

$$\Theta_3\left(\frac{\theta}{2}, \frac{-it}{\pi}\right) = 1 + 2\sum_{m=1}^{\infty} e^{-m^2t}\cos(m\theta) = \sqrt{\frac{\pi}{t}}\sum_{k=-\infty}^{\infty} e^{\frac{-(\theta-2k\pi)^2}{4t}}.$$

Since characters are central functions, if $y \in SU(2)$ is unitarily equivalent to $\omega_1(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ then

$$\begin{aligned} H(t,y) &= \sum_{m=0}^{\infty}(m+1)e^{-m(m+2)t}\chi_m(\omega_1(\theta)) \qquad (18) \\ &= \sum_{m=0}^{\infty}(m+1)e^{-m(m+2)t}\frac{\sin((m+1)\theta))}{\sin(\theta)} \end{aligned}$$

$$= \frac{-e^t}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sum_{m=1}^{\infty} e^{-m^2 t}\cos(m\theta)\right)$$
$$= \frac{-e^t}{2\sin(\theta)}\frac{\partial}{\partial\theta}\left(1+2\sum_{m=1}^{\infty} e^{-m^2 t}\cos(m\theta)\right)$$
$$= \frac{-e^t}{2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sqrt{\frac{\pi}{t}}\sum_{k=-\infty}^{\infty} e^{\frac{-(\theta-2k\pi)^2}{4t}}\right)$$
$$= \frac{-e^t}{2\sin(\theta)}\frac{\partial}{\partial\theta}\Theta_3\left(\frac{\theta}{2},\frac{-it}{\pi}\right).$$

To find an integral representation for the solution u to the Cauchy problem (17) and (18), consider the identity

$$H(t,xy^{-1})f(y) = \sum_{m=0}^{\infty}(m+1)e^{-m(m+2)t}\chi_m(xy^{-1})f(y).$$

The absolute convergence of the heat kernel and boundedness of f allows term-by-term integration of the series. This yields

$$\int_{SU(2)} H(t,xy^{-1})f(y)\mu(dy) = \sum_{m=0}^{\infty}(m+1)e^{-m(m+2)t}\left(\int_{SU(2)}\chi_m(xy^{-1})f(y)\mu(dy)\right)$$
$$= \sum_{m=0}^{\infty}(m+1)e^{-m(m+2)t}\mathrm{tr}(\widehat{f}(\pi^m)\pi^m(x))$$
$$= u(t,x).$$

A change of variables yields the integral representation for solutions to (18) and (19) :

$$u(t,x) = \int_{SU(2)} H(t,y)f(xy^{-1})\mu(dy).$$

Remark: In spherical coordinates, solutions to (18) and (19) can be expressed using Proposition 2.2.4 and (19) as follows:

$$u(x(\phi_0,\theta_0,\psi_0),t) = \int_{SU(2)} H(t,y)f(xy^{-1})\mu(dy)$$

$$= \frac{1}{2\pi^2} \int_0^{\pi} \int_0^{\pi} \int_0^{2\pi} H(t, y(\phi, \theta, \psi)) f(x(\phi_0, \theta_0, \psi_0) y^{-1}(\phi, \theta, \psi))$$
$$\times \sin^2(\theta) \sin(\phi) d\psi d\phi d\theta$$
$$= \frac{1}{2\pi^2} \int_0^{\pi} \int_0^{\pi} \int_0^{2\pi} \frac{-e^t}{2\sin(\theta)} \frac{\partial}{\partial\theta} \Theta_3 \left(\frac{\theta}{2}, \frac{-it}{\pi} \right)$$
$$\times f(x(\phi_0, \theta_0, \psi_0) y^{-1}(\phi, \theta, \psi)) \sin^2(\theta) \sin(\phi) d\psi d\phi d\theta$$
$$= \frac{-e^t}{\pi} \int_0^{\pi} \sin(\theta) \frac{\partial}{\partial\theta} \Theta_3 \left(\frac{\theta}{2}, \frac{-it}{\pi} \right)$$
$$\times \left(\frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} f(x(\phi_0, \theta_0, \psi_0) y^{-1}(\phi, \theta, \psi)) \sin(\phi) d\psi d\phi \right) d\theta$$
$$= \frac{-e^t}{\pi} \int_0^{\pi} \sin(\theta) \frac{\partial}{\partial\theta} \Theta_3 \left(\frac{\theta}{2}, \frac{-it}{\pi} \right) [Q_x f](\theta) d\theta$$

where $[Q_x f](\theta) = \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} f(x(\phi_0, \theta_0, \psi_0) y^{-1}(\phi, \theta, \psi) \sin(\phi) d\psi d\phi$ which was defined in Definition 2.2.8.

3 CONVERGENCE THEOREMS ON $SU(2)$

In this section, we will prove the main results of the thesis. We will first demonstrate that central functions in $\mathrm{Lip}_1(SU(2))$ have uniformly convergent Fourier series, but Holder continuous functions on $SU(2)$ need not have a pointwise convergent Fourier series.

3.1 ELEMENTARY CONVERGENCE OF FOURIER SERIES ON T.

Let us recall some well-known theorems on T and a classical example.

Theorem 3.1: (Riemann-Lebesgue) Let $f \in L^1(\mathrm{T})$, then $\lim_{|n|\to\infty} \hat{f}(n) = 0$.

Proof: See [K], p.13.

Theorem 3.2: If $f \in \mathrm{BV}(\mathrm{T})$, then $\hat{f}(n) = \mathcal{O}\left(\frac{1}{|n|}\right)$. If $f \in \mathrm{Lip}_\alpha(\mathrm{T})$ for some $\alpha \in (0,1]$, then $\hat{f}(n) = \mathcal{O}\left(\frac{1}{|n|^\alpha}\right)$. If f is absolutely continuous on T, then $\hat{f}(n) = o\left(\frac{1}{|n|}\right)$.

Proof: See [K], pp. 24-25.

Theorem 3.3: If $f \in \mathrm{Lip}_\alpha(\mathrm{T})$ for some $\alpha \in (0,1]$ then $S_N f \to f$ uniformly.

Proof: See [Z], p.63.

In fact we have the following theorem.

Theorem 3.4: (Bernstein) If $f \in \mathrm{Lip}_\alpha(\mathrm{T})$ for some $\alpha > \frac{1}{2}$ then $S_N f \to f$ absolutely.

Proof: See [K], p.32.

Theorem 3.5: Let f be of bounded variation on T; then $\{(S_N f)(x)\}_{N=1}^{\infty}$ converges to $\frac{f(x^+)+f(x^-)}{2}$ and, in particular, to $f(x)$ at every point of continuity. The convergence is uniform on closed intervals of continuity of f.

Proof: See [K], p. 53.

Remark: In some texts, such as [Z], p. 57 and [Pn], p. 27, this theorem is attributed to Dirichlet and Jordan.

The following function space and norm will be used in our main result.

Definition 3.6: The space $A(\mathrm{T})$ is the space of continuous functions on T having an absolutely convergent Fourier series. The norm we will use for $A(\mathrm{T})$ is given by $\|f\|_{A(\mathrm{T})} = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|$.

Example 3.7: For $\alpha \in (0,1)$ and $x \in \mathrm{T}$, let $f_\alpha(x) = \sum_{k=0}^{\infty} \frac{1}{2^{\alpha k}} e^{i2^k x}$ and observe that the series converges absolutely and uniformly on T. We will prove that $f_\alpha \in \mathrm{Lip}_\alpha(\mathrm{T})$ and the Fourier coefficients of f_α tend to zero as $N \to \infty$ at precisely the rate $\mathcal{O}\left(\frac{1}{N^\alpha}\right)$. To begin, consider the difference

$$
\begin{aligned}
f_\alpha(x+h) - f_\alpha(x) &= \sum_{k=0}^{\infty} \frac{1}{2^{\alpha k}} e^{i2^k(x+h)} - \sum_{k=0}^{\infty} \frac{1}{2^{\alpha k}} e^{i2^k x} \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{\alpha k}} \left(e^{i2^k(x+h)} - e^{i2^k x} \right)
\end{aligned}
$$

where x and $x+h$ belong to T with $h \neq 0$. We will split this sum into two pieces. Let

$$
S_1 = \sum_{2^k \leq \nu} \frac{1}{2^{\alpha k}} \left(e^{i2^k(x+h)} - e^{i2^k x} \right)
$$

and

$$
S_2 = \sum_{2^k > \nu} \frac{1}{2^{\alpha k}} \left(e^{i2^k(x+h)} - e^{i2^k x} \right)
$$

where $\nu \in \mathbb{R}^+$ will be chosen later. To estimate the finite sum S_1, note that for every $\theta \in \mathrm{T}$ we have $|e^{i\theta} - 1| = 2\left|\sin\left(\frac{\theta}{2}\right)\right| \leq |\theta|$. Therefore

$$
\begin{aligned}
|S_1| &= \left| \sum_{2^k \leq \nu} \frac{1}{2^{\alpha k}} e^{i2^k x} \left(e^{i2^k h} - 1 \right) \right| \\
&\leq \sum_{2^k \leq \nu} \frac{1}{2^{\alpha k}} |e^{i2^k h} - 1| \\
&\leq \sum_{2^k \leq \nu} \frac{1}{2^{\alpha k}} |2^k h|
\end{aligned}
$$

$$= |h| \sum_{2^k \le \nu} \left(\frac{1}{2^{\alpha-1}} \right)^k .$$

Let k_0 be the natural number for which $2^{k_0} \le \nu < 2^{k_0+1}$. Then

$$\begin{aligned} |S_1| &\le |h| \sum_{k=0}^{k_0} \left(\frac{1}{2^{\alpha-1}} \right)^k \\ &= \frac{|h| \left[1 - \left(\frac{1}{2^{(\alpha-1)}} \right)^{k_0+1} \right]}{1 - \frac{1}{2^{\alpha-1}}} \\ &\le \frac{|h| \nu^{(1-\alpha)}}{1 - 2^{\alpha-1}}. \end{aligned}$$

The sum S_2 will be bounded above by a geometric series as follows. By the triangle inequality,

$$\begin{aligned} |S_2| &= \left| \sum_{2^k > \nu} \frac{1}{2^{\alpha k}} \left(e^{i2^k(x+h)} - e^{i2^k x} \right) \right| \\ &\le \sum_{2^k > \nu} \frac{1}{2^{\alpha k}} |e^{i2^k(x+h)} - e^{i2^k x}| \\ &\le 2 \sum_{2^k > \nu} \frac{1}{2^{\alpha k}} \\ &= 2 \sum_{k=k_0+1}^{\infty} \left(\frac{1}{2^\alpha} \right)^k \\ &= 2 \left(\frac{1}{2^\alpha} \right)^{k_0+1} \left(\frac{1}{1 - \frac{1}{2^\alpha}} \right) \\ &= \frac{2^{1-\alpha k_0}}{2^\alpha - 1} \\ &\le \frac{2^{1+\alpha} \nu^{-\alpha}}{2^\alpha - 1}. \end{aligned}$$

If we choose $\nu = \frac{1}{|h|}$, then

$$\begin{aligned} |f_\alpha(x+h) - f_\alpha(x)| &\le \left(\frac{1}{1 - 2^{\alpha-1}} + \frac{2^{1+\alpha}}{2^\alpha - 1} \right) |h|^\alpha \\ &= M_\alpha |h|^\alpha. \end{aligned}$$

Therefore $f_\alpha \in \mathrm{Lip}_\alpha(\mathrm{T})$ for $\alpha \in (0,1)$. The Fourier coefficient for f_α at $n \in \mathbb{Z}$ is zero unless $n = 2^k$ for some nonnegative integer k, and in this case uniform convergence yields

$$\begin{aligned}
\widehat{f_\alpha}(2^k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i2^k t}\left(\sum_{j=0}^{\infty}\frac{1}{2^{\alpha j}}e^{i2^j t}\right)dt \\
&= \frac{1}{2\pi}\sum_{j=0}^{\infty}\frac{1}{2^{\alpha j}}\int_{-\pi}^{\pi} e^{-i(2^k-2^j)t}dt \\
&= \frac{1}{2^{\alpha k}}.
\end{aligned}$$

Hence, the Fourier coefficients $\hat{f}_\alpha$ tend to zero as $N \to \infty$ at a rate of precisely $\mathcal{O}\left(\frac{1}{N^\alpha}\right)$.

Remark: Notice that M_α is undefined when $\alpha \in \{0,1\}$. The function f_α can be shown to be nowhere differentiable and is due to Weierstrass. We refer to [SS], pp. 114-118 for a proof.

3.2 CONVERGENCE FOR CENTRAL FUNCTIONS ON $SU(2)$.

We will state and prove the main results of this thesis. We begin by showing the convergence problem of Fourier series for central functions on $SU(2)$ reduces to a convergence problem for Fourier series on T that we already know how to solve, and to analyzing the behavior of a Fourier integral on T. This will be a recurring theme throughout this section.

Theorem 3.8: If f is a central function which belongs to $\mathrm{Lip}_\alpha(SU(2))$ for some $\alpha \in (0,1)$, then:

(a) the Fourier series for f converges uniformly outside every neighborhood of $\pm e$;

(b) the Fourier series for f need not converge pointwise.

If f is a central function which belongs to $\mathrm{Lip}_1(SU(2))$, then the Fourier series for f converges uniformly.

Proof: To prove (a), recall from Remark 3 following Example 2.3.41 that the Nth Fourier partial sum of f is given by

$$(S_N f)(x) = (S_N f)(\omega_1(\theta_0)) = \sum_{m=0}^{N} (m+1)c_m \frac{\sin((m+1)\theta_0)}{\sin(\theta_0)}.$$

Here $x = x(\phi_0, \theta_0, \psi_0)$ is the spherical coordinate expression for x,

$$(m+1)c_m = \frac{2}{\pi}\int_0^{\pi} \sin((m+1)\theta) f(\omega_1(\theta)) \sin(\theta) d\theta,$$

and $\omega_1(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$. In that same Remark, we derived an integral formula for the Nth Fourier partial sum of f:

$$(S_N f)(\omega_1(\theta_0)) = \frac{1}{2\pi \sin(\theta_0)} \int_{-\pi}^{\pi} f(\omega_1(\theta)) \sin(\theta) D_{N+1}(\theta_0 - \theta) d\theta.$$

If the central function f belongs to $\mathrm{Lip}_\alpha(SU(2))$ for some $\alpha \in (0,1]$, then there exists a function F defined on $[-1,1]$ such that $f(\omega_1(\theta)) = F(\cos(\theta))$ for all $\theta \in [0,\pi]$. Hence, there exists a constant $M > 0$ such that

$$\begin{aligned} |F(\cos(\theta_2)) - F(\cos(\theta_1))| &= |f(\omega_1(\theta_2)) - f(\omega_1(\theta_1))| \\ &\leq M d^\alpha(\omega_1(\theta_2), \omega_1(\theta_1))) \\ &= M\left(2\left|\sin\left(\frac{\theta_2 - \theta_1}{2}\right)\right|\right)^\alpha \\ &\leq M|\theta_2 - \theta_1|^\alpha \end{aligned}$$

for all $\theta_1, \theta_2 \in [0,\pi]$. Therefore $\gamma(\theta) = F(\cos(\theta))$ belongs to $\mathrm{Lip}_\alpha(\mathrm{T})$ whenever $f \in \mathrm{Lip}_\alpha(SU(2))$ for $\alpha \in (0,1]$ and f is central. Since $\frac{(S_N\gamma)(\theta_0)}{\sin(\theta_0)} = (S_{N-1}f)(\omega_1(\theta_0))$ for

$0 < \theta_0 < \pi$, by Theorem 3.3 $S_N f \to f$ as $N \to \infty$ uniformly on intervals for $\theta_0 \in [\epsilon, \pi - \epsilon]$ for every $\epsilon > 0$. This completes the proof of (a).

To prove (b), we will examine the case where $\theta_0 \in \{0, \pi\}$, i.e. when $\omega_1(\theta_0) = \pm e$ respectively. If $\theta_0 = 0$, then $\chi_m(e) = m + 1$ and

$$\begin{aligned}
(S_N f)(e) &= \sum_{m=0}^{N} (m+1)^2 c_m \\
&= \sum_{m=0}^{N} (m+1) \left(\frac{2}{\pi} \int_0^{\pi} \sin((m+1)\theta) f(\omega_1(\theta)) \sin(\theta) d\theta \right) \\
&= \frac{2}{\pi} \int_0^{\pi} f(\omega_1(\theta)) \sin(\theta) \sum_{m=0}^{N} (m+1) \sin((m+1)\theta) d\theta \\
&= \frac{-1}{\pi} \int_0^{\pi} D'_{N+1}(\theta) f(\omega_1(\theta)) \sin(\theta) d\theta \\
&= \frac{1}{\pi} \int_0^{\pi} \cos^2\left(\frac{\theta}{2}\right) D_{N+1}(\theta) f(\omega_1(\theta)) d\theta \\
&\quad - \frac{\left(N + \frac{3}{2}\right)}{\pi} \int_0^{\pi} \cos\left(\frac{\theta}{2}\right) \cos\left(\left(N + \frac{3}{2}\right)\theta\right) f(\omega_1(\theta)) d\theta.
\end{aligned}$$

Define

$$J_1 = \frac{1}{\pi} \int_0^{\pi} \cos^2\left(\frac{\theta}{2}\right) D_{N+1}(\theta) f(\omega_1(\theta)) d\theta$$

and

$$J_2 = \frac{\left(N + \frac{3}{2}\right)}{\pi} \int_0^{\pi} \cos\left(\frac{\theta}{2}\right) \cos\left(\left(N + \frac{3}{2}\right)\theta\right) f(\omega_1(\theta)) d\theta.$$

The strategy for showing convergence or divergence of the Fourier partial sums on $SU(2)$ is broken down into two steps. The first step is to observe that J_1 converges to $f(e)$ as $N \to \infty$ by Theorem 3.3 and the proof of part (a). The second step is to show that J_2 can diverge as $N \to \infty$. Using the trigonometric identity

$$\cos\left(\frac{\theta}{2}\right) \cos\left(\left(N + \frac{3}{2}\right)\theta\right) = \frac{1}{2} \left(\cos((N+2)\theta) + \cos((N+1)\theta)\right),$$

we may rewrite J_2 as

$$J_2 = \frac{\left(N+\frac{3}{2}\right)}{2\pi}\int_0^\pi \left(\cos((N+2)\theta)+\cos((N+1)\theta)\right) f(\omega_1(\theta))d\theta.$$

Take $f(\omega_1(\theta)) = f_\alpha(\theta)$ in Example 3.7. We observe J_2 diverges as $N\to\infty$ because the Fourier cosine coefficients of f_α go to zero at a rate of precisely $\mathcal{O}\left(\frac{1}{N^\alpha}\right)$ as $N\to\infty$ and at most one of the two consecutative Fourier cosine coefficients of f_α whose sum is J_2 can be nonzero. This completes the proof of (b).

To prove (c) let f be a central function which belongs to $\mathrm{Lip}_1(SU(2))$. We first need to show that $(S_Nf)(\pm e)\to f(\pm e)$ as $N\to\infty$. This can be deduced from our previous work in an elementary way. When $\theta_0 = e$, J_1 in the proof of (b) tends to $f(e)$ as $N\to\infty$ by Theorem 3.3, and J_2 in the proof of (b) tends to zero as $N\to\infty$, by Theorem 3.2. A similar argument holds for $\theta_0=-e$, and so $(S_Nf)(\pm e)\to f(\pm e)$ as $N\to\infty$.

In the rest of the proof of Theorem 3.8, we will assume the central function f satisfies $|f(x)-f(y)|\le Md(x,y)$ for all $x,y\in SU(2)$ and some real number $M>0$. To complete the proof it suffices to show that $(S_Nf)(x)\to f(x)$ uniformly for $x\in SU(2)\setminus\{\pm e\}$. This clearly follows from the next sequence of four lemmas.

Lemma 3.9: If $0<\theta_0<\pi$ and $N\ge 1$ then

$$(S_Nf)(\omega_1(\theta_0)) - f(\omega_1(\theta_0)) = \int_0^\pi g(\theta_0,\theta)\sin\left(\left(N+\frac{3}{2}\right)\theta\right)d\theta$$

where

$$g(\theta_0,\theta) = \frac{(\Delta f)(\theta_0,\theta)\sin(\theta_0+\theta) + (\Delta f)(\theta_0,-\theta)\sin(\theta_0-\theta)}{2\pi\sin(\theta_0)\sin\left(\frac{\theta}{2}\right)},$$

and

$$(\Delta f)(\theta_0,\theta) = f(\omega_1(\theta_0+\theta)) - f(\omega_1(\theta_0).$$

Lemma 3.10: If $0 < \theta_0 < \pi$ and $0 < \theta < \pi$, then $|g(\theta_0, \theta)| \leq \frac{4M}{\pi}$.

Lemma 3.11: Let $\theta, \theta_0 \in (0, \pi)$ and $h \in \left(0, \frac{\pi}{2}\right)$. Then $g(\theta_0, \theta + h) - g(\theta_0, \theta) = A(\theta_0, \theta, h) + B(\theta_0, \theta, h)$ where:

(1) $|A(\theta_0, \theta, h)| \leq \frac{4M}{\pi}$;

(2) $\lim_{N \to \infty} A(\theta_0, \theta, h_N) = 0$ if $\{h_N\}_{N=1}^{\infty}$ is any sequence in $\left(0, \frac{\pi}{2}\right)$ converging to zero;

(3) $|B(\theta_0, \theta, h)| \leq \begin{cases} \left(\frac{5}{\pi} + \frac{3}{2}\right) Mh + \left(1 + \frac{3\pi}{8}\right) \frac{Mh}{\theta} & \text{if } 0 < 2h \leq \theta < \pi, \\ \frac{12M}{\pi} & \text{if } 0 < \theta < 2h < \pi. \end{cases}$

Lemma 3.12: Let $0 < \theta_0 < \pi$ and $h_N = \frac{\pi}{N + \frac{3}{2}}$ for $N = 1, 2, 3, \ldots$ For every $\epsilon > 0$, there exists $N_0 = N_0(\epsilon)$ such that

$$2|(S_N f)(\omega_1(\theta_0)) - f(\omega_1(\theta_0))| \leq \left(\frac{4M}{\pi} + \pi\right)\epsilon + \left(\frac{20}{\pi} + 5 + \frac{3\pi}{2}\right) Mh_N + \left(1 + \frac{3\pi}{8}\right) Mh_N \ln\left(\frac{\pi}{h_N}\right)$$

for all $0 < \theta_0 < \pi$ and all $N \geq N_0$.

The proofs of these four lemmas are rather technical. Moreover, the result for central functions in $\text{Lip}_1(SU(2))$ is a special case of a more general result which will appear later in this dissertation: The Fourier series of any function in $\text{Lip}_1(SU(2))$ converges uniformly on $SU(2)$. Consequently, we will relegate the proofs of Lemmas 3.9 through 3.12 to the Appendix.

Remark: Part (a) of Theorem 3.8 is a special case of the Jacobi Equiconvergence Theorem. See [Sz], p. 246 and pp.253-256.

3.3 CONVERGENCE FOR NON-CENTRAL FUNCTIONS ON $SU(2)$

We will now study pointwise convergence of Fourier series on $SU(2)$, and we will need the following function space for our next result.

Definition 3.13: [M], p.156. Let $\Omega \subset \mathbb{R}$ be a domain. The Sobolev space $W^{1,p}(\Omega)$ consists of functions which belong to $L^p(\Omega)$ and whose weak derivative also belongs to $L^p(\Omega)$.

We will also make use of the following result.

Theorem 3.14: [Le], p. 223. Let $\Omega \subset \mathbb{R}$ be an open bounded set and let $u : \Omega \to \mathbb{R}$. Then $u \in W^{1,1}(\Omega)$ if and only if it admits an absolutely continuous representative $v : \Omega \to \mathbb{R}$ and $v' \in L^1(\Omega)$.

Remarks:

1. Theorem 3.14 is actually a corollary of a more general result. See [Le], pp. 222-223 for this result and the proof.

2. We will need to extend Theorem 3.14 to the closed interval $\overline{\Omega} = [0, \pi]$. Given the conditions of Theorem 3.14 with $\Omega = (0, \pi)$, note that

$$v(\theta) = v(\theta_0) + \int_{\theta_0}^{\theta} v'(\phi) d\phi \qquad (*)$$

for all $\theta_0, \theta \in (0, \pi)$. If θ_0 is fixed then $\int_{\theta_0}^{\theta} v'(\phi) d\phi$ is defined for $\theta \in [0, \pi]$ because v' is integrable on $(0, \pi)$. Consequently, $v(0)$ and $v(\pi)$ are well-defined using $(*)$. The right hand side of (*) is also a continuous function of $\theta \in [0, \pi]$. To see this let $\{\theta_j\}_{j=1}^{\infty}$ be a sequence in $[0, \pi]$ which converges to θ. Then

$$\begin{aligned} v(\theta_j) &= v(\theta_0) + \int_{\theta_0}^{\theta_j} v'(\phi) d\phi \\ &= v(\theta_0) + \int_{0}^{\pi} \chi_{(\theta_0, \theta_j)}(\phi) v'(\phi) d\phi. \end{aligned}$$

Note that $|\chi_{(\theta_0,\theta_j)}(\phi)v'(\phi)| \leq |v'(\phi)|$ for all $j \geq 1$ and all $\phi \in [0,\pi]$, so by the Lebesgue dominated convergence theorem

$$\begin{aligned}\lim_{j\to\infty} v(\theta_j) &= v(\theta_0) + \lim_{j\to\infty} \int_{\theta_0}^{\theta_j} v'(\phi)d\phi \\ &= v(\theta_0) + \int_{\theta_0}^{\theta} v'(\phi)d\phi \\ &= v(\theta).\end{aligned}$$

Theorem 3.15: Let $f \in L^1(SU(2))$ and $x \in SU(2)$. If

(1) the function $\theta \longmapsto [Q_x f](\theta)$ belongs to $W^{1,1}(0,\pi)$, and

(2) the function $\theta \longmapsto \frac{[Q_x f](\theta)-[Q_x f](0)}{\theta}$ belongs to $L^1[0,\pi]$,

then $\lim_{N\to\infty} S_N f(x) = [Q_x f](0)$. In particular, if f is continuous at x then $\lim_{N\to\infty} S_N f(x) = f(x)$.

Proof: By Theorem 3.14 and the subsequent remarks, we may assume the function $\theta \longmapsto [Q_x f](\theta)$ is absolutely continuous on $[0,\pi]$. Making use of Remark 2 just before the end of section 2.3, we consider the difference

$$\begin{aligned}(S_N f)(x) - [Q_x f](0) &= \frac{-1}{\pi}\int_0^\pi D'_{N+1}(\theta)\sin(\theta)([Q_x f](\theta) - [Q_x f](0))d\theta \\ &= \frac{1}{\pi}\int_0^\pi D_{N+1}(\theta)\frac{d}{d\theta}\left(\sin(\theta)([Q_x f](\theta) - [Q_x f](0))\right)d\theta \\ &= \frac{1}{\pi}\int_0^\pi D_{N+1}(\theta)\cos(\theta)([Q_x f](\theta) - [Q_x f](0))d\theta \\ &+ \frac{1}{\pi}\int_0^\pi D_{N+1}(\theta)\sin(\theta)\frac{d}{d\theta}([Q_x f](\theta))d\theta\end{aligned}$$

using (1). Let

$$I_1 = \frac{1}{\pi} \int_0^{\pi} D_{N+1}(\theta) \cos(\theta)([Q_x f](\theta) - [Q_x f](0)) d\theta$$

and

$$I_2 = \frac{1}{\pi} \int_0^{\pi} D_{N+1}(\theta) \sin(\theta) \frac{d}{d\theta}([Q_x f](\theta)) d\theta.$$

As $N \to \infty$, I_1 tends to 0 by Theorem 3.5. Another way to see this is to first extend $\theta \mapsto ([Q_x f](\theta) - [Q_x f](0))$ to an even function on $\mathrm{T} = [-\pi, \pi)$. Note that

$$\begin{aligned} I_1 &= \frac{1}{\pi} \int_0^{\pi} D_{N+1}(\theta) \cos(\theta)([Q_x f](\theta) - [Q_x f](0)) d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin((N+1)\theta) \frac{\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \cos\left(\frac{\theta}{2}\right) \left(\frac{[Q_x f](\theta) - [Q_x f](0)}{\left(\frac{\theta}{2}\right)} \right) d\theta \\ &+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos((N+1)\theta)([Q_x f](\theta) - [Q_x f](0)) d\theta. \end{aligned}$$

Since

$$\theta \mapsto \frac{\theta}{2\sin\left(\frac{\theta}{2}\right)} \cos\left(\frac{\theta}{2}\right) \left(\frac{[Q_x f](\theta) - [Q_x f](0)}{\left(\frac{\theta}{2}\right)} \right)$$

and

$$\theta \mapsto Q_x f](\theta) - [Q_x f](0)$$

are L^1-functions on T by (2) and (1), respectively, it follows from Theorem 3.1 that $I_1 \to 0$ as $N \to \infty$.

To show that I_2 tends to 0 as $N \to \infty$, Theorem 3.14 implies $\theta \mapsto [Q_x f](\theta)$ is equal a.e. to an absolutely continuous function $v_x : [0, \pi] \to \mathbb{R}$ such that $v_x' \in L^1[0, \pi]$.

Extend $\theta \mapsto \sin(\theta)v_x'(\theta)$ to an even function on $[-\pi, \pi]$, and note that

$$\begin{aligned}
I_2 &= \frac{1}{\pi}\int_0^\pi D_{N+1}(\theta)\sin(\theta)\frac{d}{d\theta}([Q_x f](\theta))d\theta \\
&= \frac{1}{\pi}\int_0^\pi D_{N+1}(\theta)\sin(\theta)v_x'(\theta)d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^\pi D_{N+1}(\theta)\sin(\theta)v_x'(\theta)d\theta \\
&= \frac{1}{\pi}\int_{-\pi}^\pi \sin\left(\left(N+\frac{3}{2}\right)\theta\right)\cos\left(\frac{\theta}{2}\right)v_x'(\theta)d\theta \\
&= \frac{1}{\pi}\int_{-\pi}^\pi \sin((N+1)\theta)\cos^2\left(\frac{\theta}{2}\right)v_x'(\theta)d\theta \\
&+ \frac{1}{\pi}\int_{-\pi}^\pi \cos((N+1)\theta)\sin(\theta)v_x'(\theta)d\theta.
\end{aligned}$$

Since $\theta \mapsto \cos^2\left(\frac{\theta}{2}\right)v_x'(\theta)$ and $\theta \mapsto \sin(\theta)v_x'(\theta)$ belong to $L^1(\mathrm{T})$, I_2 tends to 0 as $N \to \infty$, by Theorem 3.1. The second conclusion follows from property (c) of Proposition 2.2.11 for the function $\theta \longmapsto [Q_x f](\theta)$.

Remark: A similar version of the previous result was proven in $[QHMS]$, pp. 144-146, but using Clifford algebra methods. Our proof requires only the expression for the Dirichlet kernel on $SU(2)$ and some elementary convergence theorems for Fourier series on T.

Corollary 3.16: Let $f \in C(SU(2))$ and $x \in SU(2)$. If

(1) the function $\theta \longmapsto [Q_x f](\theta)$ belongs to $W^{1,1}[0, \pi]$, and

(2) the function $\theta \longmapsto \frac{[Q_x f](\theta) - [Q_x f](0)}{\theta}$ belongs to $L^1[0, \pi]$,

then $\lim_{N\to\infty} S_N f(x) = f(x)$.

Corollary 3.17: If $f \in \mathrm{Lip}_1(SU(2))$ then $\lim_{N\to\infty} S_N f(x) = f(x)$ for all $x \in SU(2)$.

Proof: Let $f \in \mathrm{Lip}_1(SU(2))$ and $x \in SU(2)$. By property (d) of Proposition 2.2.11 the function $\theta \longmapsto [Q_x f](\theta)$ is in $\mathrm{Lip}_1([0,\pi])$. Pointwise convergence then follows from Corollary 3.16.

Remark: Corollary 3.17 was first proved in [Ca] using much different methods including non-orthogonal coordinates on S^3.

We will now develop the tools necessary to strengthen Corollary 3.17.

Theorem 3.18: Let C be a compact three-dimensional interval in $\mathbb{R}^3$ and let $F : C \to \mathbb{R}$ be Lipschitz continuous with Lipschitz constant K. Then for every $\epsilon > 0$ there exists a constant $K_3 > 0$, independent of C, and a continuously differentiable function $\overline{F_\epsilon} : C \to \mathbb{R}$ such that

$$(a)\,|\left\{x \in C :\ D\overline{F_\epsilon}(x) \neq (DF)(x)\right\}| \leq \epsilon,$$

$$(b)\,\sup_{x \in C}|\overline{F_\epsilon}(x) - F(x)| \leq \epsilon,\ \text{and}$$

$$(c)\,\sup_{x \in C}||D\overline{F_\epsilon}(x)|| \leq K_3 K.$$

Proof: Let $\delta > 0$. By the approximation theorem for Lipschitz continuous functions, [EG], p.81, there exists a constant $K_3 > 0$, independent of C, a C^1-function $\overline{F_\delta} : C \to \mathbb{R}$, and a closed set $E \subseteq C$ such that

$$(1)\ |C \setminus E| < \delta,$$

$$(2)\ \overline{F_\delta} = F \text{ and } D\overline{F_\delta} = DF \text{ on } E,\ \text{and}$$

$$(3)\ \sup_{x \in C}||D\overline{F_\delta}(x)|| \leq K_3 K.$$

Note: Here D denotes the gradient operator.

We assert that if $x \in C$ then

$$|F(x) - \overline{F_\delta}(x)| \leq (1 + K_3)\left(\frac{12\delta}{\pi}\right)^{\frac{1}{3}}.$$

Since F and F_δ agree on E, to prove the assertion we may assume that $x \in C \setminus E$. Consider the closed ball $B_r(x)$ in $\mathbb{R}^3$ centered at x with radius r such that $\frac{\pi r^3}{12} = \delta$. Since the volume of $C \cap B_r(x)$ is at least $\frac{1}{8}$ the volume of $B_r(x)$, it follows that $|C \cap B_r(x)| \geq 2\delta$. But $|C \setminus E| < \delta$, so the sets $C \cap B_r(x)$ and E cannot be disjoint. Let $y \in C \cap B_r(x) \cap E$. Then

$$\begin{aligned}
|F(x) - \overline{F_\delta}(x)| &\leq |F(x) - F(y)| + |F(y) - \overline{F_\delta}(y)| + |\overline{F_\delta}(y) - \overline{F_\delta}(x)| \\
&\leq Kd(x,y) + 0 + KK_3 d(x,y) \\
&\leq (1 + K_3)Kr \\
&= (1 + K_3)\left(\frac{12\delta}{\pi}\right)^{\frac{1}{3}},
\end{aligned}$$

proving the assertion. If we choose $\delta > 0$ such that $(1 + K_3)\left(\frac{12\delta}{\pi}\right)^{\frac{1}{3}} < \epsilon$, then (a), (b), and (c) are satisfied with $\overline{F_\epsilon} = \overline{F_\delta}$.

Lemma 3.19: Let f be Lipschitz continuous on $SU(2)$ with Lipschitz constant $K > 0$, and let $\epsilon > 0$ be given. Then there exist constants $K_3 > 0$ and $M > 0$ and $g_\epsilon \in C^1(SU(2))$ such that

$$|f(x) - g_\epsilon(x)| < \epsilon$$

and

$$\|[Q_x f]' - [Q_x g_\epsilon]'\|_{L^2[0,\pi]} < (K_3 + 1)KM^2\sqrt{\epsilon}$$

for all $x \in SU(2)$.

Proof: There exists a finite collection of proper regular coordinate patches $\mathbf{X}_i : C_i \to SU(2)$ for $1 \leq i \leq n$ with partial derivatives uniformly bounded by a constant $M > 0$ and such that $SU(2) = \bigcup_{i=1}^n \mathbf{X}_i[C_i]$. Let $F_i : C_i \to \mathbb{R}$ be the function induced by $\mathbf{X}_i$ and f such that

$$F_i(\Phi, \Theta, \Psi) = f(\mathbf{X}_i(\Phi, \Theta, \Psi)), \text{ for all } (\Phi, \Theta, \Psi) \in C_i$$

for $1 \leq i \leq n$. Clearly each F_i is Lipschitz continuous with Lipschitz constant which is at most KM. Let $E_1 = C_1$, for $2 \leq k \leq n$ let

$$E_k = \mathbf{X}_k^{-1}\left[\mathbf{X}_k[C_k] \setminus \bigcup_{j=1}^{k-1} \mathbf{X}_j[C_j]\right],$$

and set $E = \bigcup_{k=1}^{n} E_k$. Note that $E_j \cap E_k = \emptyset$ if $j \neq k$. Define F by

$$F(\Phi, \Theta, \Psi) = f(\mathbf{X}_i(\Phi, \Theta, \Psi))$$

if $(\Phi, \Theta, \Psi) \in E_i$ for some $i \in \{1, 2, .., n\}$. Then F is Lipschitz continuous with Lipschitz constant at most KM as well. By the approximation theorem for Lipschitz functions [EG], p. 251, and Theorem 3.18 there exists a C^1-function $\overline{F}_\epsilon : E \to \mathbb{R}$ and a constant $K_3 > 0$ such that:

(a) $E_\epsilon = \left\{(\Phi, \Theta, \Psi) \in E \mid D\overline{F}_\epsilon(\Phi, \Theta, \Psi) \neq DF(\Phi, \Theta, \Psi)\right\}$ has Lebesgue measure at most ϵ :

(b) $|\overline{F}_\epsilon(\Phi, \Theta, \Psi) - F(\Phi, \Theta, \Psi)| < \epsilon$ for all $(\Phi, \Theta, \Psi) \in E$;

(c) $\|D\overline{F}_\epsilon(\Phi, \Theta, \Psi)\| \leq K_3 KM$ for all $(\Phi, \Theta, \Psi) \in E$.

Let g_ϵ be the function on $SU(2)$ induced by $\overline{F}_\epsilon$:

$$g_\epsilon(\mathbf{X}_i(\Phi, \Theta, \Psi)) = \overline{F}_\epsilon(\Phi, \Theta, \Psi)$$

if $(\Phi, \Theta, \Psi) \in E_i$ for some $i \in \{1, 2, .., n\}$. The chain rule for Lipschitz functions [Sta], guarantees that

$$\lim_{h \to 0} \frac{f(x(\phi_0, \theta_0, \psi_0)y(\phi, \theta + h, \psi)) - f(x(\phi_0, \theta_0, \psi_0)y(\phi, \theta, \psi))}{h}$$

exists for a.e. $\theta \in [0, \pi]$ and is equal to

$$\frac{\partial F}{\partial \Phi}\frac{\partial \Phi}{\partial \theta} + \frac{\partial F}{\partial \Theta}\frac{\partial \Theta}{\partial \theta} + \frac{\partial F}{\partial \Psi}\frac{\partial \Psi}{\partial \theta}.$$

Thus, for a.e. $\theta \in [0, \pi]$, the bounded convergence theorem implies

$$\begin{aligned}
[Q_x f]'(\theta) &= \frac{1}{4\pi}\lim_{h\to 0}\int_0^{2\pi}\int_0^{\pi}\left[\frac{f(xy(\phi,\theta+h,\psi)) - f(xy(\phi,\theta,\psi))}{h}\right]\sin(\phi)d\psi d\phi \\
&= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\lim_{h\to 0}\left[\frac{f(xy(\phi,\theta+h,\psi)) - f(xy(\phi,\theta,\psi))}{h}\right]\sin(\phi)d\psi d\phi \\
&= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\frac{\partial F}{\partial \Phi}\frac{\partial \Phi}{\partial \theta} + \frac{\partial F}{\partial \Theta}\frac{\partial \Theta}{\partial \theta} + \frac{\partial F}{\partial \Psi}\frac{\partial \Psi}{\partial \theta}\right)\sin(\phi)d\psi d\phi,
\end{aligned}$$

where $x = x(\phi_0, \theta_0, \psi_0) \in SU(2)$ is parameterized in spherical coordinates. On the other hand, for every $\theta \in [0, \pi]$,

$$\begin{aligned}
[Q_x g_\epsilon]'(\theta) &= \frac{1}{4\pi}\lim_{h\to 0}\int_0^{2\pi}\int_0^{\pi}\left[\frac{g_\epsilon(xy(\phi,\theta+h,\psi)) - g_\epsilon(xy(\phi,\theta,\psi))}{h}\right]\sin(\phi)d\psi d\phi \\
&= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\lim_{h\to 0}\left[\frac{g_\epsilon(xy(\phi,\theta+h,\psi)) - g_\epsilon(xy(\phi,\theta,\psi))}{h}\right]\sin(\phi)d\psi d\phi \\
&= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\frac{\partial \overline{F}_\epsilon}{\partial \Phi}\frac{\partial \Phi}{\partial \theta} + \frac{\partial \overline{F}_\epsilon}{\partial \Theta}\frac{\partial \Theta}{\partial \theta} + \frac{\partial \overline{F}_\epsilon}{\partial \Psi}\frac{\partial \Psi}{\partial \theta}\right)\sin(\phi)d\psi d\phi.
\end{aligned}$$

For a.e. $\theta \in [0, \pi]$, we have by Jensen's inequality

$$\begin{aligned}
|[Q_x f]'(\theta) - [Q_x g_\epsilon]'(\theta)|^2 &\leq \int_0^{2\pi}\int_0^{\pi}\left|DF \cdot \frac{\partial}{\partial \theta}((\Phi, \Theta, \Psi)) - D\overline{F_\epsilon} \cdot \frac{\partial}{\partial \theta}(\Phi, \Theta, \Psi)\right|^2 dS \\
&= \int_0^{2\pi}\int_0^{\pi}\left|(DF - D\overline{F_\epsilon}) \cdot \frac{\partial}{\partial \theta}(\Phi, \Theta, \Psi)\right|^2 dS,
\end{aligned}$$

where $dS = \frac{1}{4\pi}\sin(\phi)d\psi d\phi$.Hence,

$$\begin{aligned}\|[Q_x f]' - [Q_x g_\epsilon]'\|^2_{L^2[0,\pi]} &\leq \int_0^{\pi}\int_0^{2\pi}\int_0^{\pi} \|DF - D\overline{F_\epsilon}\|^2 \left\|\frac{\partial}{\partial\theta}(\Phi,\Theta,\Psi)\right\|^2 dSd\theta \\ &\leq \iiint_{E_\epsilon} \|DF - D\overline{F_\epsilon}\|^2 \left\|\frac{\partial}{\partial\theta}(\Phi,\Theta,\Psi)\right\|^2 dSd\theta \\ &+ \iiint_{C\setminus E_\epsilon} \|DF - D\overline{F_\epsilon}\|^2 \left\|\frac{\partial}{\partial\theta}(\Phi,\Theta,\Psi)\right\|^2 dSd\theta \\ &\equiv J_1 + J_2.\end{aligned}$$

For a.e. point (Φ,Θ,Ψ) in E_ϵ, $DF_\epsilon(\Phi,\Theta,\Psi)$ exists and $\|DF_\epsilon(\Phi,\Theta,\Psi)\| \leq KM$, and $\|D\overline{F_\epsilon}\| \leq K_3KM$ so

$$\begin{aligned}J_1 &\leq (K_3+1)^2K^2M^4|E_\epsilon| \\ &\leq (K_3+1)^2K^2M^4\epsilon.\end{aligned}$$

Moreover, $DF = DF_\epsilon$ on $C \setminus E_\epsilon$ so $J_2 = 0$. Hence,

$$\|[Q_x f]' - [Q_x g_\epsilon]'\|_{L^2[0,\pi]} \leq (K_3+1)KM^2\sqrt{\epsilon}.$$

Theorem 3.20: If f is Lipschitz continuous on $SU(2)$, then $S_N f \to f$ uniformly on $SU(2)$ as $N \to \infty$.

Proof: Let f be Lipschitz continuous on $SU(2)$ with Lipschitz constant $K \geq 0$ and let $\epsilon > 0$ be given. By Lemma 3.19, there exist constants $K_3 > 0$ and $M > 0$ and $g_\epsilon \in C^1(SU(2))$ such that

$$\|[Q_x f]' - [Q_x g_\epsilon]'\|_{L^2[0,\pi]} < (K_3+1)KM^2\sqrt{\epsilon},$$

and

$$|f(x) - g_\epsilon(x)| < \frac{\epsilon}{3}$$

for all $x \in SU(2)$.

By Mayer's theorem [Ma1], the Fourier partial sums of g_ϵ converge uniformly to g_ϵ on $SU(2)$. Therefore there exists an integer $N_0 \geq 1$ such that $|(S_N g_\epsilon)(x) - g_\epsilon(x)| < \frac{\epsilon}{3}$ for all $N \geq N_0$ and all $x \in SU(2)$. By the triangle inequality

$$\begin{aligned} |(S_N f)(x) - f(x)| &\leq |(S_N f)(x) - (S_N g_\epsilon)(x)| + |(S_N g_\epsilon)(x) - g_\epsilon(x)| + |g_\epsilon(x) - f(x)| \\ &\leq |(S_N f)(x) - (S_N g_\epsilon)(x)| + \frac{2\epsilon}{3} \end{aligned}$$

for all $N \geq N_0$ and all $x \in SU(2)$. We estimate the quantity $|(S_N f)(x) - (S_N g_\epsilon)(x)|$ as follows:

$$\begin{aligned} |(S_N f)(x) - (S_N g_\epsilon)(x)| &= \left| -\frac{1}{\pi} \int_0^\pi D'_{N+1}(\theta) \sin(\theta) [Q_x(f - g_\epsilon)](\theta) d\theta \right| \\ &= \left| \frac{1}{\pi} \int_0^\pi D_{N+1}(\theta) \frac{d}{d\theta} \left(\sin(\theta) [Q_x(f - g_\epsilon)](\theta) \right) d\theta \right| \\ &\leq \left| \frac{1}{\pi} \int_0^\pi D_{N+1}(\theta) \cos(\theta) [Q_x(f - g_\epsilon)](\theta) d\theta \right| \\ &+ \left| \frac{1}{\pi} \int_0^\pi D_{N+1}(\theta) \sin(\theta) [Q_x(f - g_\epsilon)]'(\theta) d\theta \right| \\ &= \left| \frac{1}{\pi} \int_0^\pi D_{N+1}(\theta) \cos(\theta) [Q_x(f - g_\epsilon)](\theta) d\theta \right| \\ &+ \left| \frac{2}{\pi} \int_0^\pi \sin\left(\left(N + \frac{3}{2} \right) \theta \right) \cos\left(\frac{\theta}{2} \right) [Q_x(f - g_\epsilon)]'(\theta) d\theta \right| \\ &\equiv I_1 + I_2. \end{aligned}$$

To estimate I_1 set $F_x(\theta) = \cos(\theta)[Q_x(f - g_\epsilon)](\theta)$ for $\theta \in [0, \pi]$. Then

$$\begin{aligned} I_1 &= \left| \frac{1}{\pi} \int_0^\pi D_{N+1}(\theta) F_x(\theta) d\theta \right| \\ &= |(S_N F_x)(0)|. \end{aligned}$$

Since Proposition 2.2.11 (d) implies $F_x(\theta)$ is absolutely continuous on $[0, \pi]$, we have for a.e. $\theta \in [0, \pi]$

$$F_x'(\theta) = -\sin(\theta)[Q_x(f - g_\epsilon)](\theta) + \cos(\theta)[Q_x(f - g_\epsilon)]'(\theta).$$

Hence $F_x' \in L^2[0, \pi]$ by Proposition 2.2.11 (d) and Lemma 3.19, and from the triangle inequality,

$$\begin{aligned} \|F_x'\|_{L^2[0,\pi]} &= \| - \sin(\cdot)[Q_x(f - g_\epsilon)](\cdot) + \cos(\cdot)[Q_x(f - g_\epsilon)]'(\cdot)\|_{L^2[0,\pi]} \\ &\le \| - \sin(\cdot)[Q_x(f - g_\epsilon)](\cdot)\|_{L^2[0,\pi]} + \| \cos(\cdot)[Q_x(f - g_\epsilon)]'(\cdot)\|_{L^2[0,\pi]} \\ &\le \| - \sin(\cdot)\|_{L^2[0,\pi]} \|Q_x(f - g_\epsilon)\|_\infty + \| \cos(\cdot)\|_\infty \|[Q_x(f - g_\epsilon)]'\|_{L^2[0,\pi]} \\ &\le \sqrt{\frac{\pi}{2}} \|Q_x(f - g_\epsilon)\|_\infty + \|[Q_x(f - g_\epsilon)]'\|_{L^2[0,\pi]} \\ &\le \frac{\epsilon}{3}\sqrt{\frac{\pi}{2}} + (K_3 + 1)KM^2\sqrt{\epsilon}. \end{aligned}$$

Consequently, the Fourier series of F_x is absolutely convergent with

$$\begin{aligned} \|F_x\|_{A([0,\pi])} &\le \|F_x\|_{L^1[0,\pi]} + \left(2 \sum_{n=1}^\infty \frac{1}{n^2} \right)^{\frac{1}{2}} \|F_x'\|_{L^2[0,\pi]} \\ &\le 2\|Q_x(f - g_\epsilon)\|_\infty + \frac{\pi}{\sqrt{3}} \left(\frac{\epsilon}{3}\sqrt{\frac{\pi}{2}} + (K_3 + 1)KM^2\sqrt{\epsilon} \right) \\ &\le \left(\frac{2}{3} + \frac{\pi\sqrt{\pi}}{3\sqrt{6}} \right) \epsilon + \frac{\pi(K_3 + 1)KM^2\sqrt{\epsilon}}{\sqrt{3}}. \end{aligned}$$

Hence,

$$
\begin{aligned}
I_1 &= |(S_N F_x)(0)| \\
&\leq \|F_x\|_{A([0,\pi])} \\
&\leq \left(\frac{2}{3} + \frac{\pi\sqrt{\pi}}{3\sqrt{6}}\right)\epsilon + \frac{\pi(K_3+1)KM^2\sqrt{\epsilon}}{\sqrt{3}}.
\end{aligned}
$$

From the Cauchy-Schwarz inequality,

$$
\begin{aligned}
I_2 &\leq \frac{2}{\pi}\int_0^\pi \left|\sin\left(\left(N+\frac{3}{2}\right)\theta\right)\cos\left(\frac{\theta}{2}\right)\right| |[Q_x(f-g_\epsilon)]'(\theta)| d\theta \\
&\leq \frac{2}{\pi}\left(\int_0^\pi \sin^2\left(\left(N+\frac{3}{2}\right)\theta\right)\cos^2\left(\frac{\theta}{2}\right)d\theta\right)^{\frac{1}{2}} \left(\int_0^\pi |[Q_x(f-g_\epsilon)]'(\theta)|^2 d\theta\right)^{\frac{1}{2}} \\
&\leq \frac{2}{\pi}\|[Q_x(f-g_\epsilon)]'\|_{L^2[0,\pi]} \\
&\leq \frac{2}{\pi}(K_3+1)KM^2\sqrt{\epsilon},
\end{aligned}
$$

for all $N \geq 1$ and all $x \in SU(2)$.

Hence,

$$
|(S_N f - f)(x)| \leq \left(\frac{4}{3} + \frac{\pi\sqrt{\pi}}{3\sqrt{6}}\right)\epsilon + \left(\frac{\pi}{\sqrt{3}} + \frac{2}{\pi}\right)(K_3+1)KM^2\sqrt{\epsilon}
$$

for all $N \geq N_0$, and all $x \in SU(2)$. Hence $S_N f \to f$ uniformly on $SU(2)$ as $N \to \infty$. This completes section 3. In the final section we will discuss directions of further research.

4 CONCLUSIONS

There are many unsolved problems regarding convergence of Fourier series on nonabelian groups. We will first concentrate on $SU(2)$ as it is the most elementary of the compact, nonabelian Lie groups.

Mean convergence is settled on $SU(2)$. If $f \in L^2(SU(2))$ then the Peter-Weyl theorem guarantees mean convergence of the Fourier series of f and the analogous result is false in $L^p(SU(2))$ for $p \neq 2$. See [ST] for a proof of the latter result on a general compact Lie group G. A similar result holds on spheres; see [BC] for a proof.

Regarding absolute convergence of Fourier series on $SU(2)$, elementary arguments show that if $f \in C^2(SU(2))$ then the Fourier series of f converges absolutely; see [F], p.168-169 for a proof. On the other hand, [Ma1] gives a concrete example of $f \in C^1(SU(2))$ whose Fourier series does not converge absolutely. We will need the following definitions for further study of absolute convergence of Fourier series on $SU(2)$.

Definition 4.1: Let $x \in SU(2)$ and $f \in L^p(SU(2))$. The n^{th} order differences of f are defined as

$$(\triangle_{hX}^n f)(x) = \sum_{j=0}^{n} (-1)^{n+j} \binom{n}{j} f\left(x \exp\left(jhX\right)\right)$$

where $h > 0$ and $X \in su(2)$ such that $\|X\| = 1$.

Definition 4.2: Let $\alpha = k + \gamma$ where k is an nonnegative integer and $0 < \gamma \leq 1$. We say that $f \in \text{Lip}_\alpha(SU(2))$ if there exists a positive constant M such that, for every $h \in (0, 1]$ and every $X \in su(2)$ satisfying $\|X\| = 1$,

$$\|\triangle_{hX}^2 \rho^{i_1}\left(X_{\alpha_1}\right) \cdots \rho^{i_r}\left(X_{\alpha_r}\right) f\|_\infty < Mh^\gamma$$

where

$$i_1 + i_2 + \cdots + i_r = k, \text{ and } \alpha_i \in \{1, 2, 3\} \text{ for } 1 \leq i \leq r.$$

If $\alpha = 1$ in Definition 4.2, then $\gamma = 1$ and $k = 0$. Hence $\|\triangle_{hX}^2 f\|_\infty \leq Mh$, which means the second differences for f are bounded. Definition 4.2 does not imply the usual definition of a Lipschitz continuous function on a metric space, i.e. $|f(x) - f(y)| \leq Md(x, y)$ for some constant $M \geq 0$ and all x and y. However if f is Lipschitz continuous on $SU(2)$ assuming the usual definition, then Definition 4.2 is satisfied as a consequence of the triangle inequality. It is also important to note that in definition 4.2 we can have nonconstant functions when $\alpha > 1$. See [P] for a concrete example.

The connection between a function's smoothness and absolute convergence of its Fourier series on $SU(2)$ was clarified significantly in [P] where it is shown: (1) any function in $\text{Lip}_\alpha(SU(2))$ for some $\alpha > \frac{3}{2}$ has an absolutely convergent Fourier series and (2) to each $\alpha \leq \frac{3}{2}$ there corresponds an explicit function in $\text{Lip}_\alpha(SU(2))$ whose Fourier series is not absolutely convergent. However, on T additional regularity hypotheses on $f \in \text{Lip}_\alpha(\text{T})$ guarantee absolute convergence of the Fourier series for f. For instance, if $f \in \text{Lip}_\alpha(\text{T})$ for some $\alpha > 0$ and if f is also of bounded variation on T then its Fourier series will converge absolutely; see [Z], p. 241 for a proof. On $SU(2)$, the definition of a function of bounded variation is a generalization of that for T.

Definition 4.3: We say that a real function f on $SU(2)$ is of bounded variation on $SU(2)$, and write $f \in BV(SU(2))$, if f has a distributional derivative which is also a bounded Borel measure.

This leads to the first problem.

Problem 1: Can additional hypotheses regarding $f \in \text{Lip}_\alpha(SU(2))$ be made which will guarantee that f has an absolutely convergent Fourier series? In particular, if

$f \in \text{Lip}_\alpha(SU(2))$ for some $\alpha \leq \frac{3}{2}$ and $f \in BV(SU(2))$, does the Fourier series of f converge absolutely?

With regard to uniform convergence, in section 3 we proved that if $f \in \text{Lip}_1(SU(2))$, then f has a uniformly convergent Fourier series. Also, it was demonstrated that $\text{Lip}_1(SU(2))$ was the best possible result in the sense that the result fails for any other $\text{Lip}_\alpha(SU(2))$, with $\alpha \in (0,1)$.

Problem 2: Let $f \in \text{Lip}_\alpha(SU(2))$ for some $\alpha \in (0,1)$.

a) Can one or more additional hypotheses be made regarding f which will guarantee that f has a pointwise convergent Fourier series?

b) Can one or more additional hypotheses be made regarding f which will guarantee that f has a uniformly convergent Fourier series?

c) In particular, if $f \in \text{Lip}_\alpha(SU(2))$ for some $\alpha \in (0,1)$ and $f \in BV(SU(2))$, does the Fourier series of f converge pointwise or even uniformly on $SU(2)$?

We have not discussed in this dissertation sufficient conditions for a function on $SU(2)$ to have a Fourier series which converges almost everywhere. In [Ma1] it was shown that if $f \in L^2(SU(2))$ and $f \in C^1(SU(2))$ a.e. then f has an almost everywhere convergent Fourier series. In [Me2] the condition of $f \in C^1(SU(2))$ a.e. is relaxed to f having a distributional derivative which is also square integrable. However, on T, Carleson proved Lusin's conjecture: If $f \in L^2(\mathrm{T})$ then the Fourier series of f converges almost everywhere on T; see [JAdR]. This makes one wonder whether any additional hypotheses on a square integrable function on $SU(2)$ are necessary to conclude almost everywhere convergence of its Fourier series. That is, we can ask the following question.

Problem 3: If $f \in L^2(SU(2))$, then does f have an almost everywhere convergent Fourier series?

If f is a central function, then the answer to problem 3 is yes, and a proof can be found in [ST]. It was also shown in [ST] that if G is a compact, semi-simple Lie group, then to each $p < 2$ there corresponds $f \in L^p(G)$ whose Fourier series diverges almost everywhere, but the example is not constructive. We contrast this with a result in [Po] which says that if f is a central function and $f \in L^p(SU(2))$ for some $p > \frac{3}{2}$, then f has an almost everywhere convergent Fourier series on $SU(2)$. It was shown in [Me1] that there exists a central function f which belongs to $L^{\frac{3}{2}}(SU(2))$ whose Fourier series diverges almost everywhere, but the proof is not constructive. These nonconstructive existence theorems lead to the following two problems

Problem 4: Does there exist a constructive example of a noncentral function $f \in L^p(SU(2))$ for $\frac{3}{2} \leq p < 2$ whose Fourier series diverges everywhere?

The following result is proved in $[K]$, p. 59.

Theorem: Let B be a homogeneous Banach space on T satisfying the following conditions.

1. If $f \in B$ and $n \in \mathbb{Z}$ then $e^{int} f \in B$ and $\|e^{int} f\|_B = \|f\|_B$.
2. $B \supseteq C(\mathrm{T})$.

Then either there exists $f \in B$ whose Fourier series diverges at every point of T or every $f \in B$ has an almost everywhere convergent Fourier series.

Problem 5: Is there an analog of the previous theorem for nonabelian groups?

It follows from a general theorem in [CT] that the Dirichlet kernel $\mathbf{D}_N$ on $SU(2)$ diverges for almost every $x \in SU(2)$ as $N \to \infty$.

Problem 6: Is it true that $\overline{\lim}\, \mathbf{D}_N(x)$ is infinite as $N \to \infty$ for every $x \in SU(2)$?

Graphical and numerical evidence obtained by the author suggests that the limit supremum in problem 6 is infinite for every $x \in SU(2)$, but a proof has not been discovered.

It would be desirable to obtain convergence results for Fourier series for higher dimensional spheres and compact, connected, Lie groups. For example, we can pose the following questions.

Problem 7: Let $N > 2$ and $f \in C^1(SU(N))$.

a) Need f have a pointwise convergent Fourier series?

b) Need f have a uniformly convergent Fourier series?

The principal idea in the proof of the main result of this dissertation was to express the Dirichlet kernel on $SU(2)$ in terms of the Dirichlet kernel on T, and then reduce the uniform convergence problem on the curved manifold $SU(2)$ to the flat torus T where convergence results for Fourier series are well known. The Dirichlet kernels are central functions on any compact group, and so only depend on the variables used to parameterize a maximal torus in a compact group. On a compact, connected Lie group G, the following theorem is due to Cartan.

Theorem: [S], p.155. Every compact, connected Lie group has the form $G = K/H$, where K is a finite product of T′s, the spin groups, the special unitary groups, the symplectic groups and the five exceptional Lie groups, and H is a finite subgroup of the center of K.

Problem 8: For which compact, connected Lie groups G can the Nth partial sum of the Fourier series of a function $f \in L^2(G)$ be written as

$$\begin{aligned}(S_N f)(x) &= (D_N \star f)(x) \\ &= \int_G D_N(y) f(xy^{-1}) \mu(dy) \\ &= \int_T D_N(t) \int_{G \setminus T} f(xy^{-1}) \mu(d(G \setminus T)) dt\end{aligned}$$

where T is a maximal torus of G, and $\mu(d(G \setminus T))$ is a G-invariant Borel measure defined on the homogeneous space $G \setminus T$ of left cosets of T in G?

By Theorem 2.4.9 and Corollary 2.51 in [Fo], pp. 57-58,

$$\int_G g(y)\mu(dy) = \int_{G\backslash T} \int_T g(y\xi)d\xi\mu(dyT).$$

By Theorem VIII.1.1′ in [S], p. 167, to each $z \in G$ and maximal torus T of G, there corresponds $w \in G$ and $t \in T$ such that $wzw^{-1} = t$. In particular, to each $y\xi \in G$ there corresponds $w \in G$ and $t \in T$ such that $wy\xi w^{-1} = t \in T$. In light of these two results, we would like to know which compact, connected Lie groups G are isomorphic to a direct product $G \cong T \times (G \setminus T)$, and the Nth partial sum of the Fourier series of a function $f \in L^2(G)$ can be written as

$$(S_N f)(x) = \int_T K_N(t) \int_{G\backslash T} f(xw^{-1}tw)\mu(d(w^{-1}twT))dt.$$

Let $f \in L^2(G)$, fix $x \in G$, and define a function $Q_x f$ on G by

$$(Q_x f)(y) = \int_{G\backslash T} f(xy^{-1})\mu(d(G \setminus T)).$$

This is called the quotient integral formula in [DE]. For the groups G for which the answer to Problem 8 is yes, we need to study the following problem.

Problem 9: Given $f \in L^p(G)$ for some $p \geq 2$, and a fixed $x \in G$ what can be said about the properties of the function $Q_x f$?

In the case $G = SU(2)$, we have $G \setminus T \cong S^2$,

$$(Q_x f)(y(\phi, \theta, \psi)) = (Q_x f)(\omega_1(\theta)) = \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} f(xy^{-1}(\phi, \theta, \psi)) \sin(\phi) d\psi d\phi,$$

and

$$(S_N f)(x) = -\frac{1}{\pi}\int_0^\pi D'_{N+1}(\theta)\sin(\theta)(Q_x f)(\omega_1(\theta))d\theta.$$

(See Remark 2 near the end of Section 2.3.) Proposition 2.2.11 then gives many properties of $Q_x f$ on a maximal torus $T \cong [-\pi, \pi)$ that reflect the properties of f on $SU(2)$.

Finally, there are many fields in science and engineering for which noncommutative harmonic analysis is useful. For applications in fields such as robotics and tomography see $[Cr1]$ and $[Cr2]$. We have mentioned nothing about noncommutative harmonic analysis on finite groups which is important in signal processing. See $[SMJ]$ for more details. Representation theory on unitary subgroups, especially $SU(2)$ and $SU(3)$, is used in particle physics; see $[G]$ for more details.

APPENDIX

In this appendix we begin by proving that the Fourier series in Example 2.4.8 converges a.e. Consider the series

$$\sum_{m=0}^{\infty} \frac{(m+1)}{(m+1)^2-\alpha^2} \sin((m+1)\theta) \qquad (1)$$

where $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ and $0 \leq \theta \leq 2\pi$. If $a_m = \frac{m}{m^2-\alpha^2}$, for $m \in \mathbb{N}$, then $a_m > 0$ and $a_{m+1} < a_m$ for $m > |\alpha|$. Hence (1) converges uniformly in each interval $\epsilon \leq \theta \leq 2\pi - \epsilon$ for every $\epsilon > 0$ (cf. [Z], Vol. 1, p.4). Note that $\lim_{m \to \infty} m a_m \neq 0$, so the convergence is not uniform in $[0, 2\pi]$ (cf. [Z], Vol. 1, p.182). However $m a_m = \mathcal{O}(1)$ as $m \to \infty$ so the partial sums of (1) are uniformly bounded on $[0, 2\pi]$ (cf. [Z], Vol. 1, p.183). Observe that

$$\begin{aligned}
\sum_{m>|\alpha|} \triangle a_m \ln(m) &= \sum_{m>|\alpha|}^{\infty} \left(\frac{m}{m^2-\alpha^2} - \frac{(m+1)}{(m+1)^2-\alpha^2} \right) \ln(m) \\
&= \sum_{m>|\alpha|}^{\infty} \left(\frac{m((m+1)^2-\alpha^2)-(m^2-\alpha^2)(m+1)}{(m^2-\alpha^2)((m+1)^2-\alpha^2)} \right) \ln(m) \\
&= \sum_{m>|\alpha|}^{\infty} \left(\frac{m((m+1)^2-\alpha^2)-(m^2-\alpha^2)(m+1)}{(m^2-\alpha^2)((m+1)^2-\alpha^2)} \right) \ln(m) \\
&= \sum_{m>|\alpha|}^{\infty} \left(\frac{m(m+1)+\alpha^2}{(m^2-\alpha^2)((m+1)^2-\alpha^2)} \right) \ln(m) \\
&= \sum_{m>|\alpha|}^{\infty} \left(\frac{1+\frac{\alpha^2}{m(m+1)}}{\left(1+\frac{|\alpha|}{m+1}\right)(m+1+|\alpha|)\left(1+\frac{|\alpha|}{m}\right)(m-|\alpha|)} \right) \ln(m) \\
&< \sum_{m>|\alpha|}^{\infty} \frac{2\ln(m)}{(m+1+|\alpha|)(m-|\alpha|)} \\
&< \infty.
\end{aligned}$$

Therefore the sum function of (1),

$$p_\alpha(\theta) = \sum_{m=0}^{\infty} \frac{(m+1)}{(m+1)^2 - \alpha^2} \sin((m+1)\theta),$$

is integrable on $[0, 2\pi]$ and

$$\lim_{N\to\infty} \left\| p_\alpha - \sum_{m=0}^{N} \frac{(m+1)}{(m+1)^2 - \alpha^2} \sin((m+1)(\cdot)) \right\|_{L^1[0,2\pi]} = 0 \qquad (2)$$

(cf. [Z], Vol. 1, p.185).

Let the central function $g \in L^1(SU(2))$ be determined by

$$g(\omega_1(\theta)) = \frac{\sin(\alpha\pi)}{\sin(\theta)} p_\alpha(\theta)$$

for $0 < \theta < \pi$. The Fourier coefficients of g satisfy

$$\begin{aligned} (m+1)a_m &= \frac{2}{\pi} \int_0^\pi g(\omega_1(\theta))\chi_m(\omega_1(\theta)) \sin^2(\theta) d\theta \\ &= \frac{2\sin(\alpha\pi)}{\pi} \int_0^\pi p_\alpha(\theta) \sin((m+1)\theta) d\theta \\ &= \frac{\sin(\alpha\pi)}{\pi} \int_0^{2\pi} p_\alpha(\theta) \sin((m+1)\theta) d\theta \\ &= \frac{(m+1)\sin(\alpha\pi)}{(m+1)^2 - \alpha^2} \end{aligned}$$

for m a nonnegative integer by (2). Therefore the functions g and q_α of Example 2.4.8 are integrable, central functions with the same Fourier coefficients, so $g = q_\alpha$. Consequently q_α is equal to its Fourier series for a.e. $\theta \in [0, \pi]$.

Next, let $f \in C(SU(2))$. From Example 2.4.9:

$$u(x) = \lim_{N\to\infty} \frac{\pi}{2\sin(\alpha\pi)} \sum_{m=0}^{N}(m+1)a_m \int_{SU(2)} f(xy)\chi_n(y^{-1})\mu(dy)$$
$$= \frac{\pi}{2} \lim_{N\to\infty} \frac{1}{2\pi^2} \int_0^{\pi}\int_0^{\pi}\int_0^{2\pi} f(x(\phi_0,\theta_0,\psi_0)y(\phi,\theta,\psi))$$
$$\times \left(\sum_{m=0}^{N} \frac{(m+1)}{(m+1)^2-\alpha^2} \frac{\sin((m+1)(\theta))}{\sin(\theta)} \chi_n(\omega_1(\theta))\right) \sin^2(\theta)\sin(\phi)d\psi d\phi d\theta.$$

Since f is continuous on $SU(2)$ and the partial sums $\left\{\sum_{m=0}^{N} \frac{(m+1)}{(m+1)^2-\alpha^2} \sin((m+1)(\cdot))\right\}_{N=0}^{\infty}$ are uniformly bounded on $[0, 2\pi]$, it follows that the integrand is uniformly bounded on $[0, 2\pi] \times [0, \pi] \times [0, \pi]$. Hence Lebesgue's Dominated Convergence Theorem and the fact that q_α is equal a.e. to its Fourier series yields

$$u(x) = \lim_{N\to\infty} \frac{\pi}{2\sin(\alpha\pi)} \sum_{m=0}^{N}(m+1)c_m \int_{SU(2)} f(xy)\chi_n(y^{-1})\mu(dy)$$
$$= \frac{\pi}{2\sin(\alpha\pi)} \int_{SU(2)} f(xy)q_\alpha(y^{-1})\mu(dy)$$
$$= \frac{\pi}{2\sin(\alpha\pi)} \int_{SU(2)} f(y)q_\alpha(xy^{-1})\mu(dy).$$

Consequently, this rigorously verifies the integral representation of the C^2-solution u to the inhomogeneous Helmholtz equation on $SU(2)$ in Example 2.4.9.

We will now restate and prove Lemmas 3.9 through 3.12 which are used in the proof of Theorem 3.8.

Lemma 3.9: If $0 < \theta_0 < \pi$ and $N \geq 1$ then

$$(S_N f)(\omega_1(\theta_0)) - f(\omega_1(\theta_0)) = \int_0^{\pi} g(\theta_0, \theta) \sin\left(\left(N + \frac{3}{2}\right)\theta\right) d\theta$$

where

$$g(\theta_0, \theta) = \frac{(\Delta f)(\theta_0, -\theta)\sin(\theta_0 - \theta) + (\Delta f)(\theta_0, \theta)\sin(\theta_0 + \theta)}{2\pi \sin(\theta_0)\sin\left(\frac{\theta}{2}\right)},$$

and

$$(\Delta f)(\theta_0, \theta) = f(\omega_1(\theta_0 + \theta)) - f(\omega_1(\theta_0)).$$

Proof: Recall from Example 2.3.41 and Remark 3 following Example 2.3.41:

$$\begin{aligned}(S_N f)(\omega_1(\theta_0)) - f(\omega_1(\theta_0)) &= \int\limits_{SU(2)} \mathbf{D}_N(xy^{-1})(f(y) - f(x))\mu(dy) \\ &= \frac{2}{\pi}\int\limits_0^\pi (f(\omega_1(\theta)) - f(\omega_1(\theta_0)))\sin^2(\theta) \\ &\times \left(\frac{1}{4\pi}\int\limits_0^\pi\int\limits_0^{2\pi} \mathbf{D}_N(xy^{-1})\sin(\phi)d\psi d\phi\right) d\theta \\ &= \frac{2}{\pi}\int\limits_0^\pi (f(\omega_1(\theta)) - f(\omega_1(\theta_0)))\sin^2(\theta) \\ &\times \left(\frac{1}{4\sin(\theta_0)\sin(\theta)}\left(D_{N+1}(\theta_0-\theta) - D_{N+1}(\theta_0+\theta)\right)\right) d\theta \\ &= \frac{1}{2\pi\sin(\theta_0)}\int\limits_0^\pi (f(\omega_1(\theta)) - f(\omega_1(\theta_0)))\sin(\theta) \\ &\times \left(D_{N+1}(\theta_0-\theta) - D_{N+1}(\theta_0+\theta)\right) d\theta\end{aligned}$$

if $0 < \theta_0 < \pi$. Let

$$I_1 = \frac{1}{2\pi\sin(\theta_0)}\int\limits_0^\pi (f(\omega_1(\theta)) - f(\omega_1(\theta_0)))\sin(\theta)D_{N+1}(\theta_0-\theta)d\theta$$

and

$$I_2 = \frac{1}{2\pi\sin(\theta_0)}\int\limits_0^\pi (f(\omega_1(\theta)) - f(\omega_1(\theta_0)))\sin(\theta)D_{N+1}(\theta_0+\theta)d\theta.$$

Then $(S_N f)(\omega_1(\theta_0)) - f(\omega_1(\theta_0)) = I_1 - I_2$. To combine the integrals let $\phi = \theta$ in I_1 and $\phi = -\theta$ in I_2. Since f is central, on I_2 we obtain

$$I_2 = -\frac{1}{2\pi\sin(\theta_0)}\int\limits_0^{-\pi} (f(\omega_1(-\phi)) - f(\omega_1(\theta_0)))\sin(-\phi)D_{N+1}(\theta_0-\phi)d\phi$$

$$= -\frac{1}{2\pi \sin(\theta_0)} \int_{-\pi}^{0} (f(\omega_1(\phi)) - f(\omega_1(\theta_0))) \sin(\phi) D_{N+1}(\theta_0 - \phi) d\phi.$$

Therefore switching ϕ back to θ, yields

$$\begin{aligned}(S_N f)(\omega_1(\theta_0)) - f(\omega_1(\theta_0)) &= \frac{1}{2\pi \sin(\theta_0)} \int_{-\pi}^{\pi} (f(\omega_1(\theta)) - f(\omega_1(\theta_0))) \sin(\theta) D_{N+1}(\theta_0 - \theta) d\theta \\ &= \frac{1}{2\pi \sin(\theta_0)} \int_{-\pi}^{\pi} (\mathit{\Delta} f)(\theta_0, -\theta) \sin(\theta_0 - \theta) D_{N+1}(\theta) d\theta\end{aligned}$$

because of the translation property of convolutions on T. Next, we write the integral as the sum of two terms:

$$\begin{aligned}(S_N f)(\omega_1(\theta_0)) - f(\omega_1(\theta_0)) = &\frac{1}{2\pi \sin(\theta_0)} \int_{0}^{\pi} (\mathit{\Delta} f)(\theta_0, -\theta) \sin(\theta_0 - \theta) D_{N+1}(\theta) d\theta \\ &+ \frac{1}{2\pi \sin(\theta_0)} \int_{-\pi}^{0} (\mathit{\Delta} f)(\theta_0, -\theta) \sin(\theta_0 - \theta) D_{N+1}(\theta) d\theta.\end{aligned}$$

Using the change of variables $\theta \mapsto -\theta$ and the fact that D_{N+1} is even yields

$$\begin{aligned}(S_N f)(\omega_1(\theta_0)) - f(\omega_1(\theta_0)) = &\frac{1}{2\pi \sin(\theta_0)} \int_{0}^{\pi} (\mathit{\Delta} f)(\theta_0, -\theta) \sin(\theta_0 - \theta) D_{N+1}(\theta) d\theta \\ &+ \frac{1}{2\pi \sin(\theta_0)} \int_{0}^{\pi} (\mathit{\Delta} f)(\theta_0, \theta) \sin(\theta_0 + \theta) D_{N+1}(\theta) d\theta.\end{aligned}$$

Combining these two integrals, we obtain

$$g(\theta_0, \theta) = \frac{(\mathit{\Delta} f)(\theta_0, \theta) \sin(\theta_0 + \theta) + (\mathit{\Delta} f)(\theta_0, -\theta) \sin(\theta_0 - \theta)}{2\pi \sin(\theta_0) \sin\left(\frac{\theta}{2}\right)},$$

and

$$(S_N f)(\omega_1(\theta_0)) - f(\omega_1(\theta_0)) = \int_0^{\pi} g(\theta_0, \theta) \sin\left(\left(N + \frac{3}{2}\right)\theta\right) d\theta.$$

Lemma 3.10: If $0 < \theta_0 < \pi$ and $0 < \theta < \pi$, then $|g(\theta_0, \theta)| \leq \frac{4M}{\pi}$.

Proof: Let f be a central function in $\text{Lip}_1(SU(2))$ with Lipschitz constnat $M > 0$, let $0 < \theta_0 < \pi$ and $0 < \theta < \pi$, and let $g(\theta_0, \theta)$ be the function defined in Lemma 3.9. To show that g is uniformly bounded, substituting the standard trigonometric identites

$$\sin(\theta_0 \pm \theta) = \sin(\theta_0)\cos(\theta) \pm \sin(\theta)\cos(\theta_0),$$

and

$$\sin(\theta) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

into the definition for g yields

$$g(\theta_0, \theta) = \frac{(D^2 f)(\theta_0, \theta)\cos(\theta)}{2\pi\sin\left(\frac{\theta}{2}\right)} + \frac{(Df)(\theta_0, \theta)\cos\left(\frac{\theta}{2}\right)\cos(\theta_0)}{\pi\sin(\theta_0)}$$

where

$$(Df)(\theta_0, \theta) = f(\omega_1(\theta_0 + \theta)) - f(\omega_1(\theta_0 - \theta))$$

and

$$(D^2 f)(\theta_0, \theta) = f(\omega_1(\theta_0 + \theta)) + f(\omega_1(\theta_0 - \theta)) - 2f(\omega_1(\theta_0)).$$

Using the Lipschitz hypothesis, translation invariance of the metric, and Example 2.1.25(c):

$$
\begin{aligned}
|(Df)(\theta_0,\theta)| &\leq |f(\omega_1(\theta_0+\theta)) - f(\omega_1(\theta_0-\theta))| \\
&\leq Md(\omega_1(\theta_0+\theta), \omega_1(\theta_0-\theta)) \\
&= 2M\sin(\theta),
\end{aligned}
$$

and since $f(\omega_1(\theta_0-\theta)) = f(\omega_1(\theta-\theta_0))$ a similar argument yields

$$|(Df)(\theta_0,\theta)| \leq 2M\sin(\theta_0).$$

The second difference $(D^2 f)(\theta_0,\theta)$ can be estimated using the triangle inequality as follows:

$$
\begin{aligned}
|(D^2 f)(\theta_0,\theta)| &\leq |f(\omega_1(\theta_0+\theta)) + f(\omega_1(\theta_0-\theta)) - 2f(\omega_1(\theta_0))| \\
&\leq |f(\omega_1(\theta_0+\theta)) - f(\omega_1(\theta_0))| + |f(\omega_1(\theta_0-\theta)) - f(\omega_1(\theta_0))| \\
&\leq Md(\omega_1(\theta), e) + Md(\omega_1(-\theta), e) \\
&= 2M\left|\sin\left(\frac{\theta}{2}\right)\right| + 2M\left|\sin\left(-\frac{\theta}{2}\right)\right| \\
&= 4M\sin\left(\frac{\theta}{2}\right).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\left|\frac{(D^2 f)(\theta_0,\theta)\cos(\theta)}{2\pi\sin\left(\frac{\theta}{2}\right)}\right| &\leq \frac{4M\sin\left(\frac{\theta}{2}\right)}{2\pi\sin\left(\frac{\theta}{2}\right)} \\
&= \frac{2M}{\pi},
\end{aligned}
$$

and

$$\left|\frac{(Df)(\theta_0,\theta)\cos\left(\frac{\theta}{2}\right)\cos(\theta_0)}{\pi\sin(\theta_0)}\right| \le \frac{2M\sin(\theta_0)}{\pi\sin(\theta_0)}$$
$$= \frac{2M}{\pi}.$$

Therefore g is uniformly bounded on $(0,\pi)\times(0,\pi)$ by $\frac{4M}{\pi}$.

Lemma 3.11: Let $\theta,\theta_0 \in (0,\pi)$ and $h \in \left(0,\frac{\pi}{2}\right)$. Then $g(\theta_0,\theta+h) - g(\theta_0,\theta) = A(\theta_0,\theta,h) + B(\theta_0,\theta,h)$ where:

(1) $|A(\theta_0,\theta,h)| \le \frac{4M}{\pi}$;

(2) $\lim_{N\to\infty} A(\theta_0,\theta,h_N) = 0$ if $\{h_N\}_{N=1}^{\infty}$ is any sequence in $\left(0,\frac{\pi}{2}\right)$ converging to zero;

(3) $|B(\theta_0,\theta,h)| \le \begin{cases} \left(\frac{5}{\pi}+\frac{3h}{4}\right)Mh + \left(1+\frac{3\pi}{2}\right)\frac{Mh}{\theta} & \text{if } 0 < 2h \le \theta < \pi, \\ \frac{12M}{\pi} & \text{if } 0 < \theta < 2h < \pi. \end{cases}$

Proof: Let f be a central function in $\mathrm{Lip}_1(SU(2))$ with Lipschitz constnat $M > 0$. We will show that

$$g(\theta_0,\theta+h) - g(\theta_0,\theta) = \sum_{j=1}^{6} A_j(\theta_0,\theta,h)$$

where

$$A_1(\theta_0,\theta,h) = \frac{1}{2\pi}(D^2f)(\theta_0,\theta+h)\cos(\theta+h)\left[\frac{1}{\sin\left(\frac{\theta+h}{2}\right)} - \frac{1}{\sin\left(\frac{\theta}{2}\right)}\right],$$
$$A_2(\theta_0,\theta,h) = \frac{\cos(\theta)[(D^2f)(\theta_0,\theta+h) - (D^2f)(\theta_0,\theta)]}{2\pi\sin\left(\frac{\theta}{2}\right)},$$
$$A_3(\theta_0,\theta,h) = -\frac{(D^2f)(\theta_0,\theta+h)\sin^2\left(\frac{h}{2}\right)\cos(\theta)}{\pi\sin\left(\frac{\theta}{2}\right)},$$
$$A_4(\theta_0,\theta,h) = -\frac{(D^2f)(\theta_0,\theta+h)\cos\left(\frac{\theta}{2}\right)\sin(h)}{\pi},$$
$$A_5(\theta_0,\theta,h) = \frac{\cos(\theta_0)}{\pi\sin(\theta_0)}(Df)(\theta_0,\theta+h)\left(\cos\left(\frac{\theta+h}{2}\right) - \cos\left(\frac{\theta}{2}\right)\right)$$
$$A_6(\theta_0,\theta,h) = \frac{\cos(\theta_0)}{\pi\sin(\theta_0)}[(Df)(\theta_0,\theta+h) - (Df)(\theta_0,\theta)]\cos\left(\frac{\theta}{2}\right).$$

Using the notation of Lemma 3.10, and suppressing the arguments of the functions A_j for $j = 1, 2, .., 6$ we obtain

$$\begin{aligned}
g(\theta_0, \theta+h) - g(\theta_0, \theta) &= \frac{(D^2 f)(\theta_0, \theta+h)\cos(\theta+h)}{2\pi \sin\left(\frac{\theta+h}{2}\right)} - \frac{(D^2 f)(\theta_0, \theta)\cos(\theta)}{2\pi \sin\left(\frac{\theta}{2}\right)} \\
&+ \frac{\cos(\theta_0)}{\pi \sin(\theta_0)} \\
&\times \left((Df)(\theta_0, \theta+h)\cos\left(\frac{\theta+h}{2}\right) - (Df)(\theta_0, \theta)\cos\left(\frac{\theta}{2}\right) \right) \\
&= A_1 + A_5 + A_6 \\
&+ \frac{(D^2 f)(\theta_0, \theta+h)\cos(\theta+h) - (D^2 f)(\theta_0, \theta)\cos(\theta)}{2\pi \sin\left(\frac{\theta}{2}\right)} \\
&= A_1 + A_5 + A_6 \\
&+ \frac{(D^2 f)(\theta_0, \theta+h)[\cos(\theta)\cos(h) - \sin(\theta)\sin(h)]}{2\pi \sin\left(\frac{\theta}{2}\right)} \\
&- \frac{(D^2 f)(\theta_0, \theta)\cos(\theta)}{2\pi \sin\left(\frac{\theta}{2}\right)} \\
&= A_1 + A_5 + A_6 \\
&+ \frac{(D^2 f)(\theta_0, \theta+h)(\cos(h) - 1)\cos(\theta)}{2\pi \sin\left(\frac{\theta}{2}\right)} \\
&+ \frac{\cos(\theta)[(D^2 f)(\theta_0, \theta+h) - (D^2 f)(\theta_0, \theta)]}{2\pi \sin\left(\frac{\theta}{2}\right)} \\
&- \frac{(D^2 f)(\theta_0, \theta+h)\cos\left(\frac{\theta}{2}\right)\sin(h)}{\pi} \\
&= \sum_{j=1}^{6} A_j.
\end{aligned}$$

We begin by estimating $A_1(\theta_0, \theta, h)$. Our first task is to estimate the term $\frac{1}{\sin\left(\frac{\theta+h}{2}\right)} - \frac{1}{\sin\left(\frac{\theta}{2}\right)}$. The mean value theorem yields

$$\begin{aligned}
\left| \csc\left(\frac{\theta+h}{2}\right) - \csc\left(\frac{\theta}{2}\right) \right| &= \frac{\left| \sin\left(\frac{\theta+h}{2}\right) - \sin\left(\frac{\theta}{2}\right) \right|}{\sin\left(\frac{\theta+h}{2}\right)\sin\left(\frac{\theta}{2}\right)} \\
&\leq \frac{h}{2\sin\left(\frac{\theta+h}{2}\right)\sin\left(\frac{\theta}{2}\right)}.
\end{aligned}$$

We have two cases:

Case 1: If $\theta + h \leq \pi$, then using the estimate $\sin(\theta) \geq \frac{2}{\pi}\theta$ for $\theta \in [0, \frac{\pi}{2}]$ yields

$$\frac{h}{2\sin\left(\frac{\theta+h}{2}\right)\sin\left(\frac{\theta}{2}\right)} \leq \frac{h\pi^2}{2(\theta+h)\theta}.$$

Case 2: If $\theta + h > \pi$, then $\theta > \frac{\pi}{2}$ and $\theta + h \leq \frac{3\pi}{2}$, so

$$\begin{aligned}\frac{h}{2\sin\left(\frac{\theta+h}{2}\right)\sin\left(\frac{\theta}{2}\right)} &\leq \frac{h}{2\left(\frac{1}{\sqrt{2}}\right)^2} \\ &= h \\ &< \frac{3h\pi^2}{2(\theta+h)\theta}.\end{aligned}$$

Now assume $0 < 2h < \theta < \pi$. The triangle inequality yields the following estimate for $A_1(\theta_0, \theta, h)$:

$$\begin{aligned}|A_1(\theta_0, \theta, h)| &\leq \frac{4M}{2\pi}\left|\sin\left(\frac{\theta+h}{2}\right)\right|\left(\frac{3h\pi^2}{2(\theta+h)\theta}\right) \\ &\leq \frac{4M}{2\pi}\left(\frac{\theta+h}{2}\right)\left(\frac{3h\pi^2}{2(\theta+h)\theta}\right) \\ &= \frac{3Mh\pi}{2\theta}.\end{aligned}$$

We will need the following identity to estimate $A_2(\theta_0, \theta, h)$. Note that

$$\begin{aligned}(D^2f)(\theta_0, \theta+h) - (D^2f)(\theta_0, \theta) &= f(\omega_1(\theta_0+\theta+h)) - f(\omega_1(\theta_0+\theta)) \\ &\quad + f(\omega_1(\theta_0-\theta-h)) - f(\omega_1(\theta_0-\theta)),\end{aligned}$$

so

$$|(D^2f)(\theta_0, \theta+h) - (D^2f)(\theta_0, \theta)| \leq 2M\sin\left(\frac{h}{2}\right) + 2M\sin\left(\frac{h}{2}\right)$$

$$= 4M \sin\left(\frac{h}{2}\right).$$

Since $0 < 2h < \theta < \pi$, we obtain

$$\begin{aligned} 0 &< \frac{\sin\left(\frac{h}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \\ &< \frac{h\pi}{2\theta}. \end{aligned}$$

These two observations imply

$$\begin{aligned} |A_2(\theta_0, \theta, h)| &\leq \left(\frac{4M}{2\pi}\right)\left(\frac{h\pi}{2\theta}\right) \\ &= \frac{Mh}{\theta}. \end{aligned}$$

To estimate $A_3(\theta_0, \theta, h)$ observe that

$$\begin{aligned} 0 &< \frac{\sin\left(\frac{\theta+h}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \\ &< \left(\frac{\theta+h}{2}\right)\left(\frac{\pi}{\theta}\right) \\ &< \left(\frac{3\theta}{4}\right)\left(\frac{\pi}{\theta}\right) \\ &= \frac{3\pi}{4}. \end{aligned}$$

Consequently,

$$\begin{aligned} |A_3(\theta_0, \theta, h)| &\leq \left|\frac{(D^2 f)(\theta_0, \theta + h)\sin^2\left(\frac{h}{2}\right)\cos(\theta)}{\pi \sin\left(\frac{\theta}{2}\right)}\right| \\ &\leq \frac{4M}{\pi\sin\left(\frac{\theta}{2}\right)}\left|\sin\left(\frac{\theta+h}{2}\right)\right|\sin^2\left(\frac{h}{2}\right) \\ &\leq \left(\frac{4M}{\pi}\right)\left(\frac{3\pi}{4}\right)\left(\frac{h^2}{4}\right) \\ &= \frac{3Mh^2}{4}. \end{aligned}$$

Previous work yields the following estimate for $A_4(\theta_0, \theta, h)$:

$$\begin{aligned} |A_4(\theta_0, \theta, h)| &\leq \frac{|(D^2 f)(\theta_0, \theta + h)| \sin(h)}{\pi} \\ &\leq \frac{4M}{\pi} \left| \sin\left(\frac{\theta + h}{2}\right) \right| \sin(h) \\ &\leq \frac{4Mh}{\pi}. \end{aligned}$$

To estimate $A_5(\theta_0, \theta, h)$ we will need two facts. First, by the mean value theorem there exists a $\xi \in \left(\frac{\theta}{2}, \frac{\theta+h}{2}\right)$ such that

$$\left| \cos\left(\frac{\theta + h}{2}\right) - \cos\left(\frac{\theta}{2}\right) \right| = |-\sin(\xi)| \frac{h}{2}.$$

Second, our work in Lemma 3.10 implies

$$|(Df)(\theta_0, \theta + h)| \leq 2M \sin(\theta_0).$$

Hence

$$\begin{aligned} |A_5(\theta_0, \theta, h)| &\leq \frac{2M \sin(\theta_0) h}{2\pi \sin(\theta_0)} \\ &= \frac{Mh}{\pi}. \end{aligned}$$

Estimating $A_6(\theta_0, \theta, h)$ again follows from our previous work. Note that

$$\begin{aligned} |A_6(\theta_0, \theta, h)| &\leq \frac{4M \sin(\theta_0)}{\pi \sin(\theta_0)} \\ &\leq \frac{4M}{\pi}. \end{aligned}$$

Let $\{h_n\}_{n=1}^{\infty}$ be a sequence in $\left(0, \frac{\pi}{2}\right)$ converging to 0. The continuity of f implies $\lim_{n\to\infty} A_6(\theta_0, \theta, h_n) = 0$ for every $\theta, \theta_0 \in (0, \pi)$. Setting $A(\theta_0, \theta, h) = A_6(\theta_0, \theta, h)$ proves (1) and (2). Setting $B(\theta_0, \theta, h) = \sum_{j=1}^{5} A_j(\theta_0, \theta, h)$ and adding up the estimates yields

(2) for $0 < 2h \le \theta < \pi$. If $0 < \theta < 2h < \pi$, Lemma 3.10 and the triangle inequality yields

$$\begin{aligned}|B(\theta_0,\theta,h)| &= |g(\theta_0,\theta+h) - g(\theta_0,\theta) - A(\theta_0,\theta,h)| \\ &\le |g(\theta_0,\theta+h)| + |g(\theta_0,\theta)| + |A(\theta_0,\theta,h)| \\ &= \frac{4M}{\pi} + \frac{4M}{\pi} + \frac{4M}{\pi} \\ &= \frac{12M}{\pi}\end{aligned}$$

which is (3).

Lemma 3.12: Let $0 < \theta_0 < \pi$ and $h_N = \frac{\pi}{N+\frac{3}{2}}$ for $N = 1, 2, 3, \ldots$ For every $\epsilon > 0$, there exists $N_0 = N_0(\epsilon)$ such that

$$\begin{aligned}2|(S_N f)(\omega_1(\theta_0)) - f(\omega_1(\theta_0))| \le \left(\frac{4M}{\pi} + \pi\right)\epsilon + \left(\frac{20}{\pi} + 5 + \frac{3\pi}{2}\right) M h_N \\ + \left(1 + \frac{3\pi}{8}\right) M h_N \ln\left(\frac{\pi}{h_N}\right)\end{aligned}$$

for all $0 < \theta_0 < \pi$ and all $N \ge N_0$.

Proof: Replacing θ by $\theta + h_N$ in

$$(S_N f)(\omega_1(\theta_0)) - f(\omega_1(\theta_0)) = \int_0^{\pi} g(\theta_0,\theta) \sin\left(\left(N + \frac{3}{2}\right)\theta\right) d\theta$$

yields

$$\begin{aligned}(S_N f)(\omega_1(\theta_0)) - f(\omega_1(\theta_0)) &= \int_{-h_N}^{\pi - h_N} g(\theta_0,\theta + h_N) \sin\left(\left(N + \frac{3}{2}\right)(\theta + h_N)\right) d\theta \\ &= -\int_0^{\pi} g(\theta_0,\theta + h_N) \sin\left(\left(N + \frac{3}{2}\right)(\theta + h_N)\right) d\theta + E_N\end{aligned}$$

where $|E_N| \leq \frac{8M}{\pi} h_N$. To see this, E_N is the sum of two integrals, each over an interval of length h_N. Therefore the estimate is a direct consequence of Lemma 3.10. Adding the two expressions for $(S_N f)(\omega_1(\theta_0)) - f(\omega_1(\theta_0))$ yields

$$\begin{aligned} 2((S_N f)(\omega_1(\theta_0)) - f(\omega_1(\theta_0))) = &\int_0^\pi (g(\theta_0,\theta) - g(\theta_0, \theta + h_N)) \\ &\times \sin\left(\left(N + \frac{3}{2}\right)(\theta + h_N)\right) d\theta + E_N. \end{aligned}$$

Lemma 3.11 yields

$$\begin{aligned} |2((S_N f)(\omega_1(\theta_0)) - f(\omega_1(\theta_0)))| \leq &\left| \int_0^\pi A(\theta_0, \theta, h_N) \sin\left(\left(N + \frac{3}{2}\right)(\theta + h_N)\right) d\theta \right| \\ &+ \int_0^{h_N} |B(\theta_0,\theta,h_N)| \left| \sin\left(\left(N + \frac{3}{2}\right)(\theta + h_N)\right)\right| d\theta \\ &+ \int_{h_N}^{\pi} |B(\theta_0,\theta,h_N)| \left| \sin\left(\left(N + \frac{3}{2}\right)(\theta + h_N)\right)\right| d\theta \\ &= I_1 + I_2 + I_3, \end{aligned}$$

respectively. Part (3) of Lemma 3.11 yields for all $N \geq 1$,

$$I_2 + I_3 \leq \frac{12M}{\pi} h_N + \left(\frac{3}{2} + \frac{5}{\pi}\right) \pi M h_N + \left(1 + \frac{3\pi}{2}\right) M h_N \ln\left(\frac{\pi}{h_N}\right).$$

Therefore for any sequence h_N tending to zero as $N \to \infty$ we conclude $I_2 + I_3$ tends to zero uniformly for every $\theta_0 \in (0,\pi)$. Given any $\epsilon > 0$, Egoroff's theorem and the continuity of f guarantees the existence of a measurable set $E \subset (0,\pi)$ such that $m(E) < \epsilon$ and $A(\theta,\theta_0,h_N) \to 0$ as $N \to \infty$ uniformly for $\theta_0 \in (0,\pi)$ and $\theta \in (0,\pi)\setminus E$. Choose $N_0 = N(\epsilon)$ such that $|A(\theta,\theta_0,h_N)| < \epsilon$ for all $\theta_0 \in (0,\pi)$ and $\theta \in (0,\pi) \setminus E$

and $N \geq N_0$. For such N we have

$$\begin{aligned} I_1 &\leq \int_E |A(\theta,\theta_0,h_N)| \left| \sin\left(\left(N+\frac{3}{2}\right)(\theta+h_N)\right)\right| d\theta \\ &+ \int_{(0,\pi)\setminus E} |A(\theta,\theta_0,h_N)| \left| \sin\left(\left(N+\frac{3}{2}\right)(\theta+h_N))\right)\right| d\theta \\ &\leq \frac{4M}{\pi} m(E) + \pi\epsilon \\ &< \left(\frac{4M}{\pi} + \pi\right)\epsilon. \end{aligned}$$

Hence $(S_N f)(\omega_1(\theta_0)$ converges uniformly to $f(\omega_1(\theta_0))$ as $N \to \infty$.

Remark: A more general version of this theorem is proved in [Be] for Jacobi Series using different techniques.

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