Coupled Horizontal and Rocking Vibration of Block Foundations

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ABSTRACT

The computation of coupled response of block foundations subjected to a constant magnitude harmonic horizontal force or moment or both, is very important for the design of block foundations. Mathematical relationships for computing response of rigid block foundations using the elastic half-space method are available but are not in an easily understandable format. More over relationships available for embedded foundation are for the case when both horizontal force and moment are acting on the foundation. To extract the solution for the case when either horizontal force or moment alone is acting from the existing relationships is extremely difficult, if not impossible.

A unified general mathematical solution has been developed to compute the response of block foundation resting on the surface of elastic half-space and/or embedded into the elastic half space with a soil side layer, and rigid block resting on piles, subjected to a harmonic exciting function at the top of foundation. Relationships have been put in a format easily understandable by practicing engineers.

INTRODUCTION:

It has been recognized that neither rocking nor sliding alone of a rigid block foundation is an ideal condition of vibration of a system. In actuality the motion of such a foundation excited by a horizontal force and/or a moment involves simultaneous rocking and sliding i.e. coupled motion.

Natural frequencies and damped and undamped amplitudes can be obtained by solving the equations of motion, considering the limiting equilibrium of the exciting and resisting forces and moments in terms of Newton's Second Law. Analytical solutions for the elastic half-space method are available (Prakash and Puri, 1988) but are not in a format easily understandable by practicing engineers. Different relationships are being used for surface foundations and embedded foundations whereas surface foundation is a special case of embedded foundation when depth of embedment equals zero. The solution available for embedded foundation is for the case when both horizontal force and a moment are acting on the foundation. It is extremely difficult to extract the solution from the existing relationships for the case when either horizontal force or moment alone is acting.

In this paper a unified general mathematical solution has been developed to compute damped and undamped natural frequencies and amplitudes of the block foundations subjected to constant magnitude harmonic horizontal force and/or moment at the top of the foundation. Three cases have been considered:

a. Rigid block resting on elastic half-space
b. Rigid block embedded into elastic half-space with soil side layer, and
c. Rigid block resting on piles

It should be noted that only relationships for computing the natural frequencies and amplitudes have been developed and reported in this paper. For design, the stiffness and damping coefficients are required for different modes of vibrations which are beyond the scope of this paper. Reader is recommended to refer Prakash and Puri (1988), Beredugo and Novak (1972), and Gazetas (1991) to compute the stiffness and damping coefficients.

ELASTIC HALF-SPACE METHOD:

The elastic-half space method idealizes the block foundation as a vibrating mechanical oscillator with a circular base resting on the surface of the ground which is assumed to be an elastic, homogeneous, isotropic, semi-infinite body.

EQUATIONS OF MOTION:

a. <u>Rigid Block Resting on Elastic Half-Space:</u>

Figure 1 illustrates the conditions of a rigid circular block that rests on the surface of an elastic half-space and is excited by a vertical moment $M_y(t) = M_y e^{i\omega t}$ and a horizontal force $P_x(t) = P_x e^{i\omega t}$. Let the center of gravity of the footing lie on the vertical axis which passes through the center of the circular base, at a height L above the surface of the half-space.

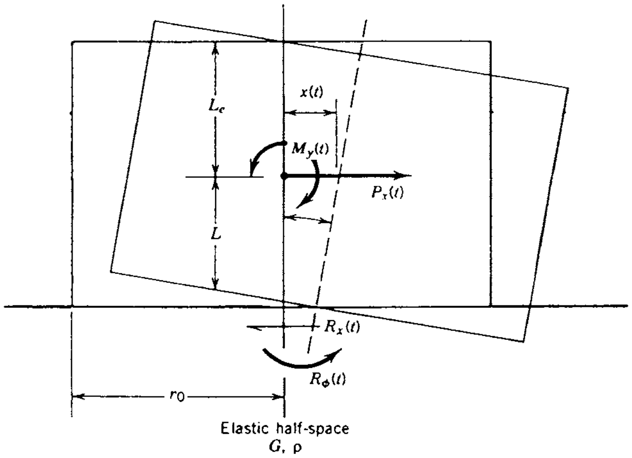

Figure 1. Mathematical model of rigid block foundation resting on elastic half-space in coupled motion

The motion of the block may be expressed in terms of translation x of the center of gravity and the rotation angle ϕ. The equation of motion may be obtained by considering the limiting equilibrium of the exciting and resisting forces and moments in terms of Newton's second law.

The equation of motion in sliding is

$$m\ddot{x} = P_x e^{i\omega t} - R_x \tag{1}$$

where,

$$R_x = c_{x_b}(\dot{x} - L\dot{\phi}) + k_{x_b}(x - L\phi)$$

Therefore, Equation 1 yields;

$$m\ddot{x} + c_{x_b}\dot{x} + k_{x_b}x - Lc_{x_b}\dot{\phi} - Lk_{x_b}\phi = P_x e^{i\omega t} \tag{2}$$

The equation of motion in rocking is

$$M_m\ddot{\phi} = M_y e^{i\omega t} - R_\phi + LR_x \tag{3}$$

where,

$$R_\phi = c_{\phi_b}\dot{\phi} + k_{\phi_b}\phi$$

Therefore, Equation 3 becomes

$$M_m\ddot{\phi} + c_{\phi_b}\dot{\phi} + k_{\phi_b}\phi - L\left[c_{x_b}(\dot{x} - L\dot{\phi}) + k_{x_b}(x - L\phi)\right] = M_y e^{i\omega t}$$

or

$$M_m\ddot{\phi} + c_{\phi_b} + L^2 c_{x_b}\dot{\phi} + \left(k_{\phi_b} + L^2 k_{x_b}\right)\phi - Lc_{x_b}\dot{x} - Lk_{x_b}x = M_y e^{i\omega t} \tag{4}$$

In the above equations

m = total mass of the footing
M_m = mass moment of inertia about a horizontal axis passing through the center of gravity
c_{x_b} = damping coefficient of base in sliding
c_{ϕ_b} = damping coefficient of base in rocking
k_{x_b} = stiffness coefficient of base in sliding
k_{ϕ_b} = stiffness coefficient of base in rocking
ω = frequency of the exciting function

Writing Equations 2 and 4 in matrix form

$$\begin{bmatrix} m & 0 \\ 0 & M_m \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} c_{x_b} & -Lc_{x_b} \\ -Lc_{x_b} & c_{\phi_b} + L^2 c_{x_b} \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{\phi} \end{Bmatrix} + \begin{bmatrix} k_{x_b} & -Lk_{x_b} \\ -Lk_{x_b} & k_\phi + L^2 k_{x_b} \end{bmatrix}\begin{Bmatrix} x \\ \phi \end{Bmatrix} = \begin{Bmatrix} P_x \\ M_y \end{Bmatrix} e^{i\omega t} \tag{5}$$

b. <u>Rigid Block Embedded into Elastic Half-Space with Soil Side Layer:</u>

For an embedded foundation, the soil resistance is mobilized both below the base and on the sides. The additional soil resistance that comes into play on the sides of an embedded footing may have a significant effect on its dynamic response.

The earlier solutions for computing the dynamic response of embedded foundations using elastic half-space method was obtained by Barnov (1967), and has been extended by Novak and Beredugo (1971), Beredugo and Novak (1971), Beredugo (1971, 1976), and Novak and Sachs (1973). The following assumptions are used as a basis for solution:

1. The foundation is rigid
2. The foundation is cylindrical
3. The base of the foundation rests on the surface of a semi-infinite elastic half space, and the soil reactions at the base are independent of the depth of embedment
4. The soil reactions on the side are produced by an independent elastic layer lying above the level of foundation base, and
5. There is a perfect bond between the sides of the foundation and soil

Consider the coupled sliding and rocking vibrations of an embedded footing shown in Figure 2. The footing is acted upon by a horizontal force, $P_x(t) = P_x e^{i\omega t}$ and a vertical moment $M_y(t) = M_y e^{i\omega t}$ about its center of gravity. The forces and moments occasioned by the soil reactions at the base and sides are shown. The equations of motion in sliding and rocking in terms of Newton's second law are;

In sliding

$$m\ddot{x} = P_x e^{i\omega t} - R_x - N_x \tag{6}$$

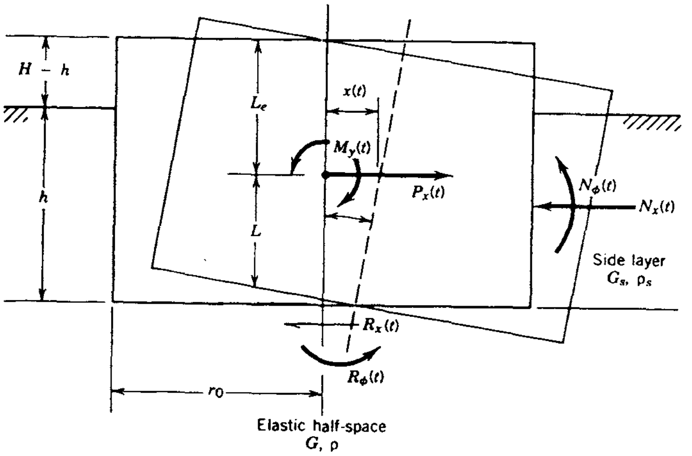

Figure 2. Mathematical model of rigid block embedded in elastic half-space with soil side layer in coupled motion

where R_x and N_x are as given below.

$$R_X = c_{x_b}(\dot{x}-L\dot{\phi}) + k_{x_b}(x-L\phi)$$

$$N_x = c_{x_s}\left[\dot{x} - \left(L-\frac{h}{2}\right)\dot{\phi}\right] + k_{x_s}\left[x - \left(L-\frac{h}{2}\right)\phi\right]$$

Substituting R_x and N_x in Equation 6, the equation of motion in sliding becomes

$$m\ddot{x} + \left(c_{x_b} + c_{x_s}\right)\dot{x} + \left(k_{x_b}+k_{x_s}\right)x - \left[Lc_{x_b} + \left(L-\frac{h}{2}\right)c_{x_s}\right]\dot{\phi} - \left[Lk_{x_b} + \left(L-\frac{h}{2}\right)k_{x_s}\right]\phi = P_x e^{i\omega t} \qquad (7)$$

In rocking, the equation of motion is

$$M_m\ddot{\phi} = M_y e^{i\omega t} - R_\phi - N_\phi + N_x\left(L-\frac{h}{2}\right) + LR_x \qquad (8)$$

where R_ϕ and N_ϕ are as given below and N_x is same as used above.

$$R_\phi = k_{\phi_b}\phi + c_{\phi_b}\dot{\phi}$$

$$N_\phi = k_{\phi_s}\phi + c_{\phi_s}\dot{\phi}$$

Therefore, the equation of motion in rocking becomes

$$M_m\ddot{\phi} + \left[\left(k_{\phi_b}+k_{\phi_s}\right) + L^2k_{x_b} + k_{x_s}\left(l-\frac{h}{2}\right)^2\right]\phi + \left[\left(c_{\phi_b}+c_{\phi_s}\right) + L^2c_{x_b} + c_{x_s}\left(L-\frac{h}{2}\right)^2\right]\dot{\phi} - \left[Lk_{x_b} + k_{x_s}\left(L-\frac{h}{2}\right)\right]x - \left[Lc_{x_b} + c_{x_s}\left(L-\frac{h}{2}\right)\right]\dot{x} = M_y e^{i\omega t} \qquad (9)$$

Writing Equations 7 and 9 in a matrix form

$$\begin{bmatrix} m & o \\ 0 & M_m \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} c_{x_b}+c_{x_s} & -\left[Lc_{x_b} + \left(L-\frac{h}{2}\right)c_{x_s}\right] \\ -\left[Lc_{x_b} + \left(L-\frac{h}{2}\right)c_{x_s}\right] & \left[\left(c_{\phi_b}+c_{\phi_s}\right) + L^2c_{x_b} + c_{x_s}\left(L-\frac{h}{2}\right)^2\right] \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{\phi} \end{Bmatrix} + \begin{bmatrix} k_{x_b}+k_{x_s} & -\left[Lk_{x_b} + \left(L-\frac{h}{2}\right)k_{x_s}\right] \\ -\left[Lk_{x_b} + \left(L-\frac{h}{2}\right)k_{x_s}\right] & \left[\left(k_{\phi_b}+k_{\phi_s}\right) + L^2k_{x_b} + k_{x_s}\left(L-\frac{h}{2}\right)^2\right] \end{bmatrix}\begin{Bmatrix} x \\ \phi \end{Bmatrix} = \begin{Bmatrix} P_x \\ M_y \end{Bmatrix} e^{i\omega t} \qquad (10)$$

In the above equations, terms with subscript s, are associated with the soil side layer.

Careful inspection of Equations 5 and 10 shows that, if the terms associated with the embedment of the footing are set to zero, then equation 10 leads to exactly the same as Equation 5. Hence it can be inferred that Equation 5 is a special case of Equation 10 when the depth of embedment is zero i.e. **rigid block resting on the elastic half-space.**

The presence of off-diagonal terms in Equations 5 and 10 shows the coupling of sliding and rocking. If the off diagonal terms are replaced with a cross coupling terms $c_{x\phi} = c_{\phi x}$ for a damping matrix and $k_{x\phi} = k_{\phi x}$ for a stiffness matrix, and the diagonal terms are replaced by corresponding single mode of vibration terms, then both the Equations 5 and 10 can be represented by just one equation i.e. Equation 11.

$$\begin{bmatrix} m & o \\ o & M_m \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} c_x & -c_{x\phi} \\ -c_{\phi x} & c_\phi \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{\phi} \end{Bmatrix} + \begin{bmatrix} k_x & -k_{x\phi} \\ -k_{\phi x} & k_\phi \end{bmatrix}\begin{Bmatrix} x \\ \phi \end{Bmatrix} = \begin{Bmatrix} P_x \\ M_y \end{Bmatrix} e^{i\omega t} \qquad (11)$$

where,

for surface block foundation

$$c_x = c_{x_b}$$

$$c_{x\phi} = c_{\phi x} = Lc_{x_b}$$

$$c_\phi = c_{\phi_b} + L^2c_{x_b}$$

$$k_x = k_{x_b}$$

$$k_{x\phi} = k_{\phi x} = Lk_{x_b}$$

$$k_\phi = k_{\phi_b} + L^2 k_{x_b}$$

and for embedded block foundation

$$c_x = c_{x_b} + c_{x_s}$$

$$c_{x\phi} = c_{\phi x} = \left[Lc_{x_b} + \left(L - \frac{h}{2}\right)c_{x_s}\right]$$

$$c_\phi = \left[\left(c_{\phi_b} + c_{\phi_s}\right) + L^2 c_{x_b} + c_{x_s}\left(L - \frac{h}{2}\right)^2\right]$$

$$k_x = k_{x_b} + k_{x_s}$$

$$k_{x\phi} = k_{\phi x} = \left[Lk_{x_b} + \left(L - \frac{h}{2}\right)k_{x_s}\right]$$

$$k_\phi = \left[\left(k_{\phi_b} + k_{\phi_s}\right) + L^2 k_{x_b} + k_{x_s}\left(L - \frac{h}{2}\right)^2\right]$$

This shows that the relationships required to compute natural frequencies and amplitudes for a rigid block foundation either resting on the surface of elastic half-space or embedded in elastic half-space with soil side layer are same. The only difference is the way the stiffness and damping terms are represented. Different researchers have recommended different relationships for computing stiffness and damping coefficients for uncoupled and coupled mode of vibrations. Simplification of relationships given by Novak and Beredugo (1972), shows that the representation of diagonal and off-diagonal terms as above is acceptable.

c. <u>Rigid Block Resting on Piles:</u>

There are many cases where it is necessary to support foundation on piles. If it is assumed that there is no contact of the footing with the bottom soil, then the stiffness and damping available at the base is only from piles. If it is further assumed that the stiffness of piles is available at the top of pile i.e. pile head, then same equations as that for surface or embedded foundations can be used with a little modification in the relationships for calculating the stiffness and damping coefficients. In conclusion, Equation 11 can be used for computing the natural frequencies and the amplitudes of all the three case of block foundations.

DAMPED AND UNDAMPED NATURAL FREQUENCIES

Damped and undamped natural frequencies,can be derived from Equation 11, by equating the right hand side of the equation to zero. So Equation 11 becomes

$$\begin{bmatrix} m & o \\ o & M_m \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} c_x & -c_{x\phi} \\ -c_{\phi x} & c_\phi \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{\phi} \end{Bmatrix} + \begin{bmatrix} k_x & -k_{x\phi} \\ -k_{\phi x} & k_\phi \end{bmatrix}\begin{Bmatrix} x \\ \phi \end{Bmatrix} = \begin{Bmatrix} o \\ o \end{Bmatrix} \quad \textbf{(12)}$$

Solution of this equation may be obtained by substituting

$$x = A_x e^{i\omega t} \quad \textbf{(13a)}$$

$$\phi = A_\phi e^{i\omega t} \quad \textbf{(13b)}$$

then

$$\dot{x} = i\omega A_x e^{i\omega t} \quad \textbf{(13c)}$$

$$\dot{\phi} = i\omega A_\phi e^{i\omega t} \quad \textbf{(13d)}$$

and

$$\ddot{x} = \omega^2 A_x e^{i\omega t} \quad \textbf{(13e)}$$

$$\ddot{\phi} = \omega^2 A_\phi e^{i\omega t} \quad \textbf{(13f)}$$

where,

A_ϕ = The rotational amplitude in radian around the combined center of gravity

A_x = linear horizontal amplitude of the combined center of gravity

Therefore equation 12 becomes

$$\begin{bmatrix} (k_x - m\omega^2) + i\omega c_x & k_{x\phi} + i\omega c_{x\phi} \\ k_{x\phi} + i\omega c_{x\phi} & (k_\phi - M_m\omega^2) + i\omega c_\phi \end{bmatrix}\begin{Bmatrix} A_x \\ A_\phi \end{Bmatrix} e^{i\omega t} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \textbf{(14)}$$

Since $e^{i\omega t} \neq 0$, for Equation 14 to be true, determinant of the matrix should be zero i.e.

$$\begin{vmatrix} (k_x - m\omega^2) + i\omega c_x & k_{x\phi} + i\omega c_{x\phi} \\ k_{x\phi} + i\omega c_{x\phi} & (k_\phi - M_m\omega^2) + i\omega c_\phi \end{vmatrix} = 0 \quad \textbf{(15)}$$

Simplification of Equation 15 gives

$$\left[\left\{\omega^4 - \left(\frac{k_x}{m} + \frac{k_\phi}{M_m} + \frac{c_x c_\phi}{mM_m} - \frac{c_{x\phi}^2}{mM_m}\right)\omega^2 + \left(\frac{k_x k_\phi}{mM_m} - \frac{k_{x\phi}^2}{mM_m}\right)\right\}^2 + \left\{\left(\frac{k_x c_\phi}{mM_m} + \frac{k_\phi c_x}{mM_m} - \frac{2k_{x\phi}c_{x\phi}}{mM_m}\right)\omega - \left(\frac{c_\phi}{M_m} + \frac{c_x}{m}\right)\omega^3\right\}^2\right]^{1/2} = 0 \quad \textbf{(16)}$$

Two positive roots of equation 16, will give two **damped natural frequencies**.

To find undamped natural frequencies, set all the terms associated with damping equal to zero and replacing ω with ω_n, Equation 16, becomes

$$\left(\omega_n^2\right)^2 - \left(\frac{k_x}{m}+\frac{k_\Phi}{M_m}\right)\omega^2 + \left(\frac{k_x k_\Phi}{mM_m}-\frac{k_{x\Phi}^2}{mM_m}\right) = 0 \qquad (17)$$

Equation 17, is quadratic in ω_n^2. Solving for ω_n^2 gives

$$\omega_{n1,2}^2 = \frac{1}{2}\left(\frac{k_x}{m}+\frac{k_\Phi}{M_m}\right) \pm \sqrt{\frac{1}{4}\left(\frac{k_x}{m}-\frac{k_\Phi}{M_m}\right)^2+\frac{k_{x\Phi}^2}{mM_m}} \qquad (18)$$

Two positive roots of Equation 18 will give two **undamped natural frequencies**.

DAMPED AMPLITUDES:

Damped amplitudes of rocking and sliding can be found by solving Equation 14 for either an exciting force $P_x(t) = P_x e^{i\omega t}$ or moment $M_y(t) = M_y e^{i\omega t}$

a. <u>When Excited by Horizontal Force Only</u>:

Solving equation 14 for only horizontal exciting force gives the amplitudes as follows

$$A_x = \frac{P_x}{mM_m}\frac{\left[\left(k_x - m\omega^2\right)^2 - \omega^2 c_x^2\right]^{1/2}}{\Delta} \qquad (19a)$$

and

$$A_\Phi = \frac{P_x}{mM_m}\frac{\left(k_{x\Phi}^2 - \omega^2 c_{x\Phi}^2\right)^{1/2}}{\Delta} \qquad (19b)$$

b. <u>When Excited by Vertical Moment Only</u>:

Solving Equation 14 for vertical moment only, gives the amplitudes as follows

$$A_x = \frac{M_y}{mM_m}\frac{\left(k_{x\Phi}^2 - \omega^2 c_{x\Phi}^2\right)^{1/2}}{\Delta} \qquad (20a)$$

and

$$A_\Phi = \frac{M_y}{mM_m}\frac{\left[\left(k_\Phi - M_m\omega^2\right)^2 - \omega^2 c_\Phi\right]^{1/2}}{\Delta} \qquad (20b)$$

where, Δ is the determinant of Equation 14 and is given by Equation 16. i.e.

$\Delta =$

$$\left[\left\{\omega^4 - \left(\frac{k_x}{m}+\frac{k_\Phi}{M_m}+\frac{c_x c_\Phi}{mM_m}-\frac{c_{x\Phi}^2}{mM_m}\right)\omega^2 + \left(\frac{k_x k_\Phi}{mM_m}-\frac{k_{x\Phi}^2}{mM_m}\right)\right\}^2 + \left\{\left(\frac{k_x c_\Phi}{mM_m}+\frac{k_\Phi c_x}{mM_m}-\frac{2k_{x\Phi}c_{x\Phi}}{mM_m}\right)\omega - \left(\frac{c_\Phi}{M_m}+\frac{c_x}{m}\right)\omega^3\right\}^2\right]^{1/2} \qquad (21)$$

When the footing is subjected to the action of both moment and horizontal force, the resulting amplitudes of sliding and rocking may be obtained by adding the corresponding solutions from Equations 19 and 20 above.

CONCLUSIONS

1. Unified general solutions for calculating damped and undamped natural frequencies and amplitudes have been developed for rigid block foundation resting on the elastic-half space, embedded into the elastic half-space with soil side layer, and rigid block resting on piles, subjected to constant harmonic horizontal force and/or moment at the top of the foundation.
2. Surface foundation is a special case for embedded foundation having depth of embedment equal to zero. So, same relationship can be used by substituting depth of embedment equal to zero.
3. Same relationships as for embedded foundation can be used for the case when rigid block is resting on piles by assuming that the stiffness and damping at the base of the block is available only from the piles at the pile head and neglecting the contribution of soil below the block.

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