A Comparison and Study on Artificial Boundaries: Conceptual Aspects

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Proceedings: Second International Conference on Recent Advances in Geotechnical Earthquake Engineering and Soil Dynamics, March 11-15, 1991, St. Louis, Missouri, Paper No. 5.63

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SYNOPSIS: The paper firstly analyzes the characteristics of various artificial boundaries (similarity and dissimilarity) and the derivation conditions of the boundaries. The problems involved in applying the artificial boundary in numerical computation and the solution to the problem are also discussed in the paper.

THE ARTIFICIAL BOUNDARIES

About thirty years have elapsed since the appearance of artificial boundaries. In these years many artificial boundaries have been proposed to simulate wave motion in infinite media by a finite model. It is therefore necessary to summarize the advantage and disadvantage, to find out the similarity and dissimilarity, to distinguish the application range and to predict the future development trend of the various artificial boundaries. Based on the above consideration, the paper tries to take a general but thorough look at these ones and to find out some useful information for future research.

Artificial bopundaries are the artifacts used to meet the need of numerical analysis of dynamic problems in infinite media. They obviously bear two basic features as following: one is the nonexistence, the other is that they have to satisfy all the continuous conditions at the boundary. In order to be suitable for numerical analysis, artificial boundaries must also satisfy the following requirements: one is that they have to be conceptually clear, simple and easy to be applied, another is that they should be highly accurate and stable, still another is that they are frequency independent. Based on these considerations, researchers have proposed many artificial boundaries up to the present. The author thinks they can be divided into the following groups: superimposing boundary (Smith,1974; Cundall, etc.,1978), transmitting boundaries (Liao and Wong, 1984; Tseng,1975; Akao and Hakuno,1983), viscous boundary (Akiyoshi,1977; Lysmer and Kulemeyer, 1969; White,1977), paraxial approximation (Clayton and Engquist,1977;Keys,1985), consistent boundary (Lysmer and Waas, 1972), infinite element (Zhao,1987) and boundary element in infinite media. The paper will discuss their similarity and dissimilarlty in various aspects in the following sections.

MATHEMATICAL RELATIONS OF ARTIFICIAL BOUNDARY

The derivation of any artificial boundary has to take these two steps: the first is the decomposition of wave field, the second is the approximation of the out-going (scattering) wave field (wave field that travels from the inner computatioal region to the outside). A traveling wave field can be decomposed into two parts: one is the input wave field that travels toward the computational region, the other is the scattering wave field that travels from the computational region. It is obvious that different decomposition or different approximation of the wave field will lead to different artificial boundary. There are many methods that can be used to decompose a wave field. Generally speaking, scattering wave field in time domain equals to the total wave field substracting the input wave field, and that in frequency domain can be deduced from the negative or positive value of the wave number. Many approximation methods could be used to approximate the scattering wave field such as Taylor expansion, Pade expansion or even the interposition approximation. Nevertheless, we can verify that in the limit of continuum, the mathematical foundation of all the artificial boundaries for SH problems may be expressed by a unified equation as following

$$\text{Basic form:} \quad \frac{\partial w}{\partial x} + p\frac{\partial w}{\partial t} = Bw = 0 \qquad (1)$$

$$\text{High order form:} \quad B^{N} w = 0 \qquad (2)$$

where w is the anti-plane displacement, B a differentiate operator, p a parameter which varies with the form of the artificial boundary, N the order of the boundary. The first term in eq.(1) expresses the elastic force while the second one expresses the viscous force. Eq.(1) demonstrates that the physical base of all artificial boundaries is that they directly or indirectly realize the dynamic equilibrium of the elastic force and viscous force, that is to say, this treatment indirectly realize the continuous condition at the artificial boundary.

The reason why all the artificial boundary have the above similarity is that they are the approximation of the scattering wave field at the boundary, their objective is to simulate the infinite media with a boundary of a finite region. It is because the common objective and base that enables the same mathematical relationship to exist among all the artificial boundaries. This common feature makes us conclude that an optimum artificial boundary must utilize the most applicable approximation method to simulate the scattering wave field under the simplest and most accurate decomposition method. This is the fundamental guide in deducing any new artificial boundaries.

ERRORS OF ARTIFICIAL BOUNDARY

In addition to the similarity of artificial boundaries, dissimilarity of them should be particularly stressed. It is only because the dissimilarity can we compare the artificial boundary, thus we are able to select the most applicable artificial boundaries. Error of artificial boundaries is one of the major aspects which should be considered in comparing artificial boundaries. Let us suppose a unit harmonic plane wave impinging on the artificial boundary from the inner computational region under certain direction, the relative error of artificial boundary can be written as

$$R = \frac{|u_i| - |u_t|}{|u_i|} = \frac{|u_r|}{|u_i|} \qquad (3)$$

where u_i, u_t and u_r represent input, transmitting and reflected wave field, respectively, R the relative error. Further analytical results demonstrate that when the impinging direction of the wave field approaches to parallel the artificial boundary the accuracy of the boundary will be greatly declined or even the boundary does not work. Specificly, for a harmonic wave directly impinging on a boundary (without multi-reflection), superimposing boundary does not produce any error. For transmitting boundaries when the ratio of the time increment Δt and wave motion period T is less than 1/6 ($\Delta t/T < 1/6$, Liao's boundary can simulate the impinging wave field under any direction. The accuracy of the boundary increases with the declination of $\Delta t/T$. Given the impinging direction of the wave field and when $\Delta t/T \leq 1/3$, Akao's boundary is sufficiently accurate, whereas, when the impinging direction is unknown, the accuracy of Akao's boundary sharply declines. Tseng's boundary doesn't produce any error when a harmonic wave directly impinging on the boundary. For viscous boundary and paraxial approximation the error is strictly dependent on the impinging direction. When impinging direction $\theta = 0$, which means that the wave field perpendicularly impinging on the boundary, $R = 0$, while $R = 1$ when $\theta = 90°$, which implies that viscous boundary and paraxial approximation have lost their effect. It is interesting to note that when $\Delta t/T \leq 1/4$, the error of Liao's boundary is always smaller than that of paraxial approximation of the same order. The error of infinite element is generally larger than that of the other artificial boundaries. For the consistent boundary proposed by Lysmer and Waas, the error is also significant. Only when discretization of the boundary satisfies certain condition can it produces good result. This conclusion contradicts that of the common recognition. The reason is that, physically speaking, the consistent boundary is only a method which uses a one-dimensional model to simulate the wave motion in two-dimensional media (Wang,1989) . This assumption greatly weakens the accurateness of the consistent boundary. If we only consider the error of the artificial boundaries the author intends to use transmitting boundary for complicatd wave field. This is because the transmitting boundaries are simple and always guarantee accuracy under specific conditions.

SOME BASIC PROBLEMS OF ARTIFICIAL BOUDARY FOR IN-PLANE PROBLEMS

Application of artificial boundary for anti-plane problems in in-plane problems will bring about several totally different subjects to be carefully studied. We will concentrate here on the following problems.

1. Artificial Wave Velocity

In numerical computation discretization of time and space causes dispersion of wave motion which lead to the introduction of artificial wave velocity. This is another numerical problem whose discussion is beyond the range of this paper. In this paper we will focus on the physically conceptual aspects. For an in-plane problem wave motin is composed of P-wave, S-wave, Rayleigh wave and other kind of waves. Therefore, in deducing the formula for transmitting boundaries we have to make the following assumption: wave motion is consisted of a series of plane waves with velosity c_a impinging on the artificial boundary, thus the artificial wave velocity c_a emerged. The appearance of c_a makes another contribution to enlarge the computational error and destablize the numerical analysis of infinite problems. Theoretical and computational results demonstrate that reasonable upper and lower limit of artificial wave velocity should be c_p and $c_s/2$, respectively, where c_p and c_s represent P-wave and S-wave velocity, respectively.

2. Stability

Numerical stability has always attracted the attention of researchers worldwide, no less to mention the stability of artificial boundaries.Generally there are two ways to consider the stability of artificial boundaries. One is the propagation of error wave. This means that numerical error produced at boundary nodes will not be enlarged in propagation. The other is the reflection coefficient (the ratio of reflected wave amplitude and impinging wave amplitude) of the artificial boundary is always less than 1. Starting from these two considerations, we can derive the stability condition for various artificial boundaries. Analytical results demonstrate that if the wave field can not be expanded by a series of plane waves all the artificial boundaries may encounter the problem of losing stability.

3. Some Consideration in Deriving Artificial Boundaries for In-Plane Problems

The complexity of wave field in in-plane problems often compells us to start from the very basic assumption in deriving the artificial boundaries, that is, to reconsider the physical interpretation of the artificial boundaries. Take the paraxial approximation as an example Owing to the difficulty in directly deriving the paraxial approximation from expanding the dispersion relations of in-plane problems, we have to reconsider the physical foundation of the paraxial approximation, namely, the expansion of wave field by a series of plane waves perpendicularly impinging on the boundary. Similarly for consistent boundary the problem of solving the eigenvalue of a second order equation arises. The appearance of and solution to these problems provide us with favorable conditions to understand some fundamental characteristics(like dispersion, accuracy of simulation) of wave motion and the physical foundation of artificial boundaries.

APPLICATION OF ARTIFICIAL BOUNDARIES IN IN-PLANE PROBLEMS

The first problem we encounter in application of artificial boundaries in in-plane problems is discretization. Discretization brings about several questions to be answered which do not exist in continuum. They include dispersion, cut-off frequency and parasitic oscillation of elastic wave motion which are caused by discretization. Their effect on the final result is systematically discussed by J.B.Liu(1989). Discretization produces error in computation and further results in destability. We should note that the result by discrete model is valid only under certain frequency (that is , the cut-off frequency). This is important in modifying transmitting boundaries in order to get stable and accurate results. Further results on these topics may be found in the article by J.B.Liu(1989).

For the clarity and concreteness of the conclusions made in this article we make the following assumptions: (1) the problem considered is a source problem with a cavity while the source is acting directly on the surface of the cavity; (2) the computational formula for inner nodes adopts the pattern by an explicit finite difference-finite element method (further description of the formula may be found in the paper by J.B.Liu(1989)); (3) the application form of the artificial boundary also takes the form of explicit finite difference in order to match that for inner nodes. We will concentrate here on the accuracy and stability in the application of artificial boundary.

1. Superimposing Boundary

Compared with free or fixed boundary the superimposing boundary increases the accuracy of computation, but wave field reflected from the boundary is still obvious, which means that the accuracy of the superimposing boundary should be improved. We think there are two reasons why the superimposing boundary is not accurate enough. One is that the superimposing boundary requires the reflected wave field to be plane waves, but the real reflected one is cylindrical in a short distance from the boundary according to Hyghens' principle. The other is that for a source problem the input wave field can not be a plane one. The advantage of the superimposing boundary is that it scarcely causes destability.

2. Transmitting Boundaries

Generally the accuracy of transmitting boundaries is very high. The problem is that they often lead to destability of the computation. For a source problem the destability is often caused by high frequency wave motion in the iscrete grids. J.B.Liu(1989) has clarified in a one-dimensional model that the destability is caused by multi-reflections in the discrete model, and the multi-reflection results from small error introduced at the boundary. This is because transmitting boundaries are highly sensitive to the change of the input wave field Presently there are two ways to treat this kind of destability. The first is the filteration of high frequency wave motion, that is , the filtering of high frequency wave motion by a modified equation in a discrete model. This modification bases on two recognitions of wave motion in a discrete model: (1) discrete model can only simulate wave motion under the cut-off frequency; (2) destability is often caused by high frequency wave motion in discrete models. The modification can be written as (valid for all the nodes in the boundary region except those at the boundary)

$$u_i = r_1 u_i + (1-r_1)(u_{i-1}+u_{i+1})/2 \qquad (4)$$

where r_1 is the modification factor with reasonable range 0.5+ 1.0, u the displacement, subscript of u indicates the node number. The second way to treat destability may be called as the modification of inhomogeneous wave field. This is to restrain the amplification of inhomogeneous wave motion at the boundary nodes. Modification of the displacement at nodes in the boundary region except those at the boundary could be put as the following

$$u_{L-1} = u_{L-1} * r'_2 \qquad (5)$$

where L is the boundary node number, r_2 a modifying factor with reasonable range 0.5+ 1.0. Numerical results demonstrat that the above two treatments have very good effect on stablizing the computation without any noticable change of the accuracy.

3. Viscous Boundary and Paraxial Approximation

For viscous boundary when the velocity is expanded by central difference method in time domain the computational formula is stable and fairly accurate, other forms of explicitly expanding the velocity cannot result in stable computation. For paraxial approximation directly expanding the approximation equation by explicit finite difference method cannot lead to stable computation. This may be due to the fact that the computational pattern at the boundary does not match that at inner nodes. Expanding the approximation by explicit finite element method still produces not very accurate results. This brings about a problem to be solved: how to implement the paraxial approximation in the explicit finite element computation.

4. Infinite Element

Application of infinite element in time domain can only consider the decaying factor but leaves out the wave propagation factor, thus the result is surely not accurate enough. Further research demonstrates that the infinite element method is not accurate compared with the other artificial boundaries. This conclusion is particularly true in time domain computation (This is valid for dynamic mapping infinite element).

SUMMARY

In the above sections we have discussed in various aspects the relations, errors, stability and application of the most widely used artificial boundaries. From the analysis we can conclude that the derivation and application of any artificial boundary have to undergo the following steps: (1) decomposition of wave field (this step is sometimes directly realized in the numerical computation, the objective is to identify the direction of wave propagation); (2) approximation of the outgoing wave field (this is the artificial boundary condition); (3) error analysis of the approximation; (4) application and stability analysis of the approximation in in-plane problems; (5) numerical test of the approxiamtion and effective method to stablize the computation. Dissatisfaction of any requirement in the five steps compells the proposer of artificial boundary to reconsider the decomposition of wave field and the approximation of the outgoing wave field thus enables researchers to propose better and more applicable artificial boundaries. The author thinks that the artificial boundaries proposed presently have basically met the requirement of numerical analysis in engineering. The main objective at the present time is to apply and implement the already existent method, especially the solution to the problems related to stability. Whereas, with the development of science and technology new artificial boundaries will be proposed. No matter what form the artificial boundaries adopt when they appear, they are always to achieve the same goal: to realize the continuous conditions at the boundary for computation.

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