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Predicting and reducing warranty costs by considering customer expectation and product performance

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PREDICTING AND REDUCING WARRANTY COSTS BY CONSIDERING CUSTOMER EXPECTATION AND PRODUCT PERFORMANCE

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ABSTRACT

This dissertation develops from quality loss function to warranty loss function in which customer expectation is also considered to be a variable. First, Taguchi's quality loss function for the larger-the-better case, which is different from the smaller-the better and nominal-the-best cases, has been assimilated into the other two by introducing a term called the *target-mean ratio.* Further topics addressed include the implications of a finite target on the classification of LTB characteristics, a new concept of a "Complementary Characteristic," operating window, complexity, and SN ratio based on complexity.

A warranty is a buyer's confidence owing to the seller's assurance to the buyer that a product will perform as stated or implied. The quality loss function only accounts for immediate issues within manufacturing facilities, whereas the warranty cost occurs during customer use. Therefore, this dissertation develops a methodology that can predict warranty probability and warranty costs on the basis of customer expectation in addition to product performance for smaller-the-better, nominal-the-best, and larger-the-better cases.

In robust engineering, the signal-to-noise ratio is used to improve the robustness of a system. Most products and processes have multiple quality characteristics or output responses. Therefore, this research has been conducted to propose a metric that can be used for multi-response experiments for minimizing quality loss and improving robustness at the same time. The methodology proposed incorporates all three types of characteristics smaller-the-better, nominal-the-best, and larger-the-better and is based on components of quality loss.

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NOMENCLATURE

Symbol	Description
A	Dollar loss when $Y^* < y^*$ in the case of LTB or when $Y^* > y^*$ in the case of STB
A_0	Cost of corrective action corresponding to the point of customer intolerance
A_1	Dollar loss when $y_1^* > Y^*$ and in the case of NTB
A_2	Dollar loss when $Y^* > y_2^*$ in the case of NTB
b_{ijk}	Normalized squared bias at *i*-th run, *j*-th response, and *k*-th signal factor level
C	Customer expectation
$\mathbf{C}$	Positive definite matrix representing associated cost of off-target performance
COP	Coefficient of performance
c	Coefficient of performance
$\overline{c}$	Mean of coefficient of performance
c'	Complementary characteristic of coefficient of performance
dr^l	Lower limit of design range
dr^u	Upper limit of design range
FR	Functional requirement
H_b	Shannon entropy
$h(f)$	Complexity in axiomatic design
I	Information content
k	Proportionality constant when $Y^* > y^*$ in the case of STB or when $Y^* < y^*$ in the case of LTB
k	Shape parameter of Gamma distribution
k_1	Proportionality constant when $y_1^* > Y^*$ in the case of NTB
k_2	Proportionality constant when $Y^* > y_2^*$ in the case of NTB
MSD	Mean squared deviation
m	Target value
m_{jk}	Target of *j*-th response, at *k*-th signal factor level
N	Potential number of complaints
n	Sample size
n	Actual number of complaints
P	Product performance
P_W	Probability of warranty complaint

P_{W1}	Warranty probability when $y_1^* > Y^*$ in the case of NTB
P_{W2}	Warranty probability when $Y^* > y_2^*$ in the case of NTB
pdf	Probability density function
Q	Heat output of the heat pump
$R(\mathbf{x})$	Expected loss at $\mathbf{x} = (x_1, x_2, \ldots, x_k)$
S	Standard deviation of performance in sample
s_{ijk}	Standard deviation of y_{ijk} for all noise factors combinations
T_{FR}	Target of functional requirement
T_P	Amount of production
T and t	Quality characteristic of interest
t	Dummy variable for y
t'	Dummy variable for y'
u_j	Weights corresponding to bias for each response j
v_{ijk}	Normalized variance at i-th run, j-th response, and k-th signal factor level
W	Reciprocal of performance or characteristic
W	Work done or input by compressor corresponding to heat output
WBV_{ijk}	Product of weighted normalized squared bias and normalized variance at i-th run, j-th response, and k-th signal factor level
WBV_{ij}	Product of weighted normalized squared bias and normalized variance at i-th run, j-th response, and all signal factor levels
$WMBV_i$	Multi-response product of weighted normalized squared bias and normalized variance at i-th run for all responses, and all signal factor levels
WL	Warranty loss
W_r	Weight factors that scale the relative importance
w_j	Weights corresponding to variance for each response j
x	STB characteristic of an operating window that needs to be minimized
x'	LTB characteristic of an operating window that needs to be maximized
Y	Performance or characteristic
Y'	Complementary characteristic of Y
$Y^* = P - C$	Transformed characteristic
$\bar{y}$	Mean performance
y^*	Restrictive value of Y^* in the case of STB or in the case of LTB
y_1^*, y_2^*	Restrictive values of Y^* in the case of NTB
y_{ijk}	Performance value at i-th run, j-th response, and k-th signal factor level
$\mathbf{y}(\mathbf{x})$	Performance $\mathbf{y}$ at $\mathbf{x} = (x_1, x_2, \ldots, x_k)$
z	Dummy variable

α	Target-mean ratio
β	Shape parameter of Weibull distribution
Δ_0	Point of customer intolerance
$\Gamma(.)$	Gamma function
$\xi(y)$	Distribution of $Y^* = P - C$ when closed for solution is not possible to obtain
η	Efficiency of a machine
η	Location parameter of an exponential distribution
η'	Complementary characteristic of efficiency of a machine
$\bar{\eta}$	Mean of efficiency of machines
η_C	Location parameter of exponential distribution for customer expectation
η_P	Location parameter of exponential distribution for product performance
$\eta(\mathbf{x})$	Expected value of $\mathbf{y}$ at control factor levels of $\mathbf{x} = (x_1, x_2, \ldots, x_k)$
θ	Scale parameter of Gamma and Weibull distributions
θ_C	Scale parameter of exponential distribution for customer expectation
θ_P	Scale parameter of exponential distribution for product performance
$\mathbf{K}$	An $m \times m$ matrix of constant elements carrying information of repair costs
λ	Scale parameter of inverse Gaussian distribution
μ	Mean of Y and location parameter of inverse Gaussian and Normal distributions
μ_C	Mean of customer expectation
μ_{FR}	Mean of functional requirement
μ_P	Mean of product performance
$\hat{\mu}_3$	Third central moment of performance
$\hat{\mu}_4$	Fourth central moment of performance
$\Sigma(\mathbf{x})$	Variance-covariance matrix of $\mathbf{y}$ at control factor levels of $\mathbf{x} = (x_1, x_2, \ldots, x_k)$
σ	Standard deviation of performance, scale parameter of Normal distribution
σ_C	Standard deviation of customer expectation
σ_P	Standard deviation of product performance
τ_{FR}	Target of functional requirement
μ_{FR}	Mean of functional requirement
$\Phi(.)$	Cumulative density function for normal distribution
ω	Complaint factors when $Y^* < y^*$ for LTB or $Y^* > y^*$ for STB

ω_1	Fraction of customers that will make a complaint when $T > t$ or complaint factor when $y_1^* > Y^*$ in the case of NTB
ω_2	Fraction of customers that will make a complaint when $T < t$ or complaint factor when $y_2^* < Y^*$ in the case of NTB

1. OVERVIEW AND OBJECTIVES

1.1. OVERVIEW AND OBJECTIVES

A warranty is a buyer's confidence owing to the seller's assurance to the buyer that a product will perform as stated or implied. The warranty costs add to the cost of the product and often require fire fighting, attention, and manpower, and still the company sometimes loses reputation, goodwill, and market share. It is a lose–lose situation for everyone. Warranty cost is, in a way, a synonym for waste; waste of money, man-power, time, and energy. All rework should be avoided and so should a warranty. An occurrence of warranty cost is a loss to society as a whole. Often the reason for a warranty can be attributed to a flaw in the process of design and development rather than to the manufacturing itself.

Warranty costs can affect the growth of a company in today's competitive world. From a strategic standpoint, reducing warranty costs is very crucial to the success of a company. High warranty cost is a good indicator of poor quality of a product. Measurement and prediction of warranty costs is an important step towards its reduction. As mathematical physicist and engineer Lord Kelvin said, "If you cannot measure it, you cannot improve it" (Lord Kelvin Quotations, 2007). Table 1.1 summarizes the benefits of a reduced warranty cost.

The role of a warranty is becoming increasingly important to both the consumer and the supplier. A warranty is a supplier's commitment to repair or replace a product during a specified time frame. Customers seek assurance that the complex and expensive products they purchase will be durable. Suppliers may utilize warranties to gain a competitive advantage when competing products are nearly indistinguishable. Warranty management must be a strategic approach that encompasses the entire product life cycle.

Table 1.1. High Warranty Cost vs. Low Warranty Cost

High WC	**Low WC**
High price	Low price
Low sales	High sales
More failures and complaints	Less failures and complaints
Higher customer dissatisfaction	Lower customer dissatisfaction
Lower employee satisfaction	Higher employee satisfaction
Higher rework	Lower rework
More fire-fighting and attention on WC management	Less fire-fighting and attention on WC management
Less creativity and growth	More creativity and growth
Loss to society	Lower loss to society

One approach to estimating warranty costs is the reliability approach, where the mean-time-to-failure (MTTF) is considered among other parameters. This approach is suitable for electrical circuits, electronic products, or even mechanical systems where on the basis of trials MTTF can be estimated fairly accurately. Another approach is to use historical warranty claims data, as evident from the following statement: "IBM estimated the warranty costs based on historical warranty claim experience for eligible products under warranty" (Lenovo Release, 2004). However, in this dissertation an attempt has been made to include customer expectation as an important variable that affects warranty cost prediction and reduction.

Under the Transportation Recall, Enhancement Accountability, and Documentation (TREAD) Act of 2000, manufacturers and suppliers of automobiles are required to report specific information regarding virtually all of their customer contact to the National Highway Traffic Safety Administration (NHTSA). Automotive companies need to record, aggregate, and report a broad collection of data regarding dozens of components and safety systems, such as production statistics, injuries and fatalities, complaints, and warranty claims. According to the Uniform Commercial Code in implied warranty, the seller is automatically responsible for the fitness of the product for use. For these two reasons and because of the problems and costs associated with lawsuits and lost sales, companies are facing intense pressure to reduce warranty costs. Companies should turn this liability into a competitive advantage.

Current warranty costs can be seen as a source of future profits. It is up to the company to strategize and convert this money into profits or use it as a tool against competitors for increasing the market share. Reducing warranty costs does not mean changing the warranty policy and offering lower warranties, but instead it means reducing the opportunities of warranty occurrences by strengthening the quality of the upstream, i.e., the design and development, suppliers, manufacturing, and assembly.

Closer tolerances are associated with better quality and lower warranty expenses. The strategic approach for warranty management and warranty avoidance must provide a balance of cost and tolerance levels. Companies cannot focus their efforts solely on warranty avoidance, but they must also learn from customer experience. This information and data must then be managed and translated back into product performance (PP), tolerance design, and process capability with regard to cost.

According to Mendiratta (2002), “Warranty costs can constitute a significant portion of a product’s cost over the product life cycle.” This author continues: “There is not much in the modeling literature relating warranty and quality costs over the product life cycle” (Mendiratta, 2002).

In general, a warranty serves as both a repair contract as well as an insurance policy (Priestt, 1981). As a repair contract, a warranty is an obligation to the manufacturer for a specific period of time to provide, without charge, any services necessary to repair a defect in the product to make it functional. As an insurance policy, in the event of a product or some part of the product becoming defective, the warranty will provide compensation from the manufacturer for the loss by repair, replacement, or refund of the purchase price if the event occurs during a specified time frame. A warranty functions as an insurance policy on the premise that a product defect occurs probabilistically. Many companies sell warranty policies and extended warranties, and this business is generally very profitable. The defects may occur in new products known as infant mortality, but the probability for this is generally low. In the case of an extended warranty, the infant mortality phase is already over and therefore the probability of defects is usually very low.

In order to reduce warranty costs long term, quality must be considered at the design conception. Approximately 70 percent of a product's total cost is determined by its design. Reliability predictions can be based on past experiences with similar products and further refined during product development (Cudney, 2008). Companies need to be able to predict failure rates as a function of time based on an analysis of a product and process testing. These data will enable companies to predict and evaluate warranty costs. Comparison with the customer expectation (CE) can give a new direction to evaluating and predicting warranty costs.

This dissertation proposes developing a new way to assess and reduce warranty costs that takes customer expectation into account in addition to the usually considered product performance.

1.1.1. Taguchi's Quality Loss Function. Before arriving at a methodology to predict warranty costs, it is important to study the quality loss function developed by Genichi Taguchi. Fowlkes (1995), Taguchi (1999), and Taguchi (2004) are good sources for a detailed discussion on Taguchi's quality loss function. The quality loss function developed by Taguchi considers three cases: nominal-the-best (NTB), smaller-the-better (STB), and larger-the-better (LTB). The methodology used to deal with the larger-the-better case is slightly different from the other two cases because of reciprocal transformation.

Section 2 attempts to study the effects of reciprocal transformation and instead proposes a linear transformation by introducing a term called the *target-mean ratio*. Section 2 also proposes a common formula for all three cases to bring about similarity among them (Sharma, 2007). The target-mean ratio can take different values representing all three cases to bring consistency and simplicity to the model. In addition, it eliminates the assumption of target performance at an infinite level and brings the model closer to reality. Characteristics such as efficiency, coefficient of performance, and percent non-defective are presently not larger-the-better characteristics due to the assumption of target performance at infinity and the subsequent necessary derivation of the formulae. These characteristics can also be brought under the category of the larger-the-better characteristics. An example of the efficiency of prime movers is discussed to illustrate that efficiency can also be considered as a larger-the-better characteristic (Sharma, 2007). A second example is presented to show the subtle differences between both methodologies. Therefore, one of the objectives of this dissertation is to unify the quality loss function for all three cases.

Another objective of this dissertation is to study the implications of a unified methodology on quality engineering. The new methodology has some implications that need to be addressed. Section 3 attempts to study the implications and effects of the new methodology on the field of quality engineering. This section presents an implied classification of (LTB) characteristics according to Taguchi on the basis of a target value at infinity and also discusses the classification of LTB characteristics based on the new methodology (Sharma, 2008). A new concept or axiom called the "Complementary Characteristic" is also suggested. It is argued that the common methodology is more suitable for the purpose of computing quality loss. What should be the most appropriate value of 'α,' the *target-mean ratio*, has also been studied.

1.1.2. Warranty Cost Considering Customer Expectation as a Variable. The role of a warranty is becoming increasingly more important to both the consumer and the supplier. A warranty is a supplier's commitment to repair or replace a product during a specified time frame. Customers seek assurance that the complex and expensive products they purchase will be durable for at least a known period of time. Suppliers may utilize warranties to gain a competitive advantage when competing products are nearly indistinguishable. Warranty management must be a strategic approach that encompasses the entire product life cycle. Companies must also learn from customer experience. This information and data must then be managed and translated back into product performance, tolerance design, and process capability with regard to cost. Tolerances have far more of an impact on cost, quality, and customer satisfaction than they have traditionally been accorded (Creveling, 1996). Tolerance design traditionally deals with geometric dimensioning and *tolerancing*, which is a specification communication process focusing on the development of a graphical model of the design after the analytical and physical model development has been completed through tolerance development (Creveling, 1996).

In order to reduce warranty costs in the long term, quality must be considered early in the product design process. Approximately 70 percent of a product's total cost is determined by its design (Cudney, 2008). Reliability predictions can be based on past experiences with similar products and further refined during product development. Manufacturing capability or producibility greatly influences how effectively the product meets the design. Companies should be able to predict failure rates or reliability as a function of time based on analysis of a product and process testing.

Warranty costs depend on product performance and warranty terms. Product performance can be determined by engineering design, manufacturing process design, raw materials, supplier performance, quality control, product use, and product maintenance. The warranty terms must be created based on the product and process testing in combination with warranty marketing strategies and warranty servicing strategies. A warranty must be considered throughout these areas.

Several key market mechanisms provide consumers with information about product durability including reputation, advertising, and product-specific investment. A warranty can also serve as a sign of product durability in an oligopolistic or perfectly competitive market (Gal-Or, 1989). Warranties can serve as a signal in this type of market because the cost of providing a warranty increases as the product durability decreases. Therefore, the cost of providing a warranty can be a decreasing function of durability. This generates a relationship between the warranty cost and the product durability, which in turn creates equilibrium in which the warranty signals the product quality (Gal-Or, 1989).

Warranty cost is a key signal of a product's or service's level of quality. Warranty cost is also simply lost revenue. In order for North American automakers to remain competitive and profitable, focus and effort must divert to predicting warranty costs and subsequently eliminating these costs (Web of The Auto Channel, 2003). The Web of The Auto Channel (2003) examined the North American warranty management programs at BMW, DaimlerChrysler, Ford, General Motors, Nissan, and Toyota. This study clearly shows the need to focus on eliminating warranty concerns.

The objective of this research is to develop a warranty cost prediction model assuming a perfectly competitive market. This research utilizes customer expectation or preference as an important variable in the methodology proposed. By predicting the origin of warranty claims, Design for Six Sigma can be utilized to significantly reduce the cost of products before production begins, therefore increasing profitability.

Actual performance of a product varies from consumer to consumer and from time to time. The product performance is measured in terms of certain characteristics that affect product performance and which need to be within a certain range in order to satisfy the customer. When product performance is measured in terms of time to failure, time to first failure, or time between failures, it is related to reliability and durability. On the other hand, customer satisfaction is a statistical phenomenon because a large number of customers purchase a given product and each customer has a different expectation from a product that is measurable in terms of a product performance unit. Customer expectation also varies from time to time. Two variables are considered in this research: performance variation and consumer expectation variation.

Estimation of warranty cost without considering customer expectation as a variable is not plausible. When the product performance matches or exceeds the customer expectation, a warranty cost should not occur. On the other hand, when product performance (PP) falls short of customer expectation (CE), a warranty cost may occur. Warranty cost is generated as a result of a clash between customer expectation and product performance. Section 4 considers customer expectation as a variable in the formulation of warranty cost models for all three types of characteristics. Each developed model has been used to predict the warranty cost for a relevant example. The objective is to develop warranty probability functions and warranty loss functions that include customer expectation along with product performance, both of which have been assumed to follow a normal distribution. Section 5 provides conceptual validation and validation of models by simulation. In Section 6, warranty probability functions have been derived for other distribution combinations.

The objective of this research is to predict and reduce warranty cost expenditures. The following are the sub-objectives / steps towards reaching the main objective.

- Relate warranty performance at the customer's hand with the work carried out at the manufacturing facility; and
- Predict the warranty costs.

Warranty costs are the indicator of performance of the manufacturing facility including design and development. To improve the performance of the manufacturers and suppliers, it is imperative to combine techniques of reliability engineering and quality engineering.

1.1.3. Methodology to Compute the Warranty Cost when PP and CE Follow Other Distributions that May or May Not Render a Closed-Form Solution. It is unnecessary for product performance and customer expectation to follow normal distributions. However, a closed-form solution can be found if they do. If other distributions are observed, then one still should be able to form estimates of warranty cost. Section 6 discusses the methodology when product performance and customer expectation follow distributions other than normal distributions. Some distribution combinations can give a closed-form solution, but many others do not. Therefore, Section 6 is dedicated to closed-form solutions as well as to a methodology for dealing with distribution combinations that cannot give closed-form solutions. A methodology is developed to estimate the warranty loss when product performance and customer expectation follow such distributions that do not render closed-form solutions.

1.1.4. Metric for Optimization of Multi-Response Systems. The quality loss function (QLF) given by Taguchi takes into account only one characteristic. Most of the components and products have more than one quality characteristic to consider simultaneously for assessing the quality of the component or product. However, the QLF does not take into account multiple characteristics simultaneously. Because more than one characteristic is to be considered, it is also imperative to consider different types of characteristics simultaneously for the purpose of determining the quality loss. This is because a product need not have only one type of characteristic, e.g., smaller-the-better (STB).

The design process should encompass all four domains as described in Suh (2000). These domains are the customer domain, functional domain, physical domain, and process domain. The customer domain is characterized by the needs or attributes that a customer wants from a product, process, or system. A customer's use of a product, warranty complaints, and warranty expenditures all occur in the customer domain. The functional domain has the customer needs specified in terms of functional requirements (FRs) and constraints thereof. Functional requirements need to be satisfied by design parameters (DPs) in the physical domain. The methodology proposed connects the customer domain with the physical domain. Therefore actions taken, such as optimization of multi-response systems in the physical domain, are supposed to affect the customer domain in terms of low warranty probability and costs.

The objective of this research is to develop a metric for the optimization of multi-response systems from a robustness point of view. A signal-to-noise ratio has been used for decades to optimize product performance. *Signal* refers to response and *noise* refers to variation in response due to variation in input conditions. Section 7 proposes a methodology that takes into account the bias of response and variance of response or noise in a different way. A product of normalized squared-bias and normalized variance has been used for optimizing the multi-response system. Therefore, in Section 7 a methodology is developed that can be used for any type or number of characteristics. The metric shown gives one number for quality loss even if a number of characteristics are taken into account.

1.2. CONCLUSION

The methodology proposed in this dissertation includes customer expectation to predict the warranty cost. The first step is to take a fresh look at Taguchi's quality loss function especially for the larger-the-better case. The second step is to develop warranty loss functions for all three types of characteristics. This research focuses on warranty cost reduction strategies through reliability engineering and quality engineering. It is also important to check the functions for their ability to suggest ways for improvement to reduce warranty costs. The objective of the models is to identify factors affecting the warranty cost as well as techniques that can be employed to improve the elements and, in turn, to reduce the warranty cost expenses for a company. One should be able to predict how much quality improvement at the production facilities will change the amount of warranty cost at the customer's hand. This dissertation covers a manufacturing firm rather than a service firm because the engineering quality characteristics are more readily measurable in the former case.

A methodology is proposed that takes into account the bias of response and variance of response or noise in a different way. A product of normalized squared-bias and normalized variance has been used for optimizing the multi-response system. The methodology developed can be used for any type or number of characteristics. The metric shown gives one number for quality loss even if several characteristics are taken into account.

2. UNIFICATION OF QUALITY LOSS FUNCTION – A UNIFIED METHODOLOGY FOR THREE CASES

2.1. INTRODUCTION

The quality loss function developed by Genichi Taguchi considers three cases: nominal-the-best, smaller-the-better, and larger-the-better. The methodology used to deal with the larger-the-better case is slightly different from the other two cases. This section employs a term called the *target-mean ratio* and proposes a common approach for all three cases. The target-mean ratio can take different values representing all three cases to bring consistency and simplicity to the model. In addition, it eliminates the assumption of target performance at an infinite level and brings the model closer to reality. Characteristics such as efficiency, coefficient of performance, and percent non-defective are presently not larger-the-better characteristics due to the assumption of target performance at infinity and the subsequent necessary derivation of the formulae. These characteristics can also be brought under the category of larger-the-better characteristics. An example of the efficiency of prime movers is discussed to illustrate that the efficiency can also be considered as a larger-the-better characteristic. A second example is presented to suggest the differences between both methodologies.

The following paragraph is taken from Taguchi's *Quality Engineering Handbook* (Taguchi, 2004):

> "The larger-the-better characteristic should be nonnegative, and its most desirable value is infinity. Even if the larger the better, a maximum of nonnegative heat efficiency, yield, or nondefective product rate is merely 1 (100%); therefore, they are not larger-the-better characteristics. On the other hand, amplification rate,

power, strength, and yield amount are larger-the-better characteristics because they do not have target values and their larger values are desirable."

Two types of performance characteristics are discussed in the paragraph above. First, the characteristics that have a maximum possible target of 100% are not larger-the-better (LTB) characteristics. Second, the characteristics that have infinity as the target value do not actually have a target value and are LTB characteristics. The LTB methodology requires infinity as the target.

It is plausible to have a target value as infinity, in which case a target value is not assigned. One of the purposes of this research is to explore why characteristics are not considered LTB when the target is known, as in the case of efficiency, yield, or non-defective. It seems that because of the assumption of infinity as the target, some of the characteristics are presently not LTB characteristics. The subsequent derivation of the formulae also supports the theory that some characteristics do not fall under the category of LTB characteristics.

Taguchi's quality loss function approximates loss based on two reasons: (1) the variation, denoted by the standard deviation, of performance about some mean; and (2) the mean performance away from the target, denoted by the distance, called bias, by which the mean performance is away from the target. For smaller-the-better (STB) and nominal-the-best (NTB) cases, both bias and variance have clearly been shown to affect the mean-squared deviation (MSD) and, in turn, the quality loss. In Taguchi's existing approach for LTB, however, it is unclear how variation of the performance affects the MSD or quality loss.

Maghsoodloo (1991) showed that MSD for LTB cases could be approximated using Taylor series expansion by Equation (2.1). The MSD given for the STB and NTB cases is exact while it is only approximate for the LTB case.

$$MSD = \frac{1}{\bar{y}^2}\left[1 + \frac{3\sigma^2}{\bar{y}^2}\right] \tag{2.1}$$

Where,

$\bar{y}$ = mean performance, and

σ = standard deviation of performance.

A better approximation can be given by Equation (2.1a):

$$MSD = \frac{1}{\bar{y}^2}\left[1 + \frac{3\sigma^2}{\bar{y}^2} - \frac{4\hat{\mu}_3}{\bar{y}^3} + \frac{5\hat{\mu}_4}{\bar{y}^4}\right] \tag{2.1a}$$

where the third central moment is given by Equation (2.1b):

$$\hat{\mu}_3 = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \bar{y}\right)^3 \tag{2.1b}$$

In which y_i represents the performance value of each product in the lot. The fourth central moment is

$$\hat{\mu}_4 = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \bar{y}\right)^4 \tag{2.1c}$$

Equations (2.1) and (2.1a) for the LTB case clearly show how variation of the performance affects the MSD or quality loss.

A discussion on the idea of an infinite target and the problems associated with the target at infinity has only begun. The next section discusses the importance of the quadratic loss function in quality engineering. More research is necessary if a target value of infinity is needed. In the theory section, relevant formulae are derived to incorporate the idea of some finite target in place of infinity. Two examples have been given to explain the methodology. Towards the end of the section, an appropriate target-mean ratio for the LTB characteristic is suggested.

2.1.1. Quadratic Quality Loss Function and Signal-To-Noise Ratio. Robust design is achieved by applying a three-step decision-making process:

1. Define the objective;
2. Define the feasible options; and
3. Select the feasible option that best meets the objective.

Taguchi suggested a signal-to-noise ratio (SN ratio) as the best measure of robust performance. The SN ratio attempts to combine variation of the mean and deviation of the mean performance from the target into a single metric. Maximum robustness means minimum quality loss and maximum customer satisfaction. The SN ratio recognizes and measures deviation from the nominal value and integrates the information into one metric (Taguchi, 1999). It is very important to define the measure of the quality loss and then incorporate the same into the design.

Suppose several performance characteristics exist, and it is important to distinguish between these when evaluating the quality. Therefore, a different SN ratio is needed for each performance characteristic. The nominal value is the best performance characteristic value for many parameters. Nominal-the-best (NTB) should be used whenever possible because this supports two-step optimization. In two-step optimization, appropriate actions are taken to first reduce variation and then to move the mean performance to the desired target. The SN ratio measures the deviation from the nominal value, thus allowing for subsequent adjustment.

A system can be represented with the help of a P-diagram, as shown in Figure 2.1. Figure 2.1 has been reproduced with permission from Ragsdell (2008). Any dynamic system or product consists of control, noise, and signal factors. A static system does not have a signal factor. A large SN ratio for a system means a lower standard deviation. In the case of dynamic signals, e.g., a steering wheel or brake pedal application, a series of dynamic SN ratios exists. A signal factor is a control factor chosen that can modify the output response in a linearly proportional way. For example, for a steering wheel, the turn angle of the steering wheel is the signal factor that adjusts the radius of curvature for vehicle motion as the output response. Similarly, for a brake system, the brake pedal pressure is the signal factor that regulates the braking distance as the output response. Figure 2.2 shows a simple linear relationship between response *y* and signal factor *M*. A linear relationship between the output or response and the input or signal is the most desirable relationship for dynamic systems (Fowlkes, 1995; Phadke, 1989). Thus, an equation that measures the robustness of a system can be obtained. The objective for achieving a robust design is to have the highest SN ratio (i.e., the smallest standard deviation or variation).

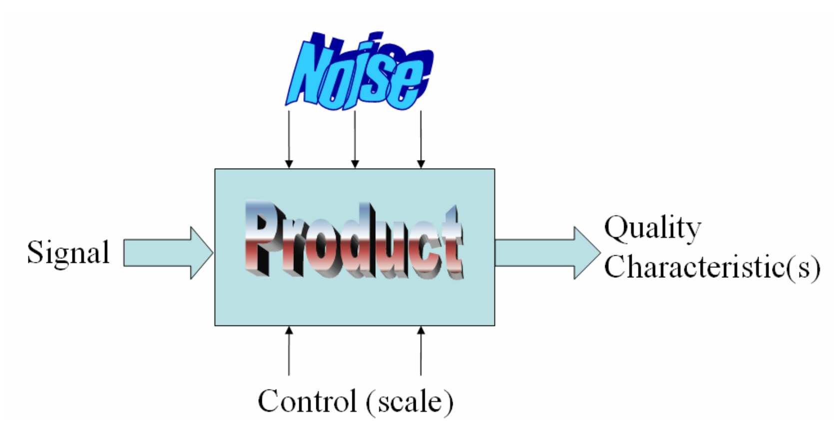


Figure 2.1. P-Diagram

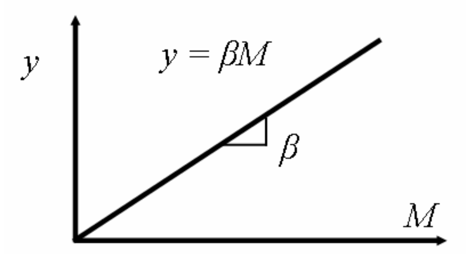
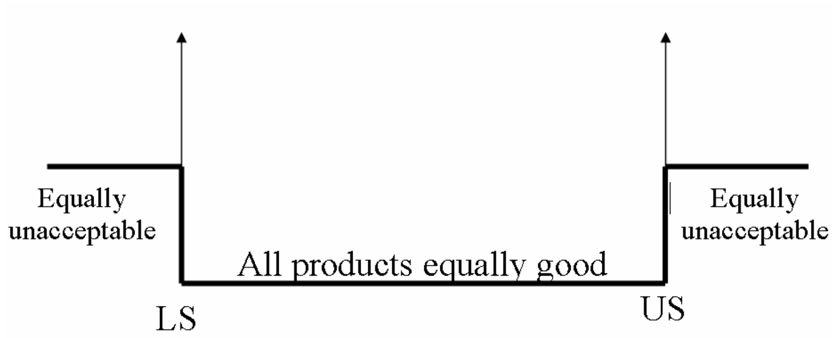
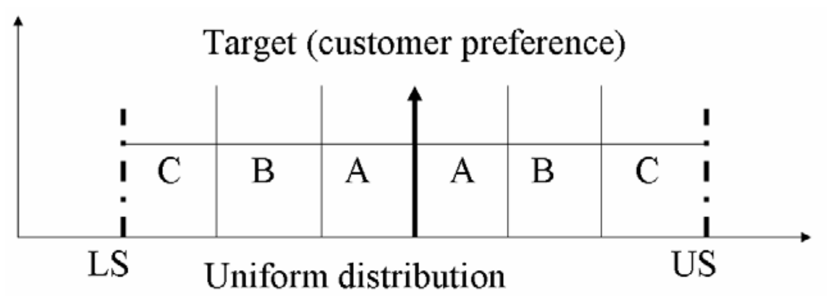


Figure 2.2. Response and Signal Relationship

Noise factors are the factors affect the output performance of a system but are chosen not to be controlled because they are either difficult or uneconomical to control. Three types of noises exist: variation in the conditions of customer use, part-to-part variation, and wear and deterioration. The main objective is to satisfy the customer and not just meet specifications. According to Figure 2.3(a), all of the parts within the specification limits are acceptable, which suggests zero defects. Figures 2.3(a) and 2.3(b) have been reproduced with permission from Ragsdell (2002). Figure 2.3(b) implies that quality loss is incurred if parts are away from the target although within the specification limits. Optimum performance is achieved when the variation is low and the mean of performance is close to the target. From the customer's viewpoint, no difference exists between products with performance just inside or just outside of the specification limits. Taguchi developed his quality loss function to convert customer satisfaction into a monetary value so that a manufacturer could estimate the loss to the company as a result of poor product performance.



(a) Goal Post Approach



(b) Product Performance Levels

Figure 2.3. Product Performance

The idea is to deliver performance near the target (customer preference), which maximizes the customer satisfaction value, thus, overriding the specification limits. Depending on the quality characteristics, this satisfaction level can be of three types: smaller-the-better (STB), nominal-the-best (NTB), or larger-the-better (LTB). When it is desirable to deliver performance near the target, the case is termed NTB. In LTB cases, these values need to be higher than and away from a certain threshold value. In STB cases, these values need to be lower than and away from a certain limiting value.

It is important to understand the relationship of performance away from the target to quality loss. Products with smaller variation from time to time and from point to point have a smaller quality loss. The quality loss function essentially translates the qualitative terms, which affect the consumer, into quantitative terms such as monetary values. Depending on the situation, the quality loss function takes three forms:

1. Nominal-the-best (NTB) - for these cases, the nominal value is best because it is the one that satisfies the customer's need. The characteristic value away on either side of the target is undesirable, such as air pressure in vehicle tires or the location of gauges on the instrument panel.

2. Smaller-the-better (STB) - for these cases, a smaller value is better and higher values are undesirable, such as vehicle emissions or fuel consumption (gallons per mile).

3. Larger-the-better (LTB) - for these cases, a larger value is better and smaller values are undesirable, such as gas mileage (distance per gallon).

2.1.2. Matching Performance to Intent. Quality can be divided into two types: design quality and production quality. As such, the overall quality of a product can be improved by improving the design quality and production quality. Improving the design quality and production quality basically minimizes the design loss and production loss in order to minimize the quality loss. Design quality is reflected in product properties that are manifested when using the product. Design loss means off-target performance due to variable conditions. Robust design is designing and manufacturing a product that enhances customer satisfaction (Taguchi, 1999). Design intent should match the customer's requirements, and production should match the design intent. All products that perform off-target produce a cost that is unnecessary. Elements of design quality include the following:

- Robust performance—Optimize the nominal values to achieve performance;
- Eliminate mistakes—Review the robust design; and

- Correct precision levels—Strike a balance between the manufacturing costs and precise tolerances.

In the development of new products, the most important step is early robust optimization, which provides on-target performance. It is best to have a world-class quality system on hand that satisfies the customer.

For LTB characteristics, the intent is not to achieve an infinitely large performance value. That is why other than the three cases of LTB, STB, and NTB, it is necessary to suggest two other cases that are "LTB to a point" and "STB to a point." In many cases, values of performance characteristics that are higher than a certain value would add very little to the quality of the product, so the loss has upper limit, Drain (1996). For example, additional tensile strength of a tire rubber may not help improve the quality of the tire. What is required is wear-resistance at higher operating speeds and temperatures. Infinity is not in a customer's perspective. For example, a customer wants the cargo space in his or her vehicle to be LTB, but only to a point. This characteristic (cargo space) requires tradeoffs with size, mass, fuel consumption, and maneuverability for parking. The LTB case assumes that the larger the value of a parameter, the better. In Taguchi's (1989) quality loss function, when the performance value approaches infinity, the loss approaches zero. Thus, it is obvious that the LTB case does not represent reality. The target should never be infinity because this goal is unachievable. One view is that infinity as the target for the LTB case is notional and is so assumed to facilitate the mathematical derivation. However, it is unnecessary for this to be only notional, and therefore the formulation needs to be improved. The mean-squared deviation (MSD) for LTB is given in Equation (2.2):

$$MSD = \frac{1}{n}\sum_{i=1}^{n}\left(1/y_i\right)^2 \tag{2.2}$$

The quality loss for LTB is given in Equation (2.3):

$$L(y) = k\left[\frac{1}{n}\sum_{i=1}^{n}\left(1/y_i\right)^2\right] \tag{2.3}$$

In the formulae above, the MSD and quality loss cannot be zero regardless of how large the performance value is. It suggests that infinity is only sought after for achieving zero quality loss. However, infinity is unachievable and impractical. Because "LTB to a point" is of interest, it is appropriate to look for a better formulation that can use a target value rather than assuming it as infinite. The formula thus derived represents a more realistic situation. But before discussing any further the topic of LTB to a point or with finite target, the next section is dedicated to studying the behavior of the reciprocal of a response in comparison with that of the response itself.

2.2. RECIPROCAL OF THE RESPONSE VALUE

LTB methodology according to Taguchi suggests that the reciprocal of a performance value be taken and the reciprocal be treated as an STB characteristic. This is so because the target assumed is infinity. This section makes an attempt to analyze and compare the algebraic results obtained by using Taguchi's methodology as well as without using the reciprocal of the data. If that is done, then the quality loss changes, which otherwise would be different. In general, quality loss has two components: loss due to bias and loss due to variation. The same two components have been used to find the change in quality loss due to reciprocal transformation. This is called the *two component approach* and is used to analyze and compare the results obtained by using the current methodology of reciprocal transformation and also without using the reciprocal of the data. It is assumed that the data follow certain distributions so that the algebraic analysis is possible instead of using actual data from the field. Four distributions have been considered: the Inverse Gaussian distribution, Gamma distribution, Weibull distribution, and Normal distribution.

2.2.1. Inverse Gaussian Distribution. Suppose some LTB characteristic Y follows an inverse Gaussian distribution as given in Equation (2.4). Notations have been taken from Chhikara (1988). Where, μ is the mean of Y and λ is a scale parameter.

$$f\left(y;\mu,\lambda\right)=\sqrt{\frac{\lambda}{2\pi}}y^{-\frac{2}{3}}e^{-\frac{1}{2}\left\{\frac{\lambda(y-\mu)^2}{\mu^2 y}\right\}} \tag{2.4}$$

When the reciprocal of Y is taken, i.e., $1/Y=W$, then W is a transformed STB characteristic. Also, W is distributed as follows (Chhikara, 1988):

$$f_1\left(w;\mu,\lambda\right)=\sqrt{\frac{\lambda}{2\pi w}}e^{-\frac{1}{2}\left\{\frac{\lambda(1-\mu w)^2}{\mu^2 w}\right\}} \tag{2.5}$$

Table 2.1 summarizes the algebraic mean, variance, and MSD of Y and W.

Table 2.1. Distribution of 1/Y = W when Y Follows an Inverse Gaussian Distribution

Inverse Gaussian Distribution of Y		**Corresponding Distribution of $1/Y=W$**	
Mean $E(y)$	μ	Mean $E(w)$	$\frac{1}{\mu}+\frac{1}{\lambda}$
$E(y^2)$	$\mu^2+\frac{\mu^3}{\lambda}$	$E(w^2)$	$\frac{1}{\mu^2}+\frac{3}{\lambda\mu}+\frac{3}{\lambda^2}$
$Var(y)$	$\frac{\mu^3}{\lambda}$	$Var(w)$	$\frac{1}{\lambda\mu}+\frac{2}{\lambda^2}$
$MSD(y)$	$\mu^2+\frac{\mu^3}{\lambda}$	$MSD(w)$	$\frac{1}{\mu^2}+\frac{3}{\lambda\mu}+\frac{3}{\lambda^2}$

2.2.2. Gamma Distribution. This section considers the two parameter Gamma distribution of the form given in Johnson (1994). However, notations have been taken from Bain (1991). Suppose an LTB characteristic Y follows a Gamma distribution as given in Equation (2.6).

$$f\left(y;\theta,k\right)=\frac{1}{\theta^{k}\Gamma\left(k\right)}y^{k-1}e^{-\frac{y}{\theta}}\ldots y,k,\theta>0 \tag{2.6}$$

Where, k is called the shape parameter and θ is called the scale parameter. When the reciprocal of Y is taken, i.e., $1/Y=W$, then W is transformed into an STB characteristic. Table 2.2 summarizes the algebraic mean, variance, and MSD of Y and W, where W is distributed as in Equation (2.7):

$$f\left(w;\theta,k\right)=\frac{1}{w^{2}}\frac{1}{\theta^{k}\Gamma\left(k\right)}\left(\frac{1}{w}\right)^{k-1}e^{-\frac{1}{\theta w}}\ldots w,k,\theta>0 \tag{2.7}$$

Table 2.2. Distribution of 1/Y = W when Y Follows a Gamma Distribution

Gamma Distribution of Y		**Corresponding Distribution of $1/Y=W$**	
Mean $E\left(y\right)$	$k\theta$	Mean $E\left(w\right)$	$\frac{1}{k\theta}$
$E\left(y^{2}\right)$	$\theta^{2}k\left(k+1\right)$	$E\left(w^{2}\right)$	$\frac{1}{\theta^{2}k\left(k-1\right)}$
$Var\left(y\right)$	$\theta^{2}k$	$Var\left(w\right)$	$\frac{1}{\theta^{2}k^{2}\left(k-1\right)}$
$MSD\left(y\right)$	$\theta^{2}k\left(k+1\right)$	$MSD\left(w\right)$	$\frac{1}{\theta^{2}k\left(k-1\right)}$

2.2.3. Weibull Distribution. This section considers a two parameter Weibull distribution of the form given in Johnson (1994). If an LTB characteristic Y follows a Weibull distribution, then the pdf of Y is given as in Equation (2.8) (Bain, 1991):

$$f\left(y;\theta,\beta\right)=\frac{\beta}{\theta^{\beta}}y^{\beta-1}e^{-\left(\frac{y}{\theta}\right)^{\beta}}\ \ldots y,\beta,\theta>0, \tag{2.8}$$

Where, β is a shape parameter and θ is a scale parameter. When the reciprocal of Y is taken, i.e., $1/Y=W$, then W is transformed into an STB characteristic. Table 2.3 summarizes the algebraic mean, variance, and MSD of Y and W, where the distribution of W is given in Equation (2.9).

$$f\left(w;\theta,\beta\right)=\frac{1}{w^{2}}\frac{\beta}{\theta^{\beta}}\left(\frac{1}{w}\right)^{\beta-1}e^{-\left(\frac{1}{w\theta}\right)^{\beta}}\ \ldots w,\beta,\theta>0 \tag{2.9}$$

Table 2.3. Distribution of 1/Y = W when Y Follows a Weibull Distribution

Weibull Distribution of Y		**Corresponding Distribution of $1/Y=W$**	
Mean $E\left(y\right)$	$\theta\Gamma\left(1+\frac{1}{\beta}\right)$	Mean $E\left(w\right)$	$\frac{\Gamma\left(1-\frac{1}{\beta}\right)}{\theta}$
$E\left(y^{2}\right)$	$\theta^{2}\Gamma\left(1+\frac{2}{\beta}\right)$	$E\left(w^{2}\right)$	$\frac{\Gamma\left(1-\frac{2}{\beta}\right)}{\theta^{2}}$
$Var\left(y\right)$	$\theta^{2}\left[\Gamma\left(1+\frac{2}{\beta}\right)-\Gamma^{2}\left(1+\frac{1}{\beta}\right)\right]$	$Var\left(w\right)$	$\frac{\left[\Gamma\left(1-\frac{2}{\beta}\right)-\Gamma^{2}\left(1-\frac{1}{\beta}\right)\right]}{\theta^{2}}$
$MSD\left(y\right)$	$\theta^{2}\Gamma\left(1+\frac{2}{\beta}\right)$	$MSD\left(w\right)$	$\frac{\Gamma\left(1-\frac{2}{\beta}\right)}{\theta^{2}}$

2.2.4. Normal Distribution. Even though a normal distribution has negative infinity as a limit on the left-hand side, most non-negative quality characteristics follow this distribution. Therefore, it is important to consider what happens when a larger-the-better response follows a normal distribution. The probability density function for a normal distribution is given as

$$f\left(y\right)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}\cdots-\infty<y<\infty \tag{2.10}$$

Assume,

$$w=\frac{1}{y}$$

So, the Jacobian is

$$J=\left|-\frac{1}{w^2}\right|$$

By transformation, the distribution of W, called the inverse normal distribution, can be given by Equation (2.11):

$$f\left(w\right)=\begin{cases}\dfrac{1}{w^2}\dfrac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2}\left(\frac{\frac{1}{w}-\mu}{\sigma}\right)^2}\cdots for,-\infty<w\leq 0\\[2ex]\dfrac{1}{w^2}\dfrac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2}\left(\frac{\frac{1}{w}-\mu}{\sigma}\right)^2}\cdots for,0\leq w<\infty\end{cases} \tag{2.11}$$

The limits are as follows:

Y	- ∞	-1000	-10	-1	0	10	μ	1000	∞
W	0	-0.001	-0.1	-1	∞ -∞	0.1	$\frac{1}{\mu}$	0.001	0

To prove that Equation (2.11) represents a pdf, put

$$\frac{1}{w} = t \text{, So,}$$

$$dw = -\frac{1}{t^2}dt$$

Therefore, the CDF is

$$F(w) = \int_{-\infty}^{0} \frac{1}{w^2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{\frac{1}{w}-\mu}{\sigma}\right)^2} dw + \int_{0}^{\infty} \frac{1}{w^2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{\frac{1}{w}-\mu}{\sigma}\right)^2} dw$$

$$F(w) = \int_{-\infty}^{0} t^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} \left(-\frac{1}{t^2}\right) dt + \int_{0}^{\infty} t^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} \left(-\frac{1}{t^2}\right) dt$$

$$F(w) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt + \int_{0}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt$$

$$F(w) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt = 1$$

This proves that Equation (2.11) is a pdf of $1/Y = W$. However, it is impossible to go further for analysis because according to Robert (1991), the mean and variance of an inverse normal distribution are undefined. Therefore, for one particular case only a numerical solution has been resorted to for understanding the distribution. Figure 2.4 shows a normally distributed quality characteristic Y plotted using Equation (2.10).

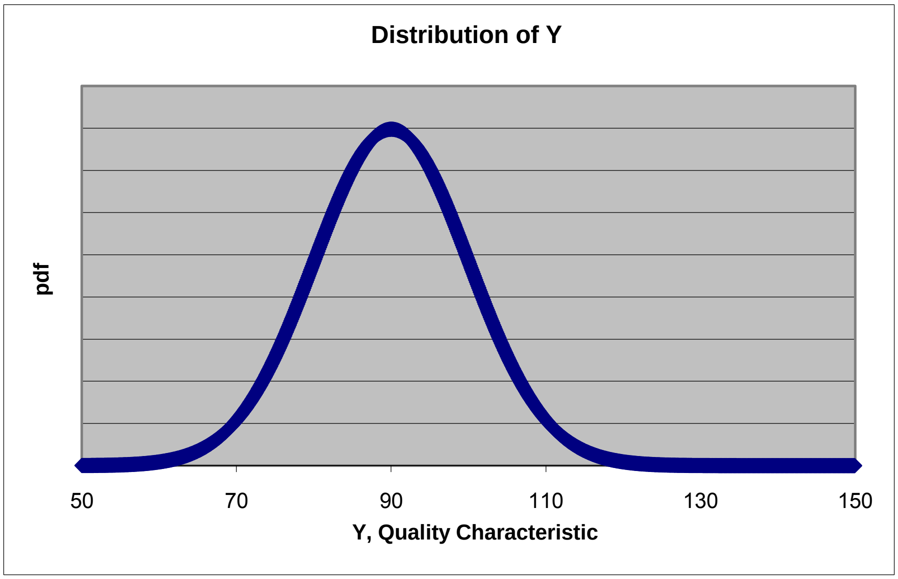


Figure 2.4. Normally Distributed Quality Characteristic Y

Figure 2.5 shows the distribution of the reciprocal of Y when Y follows a normal distribution plotted using the corresponding Equation (2.11).

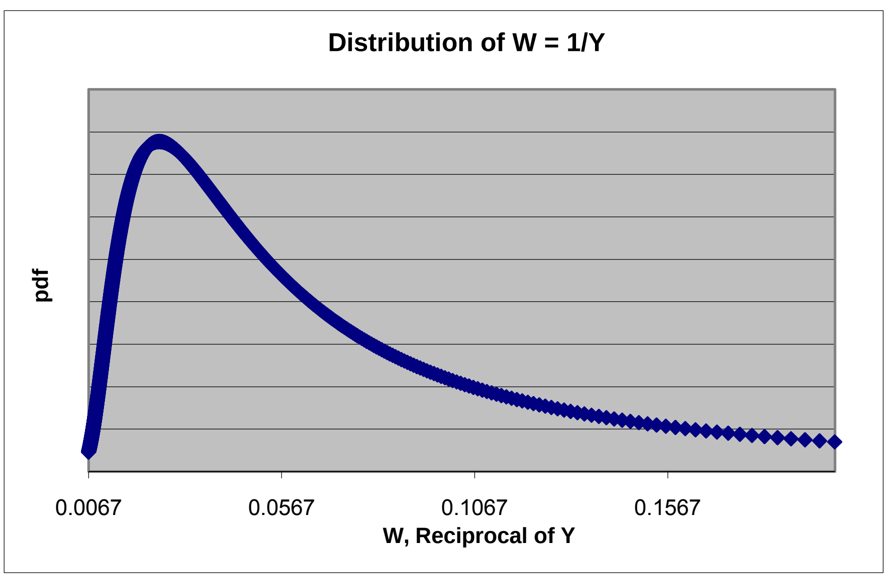


Figure 2.5. Distribution of the Reciprocal of Y when Y Follows a Normal Distribution

A simulation method has also been used to study the distribution of the reciprocal of y and to compute the expected value of w and expected value of w^2 for one particular case. The first 10,000 data points were generated following a normal distribution with the mean of 90 and standard deviation of 10. After the data were generated, the actual mean and standard deviation were computed for 10,000 data points, resulting in 89.98255 and 9.881387, respectively. Please refer to Table 2.4 for more information. The data were rounded for converting them to frequency (a probability estimate) to plot the graph as shown in Figure 2.6. Appendix A can be referred to for distribution of the data based on a normal distribution.

Table 2.4. Simulated Results for 1/Y = W when Y Follows a Normal Distribution

Distribution	**Normal Distribution of** Y	**Distribution**	**Corresponding Distribution of** $W = \frac{1}{Y}$
Mean $E(y)$	$\mu = 89.98255$	Mean $E(w)$	0.0112528
$E(y^2)$	8194.491	$E(w^2)$	0.0001283
$Var(y)$	$\sigma^2 = 97.63204$	$Var(w)$	0.000001658
$MSD(y)$	8194.491	$MSD(w)$	0.0001283

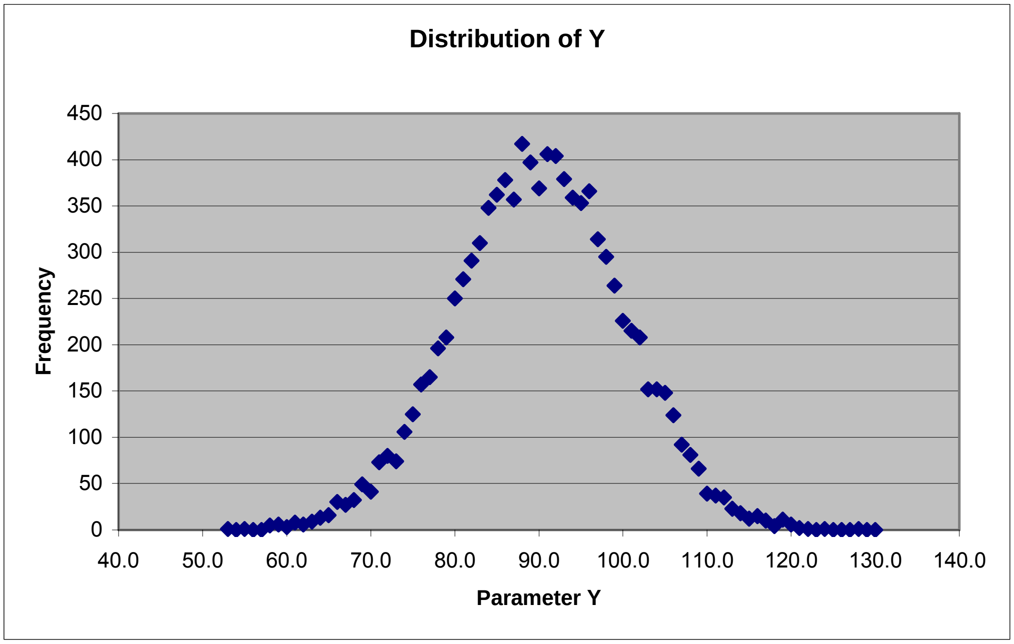


Figure 2.6. Simulated Normally Distributed Quality Characteristic Y

The next stage was to find the reciprocal of each data point and then multiply each with a suitable scaling factor, and in this case, 10,000. Then data were rounded to compute the frequency (probability estimate) and plot the graph. Figure 2.7 shows the obtained distribution of the reciprocal of response y. It is evident from the discussion above that the reciprocal of a performance value does not retain the information of the response because of the nature of the transformation. First, the unit of response changes to something that in most cases is not really useful or comprehensible. Second, the computations become complicated. Third, the distribution of a reciprocal of response changes from that for the response. Fourth, the results no longer remain comparable with STB and NTB characteristics. Fifth, if one were to combine them in the case of multivariate quality loss when LTB needs to be considered along with STB, NTB, or both, then it would not make sense to use the covariance between STB (or NTB) y_i and LTB y_j and use y_i and $1/y_j$ in computations.

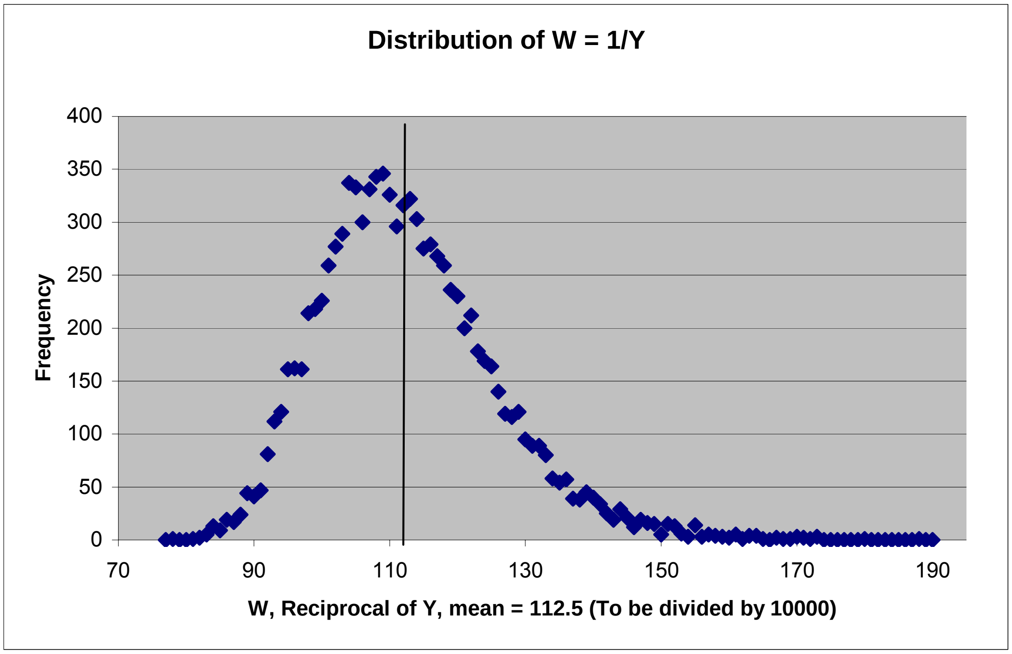


Figure 2.7. Simulated Distribution of 1/Y = W when Y Follows a Normal Distribution

This section proposes that the quality loss characteristics be comparable with one another in these three cases. In the model proposed, the case of NTB can be visualized with its target shifting from zero to a large value to generate all three cases. In the derivation when the target shifts to zero, the case becomes STB. When the target is near the mean performance value, it becomes NTB. When the target shifts to greater than the mean performance value, the case becomes LTB. For the case to be LTB, it is unnecessary for the target to shift to infinity. By introducing this methodology, an attempt has been made to streamline the mathematical as well as the practical aspects of the quality loss function.

This section discusses how to define a target value in the case of LTB characteristics. However, it is worth mentioning here that the target will be difficult to achieve but not impossible. The achievability of the target might depend upon a technology change, innovation, material, or process.

2.3. THEORY

In this section, the NTB case is considered. The derivation of Taguchi's quality loss function expressed here is given in parts in Fowlkes (1995), Taguchi et al. (1989), and Venkateswaren (2003). If *y* is an observed value of a given parameter, and *m* is the target, then the loss function *L(y)* using Taylor series expansion can be given as follows:

$$\begin{aligned} L(y) &= L(m + y - m) \\ L(y) &= L(m) + \frac{L'(m)}{1!}(y-m) + \frac{L''(m)}{2!}(y-m)^2 + \ldots\ldots. \end{aligned} \tag{2.12}$$

When $y = m$, quality loss and $L(y) = 0$, then $L(m) = 0$. Also, because *L(y)* attains its minimum value at this point, $L'(m) = 0$. If cubic and higher-order terms are neglected, then equations (2.13) and (2.14) are obtained:

$$L(y) = \frac{L''(m)}{2!}(y-m)^2 \tag{2.13}$$

$$L(y) = k(y-m)^2 \tag{2.14}$$

Δ_0 is defined as the point of intolerance as shown in Figure 2.8. It is the deviation from the target that causes an average customer to take an action. It is assumed that the corresponding monetary loss caused due to a defective component is A_0. A_0 is also defined as the cost of a corrective action. When the deviation of performance from the target of a product is Δ_0 and the corresponding loss is A_0, then for NTB and STB, $k = A_0 / \Delta_0^2$. Taguchi (2004), Maghsoodloo (1991), Fowlkes (1995), and Phadke (1989) have been referred to for this discussion. Taguchi's quality loss function for the NTB and STB takes the form of Equation (2.15).

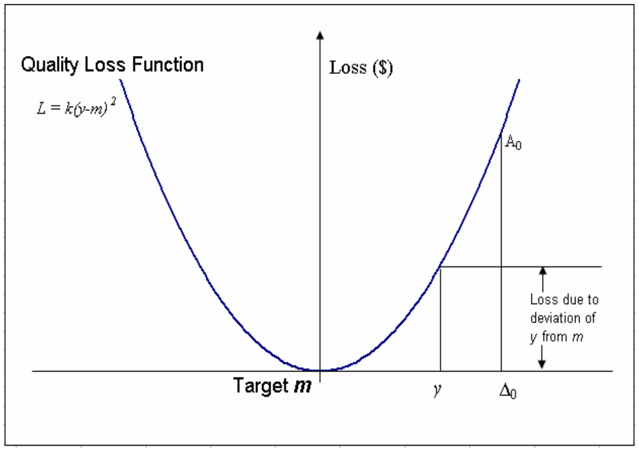


Figure 2.8. Loss Due to Off-Target Performance

$$L(y) = \frac{A_0}{\Delta_0^{\ 2}}(y-m)^2 \tag{2.15}$$

For the LTB, however, the quality loss function takes the form as in Equation (2.16) (Taguchi et al., 2004; Fowlkes, 1995; and Taguchi, 1999).

$$L(y) = A_0 \Delta_0^{\ 2} \left(\frac{1}{y_i} \right)^2 \tag{2.16}$$

The term Δ_0 is placed in the numerator instead of the denominator, which introduces an inconsistency among the NTB, STB, and LTB methods. Therefore, Δ_0 needs to be placed back to the denominator instead of the numerator in order to bring back the consistency among all three cases.

The term $(y\text{-}m)^2$ is called the mean-squared deviation (MSD). For a group of n products, if the performance readings are $y_i = y_1, y_2, y_3, \ldots, y_n$, then the MSD for this group of n products can be derived as in Equation (2.17):

$$MSD = \frac{1}{n}\left[(y_1 - m)^2 + (y_2 - m)^2 + \ldots + (y_n - m)^2 \right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(y_i - m\right)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(y_i^2 - 2y_i m + m^2\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(y_i\right)^2 + \frac{1}{n}\sum_{i=1}^{n}\left(-2y_i m + m^2\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(y_i\right)^2 - 2\overline{y}m + m^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(y_i\right)^2 - \overline{y}^2 + \overline{y}^2 - 2\overline{y}m + m^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(y_i\right)^2 - 2\overline{y}^2 + \overline{y}^2 + \left(\overline{y} - m\right)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(y_i\right)^2 - \frac{2\overline{y}}{n}\sum_{i=1}^{n} y_i + \frac{1}{n}\sum_{i=1}^{n}\overline{y}^2 + \left(\overline{y} - m\right)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(y_i\right)^2 - \frac{1}{n}\sum_{i=1}^{n} 2\overline{y}y_i + \frac{1}{n}\sum_{i=1}^{n}\overline{y}^2 + \left(\overline{y} - m\right)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \overline{y}\right)^2 + \left(\overline{y} - m\right)^2$$

$$MSD = \sigma^2 + \left(\overline{y} - m\right)^2 \qquad (2.17)$$

2.3.1. Unification of Three Cases. The chosen technique to define the target is to set it equal to a ratio, α, referred to as the target-mean ratio, times the mean performance. The advantage of doing this is that as the performance improves, the target also changes. Therefore, the target is not constant; rather it might change. By setting α equal to a ratio of the target and mean performance, Equation (2.18) is obtained, Sharma (2007).

$$\alpha = \frac{m}{\overline{y}}; or; m = \alpha.\overline{y}$$

$$MSD = \sigma^2 + \left(\overline{y} - \alpha.\overline{y}\right)^2$$

or

$$MSD = \sigma^2 + \overline{y}^2 \left(1 - \alpha\right)^2 \tag{2.18}$$

Equation (2.18) can be used interchangeably to encompass all three cases.

2.3.1.1. Quality loss function for smaller-the-better characteristics. As shown in Figure 2.9, in the case of STB, the target is zero. Therefore, by setting, $m = 0$ in Equation (2.17), Equation (2.19) is obtained. In this case, α is set at zero.

$$MSD = \sigma^2 + \overline{y}^2 \tag{2.19}$$

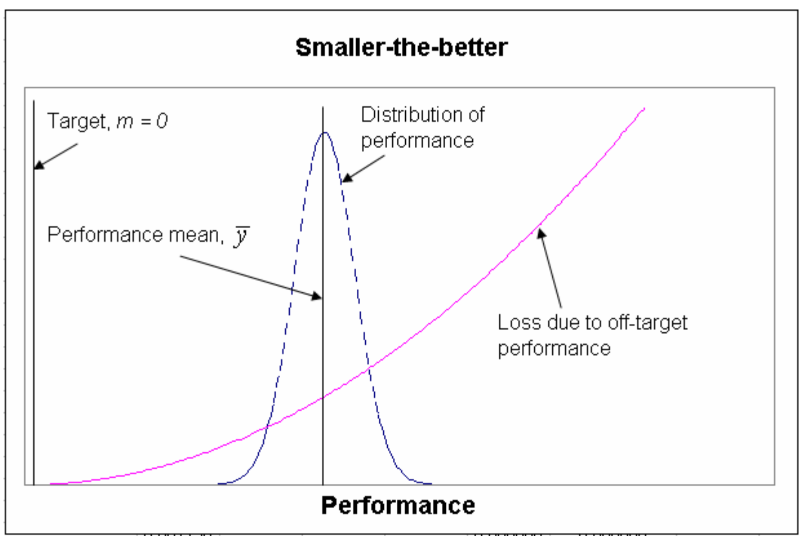


Figure 2.9. Smaller-the-Better

2.3.1.2. Quality loss function for nominal-the-best characteristics - performance on target. By setting $m = \overline{y}$ for the NTB approach where the value of α is set at 1, Equation (2.20) is obtained out of Equation (2.18):

$$MSD = \sigma^2 \tag{2.20}$$

2.3.1.3. Quality loss function for nominal-the-best characteristics—performance not on target. When the performance is not on target, Equation (2.21) or (2.22) can be used. Refer to Figure 2.10.

$$MSD = \sigma^2 + \left(\overline{y} - m\right)^2 \tag{2.21}$$

By setting $m = \alpha.\overline{y}$ in Equation (2.13), Equation (2.14) is obtained:

$$MSD = \sigma^2 + \overline{y}^2 \left(1 - \alpha\right)^2$$

Or

$$MSD = \sigma^2 + \overline{y}^2 (\alpha - 1)^2 \tag{2.22}$$

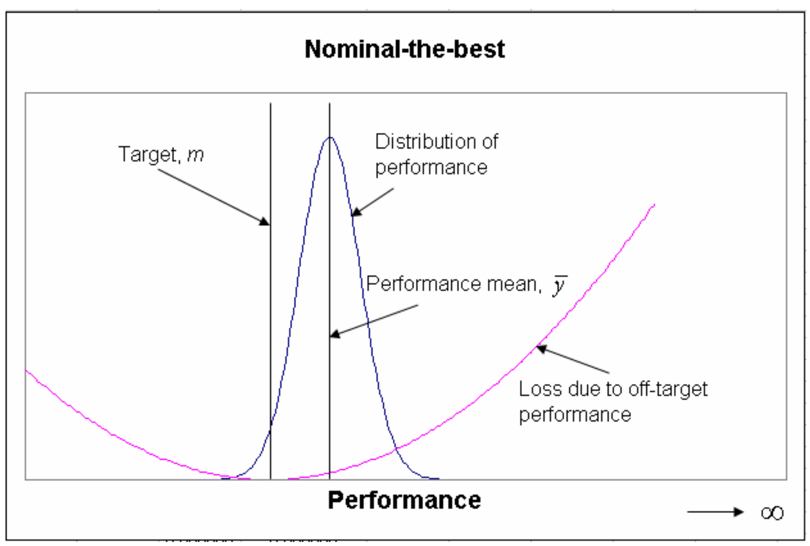


Figure 2.10. Nominal-the-Best

2.3.1.4. Quality loss function for larger-the-best characteristics. As shown in Figure 2.11, in the case of LTB, α needs to be significantly greater than 1 but not necessarily a large number or infinity.

For example, when α = 1.5, one has:

$$MSD = \sigma^2 + 0.25\bar{y}^2 \tag{2.23}$$

Equation (2.23) represents the LTB case with the target being equal to 1.5 times the mean performance.

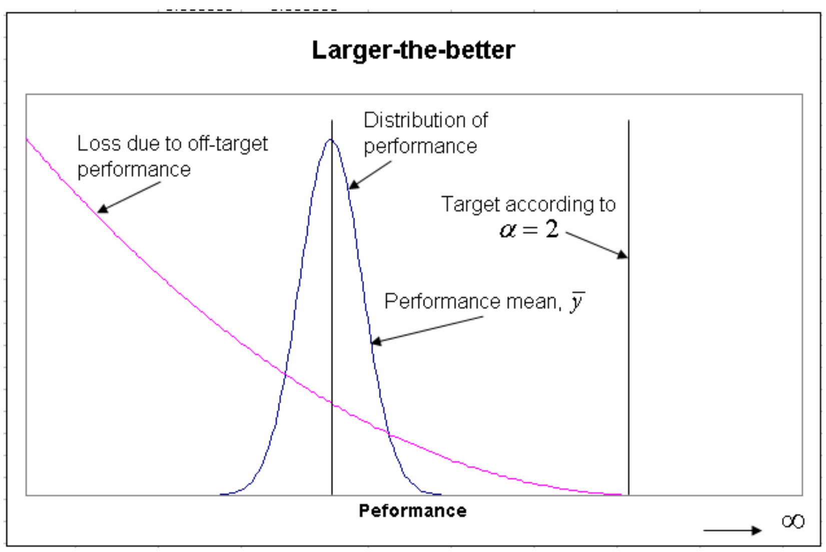


Figure 2.11. Larger-the-Better

When α = 2, Equation (2.24) is obtained:

$$MSD = \sigma^2 + \bar{y}^2 \tag{2.24}$$

This is the LTB case with the target double the mean performance. This particular situation in LTB is analogous or equivalent to the STB case. The only difference is that the target is not zero but is placed equidistant from the mean on the right-hand side. The target is also not assumed to be infinity or very large, though the case is LTB. The advantage of using this approach is that one can achieve comparable results in all three cases. The following case studies verify the results.

2.4. CASE STUDIES

A case of prime movers' efficiency is considered to illustrate how the new methodology can be used to assess the quality loss associated with the low efficiency, which according to Taguchi is not an LTB characteristic. However, this research has considered the efficiency to be an LTB characteristic simply because one would want it to be higher.

2.4.1. Case Study 1: Efficiency of Prime Movers (Engine / Electric Motor). Internal combustion engines were developed to serve as a primary source of power for automobiles, ships, airplanes, and many other mobile and stationary applications. Consider a manufacturer engaged in the manufacturing of internal combustion engines and simultaneously using engines and other prime movers for operating machines in a factory. The manufacturer has been manufacturing engines from the time of invention when the efficiency of an engine was as low as 4%. Therefore, the manufacturer has gone through almost all possible phases of improvement in engine efficiency. The efficiencies of commonly used motors are given in Table 2.5, which also has an entry for the efficiency corresponding to an ideal motor. The values of engine efficiency given in Table 2.5 have been taken from Ferreira (2005), and these numbers show the evolution of engine efficiency throughout a period of nearly a century. Engine manufacturing and use can be seen as continuous improvement in the quality of the product and its production.

Table 2.5. Evolution of Prime Mover Efficiency

Machine	Year	Efficiency
Engine	1902	0.04
Engine	1923	0.07
Engine	1935	0.10
Engine	1958	0.20
Engine	1975	0.28
Engine	2000	0.32
Hydrated alcohol Engine	2000	0.38
Most efficient Engine	2005	0.52
Motor	1937	0.76
Motor	2005	0.90
Energy efficient motor	2005	0.96
Ideal motor	2007	1.00

Suppose that eight sets of engines and four sets of electric motors are considered. Because of their year of development and manufacture, each set of machines operates at an efficiency level that is different from the other sets. The mean efficiency levels for these twelve sets are 4%, 7%, 10%, 20%, 28%, 32%, 38%, 52%, 76%, 90%, 96%, and 100%, as shown in Table 2.5. It is also given that at 50% efficiency, the estimated loss is $10.

Table 2.6 shows the calculated values of α and MSD in terms of the standard deviation and mean efficiency. It is evident from Table 2.6 that the quality loss depends on the mean efficiency and standard deviation of the performance. It is also depicted how much quality loss can be attributed to variation and how much to the distance between the performance mean and the target. With this methodology, the root cause can be identified, and by reducing the effect of the root cause the quality loss can be reduced. In contrast, the LTB case in Taguchi's methodology does not show whether the variation or performance away from the target causes a quality loss.

Table 2.6. Target-Mean Ratio and *MSD*

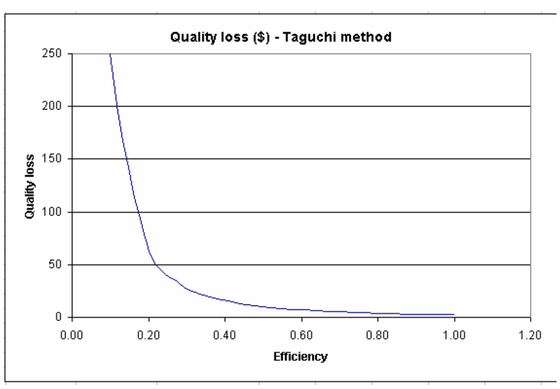
Efficiency, $\bar{y}$	**α**	***MSD***
4%	25	$\sigma^2 + 576\,\bar{y}^2$
7%	14.29	$\sigma^2 + 176.6241\,\bar{y}^2$
10%	10	$\sigma^2 + 81\,\bar{y}^2$
20%	5.00	$\sigma^2 + 16\,\bar{y}^2$
28%	3.57	$\sigma^2 + 6.6049\,\bar{y}^2$
32%	3.13	$\sigma^2 + 4.5369\,\bar{y}^2$
38%	2.63	$\sigma^2 + 2.6569\,\bar{y}^2$
52%	1.92	$\sigma^2 + 0.8464\,\bar{y}^2$
76%	1.32	$\sigma^2 + 0.1024\,\bar{y}^2$
90%	1.11	$\sigma^2 + 0.0121\,\bar{y}^2$
96%	1.04	$\sigma^2 + 0.0016\,\bar{y}^2$
100%	1.00	$\sigma^2 + 0.0000\,\bar{y}^2$

The value of k equates to $10/0.5^2 = \$40$. The associated loss in dollars in each case can be computed by multiplying MSD with k once the MSD for each efficiency level is known. Table 2.7 shows a comparison between the computations of quality loss using the new method and Taguchi's method. The standard deviation for each efficiency level is assumed to be small and constant at 0.01 except at the ideal efficiency of 100%, where it would not make sense to have variation and still have the mean efficiency at 100%.

Figure 2.12 shows two graphs of the mean efficiency vs. quality loss using Taguchi's methodology and the new methodology.

Table 2.7. Comparison of Quality Loss

k_{STB}	Mean efficiency $\bar{y}$	Target-mean ratio	Standard deviation	MSD-Variance	MSD-Bias	MSD-New	Quality loss ($) - New method	$\frac{1}{\bar{y}}$	k_{LTB}	MSD-Taguchi	Quality loss ($) - Taguchi method
40	0.04	25.00	0.01	0.0001	0.922	0.922	36.86	25.00	2.5	625.00	1562.50
40	0.07	14.29	0.01	0.0001	0.865	0.865	34.60	14.29	2.5	204.08	510.20
40	0.10	10.00	0.01	0.0001	0.810	0.810	32.40	10.00	2.5	100.00	250.00
40	0.20	5.00	0.01	0.0001	0.640	0.640	25.60	5.00	2.5	25.00	62.50
40	0.28	3.57	0.01	0.0001	0.518	0.519	20.74	3.57	2.5	12.76	31.89
40	0.32	3.13	0.01	0.0001	0.462	0.463	18.50	3.13	2.5	9.77	24.41
40	0.38	2.63	0.01	0.0001	0.384	0.385	15.38	2.63	2.5	6.93	17.31
40	0.52	1.92	0.01	0.0001	0.230	0.231	9.22	1.92	2.5	3.70	9.25
40	0.76	1.32	0.01	0.0001	0.058	0.058	2.31	1.32	2.5	1.73	4.33
40	0.90	1.11	0.01	0.0001	0.010	0.010	0.40	1.11	2.5	1.23	3.09
40	0.96	1.04	0.01	0.0001	0.002	0.002	0.07	1.04	2.5	1.09	2.71
40	1.00	1.00	0.00	0	0.000	0.000	0.00	1.00	2.5	1.00	2.50



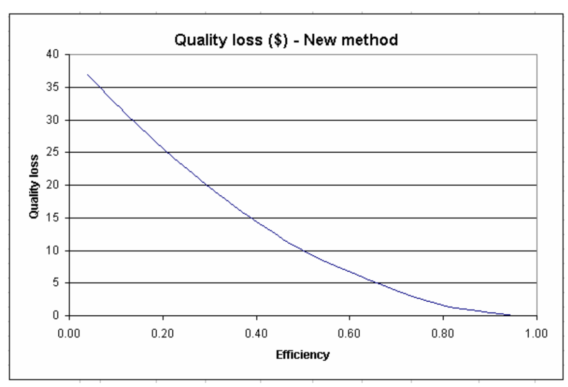


Figure 2.12. Comparison of Quality Losses

The results from Table 2.7 were used to plot the two graphs shown in Figure 2.12. It is observed from the computation as well as from the graphs that the quality loss is zero if the maximum possible efficiency is achieved when the new methodology is employed. In comparison, at the ideal situation, the Taguchi methodology delivers some small quality loss instead of zero. Moreover, the new methodology gives a finite loss when the efficiency approaches zero. On the other hand, the Taguchi methodology suggests that the loss tends to be infinite at zero efficiency. The Taguchi method does not produce realistic results at both boundary conditions. On the other hand, the proposed methodology produces realistic results at all increments including the ideal boundary conditions.

2.4.2. Case Study 2: Thermal Conductivity of Material. In the second example, a characteristic was selected that can be STB in certain situations and LTB in certain other situations. Among many such characteristics a simple and more common property, the thermal conductivity of a material, was selected to illustrate this concept, as shown in Table 2.8, A numerical problem is considered to be an STB case and is solved using the Taguchi method. The results achieved are given in the column designated as A* in Appendix B. The same STB numerical problem is then converted into an LTB numerical problem. Next, the LTB problem is solved using the Taguchi approach and the results are presented in the column designated as B* in Appendix B. The LTB problem is also solved using the proposed approach, and the results are shown in column C* in Appendix B. The results in columns A*, B*, and C* are then compared to understand the difference between the Taguchi methodology for LTB and the proposed methodology for LTB, as well as the equality or correspondence between the Taguchi methodology for STB and the proposed methodology for LTB.

2.4.2.1. Smaller-the-better numerical problem. It is assumed that stainless steel is used as a thermal insulator because of its strength and formability. Twenty sample pieces of stainless steel were drawn from a production system for inspection with regard to heat/thermal conductivity. The following readings are observed as shown in Table 2.8. The unit used for thermal conductivity is ($W \cdot m^{-1} \cdot K^{-1}$).

Table 2.8. Thermal Conductivity Readings

13.97	14.04	14.09	13.99	14.00	14.06	13.92	13.93	14.01	14.03
15.12	15.09	13.93	13.98	14.02	14.05	14.08	13.98	14.00	14.11

Thermal conductivity is required to be within (or no more than) 15 $W \cdot m^{-1} \cdot K^{-1}$. Because of heat dissipation, the estimated quality loss is $2. The quality loss due to the production is then calculated. The STB problem is solved using Taguchi's approach and the results can be seen in Column A* of Appendix B. Appendix B contains the results obtained using LTB problem as well.

2.4.2.2. Larger-the-better numerical problem. Using the same thermal conductivity readings, the quality loss due to the production is then calculated. A thermal conductivity value of 15 is chosen here to keep the data similar between the STB and LTB cases. The LTB problem is solved using Taguchi's approach where the results are shown in Column B* of Appendix B. The same LTB problem is also solved using the new approach where the results are provided in Column C* of Appendix B.

2.4.2.3. Comparison of results. When the STB problem is solved using Taguchi's approach, the quality loss computed is $1.77 per piece. After the same problem is converted to LTB, the quality loss computed using Taguchi's approach changes to $2.31 per piece, which is 30.5% higher than the previous value. This signifies inadequate consistency in Taguchi's method for STB and LTB because the formulae are different. When the converted LTB problem is solved using the new methodology, choosing $\alpha = 2$, the quality loss is restored to its original value of $1.77 per piece.

It is important to understand how α affects the quality loss. Five values of α have been considered: 1.5, 2, 2.5, 4, and 5. The quality loss increases somewhat quadratically with α. Figure 2.13 illustrates the relationship between the target-mean ratio, α, and the quality loss.

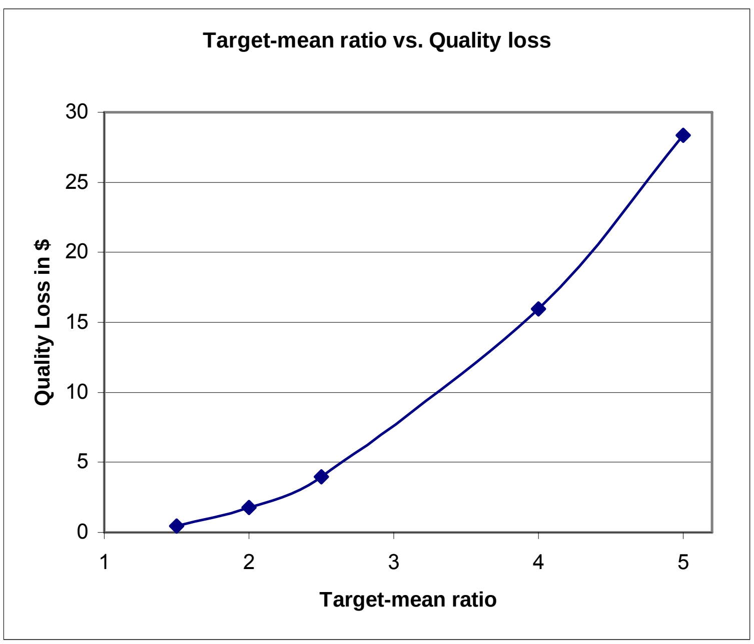


Figure 2.13: Target-Mean Ratio vs. Quality Loss

2.5. VALUE OF THE TARGET-MEAN RATIO

The target-mean ratio does away with the assumption of the target at infinity for the LTB case, and it can be used as a tool to bring about similarity among all three cases.

A similar term called the *scaling ratio* is used in robust engineering after the variation is minimized using the signal-to-noise ratio to modify the adjustment factor in order to put the system performance on target. The value of α can take different values depending on the type of case. Two types of LTB characteristics exist: (1) LTB characteristics that have a naturally available target, i.e., efficiency and coefficient of performance (COP) characteristics, and (2) characteristics that have no obvious target such as the strength of the material.

Two types of LTB performance parameters have a natural target. The first type includes those parameters that have an ideal limit of performance such as efficiency, coefficient of performance, and percent non-defective as opposed to percent defective. Here the target value is naturally available and any target value, even theoretically, more than 100% for efficiency is impossible. Similarly for the COP, a given maximum value cannot be exceeded. In such cases, it is recommended to set the target at its ideal limit. For example, if the present performance of a machine is at 33.33% and the ideal limit is 100%, then the target is set at 100%. In this case, the value of α equates to 3. Similarly, if the present performance of a machine is at 80% and the ideal limit is 100%, then the target is set at 100%. In this case, the value of α equates to 1.25.

For no obvious target-like characteristics, the target can be set at α equal to a specific value. In this case, it is theoretically possible to assume a certain higher target value. It is worthwhile to mention that the target should be difficult to achieve but not impossible. Whether the target can really be achieved might depend upon technology change, innovation, material type, process, or conditions, among other factors. The target-mean ratio α needs to be significantly greater than 1 but need not be infinity. Because the loss is equivalent to that for STB at $\alpha = 2$, it is recommended that the target be set according to $\alpha = 2$. However, a more detailed discussion on the value of α is provided in Section 3. It can easily be visualized that the target is placed equidistant from the mean on the other side as the mean is away from zero. The target also need not be assumed as infinity though the case is LTB. The examples discussed above verify the results.

2.6. CONCLUSION

This section provided a brief introduction of Taguchi's methodology for computing quality loss. The quality loss methodology for LTB characteristics is different from that for STB and NTB. Even though Taguchi's methodology is distribution independent, it was studied how the reciprocal of response behaves when the response itself follows a particular distribution. The proposed methodology is also distribution independent.

Furthermore, an attempt was made to analyze and compare the algebraic results obtained by using Taguchi's methodology and without using the reciprocal of the data. In general, quality loss has two components: loss due to bias and loss due to variation. The same two components were used to find the change in quality loss. This is called the two component approach, and it is used to analyze and compare the results obtained by using the methodology in use and without using the reciprocal of the data. It was assumed that the data follow certain distributions so that algebraic analysis is possible instead of using actual data from the field. The inverse Gaussian distribution, Gamma distribution, Weibull distribution, and Normal distribution were also considered.

This section attempted to present a similarity among all three cases of quality loss function by employing the target-mean ratio and proposing a common formula for all three cases. It was shown that the target-mean ratio can take different values to represent all three cases. The new method brings about uniformity with regard to the methodology among all three cases of STB, NTB, and LTB. It leads to consistent results, and because of this consistency, the results can be easily compared. Also, it is easy to compute the quality loss in the case of LTB using the proposed method. The proposed method eliminates the need to assume the target value as infinity.

The proposed methodology is especially suitable for characteristics such as efficiency, coefficient of performance, and percent non-defective, among many others. The same model can be converted into any of the three cases: STB, NTB, or LTB. It permits a target to be assumed for LTB cases so that improvement can be measured with reference to the target. With this approach, it is possible that the quality loss may become zero for a certain case when the target is reached. In contrast, with Taguchi's approach, this is only possible if the performance characteristic reaches an infinite value.

The proposed methodology also permits most characteristics that could not be designated as LTB characteristics to be brought under the LTB category. In the case of characteristics where the target value is naturally available, it is recommended to set the target at an ideal limit. Finally, in the case of parameters that have no obvious ideal performance limit, unlike efficiency, it is recommended to set the target at two times the present mean performance, i.e., α may be set equal to 2. By introducing the new methodology, an attempt was made to streamline the mathematical aspect as well as the practical aspect of the quality loss function.

2.7. FUTURE RESEARCH

Often higher performance is achievable but it does not make economic sense. Therefore, the proposed methodology suggests using the target-mean ratio to determine the target for LTB characteristics. More research should be done on selecting the appropriate target-mean ratio for different types of LTB characteristics such as the coefficient of performance, efficiency, percent defective, and strength of a material. Further research is also needed to study the implications of this methodology on the concept of the signal-to-noise ratio and optimization of systems using the signal-to-noise ratio. In addition, further research may be conducted as to how the signal-to-noise ratio for operating window is affected due to the new methodology. Another possibility is to extend the quality loss function thus obtained for LTB characteristics to multivariate cases wherein other types (e.g., STB and NTB) are also part of the quality loss function. Some of these implications of a finite target for LTB are studied in Section 2.

3. IMPLICATIONS OF QUALITY LOSS FUNCTION IN UNIFIED METHODOLOGY—LTB CASE WITH TARGET

3.1. INTRODUCTION

A unified methodology has been proposed in Section 2 to bring about similarity among the three cases - smaller-the-better, nominal-the-best, and larger-the-better - by introducing a term called the *target-mean ratio* and proposing a unified formula for quality loss. The new methodology has some implications that need to be addressed. This section attempts to study the implications of the new methodology on the field of quality engineering and axiomatic design. An implied classification of LTB characteristics according to Taguchi on the basis of a target value at infinity has been presented as well as the classification of LTB characteristics based on the new methodology. A new concept of a *complementary characteristic* is also suggested. It is suggested that whether a given LTB characteristic or its complementary characteristic is considered for one and the same case, the quality loss must be equal for both characteristics. It is then shown mathematically that with the use of the new methodology, any LTB characteristic and its complementary characteristic have the same mean-squared deviation (*MSD*) or quality loss when the target is set according to $\alpha = 2$, or at 100%, or at any other value, without a loss of generality and consistency. In this way, it is argued that the common methodology is better and more suitable for the purpose of computing quality loss. It is also studied and deliberated upon as to what should be the most appropriate value of α, the target-mean ratio.

The implication of a finite target on axiomatic design complexity has been studied from a mathematical point of view. It is shown that the complexity for LTB with an infinite target is not equivalent to that for STB and NTB characteristics. Therefore, complexity due to variability for LTB with a finite target is derived, and it is found that a comparable complexity measure is obtained with the unified methodology. The effect of a finite target for LTB on the mathematical relationship between the signal-to-noise ratio (SN ratio) and axiomatic measures has also been studied. It is shown that the relationship between the SN ratio and complexity of an axiomatic measure becomes equivalent to STB case. The mathematical relationship for the SN ratio for operating window with LTB having a finite target is then derived, which is different from the existing formulation.

Two types of performance characteristics have been discussed in the literature on quality engineering in Fowlkes (1995), Taguchi (2004), and Taguchi (1999). It is said that the characteristics having the maximum theoretically possible target of 100% are not larger-the-better (LTB) characteristics (Taguchi, 2004). Also, it is described that the characteristics having infinity as the target value do not really have a target value and are LTB characteristics.

In Section 2, it was discussed that it is plausible to have a target value of infinity and, therefore, not really assign a value to it (Sharma, 2007). It has been further argued that when the target is known, as in the case of the efficiency, yield, or percent non-defective, then it can also be an LTB characteristic. It has also been discussed that when the target of 100% efficiency is extremely difficult or impossible to achieve, then the target should not be considered as infinity. According to Sagan (1980), infinity is larger than what can be imagined. Because of the assumption of infinity as the target, some of the characteristics are not presently permitted to be LTB characteristics. Subsequent derivation of formulae also supports the theory that some characteristics do not fall under the category of LTB characteristics.

A little consideration shows that in Taguchi's quality loss function, the approximated loss is based on two reasons (Taguchi, 2004): the variation (denoted by the standard deviation) of performance about some mean, and the mean performance away from the target, also called bias by El-Haik (2005) (denoted by the distance by which the mean performance is away from the target). In Taguchi's approach, the cases of smaller-the-better (STB) and nominal-the-best (NTB) have both been clearly shown to affect the mean-squared deviation, *MSD*, and in turn, the quality loss. In contrast, in Taguchi's existing approach for LTB characteristics, it is unclear as to how variation of the performance affects the *MSD* or quality loss.

However, Maghsoodloo (1991) showed that the *MSD* for LTB cases can be approximated by Equation (3.1) (Maghsoodloo, 1991). The *MSD* (or quality loss) given for the STB and NTB cases is exact, while it is only approximate for the LTB case.

$$MSD = \frac{1}{\bar{y}^2}\left[1+\frac{3\sigma^2}{\bar{y}^2}\right] \tag{3.1}$$

Where $\bar{y}$ = mean performance, and σ = standard deviation of performance.

3.1.1. Taguchi's Quality Loss Function for LTB Characteristics. In practice, any performance value infinitely large is not targeted. One view can be that infinity as the target value is notional. However, it is unnecessary for the target to be infinity notionally as well. In many cases, performance characteristics higher than a certain value would add little value to the quality of the product. For example, additional tensile strength of paper for general purposes would be of little help; what is required is wear-resistance and durability. In the LTB case, it is assumed that the larger the value of a parameter, the better it is. Ideally, if the performance value approaches infinity, then the loss approaches zero. This is an ideal case, and thus does not match reality. The target should not be infinity because that is unachievable. Therefore, a new formulation for the quality loss function for the LTB case has been developed.

Taguchi's quality loss function is given in Equation (3.2). Fowlkes and Creveling (1995), Taguchi, Chowdhury, and Wu (2004), Venkateswaren (2003), and Sharma and Ragsdell (2007) may be referred to for a detailed discussion on the derivation. If y is some value of a given parameter, and *m* is the target, then the loss function $L(y)$ is given as follows:

$$L(y) = k(y - m)^2 \tag{3.2}$$

The term $(y - m)^2$ is called the mean-squared deviation (*MSD)*. For a group of *n* products, if the performance readings are $y_1 = y_1, y_2, y_3, ..., y_n$, then the *MSD* for the group of *n* products can be derived as shown in Equation (3.3).

$$MSD = \sigma^2 + (\bar{y} - m)^2 \tag{3.3}$$

3.1.2. Unified Methodology. A ratio, α, called the target-mean ratio is introduced. The advantage of defining the target equal to α multiplied by the mean performance is that as the performance improves, the target also changes. Therefore, the target is not constant for a long time; rather it might change after some time. It may remain unchanged for a short period of time. By setting α = target / mean performance, Equation (3.4) is obtained.

$$\alpha = \frac{m}{\bar{y}} \text{ or } m = \alpha.\bar{y}$$

$$MSD = \sigma^2 + \bar{y}^2 (\alpha - 1)^2 \tag{3.4}$$

This equation will be used interchangeably to encompass all three cases.

When the characteristic under consideration is smaller-the-better, the target as well as the target-mean ratio is zero. Therefore, by setting $\alpha = 0$ in Equation (3.4), Equation (3.5) is obtained.

$$\alpha = 0;$$

$$MSD = \sigma^2 + \bar{y}^2 \tag{3.5}$$

When nominal-the-best is considered and performance is on target, by setting $m = \bar{y}$, Equation (3.6) is obtained.

$$\alpha = 1$$

$$MSD = \sigma^2 \tag{3.6}$$

However, that may not be the case. Performance away from the target also needs to be considered. When the performance is not on target, then either Equation (3.7) or (3.8) can be used.

$$MSD = \sigma^2 + \left(\bar{y} - m\right)^2 \tag{3.7}$$

By setting $m = \alpha.\bar{y}$, Equation (3.8) is obtained.

$$MSD = \sigma^2 + \bar{y}^2\left(\alpha - 1\right)^2 \tag{3.8}$$

When larger-the-better is of interest as shown in Section 2, α needs to be significantly greater than unity but need not be a very large number or infinity.

$$MSD = \sigma^2 + \bar{y}^2\left(\alpha - 1\right)^2 \tag{3.9}$$

If $\alpha > 1$, say, $\alpha = 2$

$$MSD = \sigma^2 + \bar{y}^2 \tag{3.10}$$

This is the LTB case with the target double the mean performance. This particular situation in LTB is equivalent to the STB case because the quality losses for both are equal. The only difference is that the target is not zero but is equidistant from the mean on the other side. Also, the target is not assumed to be infinity or very large, though the case is LTB. The advantage of using this approach is that one can achieve comparable results in all three cases.

Having introduced the new methodology, it is now important to discuss the implications of this methodology on quality engineering. First, a classification of LTB characteristics is presented. Then, a new concept of a complementary characteristic is suggested and analyzed with regard to each type of LTB characteristic. Finally, what should be the most appropriate value of α, the target-mean ratio, is studied and discussed.

The following section discusses the classification of LTB characteristics based on the target. It also discusses in detail the implied classification of LTB characteristics based on Taguchi's methodology as well as the new methodology.

3.2. CLASSIFICATION OF LTB CHARACTERISTICS

This section presents a method of classifying LTB performance parameters on the basis of the target value. This classification helps in understanding the organization of the LTB type of characteristics and its relationship with the target value. A classification of LTB characteristics based on Taguchi's methodology as well as according to the new methodology is given following a short description of the coefficient of performance.

The coefficient of performance (COP) should be reviewed first in order to appreciate its position in the classification presented. The COP of a heat pump, for example, is the ratio of heat delivered to the input work, as shown in Equation (3.11).

$$COP = \frac{|Q|}{W} \tag{3.11}$$

Where Q is the heat output of the heat pump and W is the work input of the compressor. According to the first law of thermodynamics, the heat output of the heat pump is more than the work input, and therefore the COP of a heat pump is more than unity.

3.2.1. Taguchi's Implied Classification of LTB Characteristics. According to Taguchi's *Quality Engineering Handbook* (2004), depending on maximum possible target two types of performance characteristics exist, Figure 3.1. First, the characteristics that have the maximum possible target as 100% are not the larger-the-better characteristics.

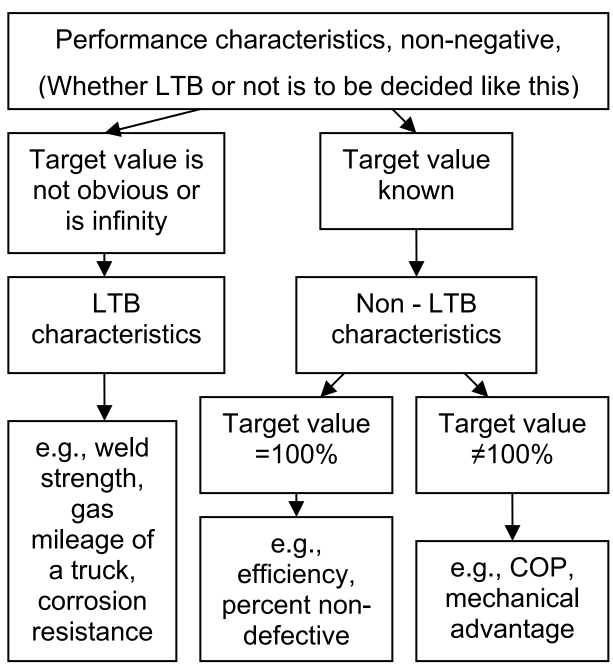


Figure 3.1. Taguchi's Implied Classification of LTB Characteristics

Second, the characteristics that have infinity as the target value do not really have a target value and are LTB characteristics. In other words, a performance characteristic should have a target as infinity for it to be designated as an LTB characteristic, and characteristics that have a limitation in the theoretical target value cannot be designated as LTB. This provides a method of classification, and the situation is represented in Figure 3.1. In effect, there are certain characteristics which need to be larger for improving quality but cannot be categorized as larger-the-better type seemingly because of infinite target.

3.2.2. New Classification of LTB Characteristics. Because the target for the LTB case in the new methodology is assumed to be a finite value instead of infinity, the LTB characteristics can be divided into two types. The first type of parameters includes those having an ideal limit of performance, such as efficiency (with the maximum being 100%), coefficient of performance (with the maximum COP being a certain value, such as 250%), percent non-defective (with the maximum being 100%), etc. The second type of performance parameters includes those parameters having no obvious ideal performance limit as opposed to efficiency (because efficiency has an ideal performance limit). Examples include the strength of a material, corrosion resistance, and wear resistance.

Because even a target of 100% efficiency is extremely difficult or impossible to achieve, it is unadvisable to consider infinity as the target. It is suggested here that the target for LTB characteristics be fixed instead of assuming it to be infinity. In the case of the efficiency of a machine, or percent non-defective, the target can be assumed to be at 100% or at a certain higher finite value, and the quality characteristic may be considered to be LTB. In this way, some of the characteristics that are presently not LTB characteristics can also be brought under the category of larger-the-better characteristics.

A performance characteristic should be non-negative, and its higher value should be desirable for it to be designated as LTB. This is the primary concern that needs to be considered. Figure 3.2 shows this situation clearly. In other words, a performance characteristic should have a target value not necessarily as infinity for it to be designated as an LTB characteristic. LTB characteristics are divided into two categories at the first level: the target value known type and the target value unknown type. For parameters where the target value is known, further division occurs at the second level: target value 100% and target value not 100%.

If parameters have a limited theoretical target value such as 100%, then these are termed *efficiency-like characteristics*. If they have a limited theoretical target value other than 100%, such as 250%, then these are termed *COP-like characteristics*. This situation is represented in Figure 3.2, which gives a new method of classification.

The efficiency-like characteristics include but are not limited to efficiencies (e.g., thermal efficiency, fuel efficiency, mechanical efficiency, energy conversion efficiency, electrical efficiency, volumetric efficiency, thermodynamic efficiency, and % profit). Furthermore, parameters in the same category that can be included are probability, relative humidity, relative strength, frequency ratio, percentage, ratio, proportion, relative amount, part, fraction, quotient or share, freedom from error or defect, degree of correctness, accuracy, precision, or exactness.

The COP-like characteristics include but are not limited to mechanical advantage, gearing ratio, and the coefficient of performance (COP). COP is mainly used in context with cooling and heating systems, such as, heat pumps, refrigeration cycles and systems, and heat exchangers.

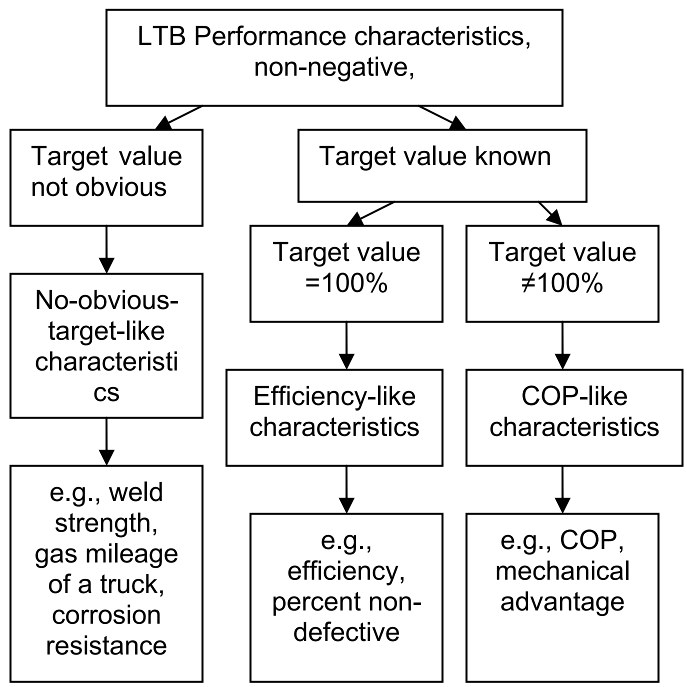


Figure 3.2. New Classification of LTB Characteristics

All other non-negative characteristics for which a higher value is desirable and which do not fall under the previous categories of efficiency-like and COP-like characteristics are termed *no-obvious-target-like characteristics*. In this case, the target is unknown but needs to be assumed. This category includes wear resistance, corrosion resistance, heat or thermal conductivity, electrical conductivity, tensile strength, reaction time, toughness, ductility, pressure, grain size, etc.

3.3. COMPLEMENTARY PROPERTY

As discussed previously, two types of LTB characteristics exist. The first type of characteristics, which have a theoretically maximum possible target, are termed *efficiency-like* or *COP-like*. The second type, which do not have an obvious target or any theoretically defined value, at least in clear terms, are termed *no-obvious-target-like*. A new concept of a *complementary characteristic* is being introduced here. A little consideration will show that the LTB characteristics such as efficiency can be viewed from two perspectives: from a target of zero and from a target of 100%. Similarly, characteristics that are similar to COP can also be viewed from two perspectives: from a target of zero and from a theoretically maximum possible target value such as 250%. Also, other characteristics can be viewed from both perspectives, i.e., from a target of zero and from a theoretically decided target value according to a certain α value such as 2. Depending on the perspective target, the characteristics can be considered to be smaller-the-better or larger-the-better. The same characteristic can be considered an LTB characteristic if the target is 100%, 250%, or a certain higher finite value, and an STB if the target is zero. Such an STB characteristic corresponding to an LTB characteristic may be called a complementary characteristic. For example, efficiency is an LTB characteristic and can be converted to inefficiency, an STB, which is a complementary characteristic of efficiency. Similarly, percent non-defective, an LTB characteristic, can be converted to percent defective, an STB, which is a corresponding complementary characteristic.

Axiomatically, it is suggested that whether a given LTB characteristic or its STB complementary characteristic is considered for one and the same case, the *MSD* and quality loss must be equal. If the LTB and its complementary characteristic have the same *MSD* or quality loss, then it is said that the complementary property is satisfied. It can also be said that quality loss equivalence to a complementary characteristic is observed. It is suggested that a methodology to compute quality loss should be such that the complementary property is satisfied. This is possible if the new methodology is used, but not with Taguchi's methodology. It is mathematically shown in the following sections that each LTB characteristic and its complementary characteristic have the same *MSD*. Because it is the same process being observed, the standard deviation of performance is assumed to be the same. It is observed that when the target is set according to $\alpha = 2$, or at 100% or any finite value, the *MSD* or quality loss is equal for both LTB and its complementary characteristic without a loss of generality and consistency. In general, if the target is assumed to be infinity, then the *MSD* or quality loss for LTB and its complementary characteristic is unequal and the complementary property is not satisfied.

3.3.1. No-Obvious-Target-Like Characteristics. Now consider the case with no-obvious-target-like characteristics. Suppose the strength of a material in a lot is $y_i = t_i$, where y_i denotes a characteristic, t_i denotes strength, and subscript i represents i th observation where $i = 1, 2, 3, \ldots\ldots,$. Therefore, the "lack of strength" is $y_i' = m - y_i$, where y_i' denotes the complementary characteristic of y_i. Also, suppose $\bar{y}$ is the mean of y_i, and $\bar{y}'$ is the mean of y_i', then $\bar{y}' = m - \bar{y}$. In addition, if $\bar{t}$ is the mean of t_i, and if $\bar{t}'$ is the mean of t_i', then it is easy to show that

$$\overline{y} = \overline{t} \tag{3.12}$$

$$\overline{y'} = \overline{t'} \tag{3.13}$$

and lack of strength

$$\bar{t}' = m - \bar{t} \tag{3.14}$$

As shown in Figure 3.3, when strength is considered, it should be as high as possible. Therefore, essentially this is an LTB characteristic. Suppose the target is set according to $\alpha = 2$, i.e., $m = 2\bar{t}$.

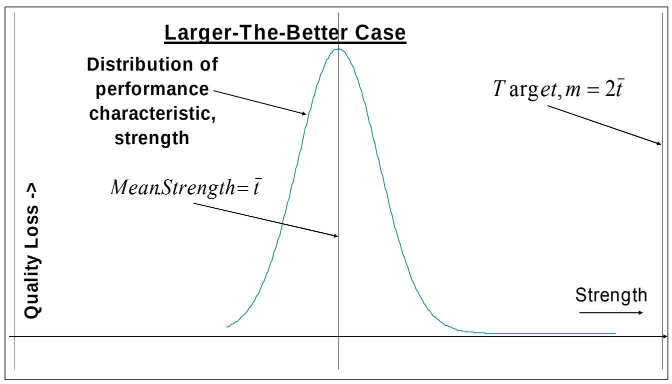


Figure 3.3. Complementary Property - No-Obvious-Target-Like Characteristics - LTB Case

Using Equation (3.3), Equation (3.12), and $m = 2\bar{t}$,

$$MSD_{LTB} = \sigma^2 + \left(\bar{t} - 2\bar{t}\right)^2$$

$$MSD_{LTB} = \sigma^2 + \bar{t}^{\,2} \tag{3.15}$$

So, for $\alpha = 2$ and $m = 2\bar{t}$

$$\bar{t}' = 2\bar{t} - \bar{t}$$

$$\bar{t}' = \bar{t} \tag{3.16}$$

When lack of strength is considered, denoted by t_i', it is desirable for this number to be as low as possible. Therefore, this essentially is an STB characteristic with the target as zero (i.e., $m = 0$) as shown in Figure 3.4. Converting Equation (3.3) to (3.3a) and using Equation (3.3a), and Equation (3.18), and $m = 0$,

$$MSD = \sigma^2 + \left(\overline{y}' - m\right)^2 \tag{3.3a}$$

$$MSD_{STB} = \sigma^2 + \left(\overline{t}' - 0\right)^2$$

$$MSD_{STB} = \sigma^2 + \left(\overline{t}'\right)^2$$

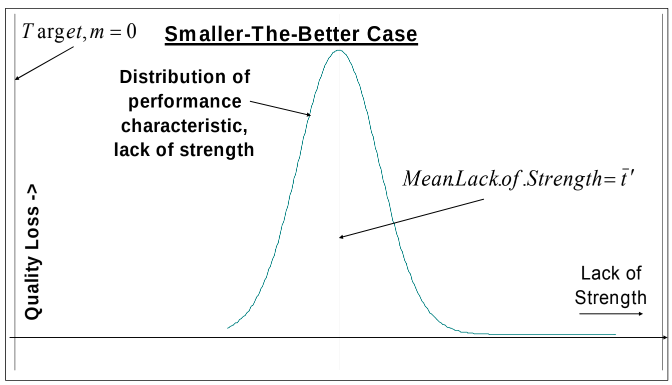


Figure 3.4. Complementary Property–No-Obvious-Target-Like Characteristics - STB Case

Using Equation (3.16),

$$MSD_{STB} = \sigma^2 + \overline{t}^{\,2} \tag{3.17}$$

Combining Equation (3.15) and Equation (3.17) yields

$$MSD_{LTB} = MSD_{STB} = \sigma^2 + \overline{t}^{\,2} \tag{3.15 \& 3.17}$$

Equations (3.15) and (3.17) show that in both cases the *MSD* is essentially the same. Whether strength t_i is considered or "lack of strength" $t_i' = m - t_i$ is considered, the results for the *MSD* are same. With the new method, the same results are obtained by either considering the strength or the lack of strength with the same performance distribution. Therefore, the generality and consistency are maintained. In this way, it is evident that the new methodology is better and more suitable for the purpose of computing quality loss. The same concept can be represented by two complementary characteristics - one STB and the other LTB - and consistent results can be obtained.

3.3.2. Efficiency-Like Characteristics. If the "efficiency" of an engine in a population is η_i, then the inefficiency is $1-\eta_i$. If $y_i = \eta_i$, where y_i denotes a characteristic and η_i denotes efficiency, and subscript i represents ith observation where $i = 1, 2, 3, \ldots\ldots$, then the "loss of efficiency" or "inefficiency" is $y_i' = 1-\eta_i$, where y_i' denotes the complementary characteristic of y_i. If $\bar{y}$ is the mean of y_i, $\bar{y}'$ is the mean of y_i', and $\bar{\eta}$ is the mean of η_i, then it can easily be shown that

$$\bar{y} = \bar{\eta} \tag{3.18}$$

And

$$\bar{y}' = 1 - \bar{\eta} \tag{3.19}$$

When efficiency is considered, it should be as high as possible. Therefore, it is essentially an LTB characteristic with the target as unity (i.e., $m = 100\%$ or $m = 1$) as shown in Figure 3.5. Using equations (3.3), (3.18), and $m = 1$,

$$MSD_{LTB} = \sigma^2 + \left(\bar{\eta} - 1\right)^2 \tag{3.20}$$

Figure 3.6 shows a "lack of efficiency" or inefficiency as STB. When inefficiency is considered, one wants it to be as low as possible. Therefore, it is essentially an STB characteristic with the target as zero (i.e., $m = 0$). Changing Equation (3.3) to (3.3a),

$$MSD = \sigma^2 + \left(\bar{y}' - m\right)^2 \tag{3.3a}$$

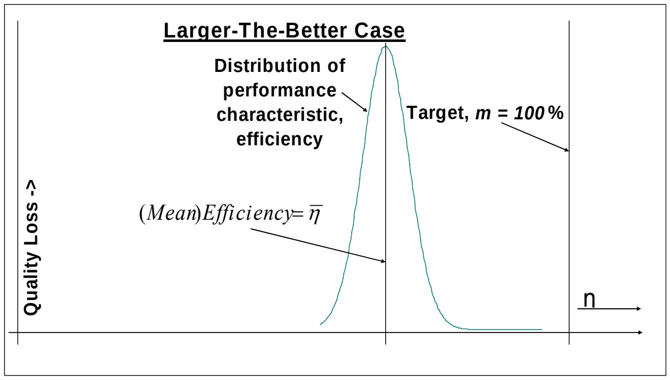


Figure 3.5. Complementary Property–Efficiency-Like Characteristics—LTB Case

Where $\bar{y}'$ is the mean of a different characteristic inefficiency, a complementary characteristic of efficiency. Using equations (3.3a), (3.19), and $m = 0$,

$$MSD_{STB} = \sigma^2 + \left(1 - \bar{\eta} - 0\right)^2$$

$$MSD_{STB} = \sigma^2 + \left(\bar{\eta} - 1\right)^2 \tag{3.21}$$

Combining equations (3.20) and (3.21) yields

$$MSD_{LTB} = MSD_{STB} = \sigma^2 + \left(\bar{\eta} - 1\right)^2 \tag{3.20 \& 3.21}$$

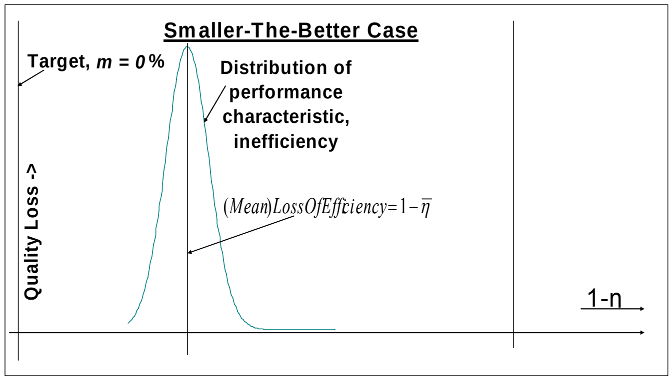


Figure 3.6. Complementary Property - Efficiency-Like Characteristics—STB Case

Equations (3.20) and (3.21) show that in both cases the *MSD* is essentially the same. With the new methodology, if the performance distributions are the same, then the result for the *MSD* will be the same whether efficiency η_i is considered or inefficiency $1-\eta_i$ is considered. Thus, it is obvious that generality and consistency are maintained. In other words, the same concept can be represented by two complementary characteristics—one STB and the other LTB—and consistent results can be obtained.

3.3.3. Coefficient-of-Performance-Like Characteristics. Suppose the observed COP of a heating system in a lot is c_i and the ideal COP is C. If $y_i = c_i$, where y_i denotes a characteristic and c_i denotes the observed COP, and subscript i represents the i th observation where $i = 1, 2, 3, \ldots\ldots$, then the "loss of COP" $y_i' = C - c_i$, $y_i' = \alpha\bar{c} - c_i$, where y_i' denotes the complementary characteristic of y_i. Suppose $\bar{y}$ is the mean of y_i, $\bar{y}'$ is the mean of y_i', and $\bar{c}$ is the mean of c_i. It can be shown that

$$\bar{y} = \bar{c} \tag{3.22}$$

$$\bar{y}' = C - \bar{c}$$

and

$$\bar{y}' = \alpha\bar{c} - \bar{c} \tag{3.23}$$

When the COP is considered, it should be as high as possible. Therefore, it is essentially an LTB characteristic with the target equivalent to the ideal COP (i.e., $m = C$ or $m = \alpha\bar{c}$) as shown in Figure 3.7. Using equations (3.3) and (3.12) as well as $m = \alpha\bar{c}$,

$$MSD = \sigma^2 + \left(\bar{y} - m\right)^2 \tag{3.24}$$

$$MSD_{LTB} = \sigma^2 + \left(\bar{c} - \alpha\bar{c}\right)^2$$

$$MSD_{LTB} = \sigma^2 + \bar{c}^2\left(1 - \alpha\right)^2 \tag{3.25}$$

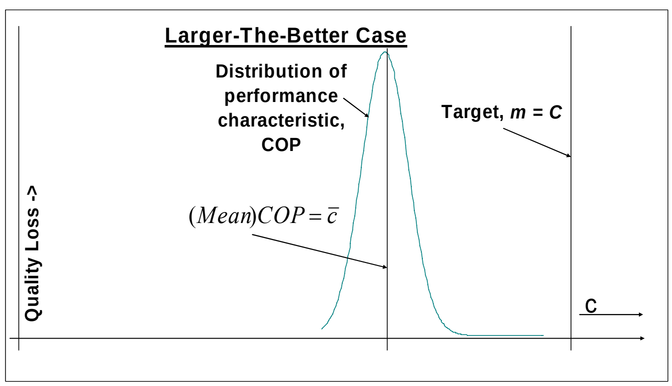


Figure 3.7. Complementary Property–COP-Like Characteristics—LTB Case

Figure 3.8 shows “lack of COP” as an STB characteristic. When considering a lack of COP, it should be as low as possible. Therefore, it is essentially an STB characteristic with the target as zero (i.e., $m = 0$), which changes Equation (3.3) to (3.3a),

$$MSD = \sigma^2 + \left(\bar{y}' - m\right)^2 \tag{3.3a}$$

Where $\bar{y}'$ is the mean of a different characteristic inefficiency, a complementary characteristic of efficiency. Using Equation (3.3a), Equation (3.23), and $m = 0$,

$$MSD_{STB} = \sigma^2 + \left(\alpha\bar{c} - \bar{c} - 0\right)^2$$

$$MSD_{STB} = \sigma^2 + \bar{c}^2\left(\alpha - 1\right)^2$$

$$MSD_{LTB} = \sigma^2 + \bar{c}^2\left(1 - \alpha\right)^2 \tag{3.26}$$

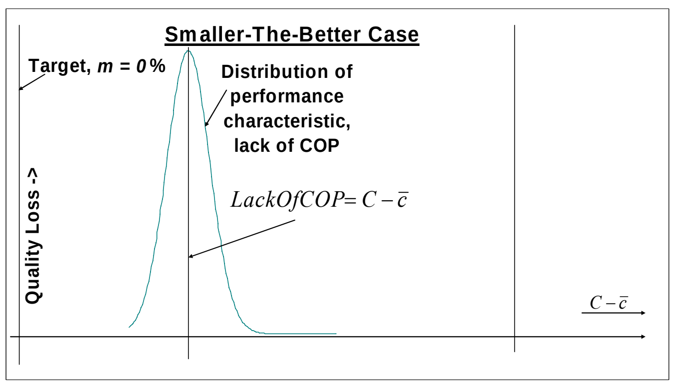


Figure 3.8. Complementary Property–Cop-Like Characteristics—STB Case

Combining equations (3.25) and (3.26) yields

$$MSD_{LTB} = MSD_{STB} = \sigma^2 + \bar{c}^2\left(1 - \alpha\right)^2 \qquad (3.25\ \&\ 3.26)$$

Equations (3.25) and (3.26) show that in both cases the *MSD* is essentially the same. With the new methodology, if the performance distributions are the same whether the COP, c_i, is considered or the lack of COP, $C - c_i$, is considered, then the result for the *MSD* will be the same. Thus, it is obvious that generality and consistency are maintained. In other words, one and the same COP type concept can be represented by two complementary characteristics—one STB and the other LTB—and consistent results can be obtained.

3.4. DISCUSSION ON THE VALUE OF ALPHA, THE TARGET-MEAN RATIO

It has been shown that with the new methodology, the quality loss for LTB is equal to the loss for STB. This property can be called *A*, that is, the loss equivalence to STB. It is also shown that with the new methodology, the quality loss for LTB is equal to that for the corresponding STB complementary characteristic. This property can be called *B*, that is, the loss equivalence to the complementary characteristic. Table 3.1 shows which of these properties are satisfied for the target under consideration for all three types of LTB characteristics. It may be seen that, in general, neither property A nor B is satisfied if the target is considered to be infinity.

Table 3.1. Properties

LTB characteristic	Target at	Which property is satisfied?	Target at	Which property is satisfied?	Target at	Which property is satisfied?
No-obvious-target-like	$\alpha = 1.5$	B, not A	$\alpha = 2$	A, B	∞	not A, not B
Efficiency-like	100%	B, not A	$\alpha = 2$	A, B	∞	not A, not B
COP-like	250% say	B, not A	$\alpha = 2$	A, B	∞	not A, not B

'A' – Loss-equivalence to STB

'B' – Loss-equivalence to complementary characteristic

3.4.1. No-Obvious-Target-Like Characteristics.

For no-obvious-target-like characteristics where the target value is not obvious, the target can be set at α equal to a certain value. In this case, it is possible to assume a certain higher target value theoretically. It is worthwhile to mention that the target should be difficult to achieve but not impossible. Whether the target can really be achieved might depend upon technology change, innovation, change of material, invention of new raw material, use of unconventional material (for example, the use of nickel or titanium in place of mild steel), process, or conditions, etc. The target-mean ratio, α, needs to be significantly greater than 1 but need not be a very large number or infinity. It has been shown before that at $\alpha = 2$,

$$MSD_{LTB} = MSD_{STB} = \sigma^2 + \bar{t}^2 \qquad (3.15 \text{ \& } 3.17)$$

In the no-obvious-target-like LTB characteristics, it is possible to assume a high but finite target value. Therefore, in such cases, it is recommended to set the target at double the present performance, or in other words, α is set equal to 2. This is a particular situation in LTB when $\alpha = 2$ gives the quality loss equal to the STB case. The target is not zero but is placed equidistant from the mean on the other side. This differentiates LTB from STB. Also, the target is not assumed to be infinity though the case is LTB. The advantage of using this approach is that comparable results can be achieved between the STB and LTB cases.

In addition, the last section demonstrates that whether an LTB or its complementary characteristic is considered to have quality loss for one and the same given case, then the *MSD* or quality loss should be the same as well. This requirement is fulfilled if the value of α is equal to 2. Because at $\alpha = 2$ the "complementary property" is also satisfied, it is recommended that the value of α be 2.

3.4.2. Efficiency-Like Characteristics. The target value of these parameters is naturally available such as the efficiency of a system. Earlier it was suggested that whether an LTB or its complementary characteristic is considered for one and the same given case, the *MSD* or quality loss should be the same, too. This requirement is fulfilled if the value of a target is 100% as well.

For efficiency-like characteristics where the target value is naturally available, if the target is set at the ideal 100%, then the property B loss equivalence to a complementary characteristic is satisfied, but property A loss equivalence to STB is not necessarily satisfied. However, if the target is set according to $\alpha = 2$, then property B loss equivalence to the complementary characteristic as well as property A loss equivalence to STB is satisfied. Therefore, it is recommended that the target value be fixed corresponding to $\alpha = 2$.

3.4.3. Coefficient-of-Performance-Like Characteristics. These parameters are ones for which the target value is naturally available such as the COP of a system. For COP-like characteristics where the target value is naturally available but is unequal to 100%, the target can be set at the ideal, highest possible theoretical value for COP-like characteristics. For example, if the present performance of a machine is 33.33% and the ideal limit is 100%, then the target is set at 100%. In this case the value of α equals 3. If the target is set at the ideal maximum, then the property B loss equivalence to the complementary characteristic is satisfied but property A loss equivalence to STB is not necessarily satisfied. However, if the target is set according to $\alpha = 2$, then the property B loss equivalence to the complementary characteristic as well as property A loss equivalence to STB is satisfied.

The target can be set either at double the mean performance or at the ideal, highest possible theoretical value. To bring about similarity among the methods used to calculate the *MSD* for efficiency-like, COP-like, and no-obvious-target-like characteristics, it is recommended to set the target according to $\alpha = 2$.

3.5. IMPLICATION OF A FINITE TARGET ON COMPLEXITY AND THE SIGNAL-TO-NOISE RATIO BASED ON COMPLEXITY

Complexity has been treated in terms of an absolute measure in many of the works. However, in axiomatic design, complexity and information are defined relative to what is being tried to achieve or wanted to know. Information is defined as a logarithmic function of the probability of achieving the specified functional requirements (FRs). The common area under the system probability density function (pdf) and design range, in turn, gives the probability of achieving a specified FR (El-Haik, 2005). Therefore, complexity is related to information. The design process is described in terms of the mapping between domains in axiomatic design (Suh, 1999). FRs in the functional domain describe the design goals for a product, e.g., systems, software, etc. In the design stage, FRs in the functional domain need to be mapped to Design Parameters (DPs) in the physical domain. Therefore, the probability (and uncertainty) of satisfying the FRs depends on the selection of DPs. When an FR is defined, its target value FR_o and tolerance are specified in the design range. The system range determines the actual pdf of the resulting design and may be different from the design range. The common range is the portion of the design range overlapped by the system range. If the system pdf for a given FR is denoted p_s(FR), then the probability P , of satisfying the FR can be given as follows (Suh, 2001):

$$P\left(dr^l \le FR \le dr^u\right) = \int_{dr^l}^{dr^u} p_s\left(FR\right) d\left(FR\right) \tag{3.27}$$

Where dr^l and dr^u are the lower and upper limits of the design range, respectively. The probability P of satisfying a given FR gives the information content I as follows:

$$I = -\log_2 P = -\log_2 \int_{dr^l}^{dr^u} p_s\left(FR\right) d\left(FR\right) \tag{3.28}$$

For the entire system, the information content for satisfying a number of FRs is simply the sum of the information content, I_i, of the separate FR_i

$$I = \sum I_i = \sum -\log_2 P = \sum -\log_2 \int_{dr^l}^{dr^u} p_s\left(FR\right) d\left(FR\right) \tag{3.29}$$

3.5.1. Complexity for LTB with a Finite Target. Complexity is unrelated to the number of FRs because the number of FRs is constant for a system. A less complex system has a low total of *I,* i.e., it has a high probability of achieving all FRs. In comparison, another system is more complex if it has a high total of *I,* i.e., a low probability of satisfying all *FRs*. *Complexity is defined as a measure of uncertainty in achieving the specified FRs* (Suh, 2001). Therefore, complexity is related to information content, which is defined as a logarithmic function of the probability of achieving.

The information axiom of axiomatic design is to "minimize the information content in a design" (Lee, 2003). The design or manufacturing of a given component needing minimum information content gives the maximum probability of achieving success. The probability of success belonging to hierarchical level, *L*, and information content are connected through entropy, Equation (3.30).

$$H_L = -\log_v\left(\Pi_{i=1}^{m} \Pr_i\right) = -\Sigma_{i=1}^{m} \log_v\left(\Pr_i\right) \tag{3.30}$$

Shannon entropy for *FRs* can be written as

$$H_b\left(\Pr_1, \Pr_2, \Pr_3, \ldots, \Pr_m\right) = -\log_v\left(\Pi_{i=1}^{m} \Pr_i\right) = -\Sigma_{i=1}^{m} \Pr_i \log_v\left(\Pr_i\right) \tag{3.31}$$

Where $v > 1$. *H* has units of bits (nats), (a nat = 1.44 bits).

$$h_v\left(f\right) = -\int_S f\left(x\right) \log_v f\left(x\right) dx \tag{3.32}$$

Equation (3.32) is valid when the integral exists. Suppose a given FR, system range, is a normal source of information, FR ~ Normal (μ_{FR}, σ^2_{FR}), (Haik, 2005), with the following example:

$$f\left(FR\right)=\frac{1}{\sqrt{2\pi\sigma_{FR}^{2}}}Exp\left\{\frac{1}{2}\left(\frac{FR-\mu_{FR}}{\sigma_{FR}}\right)^{2}\right\} \tag{3.33}$$

The probability of success means the probability of satisfying a given functional requirement (FR) in the process domain. The process domain can be viewed as the system range. Therefore, the probability of success would be the area under the curve of the probability density function for the system range bounded by the design range. The complexity due to variability for the interval [μ_{FR} - ΔFR, μ_{FR} + ΔFR] is given as follows (Haik, 2005):

$$h\left(f\right)=-\int_{\mu-\Delta FR}^{\mu+\Delta FR}f\left(FR\right)\ln f\left(FR\right)dFR$$

$$h\left(f\right)=-\int_{\mu-\Delta FR}^{\mu+\Delta FR}\frac{1}{\sqrt{2\pi\sigma_{FR}^{2}}}Exp\left\{\frac{1}{2}\left(\frac{FR-\mu_{FR}}{\sigma_{FR}}\right)^{2}\right\}\ln\left(\frac{1}{\sqrt{2\pi\sigma_{FR}^{2}}}Exp\left\{\frac{1}{2}\left(\frac{FR-\mu_{FR}}{\sigma_{FR}}\right)^{2}\right\}\right)dFR$$

$$h\left(f\right)=-\int_{\mu-\Delta FR}^{\mu+\Delta FR}\frac{1}{\sqrt{2\pi\sigma_{FR}^{2}}}Exp\left\{\frac{1}{2}\left(\frac{FR-\mu_{FR}}{\sigma_{FR}}\right)^{2}\right\}\left(-\frac{1}{2}\left(\frac{FR-\mu_{FR}}{\sigma_{FR}}\right)^{2}-\ln\sqrt{2\pi\sigma_{FR}^{2}}\right)dFR$$

The solutions for STB, NTB, and LTB given in Haik (2005) are as follows:

$$h\left(f\right)=\ln\sqrt{2\pi e\sigma_{FR}^{2}} \tag{3.34}$$

For STB,

$$L\left(FR,T\right)=k.FR^{2}$$

$$E\left[L\left(FR,T\right)\right]=k\left(\mu_{FR}^{2}+\sigma_{FR}^{2}\right)$$

$$h\left(f\right)=\ln\sqrt{2\pi e\left(\frac{E\left[L\left(FR,T\right)\right]}{k}-\mu_{FR}^{2}\right)} \tag{3.35}$$

Where,

$$E\left[L\left(FR,T\right)\right] > k.\mu_{FR}^2$$

For NTB, if the target is denoted as τ_{FR}, then

$$L\left(FR,T\right) = k\left(FR - \tau_{FR}\right)^2$$

$$E\left[L\left(FR,T\right)\right] = k\left[\left(\mu_{FR} - \tau_{FR}\right)^2 + \sigma_{FR}^2\right]$$

$$h\left(f\right) = \ln\sqrt{2\pi e\left(\frac{E\left[L\left(FR,T\right)\right]}{k} - \left(\mu_{FR} - \tau_{FR}\right)^2\right)} \tag{3.36}$$

Where,

$$E\left[L\left(FR,T\right)\right] > k\left(\mu_{FR} - \tau_{FR}\right)^2$$

For LTB,

$$L\left(FR,T_{FR}\right) = \frac{k}{FR^2}$$

$$FR \geq FR_l$$

If μ_{FR} is the mean FR of the system range, then by Taylor series expansion around FR = μ_{FR}, one has

$$L\left(FR,T_{FR}\right) = k\left(\frac{1}{\mu_{FR}^2} - \frac{2(FR-\mu_{FR})}{\mu_{FR}^3} + \frac{3(FR-\mu_{FR})^2}{\mu_{FR}^4} - \cdots\right)\Bigg|_{FR=\mu_{FR}}$$

For higher-order negligible terms

$$L\left(FR,T_{FR}\right) = k\left(\frac{1}{\mu_{FR}^2} + \frac{3}{\mu_{FR}^4}\sigma_{FR}^2\right)\Bigg|_{FR=\mu_{FR}}$$

For the normally distributed FR, one can have

$$h(f) = \ln \sqrt{2\pi e \left(\frac{2\pi\mu_{FR}^4 E\left[L(FR,T)\right]}{3k} - \frac{2\pi\mu_{FR}^2}{3} \right)} \tag{3.37}$$

Where,

$$E\left[L(FR,T)\right] > \frac{k}{\mu_{FR}^2}$$

It is obvious that the formula for complexity due to variability for LTB is different from that for the STB and NTB cases. Because the terms inside the round brackets are simply a replacement for σ_{FR}^2, the complexity due to variability for all three types of characteristics will be equal for the same standard deviation of performance. However, it is proposed that a finite target be considered for LTB. When a finite target in terms of the target-mean ratio is considered for LTB, i.e., $\alpha = \tau_{FR} / FR$, then the proposed solution is as follows:

$$L(FR,T) = k\left(FR - \tau_{FR}\right)^2 = k\left(FR - \alpha FR\right)^2 = k.FR^2\left(1-\alpha\right)^2$$

$$E\left[L(FR,T)\right] = k\left[\left(\mu_{FR} - \tau_{FR}\right)^2 + \sigma_{FR}^2\right]$$

$$E\left[L(FR,T)\right] = k\left[\mu_{FR}^2\left(1-\alpha\right)^2 + \sigma_{FR}^2\right]$$

$$h(f) = \ln \sqrt{2\pi e \left(\frac{E\left[L(FR,T)\right]}{k} - \mu_{FR}^2\left(1-\alpha\right)^2 \right)} \tag{3.38}$$

Where,

$$E\left[L(FR,T)\right] > k\mu_{FR}^2\left(1-\alpha\right)^2$$

For $\alpha = 2$

$$h(f) = \ln \sqrt{2\pi e \left(\frac{E\left[L(FR,T)\right]}{k} - \mu_{FR}^2 \right)} \tag{3.39}$$

Equations (3.39) and (3.35) are similar to each other. Therefore, the complexity for the LTB case is equivalent to that for the STB case at $\alpha = 2$. In this way, it is seen that comparable results for complexity can also be found using the new methodology—the LTB case with a target.

3.5.2. Effect of a Finite Target on the Mathematical Relationship between the Signal-To-Noise Ratio and Complexity, an Axiomatic Design Measure. For STB, the SN ratio is given as follows (Haik, 2005):

$$SN = -10\log_{10}\left(\frac{1}{N}\sum_{i=1}^{N} FR_i^2\right) \tag{3.40}$$

Therefore,

$$10^{-\frac{SN}{10}} = \frac{1}{N}\sum_{i=1}^{N} FR_i^2 \tag{3.41}$$

According to Haik (2005), the following equation relates the SN ratio for STB and complexity:

$$h(f) = \ln\sqrt{2\pi e\left(10^{-\frac{SN}{10}} - \mu_{FR}^2\right)} \tag{3.42}$$

Or,

$$SN = -10\log_{10}\left(\frac{\left[e^{h(f)}\right]^2}{2\pi e} + \mu_{FR}^2\right)$$

For LTB, the SN ratio is given as

$$SN = -10\log_{10}\left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{FR_i^2}\right) \tag{3.43}$$

Therefore,

$$10^{-\frac{SN}{10}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{FR_i^2} \tag{3.44}$$

According to Haik (2005), the following equation relates the SN ratio for LTB and complexity:

$$h(f) = \ln\sqrt{2\pi e\left(\frac{10^{-\frac{SN}{10}}\mu_{FR_i}^4 - \mu_{FR_i}^2}{3}\right)} \tag{3.45}$$

Or,

$$SN = -10\log_{10}\left(\frac{3\left[e^{h(f)}\right]^2 + 2\pi e\mu_{FR}^2}{2\pi e\mu_{FR}^4}\right)$$

However, with a finite target, the SN ratio for LTB can be given as

$$SN = -10\log_{10}\left(\frac{1}{N}\sum_{i=1}^{N}\left(FR_i - T_{FR}\right)^2\right) \tag{3.46}$$

Therefore,

$$10^{-\frac{SN}{10}} = \frac{1}{N}\sum_{i=1}^{N}\left(FR_i - T_{FR}\right)^2 \tag{3.47}$$

And

$$L(FR,T) = k\left(FR - \tau_{FR}\right)^2 = k\left(FR - \alpha FR\right)^2 = k.FR^2\left(1-\alpha\right)^2$$

$$E\left[L(FR,T)\right] = k\left[\left(\mu_{FR} - \tau_{FR}\right)^2 + \sigma_{FR}^2\right]$$

It is proposed that the SN ratio and complexity for LTB characteristics be related in a different way as follows:

$$h(f) = \ln\sqrt{2\pi e\left(\frac{E\left[L(FR,T)\right]}{k} - \mu_{FR}^2\left(1-\alpha\right)^2\right)} \tag{3.48}$$

$$h(f) = \ln\sqrt{2\pi e\left(10^{-\frac{SN}{10}} - \mu_{FR}^2(1-\alpha)^2\right)} \tag{3.49}$$

$$SN = -10\log_{10}\left(\frac{\left[e^{h(f)}\right]^2}{2\pi e} + \mu_{FR}^2(1-\alpha)^2\right)$$

For a particular case when $\alpha = 2$,

$$h(f) = \ln\sqrt{2\pi e\left(10^{-\frac{SN}{10}} - \mu_{FR}^2\right)} \tag{3.50}$$

$$SN = -10\log_{10}\left(\frac{\left[e^{h(f)}\right]^2}{2\pi e} + \mu_{FR}^2\right)$$

It may be noted that Equation (3.50) is similar to Equation (3.42). In this way, for LTB at $\alpha = 2$, the relationship between the SN ratio and the complexity measure becomes equal to that for STB. The equivalence is not obvious with Taguchi's method, as shown in Equation (3.45).

3.5.3. Case Study 3: Spring Rate Complexity and Signal-to-Noise Ratio. This is an example taken from Fowlkes (1995) and El-Haik (2005). Two sets of eight numbers of spring rates of springs manufactured by two machines, new and old, are given in Table 3.2. The spring rate is an NTB characteristic with a target value of 0.5 oz./in. Spring rate complexity and signal-to-noise ratio based on complexity for each machine is computed. Tables 3.2 and 3.3 show the computed values of complexity using equations (3.35), (3.36), and (3.37) for the three cases as if the data set pertained to STB, NTB, and LTB characteristics, respectively. Interestingly, complexity measures are the same for all three cases. The reason for this is that the complexity equations take into account the standard deviation only. In other words, complexity depends only on variability.

Table 3.2. Complexity of the Spring Rate—NTB

Machine	Spring rate data								Mean	Variance	NTB				
											Target, m	$Bias^2$	MSD	Dollar loss	Complexity
New	0.37	0.41	0.37	0.43	0.39	0.35	0.4	0.36	0.385	0.0007	0.5	0.0132	0.014	3.104	-2.184
Old	0.55	0.67	0.7	0.54	0.41	0.32	0.46	0.66	0.5388	0.0184	0.5	0.0015	0.0199	4.419	-0.579

However, the situation does not hold if the SN ratio is computed using complexity. The SN ratio computed with the assumption of the target at infinity is not the same for STB (8.26907 new machine and 5.10554 old machine) and LTB (-8.3556 new machine and -0.5792 old machine). For STB, the SN ratios are positive and for LTB they are negative with a different absolute value. NTB is not considered here as being unnecessary. This inconsistency in the results is inevitable if the target is infinity.

Table 3.3. SN Ratios Based on Complexities of the Spring Rate—STB and LTB

STB					LTB					
								SN ratio based on complexity		
$Bias^2$	MSD	Dollar loss	Complexity	SN ratio based on complexity	Target	Dollar loss	Complexity	Existing method, infinite target	Finite target at alpha=1.5	Finite target alpha=2
0.14823	0.14897	33.104	-2.1836	8.26907	∞	1521.76	-2.1836	-8.3556	14.2252	8.26907
0.29025	0.30864	68.5857	-0.5792	5.10554	∞	911.098	-0.5792	-6.1278	10.4121	5.10554

Therefore, equations (3.49) and (3.50) were used to compute the SN ratio based on complexity. As expected, the SN ratios computed with the assumption of a finite target at the target-mean ratio of 2 are the same for LTB (8.26907 new machine and 5.10554 old machine) as for STB (8.26907 new machine and 5.10554 old machine). Therefore, it can be said that the SN ratio and complexity for LTB characteristics can be related in a different way as in equations (3.49) and (3.50) to obtain consistent and comparable results.

3.6. SIGNAL-TO-NOISE RATIO AND OPERATING WINDOW WITH LTB HAVING A FINITE TARGET

The SN ratio functions as a single measure of robustness, and a gain in the SN ratio reflects the improvement (Taguchi, 2004). For it to be more useful and significant, this ratio needs to be related to lower cost, reduced time to market, and better quality. The SN ratio can also be related to the warranty cost (Taguchi, 2004). In general, the SN ratio for quality characteristics can be given as

$$S/N = -10\log(MSD) \tag{3.51}$$

For STB, it is as follows:

$$S/N_{STB} = -10\log(MSD_{STB}) \tag{3.52}$$

So,

$$S/N_{STB} = -10\log\left(\frac{\sum y^2}{n}\right) = -10\log\left(\sigma^2 + \bar{y}^2\right) \tag{3.53}$$

When $MSD = \left(\sum y^2\right)/n = \sigma^2 + \bar{y}^2$ decreases S/N_{STB} increases because of the negative sign. If a finite target *m* is considered for LTB, then the SN ratio will be as follows:

$$S/N_{LTB} = -10\log(MSD_{LTB}) \tag{3.54}$$

Here,

$$MSD_{LTB} = \frac{\sum (y-m)^2}{n} = \sigma^2 + (\overline{y}-m)^2 \tag{3.55}$$

So,

$$S/N_{LTB} = -10\log\left(\frac{\sum (y-m)^2}{n}\right) = -10\log\left(\sigma^2 + (\overline{y}-m)^2\right) \tag{3.56}$$

With LTB also, when $MSD_{LTB} = \sigma^2 + (\overline{y}-m)^2$ decreases S/N_{LTB} increases because of the negative sign. The objective of robust design is to increases $\overline{y}$ and still look for ways to decrease σ^2. The SN ratio for an operating window is a combination of STB and LTB SN ratios. Many engineering characteristics tend to be binary-like. The performance is either good or bad, and the performance can change quickly between these two extremes. In such cases, the STB and LTB cases need to be combined to form a useful SN ratio by way of establishing a concept called an *operating window*. An operating window is defined as the spread or stretch between two performance bounds (Fowlkes, 1995). The LTB analysis is employed at the upper bound, and the STB analysis is employed to the lower bound of the performance. It assumes that a critical parameter exists with bound measurements. The discontinuous behavior may be optimized using a continuous engineering parameter called the *operating window signal factor*. Quality characteristic values at these bounds form an operating window. The probability of failure can be minimized by widening the window of a critical factor important for failure. More specifically, this is achieved by increasing the SN ratio for the window.

The signal-to-noise ratio for an operating window case is given as a difference between the signal-to-noise ratio for STB and the signal-to-noise ratio for LTB. The following explains the mathematical derivation for an operating window where the performance thresholds are such that a higher performance value can be assumed as any multiple of a lower performance value. It is assumed that in an STB case x should be minimized, while in an LTB case x' should be maximized.

According to Fowlkes (1995), the signal-to noise ratio of an operating window is given as

$$S/N_{OW} = S/N_{STB(x)} + S/N_{LTB(x')} \tag{3.57}$$

Using Equation (3.57) and the new methodology (i.e., Equation (3.56)) of a finite target for LTB, the following is derived and proposed:

$$S/N_{OW} = -10\log\left(\frac{\sum x^2}{n}\right) - 10\log\left(\frac{\sum (x'-m)^2}{n}\right) \tag{3.58}$$

$$S/N_{OW} = -10\log\left(\sigma^2 + \bar{x}^2\right) - 10\log\left(\sigma'^2 + (\bar{x}' - m)^2\right) \tag{3.59}$$

$$S/N_{OW} = -10\log\left[\left\{\sigma^2 + \bar{x}^2\right\}\left\{\sigma'^2 + \left(\bar{x}' - m\right)^2\right\}\right] \tag{3.60}$$

Equation (3.60) is a different formulation for an SN ratio for an operating window. Therefore, the objective of robustness can be achieved by widening the operating window. A wide window ensures that the factor does not cause the performance to vary over a wider range of the operating window signal factor.

3.7. CONCLUSION

This section attempted to consider and deliberate on the new methodology proposed to bring about similarity among all three cases by introducing a term called the *target-mean ratio*. A common formula can represent all three cases by assuming different values of the target-mean ratio. Because the new method assumes some finite value of the target in the case of LTB also, this section has addressed the implications of this on some other issues.

A method of classifying performance parameters on the basis of the target value has been presented. Some characteristics were considered non-NTB characteristics because the ideal target limit (e.g., for an efficiency of 100%) was considered to be inadequate or inappropriate for them to be LTB characteristics. These characteristics have been reclassified as LTB by considering the ideal target limit as adequate. Merely because a characteristic needs to be greater should decide that the characteristic is an LTB characteristic. It is unnecessary to consider what the greater target is. Furthermore, efficiency-like and COP-like characteristics have been discussed. The new methodology assumes a hypothetical target for LTB cases in terms of some multiple of the present performance. An appropriate value of the multiplier has been discussed in all three cases for efficiency-like, COP-like, and other characteristics. The new methodology is suitable for all the LTB characteristics including efficiency-like and COP-like characteristics. It was also shown that it does not matter whether a characteristic or its complementary characteristic is taken into account; the results towards the quality loss will be consistent if the new methodology is employed. In contrast, in Taguchi's approach this type of consistency, called *Complementary Property*, is not visible. Finally, this section studied what should be the most appropriate value of α, the *target-mean ratio*, on the basis of two properties, i.e., quality loss equivalence to STB and loss equivalence to the complementary characteristic.

Next, the implication of a finite target on axiomatic design complexity was studied from a mathematical point of view. It was seen that the complexity for LTB with an infinite target is not in sync with that for STB and LTB characteristics. Therefore, complexity due to variability for LTB with a finite target was derived, and it was found that a comparable complexity measure is obtainable with the unified methodology. Then the effect of a finite target for LTB on the mathematical relationship between the signal-to-noise ratio and axiomatic measures was studied. It was seen that the relationship between the signal-to-noise ratio and complexity, an axiomatic measure, becomes equivalent to the STB case. The mathematical relationship for an operating window with LTB having a finite target was then derived, which is different from the existing formulation.

4. INTRODUCTION TO WARRANTY LOSS FUNCTION ON THE BASIS OF CUSTOMER EXPECTATION AND PRODUCT PERFORMANCE

4.1. INTRODUCTION

Product performance is measured in terms of certain characteristics that need to be within a certain range in order to satisfy a customer. Customer satisfaction is a statistical phenomenon because a given product has a large number of customers and each customer has a different expectation from a product that is measurable in terms of a product performance unit. The variation of expected performance value from consumer to consumer may be infinitesimally small and difficult to measure. An interesting parameter is the difference between actual performance and expected performance of a product. This section discusses mapping the difference between the performance expected by customers and the actual performance of the product. In the conventional method, the quality loss function assumes there is a fixed target, whether it is zero, some finite value, or infinite. It is also assumed that the product performance also varies about its target or designed value. This research presents a methodology to calculate the loss in which the target is fixed as usual but the consumer's expected value is considered as a variable because a large number of values are expected by a large number of customers. Two variables are considered in this research: performance variation and consumer expectation variation.

A large number of any particular type of a product is manufactured by a company. The product is designed while keeping in view the customer requirement. Customer requirement itself varies from customer to customer, and one single number can never give the true value of customer requirement with regard to any single characteristic or performance value of a product. Rather it is a statistical phenomenon. Customer expectation is assumed to follow a certain distribution. An interesting question is what kind of distribution will represent which characteristic of a product. However, the normal distribution can reasonably represent customer satisfaction while owing to its well-known properties.

A variety of products have many attributes for performance, and very few products have a single attribute. In this research, a single attribute product or a multi-attribute product with a single most important characteristic is considered.

4.1.1. Taguchi's Quadratic Loss Function for Product Performance Characteristics. Taguchi proposed a quadratic loss function to estimate quality loss, as shown in Figure 2.8 and Equation (2.15). In the proposed methodology a similar concept is used. However, Δ_0 is defined as the point of intolerance that is equal to product performance minus customer expectation. It is assumed that the corresponding monetary loss caused due to a defective component is A_0. A_0 is also defined as the cost of a corrective action. When the deviation of performance from the target of a product is Δ_0 and the corresponding loss is A_0, then for STB, NTB, and LTB, $k = A_0 / \Delta_0^2$. Taguchi (2004), Maghsoodloo (1991), Fowlkes (1995), and Phadke (1989) have been referred to for this discussion. Taguchi's quality loss function for the NTB and STB cases takes the form as in Equation (2.15).

$$L(y) = \frac{A_0}{\Delta_0^{\,2}}(y-m)^2 \tag{2.15}$$

For LTB the quality loss is given as in Equation (2.16), (Taguchi et al., 2004; Fowlkes, 1995; Taguchi, 1999).

$$L(y) = A_0 \Delta_0^{\,2} \left(\frac{1}{y_i} \right)^2 \tag{2.16}$$

Although the quadratic loss has been used for the LTB case also but instead a reciprocal form, Equation (2.16), has been used. However, formulation of warranty probability and warranty loss function does not use reciprocal transformation.

4.1.2. Customer Expectation and Warranty. A warranty is a buyer's confidence owing to the seller's assurance that a product will perform as stated or implied. Warranty costs add to the cost of the product, and they often require fire fighting, attention, and man power. Still companies lose reputation, goodwill, and market share. It can be viewed as a lose-lose situation for everyone. Warranty cost is, in a way, a synonym for waste—a waste of money, man-power, time, and energy. The principle is the same as for rework. All rework should be avoided and so should warranty cost. It does not seem logical to do a job again because it was not done well the first time.

An occurrence of warranty cost is a loss to society as a whole. In the case of product rework or warranty, the focus shifts to manufacturing and assembly. However, oftentimes manufacturing is not at fault and instead the problem lies with the design and development. When the product performance matches or exceeds customer expectation, warranty cost should not occur. On the other hand, when product performance falls short of customer expectation, warranty cost may occur. Warranty cost is generated as a result of a clash between customer expectation (CE) and product performance (PP), as shown in Figure 4.1 (Ragsdell, 2002).

4.2. LITERATURE REVIEW

Many of the currently used warranty cost models consider aggregate costs such as average material costs, labor costs, personnel costs, and inventory/logistics management costs. Supply chain managers or material managers find these models useful. However, for product engineers such models are not necessarily useful for making a design choice. Some relevant literature is reviewed as follows.

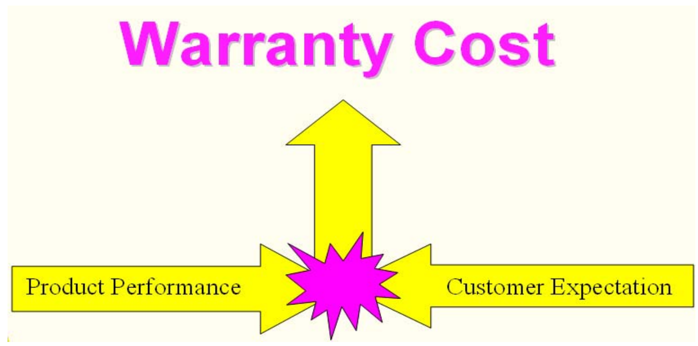


Figure 4.1. Warranty Cost Due to a Clash between the PP and CE

Venkateswaren (2003) integrates Mahalanobis distance and Ragsdell's warranty cost model. Mahalanobis distance represents multiple interrelated quality characteristics. The optimization of a number of variables is considered for predicting the warranty claim to improve the estimation of warranty cost. However, this thesis does not address the relationship between the actual warranty cost incurred at the retailer-customer interface and the manufacturing performance.

Murthy (2000) developed a strategic approach to warranty management where warranty-related decisions are made in a framework that encompasses the product life cycle and a business perspective that links the technical and commercial issues. A warranty strategy depends on the type of product, the customer, and the overall business strategy. It also depends on a number of external factors, particularly competitors' strategies. The product performance can be dependent on the engineering design, manufacturing process design, part quality, quality control, product use, and product maintenance among other factors. Warranty cost must link commercial issues at the dealers with technical issues at the manufacturers. This is because technical issues (e.g., design-related issues) affect the warranty cost, which in turn affects the commercial issues (e.g., pricing), both in turn affecting the product life cycle cost. This research considers customer expectation as a variable, which is a strategic factor in warranty cost. It also depends on a number of external factors, especially competitors' strategies (Blischke, 1994).

Cooper (1988) used a two-period model to examine inter-temporal properties of product warranties. Their research explored why the typical warranty life was considerably shorter than the expected life of a product, as well as why warranty coverage depreciates in time. They argued that a double moral hazard problem stems from this disconnect. Buyers cannot readily observe the product quality, and sellers cannot observe buyers' use of the product. The warranty must balance incentives for the seller's quality and buyer's care in product usage. It is evident from the research that both the seller's quality and buyer's usage (or expectation) affect the warranty costs.

According to Cudney (2008), consumers judge quality and performance at the system level, but important cost-effective decisions at the sub-system or component level must be made by the producer in order to economically satisfy a consumer's needs by providing affordable and high-quality products. The Mahalanobis-Taguchi System is a diagnosis and forecasting method for multivariate data. The Mahalanobis-Taguchi System is of interest because of its reported accuracy in forecasting small, correlated data sets. This paper presents the application of the Mahalanobis-Taguchi System in multidimensional systems to forecast warranty cost. The proposed methodology uses a pattern recognition scheme known as the Mahalanobis-Taguchi System to translate the performance of lower-level elements into an estimate of warranty cost at the system level. An automotive case study involving vehicle handling is provided to illustrate the proposed methodology. The purpose of this research is to develop a relationship between vehicle attributes and warranty cost for the purpose of understanding and improving customer-driven quality. The Mahalanobis-Taguchi System (MTS) enables a reduction in dimensionality and the ability to develop a scale based on the Mahalanobis distance (MD) values (Taguchi, 2002). However, a customer's expectation has not been considered explicitly in the methodology proposed.

Joseph (2004) addressed the QLF for non-negative variables. That is, Joseph derived a new set of quality loss functions for non-negative variables using Taguchi's definition of quality as a basis. The proposed quality loss functions assume that the loss is additive and employs STB, NTB, and LTB. The new quality loss function is compared with the quadratic quality loss function and is shown to be comparable because the quadratic quality loss function is meant for unrestricted variables. Joseph also proposed a multivariate extension of the QLF. However, customer expectation as a variable has not been addressed, which affects the warranty cost.

Warranty has also been shown to signal quality in oligopolistic markets. According to Gal-Or (1989), in oligopolistic markets it is only in special circumstances that warranties can serve as signals of quality. It is illustrated that warranties can act as a perfect signal for products with intrinsic attributes that are not widely spaced or clustered. In the competitive world today, warranties can be seen as an indicator of quality because the warranty costs go up with the probability of a product breakdown. Other market mechanisms providing information about product durability include reputation, advertising, and product-specific investment. Gal-Or investigated providing warranties or service contracts as a signal of product durability. Inherent durability, which depends on intrinsic characteristics, is distinguished from provided durability, which is a combination of both intrinsic characteristics and warranty terms. A warranty is a signal of quality in a perfectly competitive market because the cost of providing a warranty rises as the probability of product breakdown increases.

In a reliability context, Vintr (1999) reported a dependency among price, warranty cost, and product reliability. A warranty presents an additional cost for the manufacturer. Vintr presented two optimization methods for product reliability. In the first method, the reliability requirements are specified for a product, which leads to minimal costs for the manufacturer with respect to research, development, production, and warranty costs if the length of the warranty is firm. The second method presented specifies reliability requirements that will lead to the maximum possible length of a warranty if the manufacturer's costs are firm. This research is focused on minimizing the warranty costs based on the length of the warranty period rather than reducing warranty costs at the design conception.

In the research of Hussain (1998), the warranty costs are shown to be dependent on product reliability and quality uncertainty. The use of redundancies or quality control techniques is shown to reduce warranty costs. Hussain (1998) developed a model to determine the trade-off between manufacturing costs and warranty costs. From the model, the optimal redundancy and quality control strategies can be determined.

Bai and Pham (2008) proposed a new warranty policy, the repair-limit risk-free warranty, which includes a threshold point on the number of repairs where replacement is deemed to be more cost effective thereafter. The free repair warranty (FRW) and pro-rata warranty (PRW) policies are also discussed by Bai and Pham. Bai and Pham (2008) have presented several system warranty cost models of a renewable risk-free policy for multi-component products. Their research utilizes system structures such as series, parallel, series-parallel, and parallel-series configurations. Warranty models depend heavily on the structure of underlying warranty policies over which companies have control. Therefore, changes in a policy may provide a different cost model. This research addresses a Renewable Free Replacement Warranty (RFRW) for a pre-specified warranty period. The authors recommend future research in nonrenewable warranty policies and the time discounting effect on warranty cost. This is an important paper because warranty cost distributions, expectations, variances, and prediction intervals are then derived to facilitate practical applications. However, this dissertation makes an effort in a different direction to predict and then reduce warranty costs.

Issacson et al. (1991) presented an approach to quantify the risk associated with a warranty price. This paper discusses a useful, cost-risk model/analysis technique based on simulation that is applicable to warranty/guarantee decisions across all applicable program phases. As such, customer expectation that affects warranty expenditure has not been considered.

Frees (1988) surveyed a portion of the literature in the analysis of warranty costs. Both short-term as well as long-term perspective evaluations of warranty costs are complicated and approximations are used. New better than used (NBTU) approximations are compared with some straight line approximations introduced in the research. It is found that straight line approximations are superior in the special cases examined.

Chun (1999) considered two warranty criteria simultaneously such as age and mileage of an automobile to determine the eligibility of a warranty claim. Several decision models that estimate the expected total cost incurred under various types of two-attribute warranty policies are proposed in Chun (1999). A sensitivity analysis is performed to study the effects of several model parameters, such as the discount rate, product usage rate, and warranty terms. Warranty criteria such as age and mileage of an automobile can be considered as variables in the model proposed in this section for computing warranty claims instead of determining the eligibility of the warranty claims.

Handfield (2002) proposed that "the underlying axiom is that 20 percent of suppliers are responsible for 80 percent of the poor performance." This statement suggests that it is very important to involve suppliers in resolving the issues of high warranty costs. In addition, warranty prediction and reduction models should consider supplier and manufacturer performance because this performance affects warranty cost performance in the customers' hands.

An analysis of life cycle cost from the viewpoint of a reliability organization is presented by Kleyner et al. (2004), and optimization ways to validate the procedures are suggested. According to Kleyner et al., it is expected that improvement in reliability will reduce the warranty expenditure, but will increase the cost of product development. Reliability can be considered as one such variable in the proposed model to arrive at real-time warranty expenditure.

Kleyner (2005) presents a warranty forecasting method in which expected product warranty returns are stochastic simulated. The warranty prediction model is based on a piecewise application of two distributions; Weibull and three parameter exponential distributions. Analysis of the past warranty returns suggests that the distribution parameters stay consistent within a product line, which increases accuracy of the simulation-based warranty forecasting, whereas these parameters vary noticeably between product lines.

Yang (2007) suggested that the manufacturers are usually interested in warranty repair counts and warranty costs, and in turn costs depend on the number of warranty repair counts and the warranty policy. Murthy (2006) suggested that a warranty implies additional cost to the company throughout the period from product launch to obsolescence. This cost can be influenced by technical decisions made prior to the launch. A warranty is potentially known to the customer at the time of sale. Thus, a warranty serves two important purposes: protection and promotion. Protection is provided for the customer against defects in the product and for the manufacturer against excessive claims.

To forecast warranty cost, according to Blischke (1995), there are two distinct approaches in use. Sales over time and failure distribution for the product are used in first approach. This approach assumes that products follow a common and known distribution of failure. On the basis of past data one can estimate distribution parameters, future failure and warranty claims. The other approach is the time-series approach in which warranty expenses are plotted against time in use. This method does not require estimating the failure distribution.

Due to the uncertainties associated with failures, quality levels, and economic conditions effective allocation of funds to cover warranty expenditure against poor product performance is a major problem with the manufacturers, Thomas (1989). Model for estimating warranty reserves for non-reparable products based on exponential failure distribution allowing for discounting and price level changes over time has been extended to non-exponential cases, e.g., Gamma and Weibull distributions in Thomas (1989).

Through literature survey on predicting warranty cost it is observed that the main techniques used for forecasting warranty costs are: analysis of past data on warranty expenditure, use of reliability parameters and their distributions in predicting failure rate and associated warranty cost. Simulation of failure with the assumption of certain distribution and thereby estimation of warranty costs is also in use where there is scarcity of data. These methods first require development of warranty forecasting model and then using them for future prediction. Many of the currently used warranty cost models consider aggregate costs such as average material costs, labor costs, personnel costs, and inventory/logistics management costs, etc. Supply chain managers or material managers find these models useful. However, for product engineers such models are not necessarily useful for making a design choice.

It appears that adequate research is unavailable on warranty costs linking customer expectation or commercial issues with product performance or technical issues. This is an important area from the standpoint of the product life cycle because technical issues (e.g., design-related) affect the warranty cost, which in turn affects the commercial issues (e.g., pricing). A customer is unlikely to be aware of the distributions of customer expectation and product performance. On the other hand, a manufacturer should know the distributions of customer expectation and product performance. In this research, both customer expectation and product performance have been assumed to be normally distributed around their individual means with individual variances. Therefore, the models presented are simple and use easily obtainable parameters, e.g., mean and standard deviation. These warranty loss models and a similar one for NTB characteristics provide a basic model for computing warranty costs under the given assumptions.

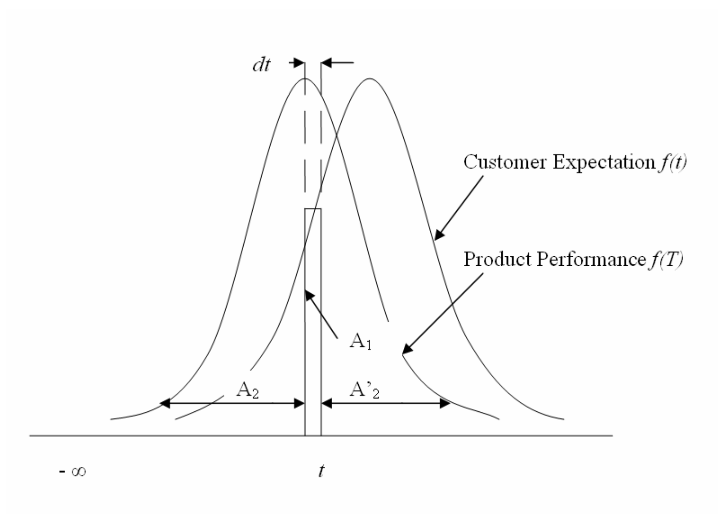
4.3. RELIABILITY THEORY APPROACH

The *actual performance* of a product is defined as the performance of a particular piece of product "as perceived" by a consumer after he or she has purchased it and used it for a short but sufficient period of time. The performance of a product during its use in the hands of a consumer is termed the *actual performance*. Some differences may exist between the perceived and actual performance in the actual practice. When this difference is significant enough to initiate a complaint, the consumer is likely to receive an explanation from the manufacturer to be convinced that no real difference exists. Therefore, for developing this model, it is assumed that no difference exists between the perceived performance and actual performance. Here it is also assumed that the customer is always right.

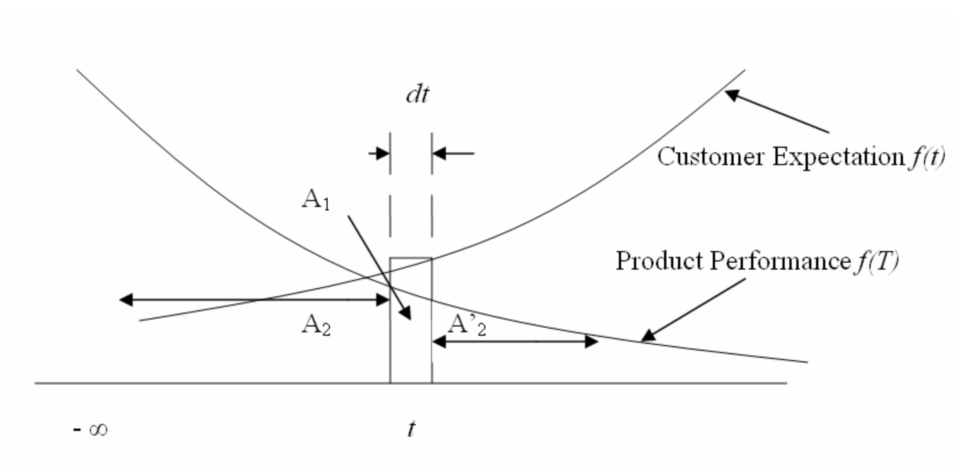
The *target performance* is the performance level for which a product lot is designed and supposed to perform. The actual performance of a product is assumed to be distributed normally about the target performance as the number of products is large enough for this assumption. The manufacturers design a product with a certain value of target performance based on experience or a market survey. This process involves defining a particular target value. However, all the customers need not be satisfied with the product performance at that fixed target, assuming no variation in the performance. However, there are three cases depending on where the Target Performance (i.e., design performance) is located with respect to the Expected Performance. This theory is based on the methodology depicted in Section 4 in Haugen (1968).

4.3.1. Smaller-the-Better Characteristics. In this case the customers are best satisfied by a smaller product performance than the customer's expectation. This case considers the level of performance at and below which a customer will be unhappy.

Therefore, the producer would like to keep the target value smaller than the Customer Expectation. Leakage of air from a tire, vibrations of a piece of equipment, and corrosion are some examples of this case. As shown in Figures 4.2 (a) & 4.2 (b), suppose customer expectation is denoted by a function $f(t)$ and product performance by $f(T)$, where t and T represent the quality characteristic of interest.



(a) Complete view



(b) Enlarged view

Figure 4.2. Probability of a Warranty Complaint (STB)

According to Figures 4.2 (a) and 4.2 (b), the probability of customer expectation value t is equivalent to the area of the element dt or area A_1. In the STB case, customer expectation can also be viewed as customer tolerance.

$$P\left(t-\frac{dt}{2}\leq t\leq t+\frac{dt}{2}\right)=f(t)dt=A_1 \tag{4.1}$$

The probability that $T < t$ is equivalent to the shaded area under the *pdf* curve T of the designed performance is shown by A_2.

$$P(T<t)=\int_{-\infty}^{t} f(T)dT=A_2 \tag{4.2}$$

Conversely, the probability that $T > t$ is equivalent to the shaded area under the *pdf* curve T of the designed performance is shown by A'_2.

$$P(T>t)=\int_{t}^{\infty} f(T)dT=A'_2 \tag{4.3}$$

Similarly, the probability that some customer complaints or some finite warranty costs will occur at t is the product of probability equivalents A_1 and A'_2.

$$P\left(t-\frac{dt}{2}\leq t\leq t+\frac{dt}{2}\right)P(T>t)=dW=f(t)dt\int_{t}^{\infty} f(T)dT \tag{4.4}$$

Suppose ω_1 is a fraction of customers that will make a complaint when the performance is greater than the customer expectation (i.e., when $T > t$). Then, the probability of a customer complaint is given as

$$\omega_1 dW=\omega_1 f(t)\,dt\int_{t}^{\infty} f(T)\,dT \tag{4.5}$$

Also, the probability of occurrence of some warranty cost of a product is the probability of product performance being greater than all possible values of customer expectation. So,

$$W=\int \omega_1 dW=\int_{-\infty}^{\infty}\omega_1 f(t)\left[\int_t^{\infty} f(T)\,dT\right]dt \tag{4.6}$$

If the warranty loss function, which gives the cost of a corrective action, is $f_1(y)$ for cases where $T > t$, then the warranty loss can be given as

$$WL=\int_t^{\infty} f_1(y)\int_{-\infty}^{\infty}\omega_1 f(t)\left[\int_t^{\infty} f(T)dT\right]dtdy \tag{4.7}$$

4.3.2. Nominal-the-Best Characteristics. In this case, a larger product performance that is not too far away from the customer's expectation is what best satisfies the customer. Common examples of this case are clothes (these should not be too loose or too tight), the shaft diameter or housing diameter for the fitment of a bearing, and the quantity of fuel injected by a fuel injection pump. Please see Figure 4.3. The probability of customer expectation value t is equivalent to the area of the element dt or area A_1.

$$P\left(t-\frac{dt}{2}\le t\le t+\frac{dt}{2}\right)=f(t)dt=A_1 \tag{4.8}$$

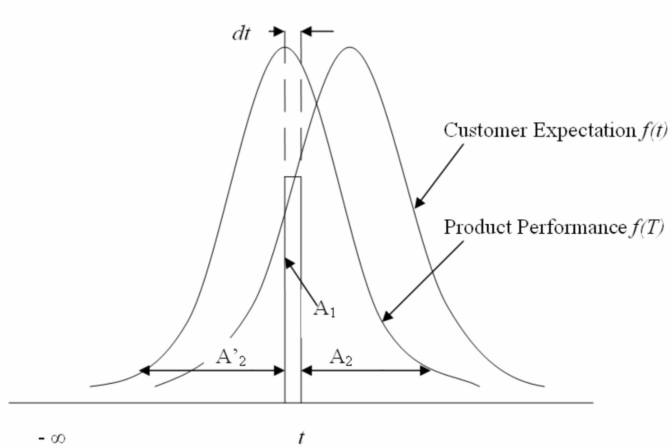


Figure 4.3. Probability of a Warranty Complaint (NTB) and (LTB)

The probability that the customer expectation value, *t,* is less than the product performance value, *T*, (i.e., $T > t$) is equivalent to the area under the *pdf* curve *T* of the designed performance, shown by A_2.

$$P(T > t) = \int_t^{\infty} f(T)dT = A_2 \tag{4.9}$$

Conversely, the probability that $T < t$ is equivalent to the shaded area under the *pdf* curve *T* of the designed performance is shown by A'_2.

$$P(T < t) = \int_{-\infty}^{t} f(T)dT = A'_2 \tag{4.10}$$

The probability that customer dissatisfaction will occur when the product performance is greater than the customer expectation (i.e., when $T > t$) is the product of probability equivalents A_1 and A_2.

$$P\left(t - \frac{dt}{2} \le t \le t + \frac{dt}{2}\right)P(T > t) = dW = f(t)dt\int_t^{\infty} f(T)dT \tag{4.11}$$

Suppose that ω_1 is a fraction of customers that will make a complaint when the performance is greater than the customer expectation (i.e., when $T > t$). Then the probability of a customer complaint is given as

$$\omega_1 dW = \omega_1 f(t)dt\int_t^{\infty} f(T)dT \tag{4.12}$$

The probability that customer dissatisfaction will occur when the product performance is less than the customer expectation (i.e., when $T < t$) is the product of probability equivalents A_1 and A_2.

$$P\left(t - \frac{dt}{2} \le t \le t + \frac{dt}{2}\right)P(T < t) = dW = f(t)dt\int_{-\infty}^{t} f(T)dT \tag{4.13}$$

Suppose that ω_2 is a fraction of customers that make a complaint when the performance is less than the customer expectation (i.e., when $T < t$). Then the probability of customer complaints is given as

$$\omega_2 dW' = \omega_2 f(t)\,dt \int_{-\infty}^{t} f(T)\,dT \tag{4.14}$$

The probability that a customer complaint will occur when a product does not perform on target (i.e., when $T \neq t$ or $T > t$ and $T < t$) is given as

$$\omega_1 dW + \omega_2 dW' = \omega_1 f(t)\,dt \int_{t}^{\infty} f(T)dT + \omega_2 f(t)\,dt \int_{-\infty}^{t} f(T)\,dT \tag{4.15}$$

The probability that a warranty cost occurs owing to a product is the probability that the product performance is not equivalent to customer expectation. So,

$$W = \int_{-\infty}^{\infty} \omega_1 f(t)\left[\int_{t}^{\infty} f(T)dT\right]dt + \int_{-\infty}^{\infty} \omega_2 f(t)\left[\int_{-\infty}^{t} f(T)\,dT\right]dt \tag{4.16}$$

If the warranty loss function, which gives the cost of corrective action, is $f_1(y)$ for cases where $T > t$ and $f_2(y)$ for the cases where $T < t$, then the warranty loss can be given as

$$WL = \int_{t}^{\infty} f_1(y) \int_{-\infty}^{\infty} \omega_1 f(t)\left[\int_{t}^{\infty} f(T)dT\right]dtdy + \int_{-\infty}^{t} f_2(y) \int_{-\infty}^{\infty} \omega_2 f(t)\left[\int_{-\infty}^{t} f(T)\,dT\right]dtdy \tag{4.17}$$

For smaller-the-better cases, the target is zero and the equation

$$WL_{NTB} = \int_{t}^{\infty} f_1(y) \int_{-\infty}^{\infty} \omega_1 f(t)\left[\int_{t}^{\infty} f(T)dT\right]dtdy + \int_{-\infty}^{t} f_2(y) \int_{-\infty}^{\infty} \omega_2 f(t)\left[\int_{-\infty}^{t} f(T)\,dT\right]dtdy \tag{4.18}$$

Can be reduced by assuming that no customer complaints will occur when the product performance is lower than the customer expectation or tolerance (i.e., $T < t$)

$$WL_{STB} = \int_{t}^{\infty} f_1(y) \int_{-\infty}^{\infty} \omega_1 f(t)\left[\int_{t}^{\infty} f(T)dT\right]dtdy \tag{4.19}$$

For larger-the-better cases, the target can be assumed to be at a certain number of times the present performance, as shown in the following equation:

$$WL_{NTB} = \int_{t}^{\infty} f_1(y) \int_{-\infty}^{\infty} \omega_1 f(t) \left[\int_{t}^{\infty} f(T) dT \right] dt dy + \int_{-\infty}^{t} f_2(y) \int_{-\infty}^{\infty} \omega_2 f(t) \left[\int_{-\infty}^{t} f(T) dT \right] dt dy \quad (4.20)$$

Can be reduced by assuming that no customer complaint will occur when the product performance is higher than the customer expectation (i.e., $T > t$).

$$WL_{LTB} = \int_{-\infty}^{t} f_2(y) \int_{-\infty}^{\infty} \omega_2 f(t) \left[\int_{-\infty}^{t} f(T) dT \right] dt dy \quad (4.21)$$

4.3.3. Larger-the-Better Characteristics. In this case, a product performance that is larger than a customer's expectation is what best satisfies a customer. The most common example of this can be seen in the strength of any product such as thread, chains, cloth, pencils, and paper. Manufacturers would like to make products with a slightly higher performance than the consumer expected value. The probability of customer expectation value t is equivalent to the area of the element dt or area A_1. Figure 4.3 may be referred to for this derivation.

$$P\left(t - \frac{dt}{2} \leq t \leq t + \frac{dt}{2} \right) = f(t) dt = A_1 \quad (4.22)$$

The probability that $T>t$ is equivalent to the shaded area under the *pdf* curve T of the designed performance is shown by A_2.

$$P(T > t) = \int_{t}^{\infty} f(T) dT = A_2 \quad (4.23)$$

Conversely, the probability that $T<t$ is equivalent to the shaded area under the *pdf* curve T of the designed performance is shown by A'_2.

$$P(T < t) = \int_{-\infty}^{t} f(T) dT = A'_2 \quad (4.24)$$

Similarly, the probability that some customer complaint or some finite warranty cost will occur at *t* is the product of probability equivalents A1 and A'2.

$$P\left(t-\frac{dt}{2}\le t\le t+\frac{dt}{2}\right)P(T<t)=dW=f(t)dt\int_{-\infty}^{t}f(T)dT \tag{4.25}$$

Suppose ω_2 is a fraction of customers that will make a complaint when the performance is less than the customer expectation (i.e., when $T < t$). Then the probability of a customer complaint is given as

$$\omega_2 dW=\omega_2 f(t)\,dt\int_{-\infty}^{t}f(T)\,dT \tag{4.26}$$

Also, the probability of occurrence of some warranty cost of a product is the probability of product performance being smaller than all possible values of customer expectation. So,

$$W=\omega_2 dW=\int_{-\infty}^{\infty}\omega_2 f(t)\left[\int_{-\infty}^{t}f(T)\,dT\right]dt \tag{4.27}$$

If the warranty loss function, which gives the cost of a corrective action, is $f_2(y)$ for cases where $T < t$, then the warranty loss can be given as

$$WL_{LTB}=\int_{-\infty}^{t}f_2(y)\int_{-\infty}^{\infty}\omega_2 f(t)\left[\int_{-\infty}^{t}f(T)\,dT\right]dtdy$$

$$WL_{LTB}=\int_{-\infty}^{t}f_2(y)\omega_2\left\{\int_{-\infty}^{\infty}f(t)\left[\int_{-\infty}^{t}f(T)\,dT\right]dt\right\}dy$$

$$WL_{LTB}=\omega_2\int_{-\infty}^{t}f_2(y)\left[\int_{-\infty}^{t}f(T)\,dT\right]dy \tag{4.28}$$

Where, $\left\{\int_{-\infty}^{\infty}f(t)\left[\int_{-\infty}^{t}f(T)\,dT\right]dt\right\}$ is a distribution combining Customer Expectation and product performance in a certain way. Existing warranty cost models do not explicitly consider customer expectation as a variable affecting the warranty expenditure. The model developed is based on the premise that the clash between customer expectation and product performance causes warranty claims and that customer expectation also varies from customer to customer. Therefore, it is necessary to link both product performance and customer expectation in some logical way. Two possibilities emerged after careful application of mind. To further the approach, another parameter y^* can be considered to depend on customer expectation and product performance. There can be two approaches in which y^* can depend on customer expectation and product performance. In the first approach, y^* can be the difference between customer expectation and product performance, and in the second approach y^* can be the ratio between product performance and customer expectation. The difference between them can give a wider window for computations than the ratio because the ratio would have a small range from about 0.3 to 3. Moreover, a ratio would be unitless. Therefore, it is recommended to choose the difference between the two over a ratio between them. In addition, the difference between product performance and customer expectation facilitated the formulation and solution of the problem. The following section depicts a methodology using the concept of the difference between product performance and customer expectation for the formulation of the problem and subsequent solution.

4.4. PREDICTING WARRANTY COST ON THE BASIS OF CUSTOMER EXPECTATION AND PRODUCT PERFORMANCE

4.4.1. Quality Loss Function to Warranty Loss Function. The quality loss function assumes a fixed target and only accounts for immediate issues within manufacturing facilities, whereas warranty loss occurs during customer use. Based on the two independent variables, product performance and consumer expectation, this research presents a methodology for predicting the probability of customer complaint. The formulation presented will serve as a basic model for predicting the warranty loss for smaller-the-better, nominal-the-best, and larger-the-better characteristics, which is dependent on both product performance (PP) and customer expectation (CE). As an example, warranty cost is estimated for automotive disc brakes to demonstrate the methodology for the smaller-the-better case. For the nominal-the-best case, warranty cost is estimated for an automotive example to demonstrate the methodology. Another example of solar panels is considered for demonstrating the prediction of warranty loss for the larger-the-better characteristic.

The quality loss function (QLF) proposed by Taguchi serves as a tool to assess loss due to poor quality. In general, poor quality means off-target performance and variation in performance. Quality loss is a concept applicable in product design, process design, and manufacturing. However, it does not directly relate customer expectation with product performance. Rather, the loss function assumes that the point of intolerance is the point where an average customer is unsatisfied with the product performance and will take action. Thus, it can be seen that no explicit distribution of customer expectation has been taken into account by Taguchi. Sometimes quality loss is viewed as additional costs up to the point of shipping a product. The customer and society bear the cost of a loss of quality. Sometimes initially, the manufacturer pays for the quality loss in warranty costs. Later the customer bears it in repair or replacement costs. From this perspective, warranty cost can be viewed as a component of quality loss.

A seller provides assurance to a buyer that a product will perform as stated or implied. This assurance is termed a *warranty*, and it gives confidence to the buyer.

Warranty expenditure should be avoided or at least reduced. An occurrence of warranty cost is a loss to society as a whole. Often a problem can be traced back to the design and development of a product rather than to manufacturing. It may be noted that warranty cost is a result of the conflict between product performance and customer expectation as depicted in Figure 4.1.

Product performance is measured in terms of characteristics that satisfy the customer. Customer satisfaction is a statistical phenomenon because every product has a large number of customers, and each customer has a different expectation of product performance. The quality loss function only accounts for immediate issues within manufacturing facilities, whereas warranty loss occurs during customer use. The quality loss function has been a very useful tool within a manufacturing facility to assess quality and improve performance. Warranty cost occurs at the downstream end (i.e., market-customer interface), and the opportunity to reduce warranty claims lies upstream (i.e., design and manufacturing) of the product life cycle. The situation can also be represented as follows:

$$Warranty\ Cost = f\begin{pmatrix} Poor\ design, Poor\ manufacturing, \\ Unforeseen\ conditions\ of\ customer\ use \end{pmatrix}$$

Warranty claims can also be viewed as an opportunity to talk to customers about products and services and gather data pertaining to customer expectation. The objective of this research is to develop a model that includes customer expectation to assess warranty loss considering customer expectation and product performance as two variables. The function so developed is called the *warranty loss function* (WLF). Warranty loss is computed using a model that will help assess the probability of a customer complaint in the marketplace. Thus, the model captures what happens downstream and translates that information to the upstream.

On the other hand, different customers have different choices. So, the population of customers does not have a fixed expected value for the performance of the product. However, a manufacturer always has to choose one particular fixed value within a certain range for a particular model to design the product. Figure 4.4 shows the warranty probability for the NTB case with a possible situation when the mean of product performance (PP) is smaller than the mean of customer expectation (CE).

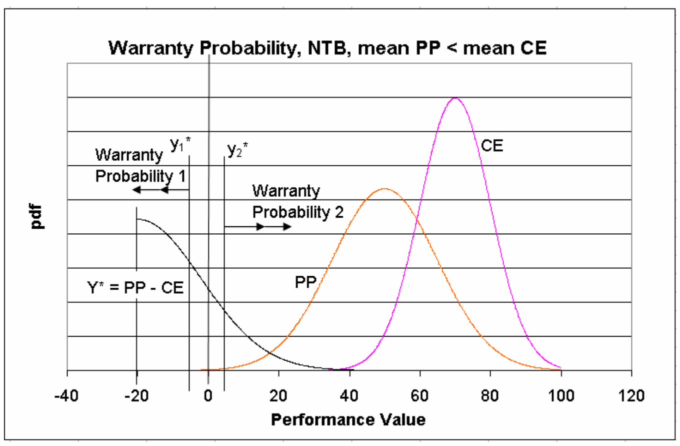


Figure 4.4. Warranty Probability—NTB—mean PP < mean CE

The rationale behind developing the Warranty Loss Function is to relate customer expectation with the product performance in such a way as to be able to predict the probability and calculate cost estimates of warranty claims. Another objective of the formulation of a warranty loss function parallel to a quality loss function is to equip an engineer with a basic tool to make decisions early in the product realization process to reduce the occurrence of a warranty. If the quality loss function is considered, then an engineer is forced to develop and improve product and processes that perform on target without much variation. In the same way if warranty loss is considered, then an engineer will be forced to take into account customer expectation and then design the variation limits and target value of product or process performance according to warranty targets.

Taguchi developed a methodology to assess quality by way of measuring the deviation of performance of a process or product from the target. In this way, quality is a phenomenon that is useful for engineers in the factory only. Today is the time where it is imperative to consider what the customer wants. Therefore, one should focus on the deviation of product or process performance from each customer's expectation. A little consideration will show that the deviation of product or process performance from the customer expectation is responsible for customer complaints and warranty expenses. In this research, a warranty loss function has been derived for smaller-the-better (STB), nominal-the-best (NTB), and larger-the-better (LTB) cases, and this warranty loss function takes into account both customer expectation and product or process performance.

The rationale behind the development of the quality loss function (QLF), proposed first by Taguchi (2004), is to direct engineers to design products and processes that perform on target. Off-target performance and variation in performance are, in general, two components of poor performance. Quality loss is a concept applicable in a manufacturing process and it does not directly relate to customer expectation. As part of a QLF, the point of intolerance is assumed to be the point QLF where an "average customer" is unsatisfied with the product performance and will initiate an action. The point of intolerance is where half of the customers will consider the product to be defective (El-Haik, 2005). This assumes that a constant customer expectation has been taken into account in the QLF.

In contrast, customer satisfaction can be considered to be a statistical phenomenon because every given product has a large number of customers, and each customer has a different expectation from a product. The QLF only accounts for current issues (i.e., variations) within manufacturing facilities, whereas the warranty loss manifests itself during customer use. A manufacturing facility can use the QLF to assess quality and improve performance by forcing an engineer to consistently perform on target. To compete in the market, it is imperative to consider what the customer wants. Therefore, one should be interested in the deviation of the product or process performance from each customer's expectation. The deviation of the product or process performance from the customer expectation is responsible for customer complaints and warranty expenses. The objective of this research is to develop a model that embodies customer expectation to assess warranty loss by considering customer expectation and product performance as two interdependent variables. The proposed models capture what is required in terms of customer expectation versus what happens in the hands of customers measured in terms of product performance.

4.4.2. Differences between Quality Loss Function and Warranty Loss Function. The warranty loss function can be used to estimate the warranty cost in the conceptual design stage as well as in the manufacturing stage. With this model engineers can estimate the warranty cost even if the product has not been developed before. Therefore engineers can draw inferences regarding the probable warranty cost of a final product developed from each product. Table 4.1 gives an overview of the differences between the quality loss function and warranty loss function. In the design stage as well as in the manufacturing stage, the WLF can be used to estimate the warranty cost. With the aid of these models, engineers can estimate the warranty cost before the production of a new product begins. Therefore, engineers can draw inferences regarding a probable warranty cost of a final product developed because of each component. Two main differences exist between QLF and WLF. In QLF, the STB or LTB deviation of performance on one side of a fixed target is considered. On the other hand, in WLF the STB or LTB deviation of performance on one side from customer expectation is considered, as shown in Table 4.1. Also, QLF does not compute the probability of the occurrence of an event (called quality loss), whereas WLF computes the probability of the occurrence of an event (called warranty loss).

Table 4.1. Differences between the Quality Loss Function and Warranty Loss Function

QUALITY LOSS FUNCTION	WARRANTY LOSS FUNCTION
Deviation of performance from a fixed target is considered	Deviation of performance from a variable customer expectation is considered
Customer expectation is not considered explicitly	Customer expectation is considered explicitly
Target is fixed	Customer expectation is variable
Deviation of performance either side of a fixed target is considered for NTB	Deviation of performance from customer expectation by certain threshold values is considered for NTB case
Probability of occurrence of mismatch of product performance from the target is not considered	Probability of occurrence of mismatch of product performance from customer expectation is considered

In actual practice, the customer expectation varies from customer to customer as well as the product performance varies about its target or designed value. A little consideration shows that reliability in terms of the mean time of first failure, mean time between failures, etc., of a product are important characteristics to be used with the proposed model and to predict warranty costs associated with the product. However, other characteristics that are measurable and controllable in the factory can also be successfully used to predict warranty loss for each work station instead of quality loss when the objective of customer satisfaction, and therefore the warranty, is of prime importance in addition to quality.

This section discusses mapping the difference between the performance expected by customers and the actual performance of the product. A methodology is presented in this section to calculate the warranty cost when the fixed target is replaced by a consumer's expected value considered as a variable. This research has considered the STB, NTB, and LTB cases, and warranty cost has been computed for each case as an example to demonstrate the methodology.

The quadratic loss function forces engineers to design and manufacture products and components that perform right on target and do not have excess variation. Still some variation is inevitable. Therefore, it is imperative to consider the distribution of product performance. For estimating the warranty cost of performance away from customer expectation, this research has used the quadratic loss function as used by Taguchi (2004).

A customer is likely to be unaware of the distributions of customer expectation and product performance. On the other hand, a manufacturer is supposed to know the distributions of customer expectation and product performance. If the distribution of customer expectation is unknown by the manufacturer, then it is imperative to approximate or assume one for it. It is easy to decide a point of warranty complaints for customers on the basis of or independent of product performance. A percentage of customer complaints that the manufacturer wants to or can handle may be chosen. This point of warranty complaint can be used to generate parameters of distribution of customer complaint.

A little consideration shows that the model should be simple to work with and should use simple and easily obtainable parameters. In addition, it should be possible to easily extend the model to a multivariate case. It may be noted that simple parameters such as the mean and standard deviation can easily be computed for any set of data not following any particular distribution. Therefore, in this research both customer expectation and product performance have been assumed to be normally distributed around their individual means with individual variances. This warranty loss model works as a basic model for computing warranty cost under the given assumptions. Choosing normal distributions will also help extension of the model to multivariate cases. It is also known that a normal distribution is the most widely followed distribution by many natural phenomena.

4.5. THEORY

4.5.1. Potential Number of Complaints. If product performance is denoted by P and customer expectation is denoted by C, then one may be interested in the differences between them because the gap between them is responsible for customer satisfaction or dissatisfaction. Therefore, a transformed characteristic $Y^* = P - C$ is defined and utilized for formulating the warranty probability and warranty cost for all three cases, i.e., STB, NTB, and LTB. Please refer to the nomenclature for more symbols used in the derivation.

The term *potential number of complaints* denoted by N is the total number of complaints where the difference between product performance and customer expectation is smaller than y_1^* and greater than y_2^* for the NTB case assuming that all the customers make a complaint. The potential number of complaints is the total number of complaints that are likely to be made because the gap between product performance and customer expectation is more than the predetermined values of y_1^* and y_2^*. The potential number of complaints for LTB is the total number of complaints where the product performance is greater than customer expectation by less than a certain threshold value y^*. For STB characteristics, it is where the product performance exceeds the customer expectation by all values greater than a certain limiting value y^* assuming that all the customers make a complaint. Tables 4.2, 4.3, and 4.4 depict scenarios for STB, NTB, and LTB cases, respectively, where a warranty complaint will and will not occur.

Table 4.2. STB Performance vs. Expectation

Product performance	**Customer expectation**	**PP-CE**	**Whether Y^* = PP-CE > y^* = - 0.01**	**Result**
0.85	0.60	0.25	Yes	Warranty complaint
0.71	0.71	0.00	Yes	Warranty complaint
0.15	0.21	- 0.06	Yes	Warranty complaint
0.59	0.46	0.13	No	No warranty complaint

Table 4.3. NTB Performance vs. Expectation

Product performance	**Customer expectation**	**PP-CE**	**Whether Y^* = PP-CE < y_1^* = - 0.1 or > y_2^* = 0.1**	**Result**
10.75	10.60	0.15	Yes	Warranty complaint
10.61	10.63	- 0.02	No	No warranty complaint
10.15	10.21	- 0.06	No	No warranty complaint
10.46	10.29	0.17	Yes	Warranty complaint

Table 4.4. LTB Performance vs. Expectation

Product performance	**Customer expectation**	**PP-CE**	**Whether Y^* = PP-CE < y^* = 0.5**	**Result**
5	6	-1	Yes	Warranty complaint
10	10	0	Yes	Warranty complaint
15	13	2	No	No warranty complaint
19	16	3	No	No warranty complaint

4.5.2. Warranty Probability. It is assumed that a fairly constant ratio exists between the potential number of complaints and the total number of products. This ratio is denoted as P_W. Thus,

$$P_W = \frac{N}{T_P} \tag{4.29}$$

Or,

$$N = P_W T_P \tag{4.30}$$

It is important to note that the term P_W can be called the *warranty probability*. Therefore, warranty probability is defined as the ratio between the potential number of complaints, N, and the total number of products, T_P. From the warranty probability, the potential number of complaints, N, can be computed by multiplying the warranty probability with the amount of production or number of products. A little consideration will show that the warranty probability, P_W, depends on the product performance falling short of the customer expectation. Therefore, the next section deals with the derivation of the formula for computation of the warranty probability as dependent on product performance and customer expectation.

4.5.3. Actual Number of Complaints. The actual number of complaints is the number of complaints actually made by customers where the product performance is smaller than the customer expectation by any value more than y^*. It is assumed that no complaints will occur when the product performance is smaller than the customer expectation by a certain value of y^* or more. The actual number of complaints is defined as the total number of complaints that are actually lodged because the product performance falls short of the customer expectation. It is denoted by the symbol n.

4.5.4. Complaint Factor. This section introduces the *complaint factor*, denoted by ω. The complaint factor is the ratio of the actual number of complaints, n, to the potential number of complaints, N, because the product performance falls short of the customer expectation. Thus, Equation (4.31) depicts the relation and Equation (4.32) can be used to compute the actual number of complaints.

$$\omega = \frac{n}{N} \tag{4.31}$$

$$n = \omega . N \tag{4.32}$$

4.5.5. Warranty Loss Function for Smaller-the-Better Characteristics. It is assumed that no complaints will occur when the product performance is smaller than the customer expectation by a certain value y^*. This section discusses the derivation of the formula for calculating the warranty probability dependent on product performance and customer expectation. Assume that the customer expectation, C, and product performance, P, are distributed normally as

$$C \sim NOR\left(\mu_c, \sigma_c^2\right) \tag{4.33}$$

$$P \sim NOR\left(\mu_p, \sigma_p^2\right) \tag{4.34}$$

The assumption is made that the customer expectation, C, and product performance, P, are independent of each other. Also, assume Y^* is a parameter that is a measure of customer satisfaction such that

$$Y^* = P - C$$

Also, suppose y^* is the maximum value for which the customer is still satisfied. This means that when $Y^* \leq y^*$, the customer is satisfied, and when $Y^* > y^*$, the customer is dissatisfied and can complain for the STB characteristic, as shown in Figures 4.5 and 4.6. From the statistics, $Y^* = P - C$ is distributed as

$$Y^* = P - C \sim NOR\left(\mu_P - \mu_C, \sigma_C^2 + \sigma_C^2\right) \tag{4.35}$$

It is assumed that a potential warranty complaint occurs. Therefore, the probability of a potential warranty (or warranty probability) is given as in Equation (4.36):

$$P_W = P\left(Y^* > y^*\right) = \int_{y^*}^{\infty} f\left(t\right) dy = \int_{y^*}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{\sigma_P^2 + \sigma_C^2}} e^{-\frac{1}{2}\left[\frac{t - \left(\mu_P - \mu_C\right)}{\sqrt{\sigma_P^2 + \sigma_C^2}}\right]^2} dt \tag{4.36}$$

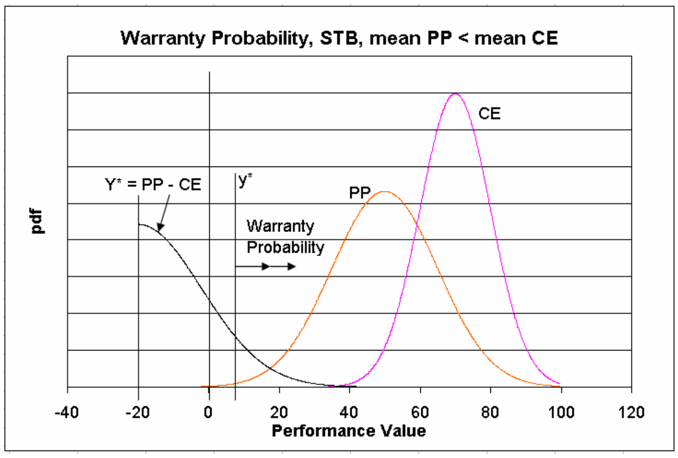


Figure 4.5. Warranty Probability—STB—mean PP < mean CE

Where, t is a dummy variable for Y^*. If Z is defined as

$$Z = \frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}} \tag{4.37}$$

Then

$$P_W = P\left(Y^* > y^*\right) = P\left[Z > z\right] = 1 - P\left[Z \le z\right]$$

$$P_W = 1 - P\left[\frac{Y^* - (\mu_P - \mu_c)}{\sqrt{\sigma_P^2 + \sigma_C^2}} \le \frac{y^* - (\mu_P - \mu_c)}{\sqrt{\sigma_P^2 + \sigma_C^2}}\right] = 1 - \Phi\left(\frac{y^* - (\mu_P - \mu_c)}{\sqrt{\sigma_P^2 + \sigma_C^2}}\right) \tag{4.38}$$

The probability of the customer not complaining would then be given by Equation (4.39):

$$P_{noW} = P\left(Y^* \le y^*\right) = P\left[Z \le z\right] = \Phi\left(\frac{y^* - (\mu_P - \mu_c)}{\sqrt{\sigma_P^2 + \sigma_C^2}}\right) \tag{4.39}$$

Using equations (4.30) and (4.32), the actual number of complaints, n, can be calculated as follows:

$$n = \omega T_P P_W \tag{4.40}$$

It is assumed that the cost of corrective action or repairs, A, is a function of the distance of the performance from y^*. The function is also quadratic, as shown in Equation (4.41):

$$A = k\left(Y^* - y^*\right)^2 \tag{4.41}$$

Where, k is proportionality constant. Therefore, the warranty cost (WC) for all the products under consideration can be estimated as in Equation (4.42):

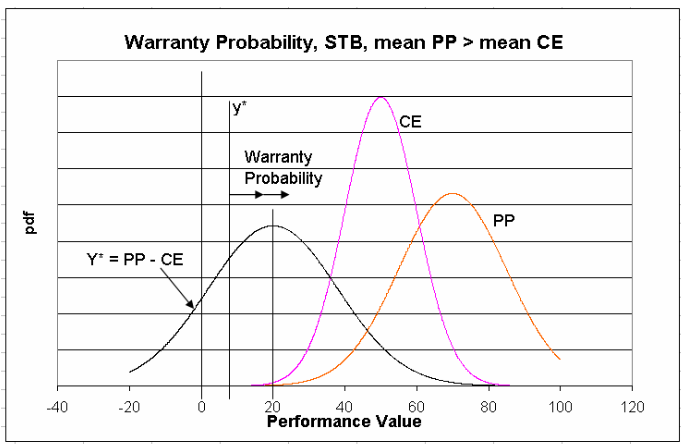


Figure 4.6. Warranty Probability—STB—Mean PP > Mean CE

$$WC = nA = k\left(Y^* - y^*\right)^2 \omega T_P P_W$$

$$WC = \int_{y^*}^{\infty} \frac{k\omega T_P \left(t - y^*\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2 + \sigma_C^2}} e^{-\frac{1}{2}\left[\frac{t-(\mu_P - \mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2} dt$$

$$WC = k\omega T_P \int_{y^*}^{\infty} \frac{\left(t - y^*\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2 + \sigma_C^2}} e^{-\frac{1}{2}\left[\frac{t-(\mu_P - \mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2} dt \tag{4.42}$$

It is more useful and easy to understand if the warranty cost is given per unit of product. Therefore, by setting $T_P = 1$, the unit warranty cost can be computed as shown in Equation (4.43).

$$WC = k\omega \int_{y^*}^{\infty} \frac{\left(t - y^*\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2 + \sigma_C^2}} e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2} dt \tag{4.43}$$

4.5.5.1. Unknown distribution of customer expectation. A general formulation of the problem has been given which can be reduced to suit cases where the distribution of customer expectation is unknown but the point of the warranty claim is known. For a special case where the distribution of customer expectation is unknown but is assumed by the manufacturer as a cutoff point, this methodology can be used. Suppose a manufacturer decides a cutoff point such that above this number some warranty loss must be assigned and below this number no warranty loss should be assigned. Therefore, it can be assumed that the customer expectation is normally distributed with the mean at the cutoff point, μ_C, and zero variance, i.e., $\sigma_C = 0$.

The point of a warranty claim can be viewed as the customer expectation distributed with the mean at the point of a warranty claim and the variance as zero. With this understanding, the same formula can be used and the probability of warranty claims can be assessed. Therefore, the formula simplifies to Equation (4.44):

$$WC = k\omega T_P \int_{y^*}^{\infty} \frac{\left(t - y^*\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2}} e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2}}\right]^2} dt \tag{4.44}$$

4.5.5.2. Case study 4: Run-out of brake rotor—smaller-the-better. This is an example of disc brakes for medium-sized vehicles. The critical to quality characteristic is axial run-out (RO) of rotors. The run-out is measured before and after mounting the wheel on the rotor. When the wheel is mounted on the rotor the RO increases causing a customer complaint. Ideally the run-out should be zero, and therefore it is essentially an STB characteristic. Based on historical data, a maximum of $40\mu M$ RO can be permitted without a warranty claim. It is also observed that less than $20\mu M$ is very expensive for the producer and not really necessary. The customer expects and accepts variation conforming to a three-sigma process. It is estimated that the mean of customer expectation, $\mu_C = 30\mu M$, and standard deviation of customer expectation is approximated as $\sigma_C = 3.33\mu M$ on the basis of the customer's request of a three-sigma process.

The total number of products is $T_P = 636,000$ for which the warranty cost must be estimated. A data set of 225 pieces was simulated and recorded for the analysis. The mean of product performance, i.e., RO, was found to be $\mu_P = 30.7\mu M$ and the standard deviation was $\sigma_P = 18.1\mu M$. On the basis of past experience, the complaint factor was estimated to be $\omega = 0.05$. The cost of corrective action was $A_0 = \$148$ at a distance of $\Delta_0 = 50\mu M$. The point of occurrence for a warranty claim at one end was $y^* = 0\mu M$. This means that if $Y^* > 0\mu M$, then potentially a warranty complaint will occur.

Therefore, $k = A_0 / \Delta_0^2 = \$148 / 50^2 = \$0.0592$, and the estimate of warranty cost WC is:

$$WC = 0.0592 \times 0.05 \times 636000 \int_0^\infty \frac{(t-0)^2}{\sqrt{2\pi}\sqrt{18.1^2 + 3.333^2}} Exp\left[-\frac{1}{2} \left\{ \frac{t-(30.7-30.0)}{\sqrt{18.1^2+3.333^2}} \right\}^2 \right] dt$$

$$= \$338,646$$

The estimate of the unit warranty cost is:

$$WC = 0.0592 \times 0.05 \int_0^\infty \frac{(t-0)^2}{\sqrt{2\pi}\sqrt{18.1^2 + 3.333^2}} Exp\left[-\frac{1}{2} \left\{ \frac{t-(30.7-30.0)}{\sqrt{18.1^2+3.333^2}} \right\}^2 \right] dt = \$0.53$$

Now suppose the customer does not specify any variation in his or her expectation. Then the standard deviation of customer expectation is $\sigma_C = 0\mu M$. In these circumstances, holding all other parameters the same, the estimate of warranty cost reduces to

$$WC = 0.0592 \times 0.05 \times 636000 \int_0^\infty \frac{(t-0)^2}{\sqrt{2\pi}\sqrt{18.1^2}} Exp\left[-\frac{1}{2}\left\{\frac{t-(30.7-30.0)}{\sqrt{18.1^2}}\right\}^2\right] dt = \$327,870$$

Furthermore, suppose the mean of customer expectation, $\mu_C = 0\mu M$, and standard deviation is assumed to be $\sigma_C = 0\mu M$, then the estimated warranty cost increases to

$$WC = 0.0592 \times 0.05 \times 636000 \int_0^\infty \frac{(t-0)^2}{\sqrt{2\pi}\sqrt{18.1^2}} Exp\left[-\frac{1}{2}\left\{\frac{t-(30.7-00.0)}{\sqrt{18.1^2}}\right\}^2\right] dt = \$2,382,640$$

4.5.6. Warranty Loss Function for Nominal-the-Best Characteristics. As mentioned before, a parameter Y^* is defined as the excess of product performance over customer expectation.

$$Y^* = P - C$$

It is assumed that product performance and customer expectation both are normally distributed. Mathematically, C is distributed as

$$C \sim NOR\left(\mu_c, \sigma_c^2\right)$$

And P is distributed as

$$P \sim NOR\left(\mu_p, \sigma_p^2\right)$$

As discussed before, using the moment-generating method given in Bain (1991), it can easily be shown in Equation (4.34) that $Y^* = P - C$ is distributed as

$$P - C = Y^* \sim NOR\left(\mu_P - \mu_C, \sigma_P^2 + \sigma_C^2\right) \tag{4.35}$$

When the performance is targeted to be equal to the average customer expectation, the case is termed *nominal-the-best*. This case considers the level of performance within certain limits for which a customer will be satisfied. Therefore, the producer keeps the target value not too far away from the average expected performance. Two restrictive values y_1^* and y_2^* for Y^* are assumed in the NTB case. These values can be zero or can take any other suitable value. The restrictive y_1^* and y_2^* can also work as a margin of safety. Please refer to Figure 4.4, which shows a situation where the mean of customer expectation is greater than the mean of product performance, and respective variances are also different. Suppose a particular customer's expectation is greater than the product performance of the product he buys by y_1^* or more. Then Equation (4.45) depicts this restrictive relation.

$$Y^* = P - C < y_1^* \tag{4.45}$$

Suppose the product performance is greater than the customer expectation by y_2^*. Then Equation (4.46) depicts this restrictive relation.

$$Y^* = P - C > y_2^* \tag{4.46}$$

The customer will be unhappy and may complain if either of these situations occurs. To encompass both situations, both of the relations are combined to find out the warranty probability as in Equation (4.47), which shows the combined restrictive relation.

$$P_W = P\left(y_1^* > Y^* > y_2^*\right) \tag{4.47}$$

Therefore, the warranty probability can be given by equations (4.48), (4.49), and (4.50).

$$P\left(y_1^* > Y^* > y_2^*\right) = \int_{-\infty}^{y_1^*} f\left(t\right)dy + \int_{y_2^*}^{\infty} f\left(t\right)dy$$
$$= P_{W1} + P_{W2} \tag{4.48}$$

It is useful to write the warranty probability function for NTB as defined in Equation (4.49).

$$W\left(y^*\right) = \int_{-\infty}^{y_1^*} f\left(t\right)dy + \int_{y_2^*}^{\infty} f\left(t\right)dy \tag{4.49}$$

$$= \int_{-\infty}^{y_1^*} \frac{1}{\sqrt{2\pi}\sqrt{\sigma_P^2+\sigma_C^2}} e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2} dt + \int_{y_2^*}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{\sigma_P^2+\sigma_C^2}} e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2} dt$$

$$= \int_{-\infty}^{y_1^*} \frac{1}{\sqrt{2\pi}\sqrt{\sigma_P^2+\sigma_C^2}} e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2} dt + 1 - \int_{-\infty}^{y_2^*} \frac{1}{\sqrt{2\pi}\sqrt{\sigma_P^2+\sigma_C^2}} e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2} dt \tag{4.50}$$

Figure 4.7 shows a situation where the mean of customer expectation and mean of product performance coincide and respective variances are different. Probability of warranty is represented by area under the normal pdf for Y^*, on left hand side up to y_1^* and on the right hand side beyond y_2^*.

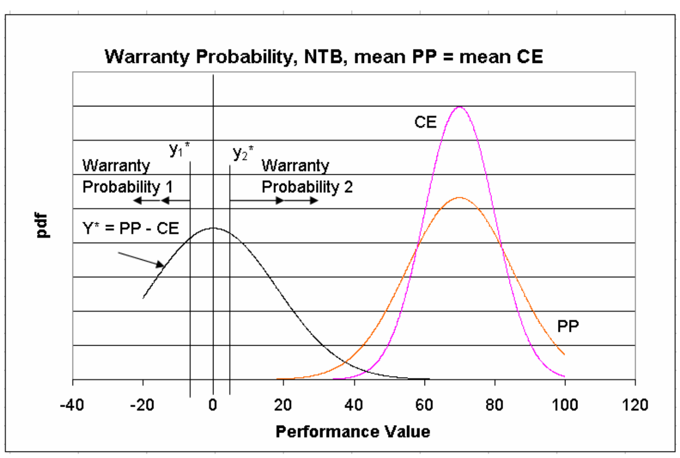


Figure 4.7. Warranty Probability—NTB—Mean PP = Mean CE

Because,

$$Z = \frac{y-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}$$

$$P_W = P\left(y_1^* > Y^* > y_2^*\right) = P\left[z_1 > Z > z_2\right] \tag{4.51}$$

$$\begin{aligned} P_W &= P_{W1} + P_{W2} = P\left[Z < z_1\right] + P\left[Z > z_2\right] \\ &= P\left[Z < z_1\right] + 1 - P\left[Z < z_2\right] \end{aligned} \tag{4.52}$$

or

$$F_Z\left(z\right) = P\left[\frac{Y^* - \left|\mu_P - \mu_c\right|}{\sqrt{\sigma_P^2 + \sigma_C^2}} \le \frac{y_1^* - \left|\mu_P - \mu_c\right|}{\sqrt{\sigma_P^2 + \sigma_C^2}}\right] + 1 - P\left[\frac{Y^* - \left|\mu_P - \mu_c\right|}{\sqrt{\sigma_P^2 + \sigma_C^2}} \le \frac{y_2^* - \left|\mu_P - \mu_c\right|}{\sqrt{\sigma_P^2 + \sigma_C^2}}\right] \tag{4.53}$$

$$P_W = 1 + \Phi\left(\frac{y_1^* - \left(\mu_P - \mu_c\right)}{\sqrt{\sigma_P^2 + \sigma_C^2}}\right) - \Phi\left(\frac{y_2^* - \left(\mu_P - \mu_c\right)}{\sqrt{\sigma_P^2 + \sigma_C^2}}\right) \tag{4.54}$$

An important equation, Equation (4.54), has thus been obtained, which gives a method to compute warranty probability. Now, from equations (4.30) and (4.32), the number of actual complaints can be found in the form of Equation (4.55).

$$n = \omega T_P P_W \tag{4.55}$$

It is assumed that the cost of corrective action or repairs, A_1 and A_2, are functions of the distance of the performance from y^*, and the function is quadratic similar to Taguchi's quadratic loss function. Then equations (4.56) and (4.57) can be used in the formulation of the warranty loss function.

$$A_1 = k_1\left(y_1^* - Y^*\right)^2 \tag{4.56}$$

$$A_2 = k_2\left(Y^* - y_2^*\right)^2 \tag{4.57}$$

Where k_1 and k_2 are proportionality constants as described in the notation section. The cost of repairs when $y_1^* > Y^*$ is given by Equation (4.58).

$$nA_1 = k_1\left(y_1^* - Y^*\right)^2 .\omega_1.T_P.P_{W1} \tag{4.58}$$

The cost of repairs when $Y^* > y_2^*$ is given by Equation (4.59).

$$nA_2 = k_2\left(Y^* - y_2^*\right)^2 .\omega_2.T_P.P_{W2} \tag{4.59}$$

Therefore, the total cost of repairs or warranty cost when $y_1^* > Y^* > y_2^*$ is given as the sum of equations (4.58) and (4.59).

$$WC = WC_1 + WC_2 = nA_1 + nA_2 = k_1\omega_1 T_P P_{W1}\left(y_1^* - Y^*\right)^2 + k_2\omega_2 T_P P_{W2}\left(Y^* - y_2^*\right)^2$$

When the respective integrals from Equation (4.49) are used, equations (4.60) and (4.61) are obtained. Equation (4.61) is the general form for estimating warranty loss and is named the *warranty loss function*.

$$WC = \int_{-\infty}^{y_1^*} \frac{k_1\omega_1 T_P\left(y_1^* - t\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2 + \sigma_C^2}} e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2} dt + \int_{y_2^*}^{\infty} \frac{k_2\omega_2 T_P\left(t - y_2^*\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2 + \sigma_C^2}} e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2} dt \tag{4.60}$$

$$WC = k_1\omega_1 T_P \int_{-\infty}^{y_1^*} \frac{\left(y_1^* - t\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2 + \sigma_C^2}} e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2} dt + k_2\omega_2 T_P \int_{y_2^*}^{\infty} \frac{\left(t - y_2^*\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2 + \sigma_C^2}} e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2} dt$$

$$WC = T_P\left[k_1\omega_1 \int_{-\infty}^{y_1^*} \frac{\left(y_1^* - t\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2 + \sigma_C^2}} e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2} dt + k_2\omega_2 \int_{y_2^*}^{\infty} \frac{\left(t - y_2^*\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2 + \sigma_C^2}} e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2} dt\right] \tag{4.61}$$

Many times it is more useful and easy to understand if the warranty cost is given per unit of product. In this case, $T_P = 1$ can be used to compute the warranty cost per unit of product. If complaint factor $\omega_1 = \omega_2 = \omega$

$$WC = T_P\omega\left[k_1\int_{-\infty}^{y_1^*}\frac{\left(y_1^*-t\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2+\sigma_C^2}}e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2}dt + k_2\int_{y_2^*}^{\infty}\frac{\left(t-y_2^*\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2+\sigma_C^2}}e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2}dt\right] \quad (4.62)$$

Also if the quadratic loss is symmetric on both sides. i.e., $k_1 = k_2 = k$, then

$$WC = T_P\omega k\left[\int_{-\infty}^{y_1^*}\frac{\left(y_1^*-t\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2+\sigma_C^2}}e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2}dt + \int_{y_2^*}^{\infty}\frac{\left(t-y_2^*\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2+\sigma_C^2}}e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2}dt\right] \quad (4.63)$$

And also if $y_1^* = y_2^* = y^*$

$$WC = T_P\omega k\left[\int_{-\infty}^{y_1^*}\frac{\left(y^*-t\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2+\sigma_C^2}}e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2}dt + \int_{y_2^*}^{\infty}\frac{\left(t-y^*\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2+\sigma_C^2}}e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2}dt\right] \quad (4.64)$$

Figure 4.8 shows a situation where the mean of customer expectation is less than the mean of product performance, and respective variances are also different.

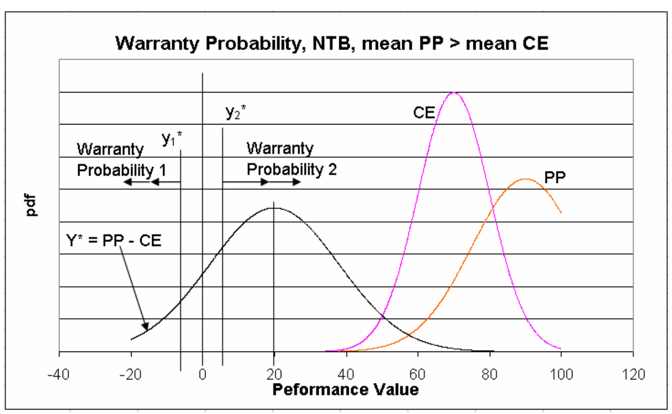


Figure 4.8. Warranty Probability—NTB—Mean PP > Mean CE

4.5.6.1. Unknown distribution of customer expectation. For a special case where the distribution of customer expectation is unknown but is assumed by the manufacturer as cutoff points, this methodology works fine. For the NTB case, cutoff points are two points decided by the manufacturer at distances from the mean performance of the product below/inside which no warranty loss needs to be assigned and beyond which some warranty loss needs to be assigned. It assumed that the customer expectation is normally distributed with means at the cutoff points and zero variance. If the distances of cutoff points from the mean of product performance are equal, then the formulation simplifies to

$$WC = T_P \omega k \left[\int_{-\infty}^{y_1^*} \frac{\left(y^* - t\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2 + \sigma_C^2}} e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_C)}{\sqrt{\sigma_P^2}}\right]^2} dt + \int_{y_2^*}^{\infty} \frac{\left(t - y^*\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2 + \sigma_C^2}} e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_C)}{\sqrt{\sigma_P^2}}\right]^2} dt \right] \tag{4.65}$$

4.5.6.2. Case study 5: Bore of gear housing—nominal-the-best. This is an example of gear housings used in an automotive. A total production of 20,000 units, i.e., $T_P = 20000$, is under consideration for the purpose of estimating the warranty cost. A sample of 100 pieces is withdrawn from the production system. Each gear housing has to have a machined bore of a customer-defined size of $\mu_C = 1.0000"$ with a tolerance of $\pm 0.0030"$. The customer also expects and is ready to accept a variation conforming to the three sigma process without shift. The six sigma process means that the customer does not like the products manufactured with a goal post mentality and that customer expectation is distributed normally. With this information that the customer expectation is met at a three sigma level, it can be approximated that the standard deviation of customer expectation is $\sigma_C = 0.0010"$.

The point of occurrence of a warranty claim at the lower end is $y_1^* = -0.0020"$ and the cost of corrective action when $P\left(Y^* < y^*\right)$ is $A_{01} = \$65$ at a deviation of

$\Delta_{01} = 0.0010"$. Similarly, the point of occurrence of a warranty claim at the upper end is $y_2^* = 0.0020"$ and the cost of corrective action when $P\left(Y^* > y^*\right)$ is $A_{02} = \$130$ at a deviation of $\Delta_{02} = 0.0010"$. It is observed that on average, 5% of the buyers complain, and of those who complain, 40% have $Y^* < y^*$ and 60% have $Y^* > y^*$. This means that the complaint factor at the lower end is given as $\omega_1 = 0.02$, and the complaint factor at the upper end is given as $\omega_2 = 0.03$.

The cost of corrective action at the lower end is less because the housing can be reworked by re-boring or reaming, whereas the cost of corrective action at the upper end is high because the housing cannot be reworked and needs to be rejected.

From the data, the following is computed:

$$k_1 = \frac{A_{01}}{\Delta_{01}^2} = \frac{\$65}{0.0010^2} = \$65000000$$

And

$$k_2 = \frac{A_{02}}{\Delta_{02}^2} = \frac{\$130}{0.0010^2} = \$130000000$$

In this example, each of 100 housings were measured for the bore diameter. Therefore, 100 data points have been analyzed and product parameters estimated. The mean of the product performance was $\mu_P = 0.9996"$, and the standard deviation of the product performance was $\sigma_P = 0.0031"$. Using this data, the estimate of the warranty cost is worked out as

$$WC = 65000000 \times 0.02 \times 20000 \int_{-\infty}^{-0.0020} \frac{(-0.0020-t)^2}{\sqrt{2\pi}\sqrt{0.0031^2+0.0010^2}} Exp\left[-\frac{1}{2}\left\{\frac{t-(0.9996-1.0000)}{\sqrt{0.0031^2+0.0010^2}}\right\}^2\right]dt$$

$$+130000000 \times 0.03 \times 20000 \int_{0.0020}^{\infty} \frac{(t-0.0020)^2}{\sqrt{2\pi}\sqrt{0.0031^2+0.0010^2}} Exp\left[-\frac{1}{2}\left\{\frac{t-(0.9996-1.0000)}{\sqrt{0.0031^2+0.0010^2}}\right\}^2\right]dt$$

$$= \$167834$$

In this case, the estimated warranty cost per unit is $= \$8.39$.

Now suppose that the customer does not specify any variation in his expectation. Then the standard deviation of customer expectation $\sigma_C = 0.0000"$. In these circumstances, with the other parameters remaining the same, the estimate of warranty cost is worked out as

$$WC = 65000000 \times 0.02 \times 20000 \int_{-\infty}^{-0.0020} \frac{(-0.0020-t)^2}{\sqrt{2\pi}\sqrt{0.0031^2+0.0000^2}} Exp\left[-\frac{1}{2}\left\{\frac{t-(0.9996-1.0000)}{\sqrt{0.0031^2+0.0000^2}}\right\}^2\right]dt$$

$$+130000000 \times 0.03 \times 20000 \int_{0.0020}^{\infty} \frac{(t-0.0020)^2}{\sqrt{2\pi}\sqrt{0.0031^2+0.0000^2}} Exp\left[-\frac{1}{2}\left\{\frac{t-(0.9996-1.0000)}{\sqrt{0.0031^2+0.0000^2}}\right\}^2\right]dt$$

$$= \$142282$$

In this case, the estimated warranty cost per unit is $= \$7.11$.

It can be observed that when variation in customer expectation reduces, the estimate of the warranty cost also reduces. This is consistent with the thought that variation in both customer expectation and product performance contributes to warranty expenditure. Therefore, between variation of customer expectation and variation of product performance, when the target is fixed the only thing that causes warranty expenditure is the variation of product performance. On the other hand, when customer expectation has variation, it increases the warranty expenditure. Hypothetically, if the product performance has no variation, then the only reason to produce a warranty cost is the shift of product performance from the customer expectation. However, it is assumed in this research that the variation in product performance is never zero.

4.5.7. Warranty Loss Function for Larger-the-Better Characteristics. It is assumed that no complaints will occur when the product performance is greater than the customer expectation by a specified value y^*. This section includes the derivation of the formula for calculating the warranty probability dependent upon product performance and customer expectation. Suppose customer expectation C and product performance P are normally distributed as

$$C \sim NOR\left(\mu_c, \sigma_c^2\right)$$

$$P \sim NOR\left(\mu_p, \sigma_p^2\right)$$

It is assumed that the customer expectation, C, and product performance, P, are independent of each other. Also, suppose that Y^* is a parameter that is a measure of customer satisfaction such that

$$Y^* = P - C$$

And, suppose y^* is a minimum value that ensures customer satisfaction, such that when $Y^* \geq y^*$ the customer is satisfied for LTB. Alternatively, when $Y^* < y^*$, the customer is dissatisfied and will complain for an LTB characteristic. The distribution of $Y^* = P - C$ is given as

$$Y^* = P - C \sim NOR\left(\mu_P - \mu_C, \sigma_C^2 + \sigma_C^2\right) \tag{4.34}$$

Figure 4.9 shows the warranty probability for LTB characteristics when the mean of product performance is smaller than the mean of customer expectation. Also, Figure 4.10 shows the warranty probability for LTB characteristics when the mean of product performance is greater than the mean of customer expectation. Therefore, the probability of customer complaint is given as in Equation (4.66):

$$P_W = P\left(Y^* < y^*\right) = \int_{-\infty}^{y^*} f\left(t\right)dy = \int_{-\infty}^{y^*} \frac{1}{\sqrt{2\pi}\sqrt{\sigma_P^2+\sigma_C^2}} e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2} dt \tag{4.66}$$

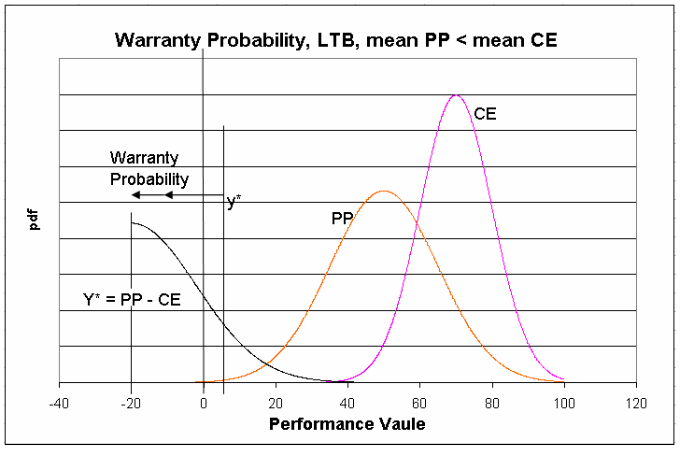


Figure 4.9. Warranty Probability—LTB—Mean PP < Mean CE

Where, t is a dummy variable for Y^*. If Z is defined as

$$Z = \frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}$$

Then the probability of a potential warranty complaint or warranty probability is given as in Equation (4.67):

$$P_W = P\left(Y^* < y^*\right) = P\left[Z < z\right]$$

$$P_W = P\left[\frac{Y^*-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}} < \frac{y^*-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right] = \Phi\left(\frac{y^*-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right) \tag{4.67}$$

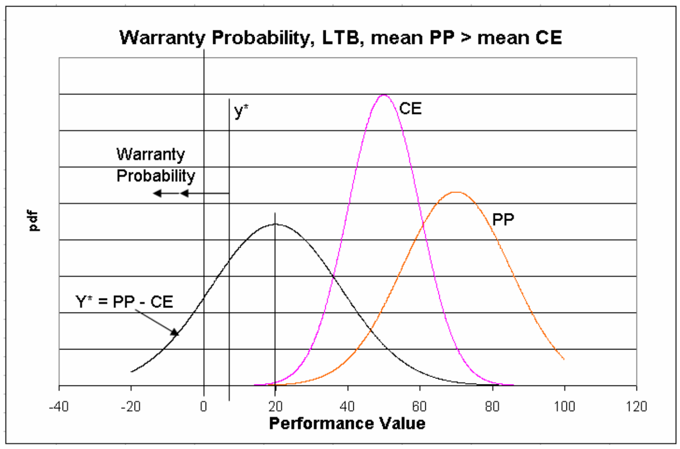


Figure 4.10. Warranty Probability—LTB—Mean PP > Mean CE

The probability of the customer not submitting a complaint would then be as shown in Equation (4.68):

$$P_{noW} = 1 - P\left(Y^* < y^*\right) = 1 - P\left[Z < z\right] = 1 - \Phi\left(\frac{y^* - \left(\mu_P - \mu_c\right)}{\sqrt{\sigma_P^2 + \sigma_C^2}}\right) \tag{4.68}$$

It is assumed that the cost of corrective action or repairs, A, is a function of the distance of the performance from $y*$, and the function is quadratic:

$$A = k\left(y^* - Y^*\right)^2 \tag{4.69}$$

Where, k is proportionality constant. Therefore, using Equation (4.69), the warranty cost (WC) for all the products in question can be estimated as in Equation (4.70):

$$WC = nA = k\left(y^* - Y^*\right)^2 \omega T_P P_W$$

$$WC = \int_{-\infty}^{y^*} \frac{k\omega T_P\left(y^* - t\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2 + \sigma_C^2}} e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2} dt$$

$$WC = k\omega T_P \int_{-\infty}^{y^*} \frac{\left(y^* - t\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2 + \sigma_C^2}} e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_c)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2} dt \tag{4.70}$$

Often it is more useful and easy to understand if the warranty cost is given per unit of product. Therefore, by setting $T_P = 1$, the unit warranty cost can be calculated as shown in Equation (4.71).

$$WC = k\omega \int_{-\infty}^{y^*} \frac{\left(y^* - t\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2 + \sigma_C^2}} e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_C)}{\sqrt{\sigma_P^2+\sigma_C^2}}\right]^2} dt \qquad (4.71)$$

4.5.7.1. Unknown distribution of customer expectation. A general formulation of the problem given above can be reduced to suit cases where the distribution of customer expectation is unknown but the point of warranty claim is known. When the distribution of customer expectation is unknown but can be assumed by the manufacturer as a cutoff point, then this methodology can be used. Suppose a manufacturer decides a cutoff point such that below which some warranty loss needs to be assigned and above which no warranty loss needs to be assigned. It can, therefore, be assumed that the customer expectation is normally distributed with the mean at the cutoff point μ_C and zero variance, i.e., $\sigma_C = 0$.

The point of the warranty claim can be viewed as customer expectation distributed with the mean at the point of the warranty claim and the variance as zero. With this understanding, the same formula can be used and the probability of warranty claims can be assessed. Therefore, the formulation simplifies to

$$WC = k\omega T_P \int_{-\infty}^{y^*} \frac{\left(y^* - t\right)^2}{\sqrt{2\pi}\sqrt{\sigma_P^2}} e^{-\frac{1}{2}\left[\frac{t-(\mu_P-\mu_C)}{\sqrt{\sigma_P^2}}\right]^2} dt \qquad (4.72)$$

4.5.7.2. Case study 6: Solar panel—larger-the-better. This example involves solar panel modules (Web of First Solar, 2007). The grid-connected solar power plants are used in commercial PV projects by leading solar project developers. An advanced film design provided on panels is claimed to produce high energy over a wide range of varying climatic conditions with excellent low light response and temperature coefficient.

The minimum power expected is 67.5 W $\pm 5\%$ for each solar panel. The characteristic is considered as larger-the-better because higher power given by the panel does not invite customer complaint. Therefore, the mean of customer expectation is $\mu_C = 67.5W$ with a tolerance of $\pm 3.375W$. The customer also expects and accepts a variation conforming to a three sigma process. Given this information, it is approximated that the standard deviation of customer expectation is $\sigma_C = 1.125W$. The total number of products is $T_P = 50,000$.

A data set for 500 pieces was simulated and recorded for analysis. The mean of product performance was found to be $\mu_P = 69.0272W$ and the standard deviation was $\sigma_P = 2.9298W$. On the basis of past experience, the complaint factor was estimated to be $\omega = 0.30$. The cost of corrective action was $A_0 = \$2000$ at a distance of $\Delta_0 = 3.375W$. The point of occurrence of a warranty claim at one end was $y^* = 1W$. Therefore, if $Y^* < 1W$, then potentially a warranty complaint will occur. Therefore,

$$k = \frac{A_0}{\Delta_0^2} = \frac{\$2000}{3.375^2} = \$175.583$$

$$WC = 175.583 \times 0.3 \times 50000 \int_{-\infty}^{1} \frac{(1-t)^2}{\sqrt{2\pi}\sqrt{2.9298^2 + 1.125^2}} Exp\left[-\frac{1}{2}\left\{ \frac{t-(69.0272-67.5000)}{\sqrt{2.9298^2+1.125^2}} \right\}^2 \right] dt$$

$$= \$9,843,120$$

Estimated unit warranty cost:

$$WC = 175.583 \times 0.3 \int_{-\infty}^{1} \frac{(1-t)^2}{\sqrt{2\pi}\sqrt{2.9298^2 + 1.125^2}} Exp\left[-\frac{1}{2}\left\{ \frac{t-(69.0272-67.5000)}{\sqrt{2.9298^2+1.125^2}} \right\}^2 \right] dt$$

$$= \$196.86$$

Now suppose that the customer does not specify any variation in his or her expectation. Then the standard deviation of customer expectation is $\sigma_C = 0.000W$. In these circumstances, with the other parameters remaining the same, the estimate of warranty cost is calculated as

$$WC = 175.583 \times 0.3 \times 50000 \int_{-\infty}^{1} \frac{(1-t)^2}{\sqrt{2\pi}\sqrt{2.9298^2}} Exp\left[-\frac{1}{2}\left\{\frac{t-(69.0272-67.5000)}{\sqrt{2.9298^2}}\right\}^2\right] dt$$
$$= \$8,406,360$$

4.5.7.3. A note on the results of case studies. When the variation in customer expectation reduces, the estimate of warranty cost also reduces. This is consistent with the idea that variation in both customer expectation and product performance contributes to higher warranty expenditures. Therefore, between variation of customer expectation and variation of product performance, if the target is fixed (meaning the customer expectation is fixed), them the only factor that causes warranty expenditure is the variation of product performance. On the other hand, when customer expectation has variation, it increases the warranty expenditure. Hypothetically, if the product performance has no variation, then the only reason to cause a warranty cost is the shift of product performance from the customer expectation. However, it was assumed in this research that the variation in product performance is never zero. Therefore, it can be said that if a process is designed according to a fixed target, then it gives a lower warranty loss than can actually occur. In light of this, it is advisable to consider customer expectation in addition to product performance.

4.6. CONCLUSION

In this section an attempt has been made to develop a warranty loss function that can predict warranty expenditure and that takes into account customer expectation as a variable in addition to product performance. The quality loss function only accounts for immediate issues within manufacturing facilities, whereas warranty loss occurs during customer use. This section presented a methodology to predict the warranty probability or the probability of customer complaint on the basis of two independent variables—product performance and consumer expectation. Because CE and PP both are considered in the models the reduction in the warranty cost can more reasonably be planned and executed by controlling or reducing PP in the factory in early product development stage. The rationale of developing this model is to relate customer expectation with the product performance in such a way as to be able to predict the probability and calculate cost estimates of warranty claims.

The formulation presented is supposed to serve as a basic model for predicting warranty loss dependent on both product performance and customer expectation. An attempt has been made to develop a warranty loss function for STB, NTB, and LTB characteristics that takes into account customer expectation as a variable in addition to product performance to predict warranty expenditure. Warranty cost can be estimated from the probability of customer complaint. An example of disc brakes was provided for STB, and warranty cost was estimated to demonstrate this methodology. Furthermore, an example of gear housing was shown for NTB in this section, and warranty cost was estimated to demonstrate this methodology. In addition, an example of solar panels was also given for LTB, and warranty cost was estimated to demonstrate this methodology.

4.7. FUTURE RESEARCH

This section considered all three types of characteristics: smaller-the-better, nominal-the-best, and larger-the-better. The measurable and controllable product characteristics in a factory can be used to successfully predict warranty loss using the proposed models when the objective of customer satisfaction is of prime importance. Therefore, both customer expectation and product performance were considered to be normally distributed in this methodology. Although the approach is novel, it may be argued that because almost all the characteristics are non-negative, a normal distribution is not the best choice for the formulation of the problem. Therefore, in future research other distributions ranging from zero to infinity may be considered in place of normal distribution in this novel approach.

The reliability in terms of mean time to failure (MTTF), mean time between failures (MTBF), etc., of a product is an important characteristic that can be used with the proposed methodology to predict the warranty cost. Therefore, in future research other distributions spreading from zero to infinity should be considered in place of the normal distribution. Also in future research, a different distribution may be considered for customer expectation than for product performance.

5. VALIDATION OF MODELS

5.1. INTRODUCTION

Validation is concerned with building the right model, and it is required to determine that a model is an accurate representation of the real system. Validation can be achieved through calibration of the model, which is an iterative process of comparing the model to the actual system. According to lecture notes from a class at Michigan State University, discrepancies between the real system and the model as well as insight gained can be used to improve the model (Michigan State University, 1999).

A related topic is model credibility, which is concerned with sufficiently developing the confidence that users have in a model and in the information derived from the model so they are willing to use the model and the derived information. A model should be developed for a specific purpose or application and its validity determined with respect to that purpose or application. A model may be valid for one set of experimental conditions and invalid for another. It is often too costly and time consuming to determine that a model is absolutely valid over the complete domain of its intended applicability (Sargent, 1998).

A “model” is a simplified representation of a system at some particular point in time or space that is intended to promote understanding of the real system (Web of Systems Thinking, 2007). A “simulation” is the manipulation of a model in such a way that it operates on time or space to compress it, thus enabling one to perceive the interactions that would not otherwise be apparent because of their separation in time or space (Web of Systems Thinking, 2007).

Whether a model is a good model depends on the extent to which it promotes understanding. Because all models are simplifications of reality, a trade-off always occurs as to what level of detail is included in the model. If too little detail is included in the model, then one runs the risk of missing relevant interactions and the resultant model may not promote understanding. However, if too much detail is included in the model, then the model may become overly complicated and actually preclude the development of understanding. According to the Web of Systems Thinking (2007), one simply cannot develop all models in the context of the entire universe.

In general, the process of checking whether something satisfies a certain criterion is called *validation*. In a quality management system, the needs of external users of the product or system can be related to validation, whereas the internal process of checking conformance to specifications or regulations can be related to verification. Validation of newly developed warranty loss models would determine whether the model developed is a good representation. Validation also includes evidence that a process or model that is used within stated parameters can produce effective and reproducible results.

5.2. VALIDATION APPROACHES

Four possible approaches may be applicable in such a case where a new model has been proposed. The first approach is to conceptually validate the model developed. The second approach is to collect data for a case from the field through an actual survey or to use already available data and check for the validity of the models. The third approach is to compare the results obtained from the new model with the results obtained using existing models. The fourth approach is to simulate a real situation by taking into account the given assumptions and comparing the simulated results with the results obtained from the model. In the case of warranty cost models, there seems to be a scarcity of data on customer expectation with regard to any given product and simultaneously of data on product performance and warranty expenditure. Therefore, it is planned to validate the models using two methods: conceptual validation and validation of models by simulation.

5.3. CONCEPTUAL VALIDATION OF MODELS

Conceptual model validity is defined as determining that the theories and assumptions underlying the conceptual model are correct and that the model representation of the problem entity is "reasonable" for the intended purpose of the model (Sargent, 1998). The following paragraphs discuss the theories and assumptions behind the warranty cost model and their validity.

5.3.1. Reasonableness of Normal Distribution for Product Performance and Customer Expectation. Both product performance and customer expectation are assumed to follow normal distribution. Both product performance and customer expectation are nonnegative parameter, such as failure time, fuel consumption, strength of a material, or shaft diameter. Normal distribution ranges from negative infinity to positive infinity. However, 99.7300204% of the area under the curve is covered by three standard deviations (SD) of the data, 99.9936658% of the area under the curve is covered by four standard deviations of the data, and 99.9999427% of the area under the curve is covered by five standard deviations of the data. Most random variables are known to be distributed normally around a mean with a standard deviation. Because an engineering parameter is nonnegative, the normal distribution representing the parameter needs to be truncated by a vertical line at zero. This truncation (Figure 5.1) leaves 0.000000287% of the area for five standard deviations, 0.000031671% of the area for four standard deviations, and 0.001349898% of the area for three standard deviations unaccounted for. This means that the possibility of error is small and can be ignored for all practical purposes provided that the parameter value is large enough to accommodate three to five standard deviations between zero and the mean value.

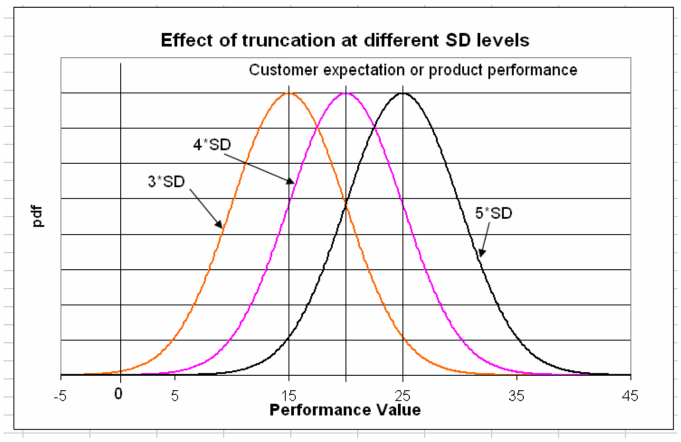


Figure 5.1. Effect of Truncation of a Normal Curve for Different SD Levels

Reliability parameters such as failure to time do not follow a normal distribution, but instead they follow distributions such as lognormal, Weibull, and exponential. Section 6, therefore, is dedicated to include distributions other than normal distributions for product performance and/or customer expectation.

5.3.2. Reasonableness of Transformed Parameter, the Difference between Product Performance and Customer Expectation. Existing warranty cost models do not consider customer expectation explicitly as a variable affecting warranty expenditure. The model developed is based on the premise that the clash between customer expectation and product performance causes warranty claims and that customer expectation varies from customer to customer. Therefore, it was necessary to link both product performance and customer expectation in some logical way. Two possibilities emerged after careful application of mind. The first is the ratio between product performance and customer expectation, and the second is the difference between them. The difference between them gave a wider window for computations than the ratio because the ratio had a small range from about 0.3 to 3. Moreover, a ratio would be unitless. Therefore, it is reasonable to choose the difference between the two over the ratio between them. The difference between product performance and customer expectation facilitated the formulation and solution of the problem in addition to having the same unit.

5.3.3. Reasonableness of the Quadratic Loss Function. The cost associated with repair of a product is assumed to be quadratic with the deviation from the threshold limit, i.e., with the loss being zero at the threshold limit and increasing quadratically on its one side. The quadratic cost function is in sync with Taguchi's quality loss function, which suggests that the loss is proportional to the square of the deviation of performance from the target. Therefore, the assumption of quadratic cost function is reasonable.

5.3.4. Reasonableness of the Concept of Potential Number of Complaints. The term *potential number of complaints* as denoted by N is the total number of complaints possible where the product performance falls short of customer expectation by all values greater than a certain value y^* assuming that all the customers make a complaint. In other words, the potential number of complaints is the total number of complaints that are likely to be made because the difference between product performance and customer expectation is more than the predetermined values y^*.

Assuming that all the customers make a complaint seems unreasonable, but it has its notional importance, hence the term *potential complaints*. It is necessary to estimate what fraction of customers will actually complain based on the available data. The said fraction is termed the *complaint factor*, which is discussed in Section 5.3.7.

5.3.5. Reasonableness of Warranty Probability. It is assumed that a fairly constant ratio exists between the potential number of complaints and the total number of products. This ratio is denoted as P_W. Thus,

$$P_W = \frac{N}{T_P}$$

Or,

$$N = P_W T_P$$

It is important to note that the term P_W can be called the *warranty probability*. Therefore, warranty probability is defined as the ratio between thenpotential number of complaints, N, and the total number of products, T_P. From the warranty probability, the potential number of complaints, N, can be computed by multiplying the warranty probability with the amount of production or number of products. A little consideration will show that the warranty probability, P_W, depends on product performance falling short of the customer expectation. The concept or assumption of a ratio between the potential number of complaints and the total number of products is simple and therefore reasonable.

5.3.6. Reasonableness of the Actual Number of Complaints. The term *actual number of complaints* is the number of complaints actually made by customers where the product performance is smaller than the customer expectation by any value more than y^*.

It is assumed that there will be no complaints when the product performance is greater than the customer expectation by a certain value y^* or more for LTB. The actual number of complaints is defined as the total number of complaints that are actually lodged because product performance falls short of the customer expectation. It is denoted by the symbol n. The assumption that there will be no complaints when the product performance is greater than customer expectation by a certain value y^* or more is logical and reasonable.

5.3.7. Reasonableness of the Complaint Factor. A factor called the *complaint factor*, as denoted by ω, is introduced in the models. The complaint factor is the ratio of the actual number of complaints, n, to the potential number of complaints, N, because product performance falls short of the customer expectation. Thus, the following equations show the relation and can be used to compute the actual number of complaints:

$$\omega = \frac{n}{N}$$

Or

$$n = \omega . N$$

5.3.8. Reasonableness of the Three Types of Characteristics: STB, NTB, and LTB. Three independent models using the same methodology have been developed to suit to three types of engineering parameters: STB, NTB, and LTB.

5.3.9. Calibration of the Model. The models developed have sufficient flexibility to render suitability to real situations under given assumptions. In other words, one can calibrate the model by assigning appropriate values to the complaint factor and the predetermined value, y^*, called the threshold limit.

5.4. VALIDATION OF MODELS BY SIMULATION

The simulation method attempts to simulate a real situation that is either difficult to reproduce or is not cost effective. The model developed can be validated by simulating a real situation according to given assumptions parallel to the model. Then results obtained by the model and simulation can be compared and discussed (Web of Systems Thinking, 2007).

A simulation generally refers to a computerized version of the model that is run over time to study the implications of the defined interactions. Simulations are generally iterative in their development. One develops a model, simulates it, learns from the simulation, revises the model, and continues the iterations until an adequate level of understanding is developed. Therefore, the approach taken in the validation of the models is to compare the results produced by three simulation models with the results obtained using a new model.

In this dissertation, situations are simulated and warranty costs are estimated for the three cases STB, NTB, and LTB. The warranty cost has also been estimated using the new model, and results have been compared. The simulation is conducted as follows. To begin, it is assumed that a large number of products and a large equal number of customers are present. Sufficient data are simulated for product performance assuming the mean and standard deviation. Similarly, sufficient data are simulated for customer expectation assuming the mean and standard deviation. To be precise, 30,000 data sets for STB, 20,000 data sets for NTB, and 10,000 data sets for LTB have been generated. Then the products with different performances are randomly handed over to all the customers and the difference between product performance and customer expectation is computed for each customer. The customers with the difference beyond given limit(s) (causing potential complaints) are then segregated from the customers with this difference within the given limit(s) (causing no complaint).

With the assumption of the quadratic loss function, the cost of repair (i.e., warranty cost) is computed for each customer lodging a complaint. It is unnecessary to compute the warranty cost for customers who do not lodge a complaint. The warranty costs thus obtained are summed up and multiplied by the complaint factor to get the warranty cost for the lot. In this method all the potential complaints are averaged (marked by *) to obtain the actual warranty expenditure.

Alternatively, one can also find actual complaints from potential complaints by picking random customers according to the complaint factor, and with the assumption of quadratic loss function the cost of repair (i.e., warranty cost) is computed for each customer lodging a complaint. Then all such costs of repair are summed up to get the warranty cost for the lot. In this method, random actual complaints (marked by †) are found from all the potential complaints.

Warranty costs for the three cases are obtained from the newly developed formulae, and the results obtained from the simulation and the developed models are reproduced in Tables 5.1, 5.2, and 5.3 for comparison. The simulation method will be used for all three types of characteristics—STB, NTB, and LTB. The simulations are done in Microsoft Excel. Appendix C shows a snapshot of each simulation case.

5.4.1. Comparison of the Results from the Model and Simulation (STB Case). Table 5.1 shows all the relevant details for comparison. Appendix C may also be referred to for simulated data. The simulation results for the warranty cost are slightly higher than for the model results. When the cost is averaged considering all the potential complaints and then calculating the fraction equivalent to the complaint factor, the warranty cost is

1.421% higher than the estimated cost using the model. When random actual complaints are considered, the warranty cost is 8.382% higher than the model's estimated cost.

Table 5.1. Comparison of Results from the Model and Simulation, STB Case

	Model	**Simulation: Averaged To Obtain Actual Complaints, *WC**	**Simulation: Random Actual Complaints, †WC**
Mean PP μ_P	30.7 μM	30.721 μM	30.721 μM
SD PP σ_P	18.1 μM	18.141 μM	18.141 μM
Mean CE μ_C	30.0 μM	29.984 μM	29.984 μM
SD CE σ_C	3.333 μM	3.337 μM	3.337 μM
Total Production T_P	636000	--	--
Number of Simulation Data Sets	--	30000	30000
Coefficient of Proportionality k	0.0592	0.0592	0.0592
Complaint factor ω	0.05	0.05	0.05
Threshold Limit y^*	0.00	0.00	0.00
Results			
Total Warranty Cost, WC	$338,646	$339195.10	$371038.70
Unit Warranty Cost, WC_{unit}	$0.53	$0.53	$0.58
Difference Between Model and Simulation	--	1.421%	8.382%

5.4.2. Comparison of Results from the Model and Simulation (NTB Case). All the relevant details for comparison can be seen in Table 5.2. Appendix C may also be referred to for simulated data. The simulation results for the warranty cost are slightly lower than for the model results. When the cost is averaged considering all the potential complaints and then calculating the fraction equivalent to the complaint factor, the warranty cost is 0.78% lower than the estimated cost using the model. When random actual complaints are considered, the warranty cost is 16.03% lower than the model's estimated cost.

Table 5.2. Comparison of Results from the Model and Simulation, NTB Case

	Model	**Simulation: Averaged To Obtain Actual Complaints, *WC**	**Simulation: Random Actual Complaints, †WC**
Mean PP μ_P	0.9996 μM	0.9996 μM	0.9996 μM
SD PP σ_P	0.0031 μM	0.0031 μM	0.0031 μM
Mean CE μ_C	1.0000 μM	1.0000 μM	1.0000 μM
SD CE σ_C	0.0010 μM	0.00101 μM	0.00101 μM
Total Production T_P	20000		
Number of Simulation Data Sets		20000	20000
Coefficients of Proportionality k_1, k_2	65000000, 130000000	65000000, 130000000	65000000, 130000000
Complaint factors ω_1, ω_2	0.02, 0.03	0.02, 0.03	0.02, 0.03
Threshold Limit y_1^*, y_2^*	- 0.002, 0.002	- 0.002, 0.002	- 0.002, 0.002
Results			
Total Warranty Cost, WC	\$167834	\$166534.16	\$140924.50
Unit Warranty Cost, WC_{unit}	\$8.39	\$8.33	\$7.05
Difference Between Model and Simulation	--	- 0.78%	- 16.03%

5.4.3. Comparison of Results from the Model and Simulation (LTB Case). All the relevant details for comparison are shown in Table 5.3. The simulation results for the warranty cost are slightly different than for the model results. When the cost is averaged considering all the potential complaints and then calculating the fraction equivalent to the complaint factor, the warranty cost is 1.617% lower than the estimated cost using the model. When random actual complaints are considered, the warranty cost is 0.0028% higher than the model's estimated cost. In both simulations, the difference seems to be negligible in the case of LTB. Appendix C shows the simulated data.

Table 5.3. Comparison of Results from the Model and Simulation, LTB Case

	Model	**Simulation: Averaged To Obtain Actual Complaints, *WC**	**Simulation: Random Actual Complaints, †WC**
Mean PP μ_P	69.0272 W	69.016 W	69.016 W
SD PP σ_P	2.9298 W	2.928 W	2.928 W
Mean CE μ_C	67.5 W	67.483 W	67.483 W
SD CE σ_C	1.125 W	1.109 W	1.109 W
Total Production T_P	50000	--	--
Number of Simulation Data Sets	--	10000	10000
Coefficient of Proportionality k	175.583	175.583	175.583
Complaint factor ω	0.3	0.3	0.3
Threshold Limit y^*	1 W	1 W	1 W
Results			
Total Warranty Cost, WC	$9,843,120	$9683931	$9843395
Unit Warranty Cost, WC_{unit}	$196.86	$193.68	$196.87
Difference Between Model and Simulation	--	- 1.617%	0.0028%

5.5. CONCLUSION

In this section, two validation approaches have been discussed and used to validate warranty cost models for STB, NTB, and LTB that consider customer expectation as a variable along with product performance: conceptual validation of models and validation through simulation. The reasons discussed in conceptual validation and the minor differences between the results obtained by the models themselves and the simulations suggest that the models can be validated.

6. WARRANTY LOSS METHODOLOGY WITH OTHER DISTRIBUTION COMBINATIONS

6.1. INTRODUCTION

In actual practice, the customer expectation varies from customer to customer just as the product performance varies about its target or designed value. The product performance is measured in terms of certain characteristics that need to be within a certain range in order to satisfy the customer. This section presents a methodology to calculate the warranty cost when the product performance and consumer's expected value are assumed to follow distributions such as one parameter exponential distribution, two parameter exponential distribution, Weibull distribution, and normal distribution.

6.1.1. Importance of Other Distributions. This section considers distributions other than normal for performance characteristics. The measurable and controllable product characteristics in the factory can be used to successfully predict warranty loss using the proposed models when the objective of customer satisfaction is of prime importance. Therefore, both customer expectation and product performance were considered to be normally distributed in the previous sections. Although the approach is novel, it may be argued that because almost all the characteristics are non-negative, normal distribution is not the best choice for the formulation of the problem. Therefore, in this research other distributions that have a spread from zero to infinity may be considered in place of normal distribution in this approach.

The reliability in terms of mean time to failure (MTTF), mean time between failures (MTBF), and other similar characteristics of a product are important descriptions that can be used with the proposed methodology to predict warranty cost. Therefore, in this section other distributions spreading from zero to infinity are considered in place of the normal distribution. In some cases, different distribution may be considered for customer expectation and a different distribution for product performance. Many characteristics tend to follow a certain distribution. For example, the characteristic of time to failure tends to follow Weibull, exponential, and lognormal distributions depending on the type of product, but it does not follow a normal distribution. To estimate warranty costs when a parameter follows a particular distribution, it is necessary to accommodate that distribution in the methodology for a better estimation.

6.1.2. Problems Associated with Other Distributions. Non-negative performance parameters follow other distributions than normal. When distributions other than normal are considered, it is sometimes impossible to find a closed-form solution in every case. Very few distribution combinations exist where a closed-form solution can be found. In some cases, it may only be possible to find a closed-form solution after certain assumptions or simplifications are made. However, a numerical solution can be applied to a combination of any two distributions. Section 6.2.2 demonstrates a method with an example using two such distributions.

6.2. POSSIBILITIES

Two possibilities emerge after careful consideration. In the first possibility, one can look for distribution combinations that can give a closed-form solution and are close to realistic parameter distributions. In the second possibility, one need not look for distribution combinations that can give a closed-form solution, but instead it is necessary only to figure out what distributions are closest to the realistic parameter (i.e., product performance and customer expectation) distributions. In this case, it is necessary to evaluate the problem numerically as explained in Section 6.2.2. However, the following section derives warranty probability for distribution combinations that can give a closed-form solution. Furthermore, warranty costs can be computed on a case-by-case basis under the given assumptions of the complaint factor, threshold limit for complaint, total production, and cost of corrective action.

6.2.1. Distribution Combinations that Can Give a Closed-Form Solution. Closed form solution can be obtained for a very few distribution combinations. One such distribution combination is discussed below from mathematical point of view. Further case studies can be conducted to use and validate the solutions. The methodology used for normal distributions in Section 4 can be used in the same way for the combination given below to formulate the problem and compute the warranty cost for a given case.

6.2.1.1. PP two parameter exponentially distributed and CE two parameter exponentially distributed. P and C are distributed as in Equations 6.1 and 6.2, respectively:

$$P \sim EXP\left(\theta_p, \eta_P\right) = \frac{1}{\theta_p} e^{-\frac{p-\eta_P}{\theta_p}}, \quad \theta_P > 0; p > \eta_P \tag{6.1}$$

$$C \sim EXP\left(\theta_C, \eta_C\right) = \frac{1}{\theta_C} e^{-\frac{c-\eta_C}{\theta_C}}, \quad \theta_C > 0; c > \eta_C \tag{6.2}$$

The transformation parameters are

$$w = p - c \ldots p = \frac{w+z}{2} > \eta_P$$
$$z = p + c \ldots c = \frac{z-w}{2} > \eta_C$$

And the new limits are

$$\eta_P < p; \cdots \eta_C < c$$
$$w + z > 2\eta_P$$
$$z > 2\eta_P - w$$
$$z > \eta_P + \eta_C$$
$$z - w > 2\eta_C$$
$$z > 2\eta_C + w$$
$$2\eta_P - z < w < -2\eta_C + z$$
$$\eta_P - \eta_C < w < \eta_P - \eta_C$$

The Jacobian of the transformation is

$$J = \begin{vmatrix} \frac{\partial p}{\partial w} & \frac{\partial p}{\partial z} \\ \frac{\partial c}{\partial w} & \frac{\partial c}{\partial z} \end{vmatrix} = \begin{vmatrix} +\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

Therefore,

$$f_{w,z}(w,z) = f_{p,c}(p,c) = f_{p,c}\left(\frac{w+z}{2}, \frac{z-w}{2}\right) = \frac{1}{2\theta_P\theta_C} e^{-\frac{w+z-2\eta_P}{2\theta_P} - \frac{z-w-2\eta_C}{2\theta_C}}$$

$$f_w(w) = \int_{2\eta_P - w}^{\infty} \frac{1}{2\theta_P\theta_C} e^{-\frac{w+z-2\eta_P}{2\theta_P} - \frac{z-w-2\eta_C}{2\theta_C}} dz$$

$$= \frac{1}{2\theta_P\theta_C} \int_{2\eta_P - w}^{\infty} e^{-\frac{w}{2\theta_P} - \frac{z}{2\theta_P} + \frac{\eta_P}{\theta_P} - \frac{z}{2\theta_C} + \frac{w}{2\theta_C} + \frac{\eta_C}{\theta_C}} dz$$

$$= \frac{1}{2\theta_P\theta_C} e^{-\frac{w}{2\theta_P} + \frac{w}{2\theta_C} + \frac{\eta_P}{\theta_P} + \frac{\eta_C}{\theta_C}} \int_{2\eta_P - w}^{\infty} e^{-\frac{z}{2\theta_P} - \frac{z}{2\theta_C}} dz$$

$$= \frac{1}{2\theta_P\theta_C} e^{-\frac{w}{2\theta_P} + \frac{\eta_P}{\theta_P} + \frac{w}{2\theta_C} + \frac{\eta_C}{\theta_C}} \left[-\frac{2\theta_P\theta_C}{\theta_P + \theta_C} e^{-\frac{z}{2\theta_P} - \frac{z}{2\theta_C}} \right]_{2\eta_P - w}^{\infty}$$

$$= \frac{1}{2\theta_P\theta_C} e^{-\frac{w}{2\theta_P} + \frac{w}{2\theta_C} + \frac{\eta_P}{\theta_P} + \frac{\eta_C}{\theta_C}} \left[\frac{2\theta_P\theta_C}{\theta_P + \theta_C} e^{-\frac{2\eta_P - w}{2\theta_P} - \frac{2\eta_P - w}{2\theta_C}} \right]$$

$$= \frac{1}{\theta_P + \theta_C} e^{-\frac{w}{2\theta_P} + \frac{w}{2\theta_C} + \frac{\eta_P}{\theta_P} + \frac{\eta_C}{\theta_C} - \frac{2\eta_P}{2\theta_P} - \frac{2\eta_P}{2\theta_C} + \frac{w}{2\theta_P} + \frac{w}{2\theta_C}}$$

$$= \frac{1}{\theta_P + \theta_C} e^{\frac{w}{\theta_C} + \frac{\eta_C}{\theta_C} - \frac{\eta_P}{\theta_C}}$$

$$= \frac{1}{\theta_P + \theta_C} e^{\frac{w + \eta_C - \eta_P}{\theta_C}}$$

And

$$f_w(w) = \int_{2\eta_C + w}^{\infty} \frac{1}{2\theta_P \theta_C} e^{-\frac{w + z - 2\eta_P}{2\theta_P} - \frac{z - w - 2\eta_C}{2\theta_C}} dz$$

$$= \frac{1}{2\theta_P \theta_C} \int_{2\eta_C + w}^{\infty} e^{-\frac{w}{2\theta_P} - \frac{z}{2\theta_P} + \frac{\eta_P}{\theta_P} - \frac{z}{2\theta_C} + \frac{w}{2\theta_C} + \frac{\eta_C}{\theta_C}} dz$$

$$= \frac{1}{2\theta_P \theta_C} e^{-\frac{w}{2\theta_P} + \frac{w}{2\theta_C} + \frac{\eta_P}{\theta_P} + \frac{\eta_C}{\theta_C}} \int_{2\eta_C + w}^{\infty} e^{-\frac{z}{2\theta_P} - \frac{z}{2\theta_C}} dz$$

$$= \frac{1}{2\theta_P \theta_C} e^{-\frac{w}{2\theta_P} + \frac{w}{2\theta_C} + \frac{\eta_P}{\theta_P} + \frac{\eta_C}{\theta_C}} \left[-\frac{2\theta_P \theta_C}{\theta_P + \theta_C} e^{-\frac{z}{2\theta_P} - \frac{z}{2\theta_C}} \right]_{2\eta_C + w}^{\infty}$$

$$= \frac{1}{2\theta_P \theta_C} e^{-\frac{w}{2\theta_P} + \frac{w}{2\theta_C} + \frac{\eta_P}{\theta_P} + \frac{\eta_C}{\theta_C}} \left[\frac{2\theta_P \theta_C}{\theta_P + \theta_C} e^{-\frac{2\eta_C + w}{2\theta_P} - \frac{2\eta_C + w}{2\theta_C}} \right]$$

$$= \frac{1}{\theta_P + \theta_C} e^{-\frac{w}{2\theta_P} + \frac{w}{2\theta_C} + \frac{\eta_P}{\theta_P} + \frac{\eta_C}{\theta_C} - \frac{2\eta_C}{2\theta_P} - \frac{2\eta_C}{2\theta_C} - \frac{w}{2\theta_P} - \frac{w}{2\theta_C}}$$

$$= \frac{1}{\theta_P + \theta_C} e^{-\frac{w}{\theta_P} + \frac{\eta_P}{\theta_P} - \frac{\eta_C}{\theta_P}}$$

$$= \frac{1}{\theta_P + \theta_C} e^{-\frac{w+\eta_C-\eta_P}{\theta_P}}$$

Therefore, the pdf of $W = P - C$ is as given in Equation 6.3:

$$f_w(w) = \begin{cases} = \frac{1}{\theta_P + \theta_C} e^{\frac{w+\eta_C-\eta_P}{\theta_C}} \cdots \text{for}; w < \eta_P - \eta_C \\ = \frac{1}{\theta_P + \theta_C} e^{-\frac{w+\eta_C-\eta_P}{\theta_P}} \cdots \text{for}; \eta_P - \eta_C < w \end{cases} \tag{6.3}$$

Recognizing the probability density function for $Y^* = w = P - C$, one can say that $Y^* = P - C$ is non-symmetric double exponentially distributed. Proof that it is a pdf is as follows:

$$F_w(w) = \int_{-\infty}^{\eta_P-\eta_C} \frac{1}{\theta_P + \theta_C} e^{\frac{w+\eta_C-\eta_P}{\theta_C}} dw + \int_{\eta_P-\eta_C}^{\infty} \frac{1}{\theta_P + \theta_C} e^{-\frac{w+\eta_C-\eta_P}{\theta_P}} dw$$

$$= \left. \frac{e^{\frac{w+\eta_C-\eta_P}{\theta_C}} \theta_C}{\theta_P + \theta_C} \right|_{-\infty}^{\eta_P-\eta_C} - \left. \frac{e^{-\frac{w+\eta_C-\eta_P}{\theta_P}} \theta_P}{\theta_P + \theta_C} \right|_{\eta_P-\eta_C}^{\infty}$$

$$= \frac{\theta_C}{\theta_P + \theta_C} + \frac{\theta_P}{\theta_P + \theta_C} = 1$$

Therefore, warranty probability for STB is given as in Equation 6.4:

$$P_w = F_w(w) = \begin{cases} \int_{y^*}^{\eta_P-\eta_C} \frac{e^{\frac{w+\eta_C-\eta_P}{\theta_C}}}{\theta_P + \theta_C} dw + \int_{\eta_P-\eta_C}^{\infty} \frac{e^{-\frac{w+\eta_C-\eta_P}{\theta_P}}}{\theta_P + \theta_C} dw \cdots \text{for}; y^* < \eta_P - \eta_C \\ \int_{y^*}^{\infty} \frac{e^{-\frac{w+\eta_C-\eta_P}{\theta_P}}}{\theta_P + \theta_C} dw \cdots \text{for}; \eta_P - \eta_C < y^* \end{cases} \tag{6.4}$$

Warranty probability for NTB can be given as in Equation 6.5:

$$P_w = F_w\left(w\right) = \int_{-\infty}^{y_1^*} \frac{e^{\frac{w+\eta_C-\eta_P}{\theta_C}}}{\theta_P+\theta_C}dw + \int_{y_2^*}^{\infty} \frac{e^{-\frac{w+\eta_C-\eta_P}{\theta_P}}}{\theta_P+\theta_C}dw \cdots \text{for; } y_1^* < \eta_P - \eta_C < y_2^* \tag{6.5}$$

And warranty probability for LTB is given as in Equation 6.6:

$$P_w = F_w\left(w\right) = \begin{cases} \displaystyle\int_{-\infty}^{y^*} \frac{e^{\frac{w+\eta_C-\eta_P}{\theta_C}}}{\theta_P+\theta_C}dw \cdots \text{for; } y^* < \eta_P - \eta_C \\ \displaystyle\int_{-\infty}^{\eta_P-\eta_C} \frac{e^{\frac{w+\eta_C-\eta_P}{\theta_C}}}{\theta_P+\theta_C}dw + \int_{\eta_P-\eta_C}^{y^*} \frac{e^{-\frac{w+\eta_C-\eta_P}{\theta_P}}}{\theta_P+\theta_C}dw \cdots \text{for;} \eta_P - \eta_C < y^* \end{cases} \tag{6.6}$$

Using equations 6.4, 6.5, and 6.6 warranty cost can conveniently be computed with the similar assumptions and method as discussed in Section 4. However, no case study has been conducted for two parameter exponentially distributed PP and two parameter exponentially distributed CE combination.

6.2.1.2. PP two parameter exponentially distributed and CE one parameter exponentially distributed. P and C are distributed as in Equations 6.7 and 6.8, respectively,

$$P \sim EXP\left(\theta_p, \eta_P\right) = \frac{1}{\theta_p} e^{-\frac{p-\eta_P}{\theta_p}}, \quad \theta_p > 0; p > \eta_P \tag{6.7}$$

$$C \sim EXP\left(\theta_C\right) = \frac{1}{\theta_C} e^{-\frac{c}{\theta_C}}, \quad \theta_C > 0; c > 0 \tag{6.8}$$

By putting $\eta_C = 0$ in the entire derivation given in Section 6.2.1.1 one can obtain as follows. The transformation parameters are:

$$w = p - c \ldots p = \frac{w+z}{2} > \eta_P$$
$$z = p + c \ldots c = \frac{z-w}{2} > 0$$

The limits are

$$
\begin{aligned}
&\eta_P < p; \cdots 0 < c \\
&w + z > 2\eta_P \\
&z > 2\eta_P - w \\
&z > \eta_P \\
&z - w > 0 \\
&z > w \\
&2\eta_P - z < w < z
\end{aligned}
$$

Therefore, the pdf of $W = P - C$ is

$$
f_w(w) = \begin{cases} \dfrac{e^{\frac{w-\eta_P}{\theta_C}}}{\theta_P + \theta_C} \cdots \text{for}; w < \eta_P \\ \dfrac{e^{-\frac{w-\eta_P}{\theta_P}}}{\theta_P + \theta_C} \cdots \text{for}; \eta_P < w \end{cases} \tag{6.9}
$$

Proof that it is a pdf

$$
F_w(w) = \int_{-\infty}^{\eta} \frac{e^{\frac{w-\eta_P}{\theta_C}}}{\theta_P + \theta_C} dw + \int_{\eta}^{\infty} \frac{e^{-\frac{w-\eta_P}{\theta_P}}}{\theta_P + \theta_C} dw
$$

$$
= \left. \frac{e^{\frac{w-\eta_P}{\theta_C}} \theta_C}{\theta_P + \theta_C} \right|_{-\infty}^{\eta_P} - \left. \frac{e^{-\frac{w-\eta_P}{\theta_P}} \theta_P}{\theta_P + \theta_C} \right|_{\eta_P}^{\infty}
$$

$$
= \frac{\theta_C}{\theta_P + \theta_C} + \frac{\theta_P}{\theta_P + \theta_C} = 1
$$

Therefore, warranty probability for STB is given as in Equations 6.10:

$$
P_w = F_w(w) = \begin{cases} \displaystyle\int_{y^*}^{\eta_P} \frac{e^{\frac{w-\eta_P}{\theta_C}}}{\theta_P + \theta_C} dw + \int_{\eta_P}^{\infty} \frac{e^{-\frac{w-\eta_P}{\theta_P}}}{\theta_P + \theta_C} dw \cdots \text{for}; y^* < \eta_P \\ \displaystyle\int_{y^*}^{\infty} \frac{e^{-\frac{w-\eta_P}{\theta_P}}}{\theta_P + \theta_C} dw \cdots \text{for}; \eta_P < y^* \end{cases} \tag{6.10}
$$

Warranty probability for NTB can be given as shown in Equation 6.11:

$$P_w = F_w\left(w\right) = \int_{-\infty}^{y_1^*} \frac{e^{\frac{w-\eta_P}{\theta_C}}}{\theta_P + \theta_C} dw + \int_{y_2^*}^{\infty} \frac{e^{-\frac{w-\eta_P}{\theta_P}}}{\theta_P + \theta_C} dw \cdots \text{for; } y_1^* < \eta_P < y_2^* \tag{6.11}$$

And warranty probability for LTB is given as in Equations 6.12:

$$P_w = F_w\left(w\right) = \begin{cases} \displaystyle\int_{-\infty}^{y^*} \frac{e^{\frac{w-\eta_P}{\theta_C}}}{\theta_P + \theta_C} dw \cdots \text{for; } y^* < \eta_P \\ \displaystyle\int_{-\infty}^{\eta_P} \frac{e^{\frac{w-\eta_P}{\theta_C}}}{\theta_P + \theta_C} dw + \int_{\eta_P}^{y^*} \frac{e^{-\frac{w-\eta_P}{\theta_P}}}{\theta_P + \theta_C} dw \cdots \text{for; } \eta_P < y^* \end{cases} \tag{6.12}$$

Using equations 6.10, 6.11, and 6.12 warranty cost can conveniently be computed with the similar assumptions and method as discussed in Section 4. However, no case study has been conducted for two parameter exponentially distributed PP and one parameter exponentially distributed CE combination.

6.2.1.3. PP one parameter exponentially distributed and CE two parameter exponentially distributed. P and C are distributed as in Equations 6.13 and 6.14, respectively,

$$P \sim EXP\left(\theta_p\right) = \frac{1}{\theta_p} e^{-\frac{p}{\theta_p}} \ldots, for\ p > 0, \theta_p > 0 \tag{6.13}$$

$$C \sim EXP\left(\theta_C\right) = \frac{1}{\theta_C} e^{-\frac{c-\eta_C}{\theta_C}}, \quad \theta_C > 0; c > \eta_C \tag{6.14}$$

By putting $\eta_P = 0$ in the entire derivation given in Section 6.2.1.1 one can obtain as follows. The transformation parameters are:

$$w = p - c \ldots p = \frac{w+z}{2} > 0$$

$$z = p + c \ldots c = \frac{z-w}{2} > \eta_C$$

The limits are

$$
\begin{aligned}
&0 < p; \cdots \eta_C < c \\
&w + z > 0 \\
&z > -w \\
&z > \eta_C \\
&z - w > 2\eta_C \\
&z > 2\eta_C + w
\end{aligned}
$$

Therefore, the pdf of $w = P - C$ is

$$
f_w(w) = \begin{cases} \dfrac{e^{\frac{w+\eta_C}{\theta_C}}}{\theta_P + \theta_C} \cdots \text{for}; w < -\eta_C \\ \dfrac{e^{-\frac{w+\eta_C}{\theta_P}}}{\theta_P + \theta_C} \cdots \text{for}; -\eta_C > w \end{cases} \tag{6.15}
$$

Proof that it is a pdf

$$
F_w(w) = \int_{-\infty}^{-\eta_C} \frac{e^{\frac{w+\eta_C}{\theta_C}}}{\theta_P + \theta_C} dw + \int_{-\eta_C}^{\infty} \frac{e^{-\frac{w+\eta_C}{\theta_P}}}{\theta_P + \theta_C} dw
$$

$$
= \left. \frac{e^{\frac{w+\eta_C}{\theta_P}} \theta_P}{\theta_P + \theta_C} \right|_{-\infty}^{-\eta_C} + \left. \frac{-e^{-\frac{w+\eta_C}{\theta_P}} \theta_C}{\theta_P + \theta_C} \right|_{-\eta_C}^{\infty}
$$

$$
= \frac{\theta_P}{\theta_P + \theta_C} + \frac{\theta_C}{\theta_P + \theta_C} = 1
$$

Therefore, warranty probability for STB is given as in Equations 6.16:

$$
P_w = F_w(w) = \begin{cases} \displaystyle\int_{y^*}^{-\eta_C} \frac{e^{\frac{w+\eta_C}{\theta_C}}}{\theta_P + \theta_C} dw + \int_{-\eta_C}^{\infty} \frac{e^{-\frac{w+\eta_C}{\theta_P}}}{\theta_P + \theta_C} dw \cdots \text{for}; y^* < -\eta_C \\ \displaystyle\int_{y^*}^{\infty} \frac{e^{\frac{w+\eta_C}{\theta_C}}}{\theta_P + \theta_C} dw \cdots \text{for}; -\eta_C < y^* \end{cases} \tag{6.16}
$$

Warranty probability for NTB can be given as in Equation 6.17:

$$P_w = F_w\left(w\right) = \int_{-\infty}^{y_1^*} \frac{e^{\frac{w+\eta_C}{\theta_C}}}{\theta_P + \theta_C} dw + \int_{y_2^*}^{\infty} \frac{e^{-\frac{w+\eta_C}{\theta_P}}}{\theta_P + \theta_C} dw \cdots \text{for; } y^* < -\eta_C < y^* \tag{6.17}$$

And, warranty probability for LTB is given as shown in Equations 6.18:

$$P_w = F_w\left(w\right) = \begin{cases} \displaystyle\int_{-\infty}^{y^*} \frac{e^{\frac{w+\eta_C}{\theta_C}}}{\theta_P + \theta_C} dw \cdots \text{for; } y^* < -\eta_C \\ \displaystyle\int_{-\infty}^{-\eta_C} \frac{e^{\frac{w+\eta_C}{\theta_C}}}{\theta_P + \theta_C} dw + \int_{-\eta_C}^{y^*} \frac{e^{-\frac{w+\eta_C}{\theta_P}}}{\theta_P + \theta_C} dw \cdots \text{for; } -\eta_C < y^* \end{cases} \tag{6.18}$$

Using equations 6.16, 6.17, and 6.18 warranty cost can conveniently be computed with the similar assumptions and method as discussed in Section 4. However, no case study has been conducted for one parameter exponentially distributed PP and two parameter exponentially distributed CE combination.

6.2.1.4. PP one parameter exponentially distributed and CE one parameter exponentially distributed. P and C are distributed as in Equations 6.19 and 6.20, respectively,

$$P \sim EXP\left(\theta_p\right) = \frac{1}{\theta_p} e^{-\frac{p}{\theta_p}}, \quad \theta_p > 0; p > 0 \tag{6.19}$$

$$C \sim EXP\left(\theta_C\right) = \frac{1}{\theta_C} e^{-\frac{c}{\theta_C}}, \quad \theta_C > 0; c > 0 \tag{6.20}$$

By putting $\eta_P = 0$ and $\eta_C = 0$ in the entire derivation given in Section 6.2.1.1 one can obtain as follows. The transformation parameters are:

$$w = p - c \ldots p = \frac{w+z}{2} > 0$$
$$z = p + c \ldots c = \frac{z-w}{2} > 0$$

And the new limits are

$$0 < p; \cdots 0 < c$$
$$-z < w < z, z > 0$$

The Jacobian of the transformation is

$$J = \begin{vmatrix} \frac{\partial p}{\partial w} & \frac{\partial p}{\partial z} \\ \frac{\partial c}{\partial w} & \frac{\partial c}{\partial z} \end{vmatrix} = \begin{vmatrix} +\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

Therefore, the pdf of $w = P - C$ is as given in Equation 6.21

$$f_w(w) = \begin{cases} \frac{1}{\theta_P + \theta_C} e^{\frac{w}{\theta_C}} \cdots \text{for}; w < 0 \\ \frac{1}{\theta_P + \theta_C} e^{-\frac{w}{\theta_P}} \cdots \text{for}; 0 < w \end{cases} \tag{6.21}$$

Recognizing the probability density function for $Y^* = W = P - C$ one can say that $Y^* = P - C$ is non-symmetric double exponentially distributed. Proof that it is a pdf:

$$F_w(w) = \int_{-\infty}^{0} \frac{1}{\theta_P + \theta_C} e^{\frac{w}{\theta_C}} dw + \int_{0}^{\infty} \frac{1}{\theta_P + \theta_C} e^{-\frac{w}{\theta_P}} dw$$

$$= \frac{\theta_C}{\theta_P + \theta_C} e^{\frac{w}{\theta_C}} \Bigg|_{-\infty}^{0} - \frac{\theta_P}{\theta_P + \theta_C} e^{-\frac{w}{\theta_P}} \Bigg|_{0}^{\infty}$$

$$= \frac{\theta_C}{\theta_P + \theta_C} + \frac{\theta_P}{\theta_P + \theta_C} = 1$$

Therefore, warranty probability for STB can be given as in Equations 6.22:

$$P_W = F_w(w) = \begin{cases} \int_{y^*}^{0} \frac{1}{\theta_P + \theta_C} e^{\frac{w}{\theta_C}} dw + \int_{0}^{\infty} \frac{1}{\theta_P + \theta_C} e^{-\frac{w}{\theta_C}} dw \cdots \text{for}; y^* < 0 \\ \int_{y^*}^{\infty} \frac{1}{\theta_P + \theta_C} e^{-\frac{w}{\theta_C}} dw \cdots \text{for}; 0 < y^* \end{cases} \tag{6.22}$$

Warranty probability for NTB can be given as in Equation 6.23:

$$P_W = F_w\left(w\right) = \int_{-\infty}^{y_1^*} \frac{1}{\theta_P + \theta_C} e^{\frac{w}{\theta_C}} dw + \int_{y_2^*}^{\infty} \frac{1}{\theta_P + \theta_C} e^{-\frac{w}{\theta_C}} dw \cdots \text{for}; y_1^* < 0 < y_2^* \tag{6.23}$$

And warranty probability for LTB is given as in Equations 6.24:

$$P_W = F_w\left(w\right) = \begin{cases} \displaystyle\int_{-\infty}^{y^*} \frac{1}{\theta_P + \theta_C} e^{\frac{w}{\theta_C}} dw \cdots \text{for}; y^* < 0 \\ \displaystyle\int_{-\infty}^{0} \frac{1}{\theta_P + \theta_C} e^{\frac{w}{\theta_C}} dw + \int_{0}^{y^*} \frac{1}{\theta_P + \theta_C} e^{-\frac{w}{\theta_C}} dw \cdots \text{for}; 0 < y^* \end{cases} \tag{6.24}$$

Using the equations 6.22, 6.23, and 6.24 warranty cost can conveniently be computed with the similar assumptions and method as discussed in Section 4. However, no case study has been conducted for one parameter exponentially distributed PP and one parameter exponentially distributed CE combination.

6.2.2. Distribution Combinations that Cannot Give a Closed-Form Solution. The following example is considered to demonstrate the methodology when distribution combinations cannot give a closed-form solution. In this case, it is assumed that product performance follows a Weibull distribution and customer expectation follows a normal distribution.

6.2.2.1. Product performance follows a Weibull distribution and customer expectation follows a normal distribution. The warranty loss function proposed in Section 4 considers product performance and customer expectation both distributed normally. The formulation presented serves as a basic model for predicting warranty loss that is dependent on both product performance and customer expectation. Using the same methodology but different distributions for product performance and customer expectation, warranty cost can be predicted. In order to predict the actual warranty cost, it is imperative that reliability parameters be considered as product performance. In this research, product performance is assumed to follow a Weibull distribution and customer expectation assumes a normal distribution with appropriate mean and variance. Failure data of shock absorbers have been considered and distribution of product performance and parameters of the distribution are thus determined. Using the proposed methodology, the first step will be to predict the probability of customer complaint on the basis of two independent variables: product performance and customer expectation. The second step will be to predict the warranty cost under certain assumptions such as the cost of repairs, and the complaint factor.

Product performance is measured in terms of characteristics that satisfy the customer. Such important parameters could include time-to-failure, distance-to-failure, the number of operations or cycles before failure, mean-time-to-failure, and time-to-first-failure. In this section an example is taken from O'Connor (2004), which considers distance to failure of a vehicle shock absorber. For the shock absorber, the distance to failure is found to follow a Weibull distribution. A little consideration shows that the distance to failure is a larger the better (LTB) characteristic. The data are considered to be Weibull distributed. Table 6.1 is taken from the book mentioned above showing the failure data.

6.2.2.2. Case study 7: Distance-to-failure of a shock absorber. These data have been taken from O'Connor (2004). The table below gives the distance-to-failure on a vehicle shock absorber, as taken from fleet records. F1 is the failure mode considered. Note that compared with the method of dealing with censored data, the calculation of plotting positions is much easier (O'Connor, 2004).

In this example, for the failure mode F1, $\widehat{\beta}$ = 2.6, $\widehat{\eta}$ = 29000 km (the life equivalent to 100 percent cumulative hazard). (This is the 63.2 percent c.d.f. on the probability scale.)

Table 6.1. Distance-to-Failure on a Vehicle Shock Absorber

No.	**Distance (km)**	**F1 Δ Hazard percent**	**F1 Cumulative hazard percent**
1	6700 (F1)	2.63 (1/38)	2.63
2	6950		
3	7820		
4	8790		
5	9120 (F1)	2.94 (1/34)	5.57
6	9660		
7	9820		
8	11310		
9	11690		
10	11850		
11	11880		
12	12140		
13	12200 (F1)	3.85	9.42
14	12870		
15	13150 (F1)	4.17	13.59
16	13330		
17	13470		
18	14040		
19	14300 (F1)	5.00	18.59
20	17520 (F1)	5.26	23.85
21	17540		
22	17890		
23	18450		
24	18960		
25	18980		
26	19410		
27	20100 (F1)	8.33	32.18
28	20100		
29	20150		
30	20320		
31	20900 (F1)	12.50	44.68
32	22700 (F1)	14.29	58.96
33	23490		
34	26510 (F1)	20.0	78.96
35	27410		
36	27490 (F1)	33.3	112.29
37	27890		
38	28100		

The warranty cost occurs at the downstream end (i.e., the market-customer interface), and the opportunity to reduce warranty claims lies upstream (i.e., design and manufacturing) of the product life cycle. In this study, customer expectation assumes a normal distribution because of its widespread use and simplicity.

The term *potential number of complaints* denoted by N is the total number of complaints where the product performance is not in excess of customer expectation by at least y^* assuming that all the customers make a complaint. The potential number of complaints is as such the total number of complaints that are likely to be made because the gap between product performance and customer expectation is more than the predetermined values y^*.

Once again it is assumed that a fairly constant ratio exists between the potential number of complaints and the total number of products. This ratio is denoted as P_W. Thus,

$$P_W = \frac{N}{T_P} \tag{6.25}$$

Or,

$$N = P_W T_P \tag{6.26}$$

The term P_W can be termed the *warranty probability*. Therefore, warranty probability is defined as the ratio between the potential number of complaints, N, and the total number of products, T_P. From warranty probability, the potential number of complaints, N, can be computed by multiplying the warranty probability with the amount of production or number of products. A little consideration will show that warranty probability, P_W, depends on the product performance falling short of the customer expectation.

The actual number of complaints is the number of complaints made by customers where the product performance is smaller than the customer expectation by any value more than y^*. It is assumed that no complaint will occur when the product performance is smaller than the customer expectation by a certain value y^* or more. The actual number of complaints is defined as the total number of complaints that are actually lodged because product performance falls short of the customer expectation. This is denoted by the symbol n.

A factor called the *complaint factor* denoted by ω is introduced here. The complaint factor is the ratio of the actual number of complaints, n, to the potential number of complaints, N, because product performance falls short of the customer expectation. Thus, Equation (6.27) depicts this relationship and Equation (6.28) can be used to compute the actual number of complaints.

$$\omega = \frac{n}{N} \tag{6.27}$$

$$n = \omega . N \tag{6.28}$$

The distance-to-failure is a larger-the-better characteristic. In this case study, product performance, i.e., distance-to-failure (PP), is assumed to be Weibull distributed, while customer expectation, i.e., distance-to-failure (CE), is also assumed to be normally distributed, as shown in Figure 6.1. Figure 6.2 and Appendix D show how Product Performance and Customer Expectation are distributed. Customer expectation, C, is distributed as

$$C \sim f(t) \sim NOR\left(\mu_c, \sigma_c^2\right) \tag{6.29}$$

Product performance, P, is distributed as

$$P \sim WEI(\eta, \beta) \tag{6.30}$$

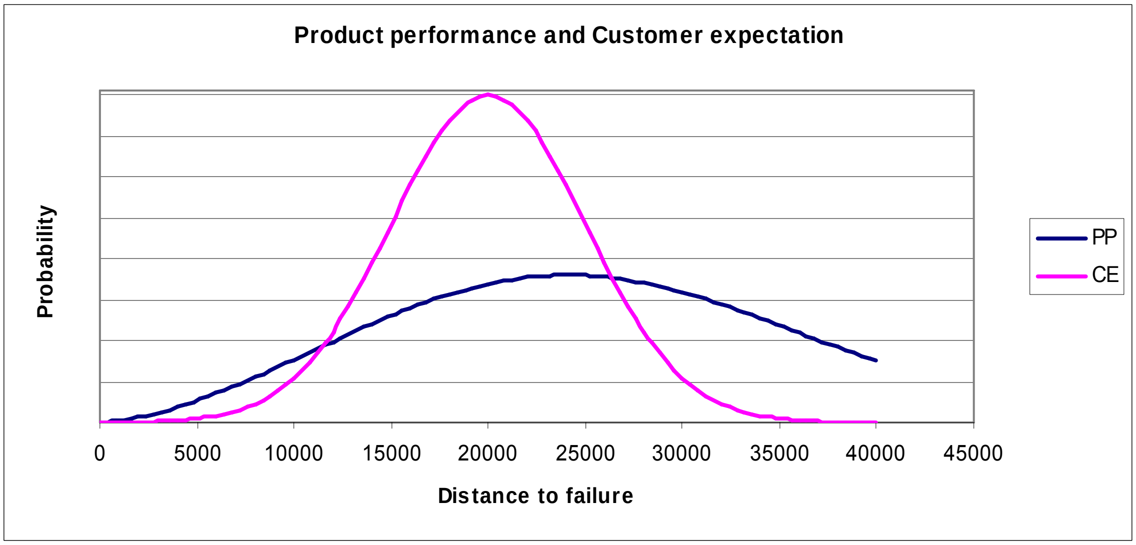


Figure 6.1. Product Performance and Customer Expectation

Now the distribution of $Y^* = P - C$ is of interest, which cannot be found in closed form. Therefore, it should be found numerically using the given distributions for PP and CE.

Suppose the resulting distribution is

$$Y^* = P - C = \xi(y) \tag{6.31}$$

It is assumed that the customer expectation, C, and product performance, P, are independent of each other. Also, suppose Y^* is a parameter that is some measure of customer satisfaction such that

$$Y^* = P - C$$

And, suppose y^* is a minimum value that ensures customer satisfaction, so when $Y^* \geq y^*$ the customer is satisfied for LTB, and when $Y^* < y^*$ the customer is dissatisfied and will make a complaint. Figure 6.2 shows the distribution of product performance minus customer expectation. Because the distance-to-failure is an LTB, it can be verified from the theory of probability that the probability of the customer complaint is given as

$$P_W = P\left(Y^* < y^*\right)$$

$$P\left(Y^* < y^*\right) = \int_{-\infty}^{y^*} \xi(t)dt \tag{6.32}$$

The function given above can be estimated from the data.

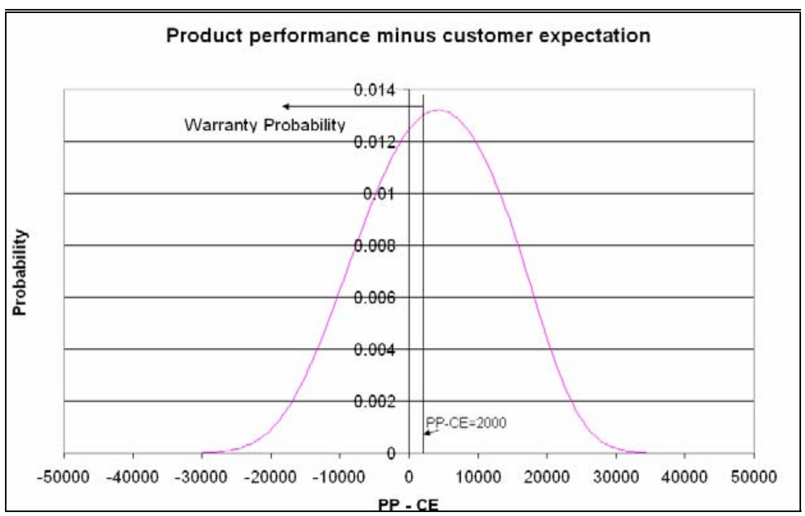


Figure 6.2. Distribution of Product Performance minus Customer Expectation

Now,

$$n = \omega T_P P_W \tag{6.33}$$

It is assumed that the cost of corrective action or repairs, A, is a function of the distance of the performance from $y*$ and the function is quadratic:

$$A = f\left(y^* - Y^*\right)$$

$$A = k\left(y^* - Y^*\right)^2 \tag{6.34}$$

Where k is a constant

$$WC = nA = k\left(y^* - Y^*\right)^2 \omega T_P P_W$$

$$WC = \int_{-\infty}^{y^*} k\omega T_P \left(y^* - t\right)^2 \xi\left(t\right) dt \tag{6.35}$$

Often, it is more useful and easy to understand if the warranty cost is given per unit of product. For that, $T_P = 1$ can be used to compute the warranty cost per unit product. So,

$$WC_{unit} = \int_{-\infty}^{y^*} k\omega\left(y^* - t\right)^2 \xi\left(t\right) dt$$

$$WC_{unit} = k\omega\int_{-\infty}^{y^*} \left(y^* - t\right)^2 \xi\left(t\right) dt$$

$$WC_{unit} = \omega\int_{-\infty}^{y^*} \frac{A_0\left(y^* - t\right)^2}{\Delta_0^2} \xi\left(t\right) dt \tag{6.36}$$

Suppose that the cost of repair is constant and is estimated to be \$100 per complaint. Also suppose that the complaint factor $\omega = 0.2$ and $y^* = 2000$. Then the above equation reduces to

$$WC_{unit} = \omega A_0 \int_{-\infty}^{y^*} \xi\left(t\right) dt \tag{6.37}$$

From the data, reading number 106, Appendix D, it is estimated that

$$\int_{-\infty}^{2000} \xi\left(t\right) dt = 0.3987$$

$$WC_{unit} = 0.2 \times 100 \times 0.3987 = \$7.974$$

Therefore, the estimated warranty cost per unit is \$7.97, under the given assumptions.

6.3. CONCLUSION

This section considers the possibility of inclusion of other distributions for product performance and customer expectation in the formulation of warranty probability and warranty cost. Section 6.2.1 derives the warranty probability for STB, NTB, and STB characteristics for distribution combinations that can give a closed-form solution. The second possibility is also explored in which distribution combinations that cannot give a closed-form solution are researched. It is necessary to find out what distributions are closest to the realistic parameter (i.e., product performance and customer expectation) distributions. In this case, one only needs to evaluate the problem numerically as explained in Section 6.2.2.

Section 6.2.2 considers the possibility of customer expectation following a normal distribution and product performance following a Weibull distribution as an example. In this method one is still able to find out how the difference between the two would be distributed. In such cases, when the distributions are not similar, it is difficult to find the distribution of another parameter that includes any two dissimilar distributions. Therefore, the numerical method has been used to find out the distribution of the difference between PP and CE. On the basis of the distribution of difference between PP and CE, the probability of customer complaints has been estimated. Considering this estimated probability of customer complaints and the cost of repair and complaint factor, the warranty cost per unit has been computed. In this way, a reliability parameter that is distance-to-failure has been used to predict the warranty cost of a shock absorber.

7. SIMULTANEOUS OPTIMIZATION OF DYNAMIC MULTI-RESPONSE SYSTEMS USING THE PRODUCT OF NORMALIZED SQUARED-BIAS AND VARIANCE

7.1. INTRODUCTION

The quality loss function (QLF) given by Taguchi takes into account only one characteristic. Most of the components and consequently the products have more than one quality characteristic to consider simultaneously for assessing the quality of the component or the product. However, the QLF does not take into account multiple characteristics simultaneously. Because more than one characteristic is to be considered, it is also imperative to consider different types of characteristics simultaneously for the purpose of determining quality loss. This is because a product need not necessarily have only one type of characteristic, e.g., smaller-the-better (STB). Therefore, the proposed model developed incorporates all three types of characteristics. The methodology shown gives one number for quality loss even if a number of characteristics are taken into account.

Some authors have given a natural extension of a single characteristic quality loss function for multi-characteristic cases. However, the larger-the-better characteristics need to be converted to smaller-the better type by using reciprocal transformation because the target is assumed to be infinity. Second, the multi-response quality loss function (MQLF) thus given does not satisfy the two boundary conditions discussed in Section 2.4.1 (Figure 2.12). Therefore, a better methodology that does not use reciprocal transformation has been proposed in this section. The methodology proposed first takes into account the target set for a larger-the-better characteristic.

When more than one characteristic is considered, one of the questions that needs to be answered is how these characteristics are related to each other. Characteristics can be related to each other in two different ways, with the first occurrence associated with manufacturing. That would include cases of any two characteristics occurring at the higher-higher, lower-lower, or higher-lower level. The second way occurs when the cost or loss is associated with any two characteristics simultaneously. That would include situations such as additional loss when any two characteristics are at the higher-higher, lower-lower, or higher-lower level.

Multi-characteristic quality loss function takes into account the quality loss caused because of each characteristic performing away from the target and having dispersion. A natural extension of the single characteristic quality loss function to multi-characteristic loss function has been given by Pignatiello (1993) as

$$\mathrm{loss}\left(\mathrm{y}(\mathrm{x})\right)=\left(\mathrm{y}(\mathrm{x})-\tau\right)'\mathrm{C}\left(\mathrm{y}(\mathrm{x})-\tau\right) \tag{7.1}$$

Where, $\mathbf{C}$ is a positive definite matrix representing the associated cost of off-target performance $\mathbf{y}$. $\mathbf{y}(\mathbf{x})$ represents the performance of $\mathbf{y}$ at $\mathbf{x}=(x_1,x_2,\ldots,x_k)$, a vector of k controllable factors. Furthermore, the expected loss $R(\mathbf{x})$ is given as

$$R(\mathbf{x})=trace\left(\mathbf{C}\Sigma(\mathbf{x})\right)+\left(\eta(\mathbf{x})-\tau\right)'\mathbf{C}\left(\eta(\mathbf{x})-\tau\right) \tag{7.2}$$

Where $\Sigma(\mathbf{x})$ is a variance-covariance matrix of $\mathbf{y}$ at the control factor levels of $\mathbf{x}$, and $\eta(\mathbf{x})$ is the expected value of $\mathbf{y}$ at the control factor levels of $\mathbf{x}$.

Ames et al. (1997) discusses the global quality loss function (GQL), which describes the loss to society resulting from both random variation and systematic error. GQL uses quadratic approximation for the loss function and is given as

$$GQL = \sum_{r=1}^{R_g} W_r \left\{ V_r - T_r \right\}^2 \tag{7.3}$$

Where W_r are weight factors that scale the relative importance of R different criteria and the sum is taken over R_g responses that contribute to quality. V_r are measured responses and are the target value for the respective V_r.

Yang (2007) has also made a reference to multi-characteristic loss function. For nominal-the-best (NTB) characteristics, the loss function is given as

$$L = \left(\mathrm{Y} - \mathbf{m_y}\right)^{\mathrm{T}} \mathbf{K} \left(\mathrm{Y} - \mathbf{m_y}\right) \tag{7.4}$$

Where $\mathbf{Y} = (y_1, y_2, \ldots, y_m)$ is the response vector and $\mathbf{m_y} = (m_{y_1}, m_{y_2}, \ldots, m_{y_m})$ is the target vector, and $\mathbf{K}$ is a $m \times m$ matrix of constant elements carrying information of repair costs. Tsui (1999) suggested that Equation (7.4) can be used for STB as well as larger-the-better (LTB) responses. In this approach, an LTB response needs to be converted to STB responses by taking its reciprocal and the target as zero.

$$E\left(\mathbf{L}\right) = \left(\mu - \mathbf{m_y}\right)^{\mathrm{T}} \mathbf{K} \left(\mu - \mathbf{m_y}\right) + trace(\mathbf{K\Sigma}) \tag{7.5}$$

Equation (7.5) essentially gives the same value as independent loss because each characteristic is calculated and added together. It does not take into account the covariance among the responses because only the trace is calculated instead of the actual determinant of $\mathbf{K\Sigma}$. Diagonal elements of $\mathbf{\Sigma}$ are the same as variances, so therefore it can be concluded that covariance does not matter. In other words, "The trace of multiplication of variance-covariance and K matrices" approach is unsuitable because it does not take into account the directionality of characteristics.

Very few techniques are available for tackling dynamic parameter design problems. Chang (2006) proposed a technique that uses a different desirability function for a different type of characteristic, e.g., STB, NTB, and LTB. The desirability function uses exponential transformation. This approach requires a software package of BPN. In comparison, the method proposed in this section of the dissertation does not need any special software package, but rather a commonly used Excel spreadsheet can do the work.

The use of desirability functions for simultaneous optimization of multi-response experiments has been proposed by Derringer (1980). The functions are based on one-sided (e.g., STB and LTB) and two-sided (e.g., NTB) transformations. After multiplying the individual desirability values for all the responses, the geometric mean is computed to arrive at the overall desirability. This methodology is widely used. In Derringer (1980), a static example is discussed that has four responses—two LTB and two NTB. However, it is unclear how this methodology of desirability function can be used to optimize dynamic multi-response experiments.

Because of the reasons explained in Section 2, the target for LTB should be set at a certain α value called the target-mean ratio. In other words, it is unnecessary to convert LTB with the target infinity into STB with the target zero by taking the reciprocal of the LTB response. As such, the following methodology has been developed for multi-response systems to compute the quality loss.

An approach based on back-propagation neural networks and desirability functions to optimize a dynamic multi-response example has been discussed in Chang (2006). Desirability functions have been used to integrate three dynamic responses into a single index. The author obtained the data related with a dynamic system by simulation using the Monte Carlo simulation and the procedure of Park (2003). The data used in this paper have been used to demonstrate the methodology proposed for optimizing multi-response systems. The next sections discuss the signal-to-noise ratio methodology and propose a new methodology.

7.2. SIGNAL-TO-NOISE RATIO METHODOLOGY

The best-known criterion to measure robust design is the signal-to-noise (SN) ratio. This is why the signal-to-noise ratio is a metric widely used for achieving robustness. Maximum robustness means minimum quality loss and maximum customer satisfaction. The SN ratio recognizes and measures deviation from the nominal value and integrates the information into one metric (Taguchi, 1999).

Several performance characteristics exist and it is important to distinguish between these when evaluating quality. Therefore, a different SN ratio is needed for each performance characteristic. The nominal value is the best performance characteristic value for many parameters. Nominal-the-best (NTB) should be used whenever possible because this allows two-step optimization. The SN ratio measures deviation from the nominal value allowing for subsequent adjustment.

A large SN ratio means a lower standard deviation. In the case of dynamic signals, e.g., steering wheel or brake pedal application, a series of dynamic SN ratios exists. A signal factor is a control factor chosen that can modify the output response in a linearly proportional way. For example, for a steering wheel, the turn angle of the steering wheel is the signal factor that adjusts the radius of curvature for vehicle motion as the output response. Similarly for a brake system, the brake pedal pressure is the signal factor that regulates the braking distance as the output response. Thus, an equation that measures the robustness of a system can be obtained. The objective for achieving a robust design is to have the highest SN ratio (i.e., the smallest standard deviation or variation). A linear relationship between the output or response and input or signal is the most desirable relationship for dynamic systems (Fowlkes, 1995; Phadke, 1989).

The idea is to deliver a performance near the target (customer preference) that will maximize the customer satisfaction value, thus overriding the specification limits. Depending on the quality characteristics, this satisfaction level can be of three types: LTB, STB, or NTB. The SN ratio functions as a single measure of robustness, and the gain in the SN ratio reflects the improvement (Taguchi, 2004). For it to be more useful and significant, this ratio needs to be related to lower cost, reduced time to market, and better quality. A gain in the SN ratio can also be related to a reduction in the warranty cost (Taguchi, 2004).

Products with smaller variation have smaller quality loss. The quality loss function essentially translates the qualitative terms, which affect the consumer, into quantitative terms such as monetary values. Depending on the situation, the signal-to-noise ratio has three forms (Fowlkes, 1995; Phadke, 1989):

1. Smaller-the-Better (STB) – a smaller value is better and higher values are undesirable, such as vehicle emissions or fuel consumption (dollar per distance). The SN ratio for STB is given as follows:

$$S/N_{STB} = -10\log\left(MSD\right) = -10\log\left(\frac{1}{n}\sum_{i=1}^{n} y_i^2\right) = -10\log\left(S^2 + \overline{y}^2\right) \tag{7.6}$$

Because, for STB:

$$MSD = S^2 + \overline{y}^2 \tag{7.7}$$

Thus, it is obvious that MSD is broken into two components: MSD-bias $\overline{y}^2$ and MSD-variation S^2. STB methodology basically supports one-step optimization. The other option, which involves reducing the variance first and then reducing the bias, will be called two-step optimization.

2. Nominal-the-best (NTB)—the nominal value is best because it is the one that satisfies the customer's need. The characteristic value away on either side of the target is undesirable, such as air pressure in vehicle tires or the location of gauges on the instrument panel. When the mean and standard deviation scale together, a type I - SN ratio for NTB is used, which is given as follows (Fowlkes, 1995):

$$S/N_{Type\,I-NTB} = 10\log\left(\frac{\overline{y}^2}{S^2}\right) \tag{7.8}$$

For many quality characteristics, the mean and standard deviation are independent of each other or they do not scale together. When the mean and standard deviation are independent of each other, the type II - SN ratio for NTB is used, which is given as follows:

$$S/N_{Type\,I-NTB} = 10\log\left(S^2\right) \tag{7.9}$$

In this type of SN ratio, only MSD due to variation has been used. For a multivariate case, it is assumed that the mean and standard deviation are independent, but bias also needs to be included in the metric being developed. Therefore, it is proposed to use following relationship:

$$MSD = S^2 + \left(\overline{y} - m\right)^2 \tag{7.10}$$

3. Larger-the-better (LTB)—a larger value is better and smaller values are undesirable, such as gas mileage (distance per gallon). The SN ratio for STB is given as follows:

$$S/N_{LTB} = -10\log\left(MSD\right) = -10\log\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{y_i^2}\right) \tag{7.11}$$

It has already been discussed and recommended in sections 1 and 2 that a finite target be assumed in the case of LTB quality loss. Therefore, it is proposed to utilize the following relationship with usual notations:

$$MSD = S^2 + \overline{y}^2 \left(1-\alpha\right)^2 \tag{7.12}$$

It can be observed that the MSD for all three types of characteristics can be expressed in two component—one due to bias and the other due to dispersion or variation. The next sub-section discusses the new methodology in detail.

7.3. NORMALIZED SQUARED-BIAS AND VARIANCE PRODUCT FOR MULTI-RESPONSE SYSTEMS

The present methodology proposes the use of multiplication of bias and variances in place of addition so that it is easy to see the effect of change of any component (bias or variation) of any characteristic (of any type) in the final metric. However, it is imperative to normalize each component of each characteristic between 0 and 1 within itself before multiplication. Further weights need to be given to each component of each characteristic between 0 and 1. Thus, it is reasonable to guess that a normalized squared-bias and variance product are being introduced in place of a signal-to-noise ratio. The following section describes the procedure for normalized squared-bias and variance product methodology.

First, an experiment may be designed with noise factors (outer array) with respect to each response. Control factors may be used in the inner array. Because the methodology can be used for dynamic experiments as well, the experiment may also have a signal factor. The next step is to run the experiments and collect data.

It is assumed that α is the target-mean ratio for LTB characteristics; i = run number = 1, 2,…, r; j = response number = 1, 2,…, n; and k = signal factor position or level number = 1, 2,…, p. If $\overline{y}_{ijk}$ is the mean of y_{ijk} for all noise factor combinations and s_{ijk} is the standard deviation of y_{ijk} for all noise factor combinations, then the computation for a normalized squared bias, b_{ijk}, for the respective run number, response number, and signal factor level number is shown as follows:

$$b_{ijk} = \frac{\left(\overline{y}_{ijk} - m_{jk}\right)^2}{\max_{i=1}^{r}\left(\overline{y}_{ijk} - m_{jk}\right)^2} \quad \text{for any type of response} \tag{7.13}$$

This relation can be simplified for each type of response in the following manner:

$$b_{ijk} = \frac{\overline{y}_{ijk}^2}{\max_{i=1}^{r}\left(\overline{y}_{ijk}^2\right)} \quad \text{for STB responses} \tag{7.14}$$

$$b_{ijk} = \frac{\left(\overline{y}_{ijk} - m_{jk}\right)^2}{\max_{i=1}^{r}\left(\overline{y}_{ijk} - m_{jk}\right)^2} \quad \text{for NTB responses} \tag{7.15}$$

$$b_{ijk} = \frac{\overline{y}_{ijk}\left(1-\alpha\right)^2}{\max_{i=1}^{r}\left\{\overline{y}_{ijk}\left(1-\alpha\right)^2\right\}} \quad \text{for LTB responses} \tag{7.16}$$

Compute normalized variance, v_{ijk}, for the respective run number, response number, and signal factor level number as follows:

$$v_{ijk} = \frac{s_{ijk}^2}{\max_{i=1}^{r}\left(s_{ijk}^2\right)} \quad \text{for STB, NTB, and LTB responses} \tag{7.17}$$

For the respective run number, response number, and signal factor level number, the product of normalized weighted squared bias and normalized weighted variance at each signal factor level is computed as follows:

$$WBV_{ijk} = WBV_{iSTBk} = WBV_{iNTBk} = WBV_{iLTBk} = b_{ijk}^{1-u_j} v_{ijk}^{1-w_j} \tag{7.18}$$

Weights u_j correspond to bias, and weights w_j correspond to the variance of each response, *j,* and are constants for all *i* and *k*. Give appropriate weights between 0 and 1 to each normalized MSD bias (i.e., b_{ij}) and each normalized MSD variance (i.e., v_{ij}). Because there are n responses, $2n$ weights will have to be given. Weights u_j and w_j must satisfy the following constraints:

$$0 \le u_j, w_j \le 1 \tag{7.19}$$

And

$$\left(u_1 + w_1\right) + \left(u_2 + w_2\right) + \cdots + \left(u_n + w_n\right) = \sum_{j=1}^{n}\left(u_j + w_j\right) = \frac{n}{2} \tag{7.20}$$

$$u_j + w_j \le 1 \tag{7.21}$$

For a given run number and response number, all signal factor levels must be combined as follows:

$$WBV_{ij} = b_{ij1}^{1-u_j} v_{ij1}^{1-w_j} b_{ij2}^{1-u_j} v_{ij2}^{1-w_j} \ldots b_{ijp}^{1-u_j} v_{ijp}^{1-w_j} = \prod_{k=1}^{p} b_{ijk}^{1-u_j} v_{ijk}^{1-w_j} \tag{7.22}$$

Compute normalized-weighted-multivariate-bias-variance $\left(WMBV\right)$ for each run by combining all the responses using the following general relationship:

$$WMBV_i = \prod_{k=1}^{p} b_{i1k}^{1-u_1} v_{i1k}^{1-w_1} \prod_{k=1}^{p} b_{i2k}^{1-u_2} v_{i2k}^{1-w_2} \ldots \prod_{k=1}^{p} b_{ink}^{1-u_n} v_{ink}^{1-w_n} = \prod_{j=1}^{n}\prod_{k=1}^{p} b_{ijk}^{1-u_j} v_{ijk}^{1-w_j} \tag{7.23}$$

Finally, the negative logarithm of $WMBV_i$ to the base 10 is taken for each experimental run for better scaling and positive directionality:

$$\text{Squared Bias-Variance Product} = -Log_{10}\left(WMBV_i\right) = -Log_{10}\left(\prod_{j=1}^{n}\prod_{k=1}^{p} b_{ijk}^{1-u_j} v_{ijk}^{1-w_j}\right) \tag{7.24}$$

This metric will be used for optimizing a dynamic multi-response experiment discussed in Section 7.4.

7.3.1. Weights Determination for Equal Weights. For equal weights among responses, the weights may be determined as follows:

$$u_j + w_j = \frac{n}{2n} = 0.5$$

Also, for equal weights among responses and between bias and variance, the following equation can be used:

$$u_j = w_j = \frac{n}{2 \times 2n} = 0.25$$

7.3.2. Weights Determination for Unequal Weights. The rank method can be used for unequal weights. Often it is difficult to give weight values even relative to each other, but it is easy to decide the relative ranking of the performance characteristics. In these cases, the first step is to determine the relative rank for all the characteristics, with the most important one being first. Therefore, the first stage is to decide the weights among the characteristics. Two examples are given in Table 7.1.

Table 7.1. Possible Weight Combinations among the Characteristics

Two responses	**Y_1**		**Y_2**	
Equal weights	0.5		0.5	
Intermediate unequal	0.3		0.7	
Extreme unequal	0.25		0.75	
Four responses	**Y_1**	**Y_2**	**Y_3**	**Y_4**
Equal weights	0.5	0.5	0.5	0.5
Intermediate unequal	0.3	0.4	0.6	0.7
Extreme unequal	0.25	0.3	0.7	0.75

The second stage is to decide the weights among the normalized squared bias and normalized variance of a given characteristic. Four examples are given in Table 7.2.

Table 7.2. Possible Weights among the Normalized Squared-Bias and Normalized Variance

Weight for a Characteristic	Weights for Normalized Squared Bias or Normalized Variance	
0.25	0.125	0.125
	0.100	0.150
	0.050	0.200
	0.000	0.250
0.40	0.200	0.200
	0.100	0.300
	0.050	0.350
	0.000	0.400
0.50	0.250	0.250
	0.200	0.300
	0.100	0.400
	0.000	0.500
0.75	0.375	0.375
	0.300	0.450
	0.200	0.550
	0.100	0.650
	0.000	0.750

7.4. CASE STUDY 8: OPTIMIZATION OF THE DYNAMIC MULTI-RESPONSE EXPERIMENT

This dynamic multi-response example has been taken from Chang (2006). Chang obtained the data related with a dynamic system by simulation using a Monte Carlo simulation and the procedure of Park (2003). Refer to Appendix E for more information. This problem has six control factors ($x_1, x_2, x_3, x_4, x_5, x_6$) all having three levels, a noise factor having two levels (N_1, and N_2), and a signal factor at three levels ($M_1 = 10$, $M_2 = 20$, $M_3 = 30$). It also has three responses: Y_1 (LTB), Y_2 (NTB), and Y_3 (STB). L_{18} has been used as an inner array, and the outer array has only one noise factor at two levels. Two replications have been used in the experimental setup. Computations have been carried out using the methodology proposed, and intermediate and final results with equal weights have been reproduced in Appendix E. With unequal weights the computations are done again and results have been reproduced in later parts of Appendix E. Both when equal weights are given and when unequal weights are given, regression analysis is conducted using the metric. The purpose of conducting regression analysis is to find out at what factor level combination the metric is maximized (3^6 = 729 runs of all possible combinations). Table 7.3 shows the metric values or factor effects when weights are equal for all three responses and their squared biases and variances. These results are computed by averaging the metric over the factor levels. Figure 7.1 shows the factor effects when the weights are equal.

Table 7.3. Metric Values or Factor Effects when Weights are Equal vs. Factor Levels

	Factors					
Levels	**x1**	**x2**	**x3**	**x4**	**x5**	**x6**
1	8.4396	7.522	8.3499	7.9495	7.438	6.0543
2	7.5866	8.8057	6.8852	7.8609	8.3478	9.5393
3	8.0558	7.7542	8.8468	8.2715	8.2962	8.4883

The following results (Metric versus x1, x2, x3, x4, x5, x6) were obtained from regression analysis when the weights were equal. The regression equation is as follows:

Metric = 4.07 - 0.192 x1 + 0.116 x2 + 0.248 x3 + 0.161 x4 + 0.429 x5 + 1.22 x6

Predictor	Coefficient	SE Coefficient	T	P
Constant	4.068	3.268	1.24	0.239
x1	-0.1919	0.6580	-0.29	0.776
x2	0.1161	0.6580	0.18	0.863
x3	0.2485	0.6580	0.38	0.713
x4	0.1610	0.6580	0.24	0.811
x5	0.4291	0.6580	0.65	0.528
x6	1.2170	0.6580	1.85	0.091

S = 2.27941   R-Sq = 27.5%   R-Sq(adj) = 0.0%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	6	21.637	3.606	0.69	0.660
Residual Error	11	57.153	5.196		
Total	17	78.790			

Source	DF	Seq SS
x1	1	0.442
x2	1	0.162
x3	1	0.741
x4	1	0.311
x5	1	2.209
x6	1	17.772

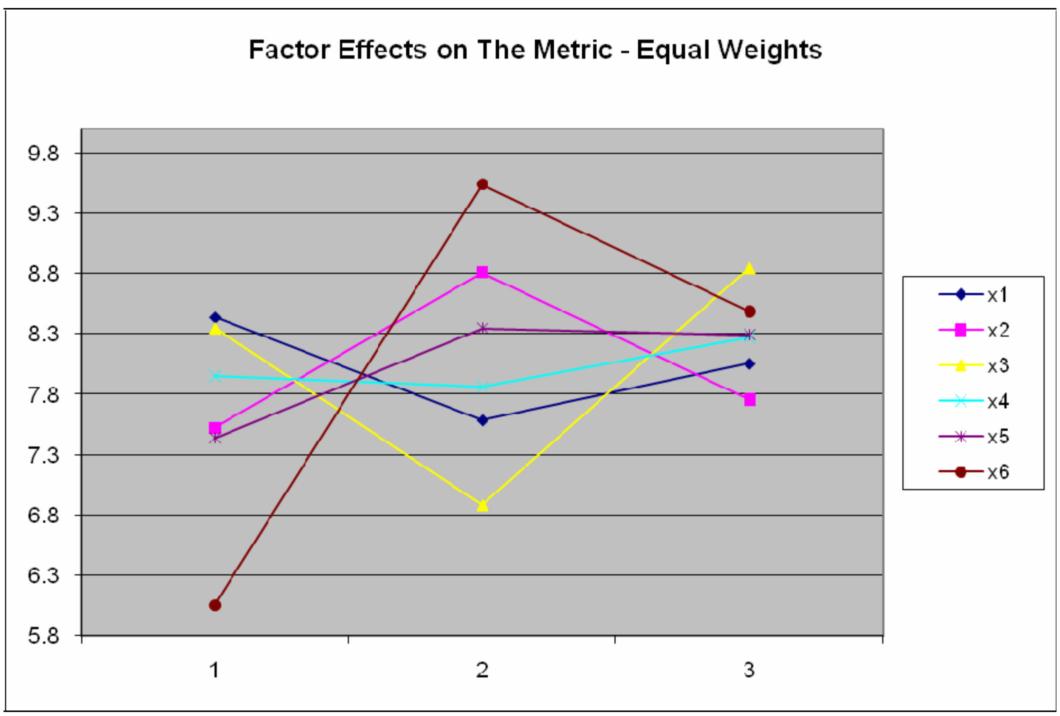


Figure 7.1. Factor Effects Plot when the Weights are Equal

The regression analysis results indicate that the R-square value is 27.5%. On the basis of the regression equation, the metric value is computed for the 729 possible runs to find out the best combination of factor levels for the maximum metric. Table 7.4 summarizes all three criteria of looking at the results.

Table 7.4. Factor Levels for the Maximized Metric when the Weights are Equal

	Factors					
Criterion	**x1**	**x2**	**x3**	**x4**	**x5**	**x6**
Optimum among runs	1	2	3	1	3	2
Optimum according to graph	1	2	3	3	2	2
Optimum according to prediction model	1	3	3	3	3	3

The first criterion is which of the 18 experimental runs will maximize the metric. The run combination is 1, 2, 3, 1, 3, 2. The second criterion is what combination of factor levels is predicted by the factor effects plot. It is 1, 2, 3, 3, 2, 2. The third criterion is what combination of factor levels is predicted by the regression equation. It is 1, 3, 3, 3, 3, 3. The results of the third criterion are dropped because of the low R-squared value of 27.5%. It can be said that the combination 1, 2, 3, 3, 2, 2 can give the best results on the basis of the second criterion, that is, the factor effects plot, because the first criterion limits the choice to only 18 possibilities. In this way a multi-response experiment can be optimized.

Table 7.5 shows the metric values or factor effects when the weights are unequal for the three responses and their squared biases and variances. These results are again computed by averaging the metric over the factor levels. Figure 7.2 shows the factor effects when the weights are unequal.

Table 7.5. Metric Values when the Weights are Unequal vs. Factor Levels

	Factors					
Levels	x1	x2	x3	x4	x5	x6
1	8.3291	7.6923	9.0743	8.4271	7.5567	6.3314
2	8.0387	9.0537	6.804	7.8531	8.5242	9.9324
3	8.3172	7.939	8.8066	8.4048	8.6041	8.4211

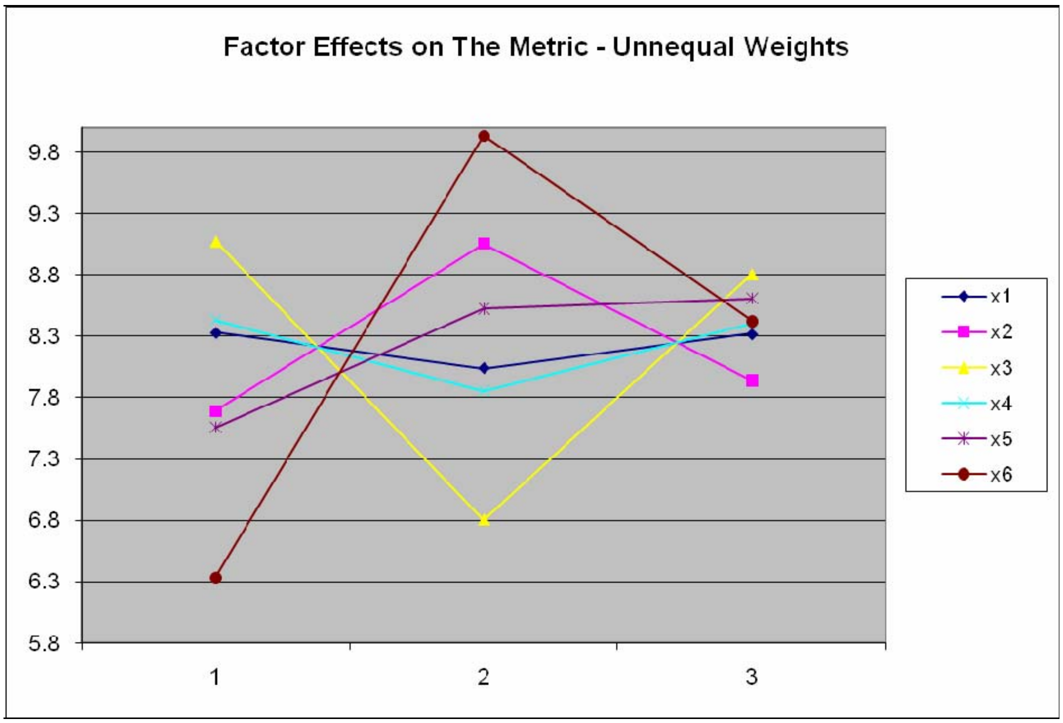


Figure 7.2. Factor Effects Plot when the Weights are Unequal

The following results (Metric versus x1, x2, x3, x4, x5, x6) were obtained from regression analysis when the weights were unequal. The regression equation is as follows:

Metric = 5.15 - 0.006 x1 + 0.123 x2 - 0.134 x3 - 0.011 x4 + 0.524 x5 + 1.04 x6

Predictor	Coefficient	SE Coefficient	T	P
Constant	5.146	3.731	1.38	0.195
x1	-0.0059	0.7512	-0.01	0.994

Predictor	Coefficient	SE Coefficient	T	P (Contd.)
x2	0.1233	0.7512	0.16	0.873
x3	-0.1339	0.7512	-0.18	0.862
x4	-0.0112	0.7512	-0.01	0.988
x5	0.5237	0.7512	0.70	0.500
x6	1.0448	0.7512	1.39	0.192

S = 2.60227   R-Sq = 18.4%   R-Sq(adj) = 0.0%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	6	16.791	2.798	0.41	0.855
Residual Error	11	74.490	6.772		
Total	17	91.280			

Source	DF	Seq SS
x1	1	0.000
x2	1	0.183
x3	1	0.215
x4	1	0.001
x5	1	3.291
x6	1	13.100

The regression analysis results indicate that the R-square value is 18.4%. On the basis of the regression equation, the metric value is computed for the 729 possible runs to find out the best combination of factor levels for the maximum metric. Table 7.6 summarizes all three criteria of looking at the results.

Table 7.6. Factor Levels for a Maximized Metric when the Weights are Unequal

	Factors					
Criterion	**x1**	**x2**	**x3**	**x4**	**x5**	**x6**
Optimum among runs	1	2	3	1	3	2
Optimum according to graph	1	2	1	1	3	2
Optimum according to prediction model	1	3	1	1	3	3

The first criterion is which of the 18 experimental runs maximizes the metric. The run combination is 1, 2, 3, 1, 3, 2, which is same as for equal weights. The second criterion is what combination of factor levels is predicted by the factor effects plot. It is 1, 2, 1, 1, 3, 2, which is different from the combination for equal weights because of different weights. The third criterion is what combination of factor levels is predicted by the regression equation. It is 1, 3, 1, 1, 3, 3, which is also different from the equal weight case because of changed weights. The results of the third criterion are dropped because of the low R-squared value of 18.4%. It can be said that the combination 1, 2, 3, 3, 2, 2 can give the best results on the basis of the second criterion, that is, the factor effects plot, because the first criterion limits the choice to only 18 possibilities. In this way a multi-response experiment can be optimized even when weights are unequal.

7.5. CONCLUSION

Many components and products have more than one quality characteristic to consider simultaneously for assessing the quality of the component or the product. In Taguchi's methodology, the larger-the-better characteristics need to be converted to smaller-the better type by using reciprocal transformation because the target is assumed to be infinity. The multi-characteristic optimization methodology proposed takes into account the performance away from the target and its variance, both of which are in turn components of quality loss.

As such, the following methodology has been developed for multi-response systems to optimize performance and minimize quality loss. The methodology uses the product of normalized squared-bias and variance, which are components of quality loss.

To demonstrate this methodology proposed, the data used in this section have been taken from Chang (2006) for optimizing dynamic multi-response systems. It is evident that a single metric can be computed and used for optimization of a multi-response experiment.

8. CLOSING REMARKS AND FUTURE RESEARCH

This dissertation starts with proposing quality loss function for larger-the-better characteristics and unifying the quality loss function for all three cases. Implications of this new methodology are then studied. After that, the dissertation develops warranty loss function in which customer expectation is also considered to be a variable. Towards the end of the dissertation, a metric has been proposed to optimize multi-response systems using product methodology. Taguchi's quality loss function for larger-the-better case, which is different from the other two cases, has been assimilated into the other two. A term called the *target-mean ratio* has been introduced, and a common formula for all three cases has been proposed. This not only brings about consistency and simplicity in the model, but also it brings the model closer to reality. Because of this unification, some non-larger-the-better characteristics also become larger-the-better type. An implied classification of LTB characteristics according to Taguchi on the basis of a target value at infinity and the classification of LTB characteristics based on the new methodology are discussed. A new concept of "Complementary Characteristic" is proposed. It is suggested that whether a given LTB characteristic or its complementary characteristic is considered for one and the same case, the quality loss must be equal for both characteristics. Although some of the implications of finite target for LTB have been studied, further research is also needed to examine, with examples, the implications of this methodology on the concept of a signal-to-noise ratio and optimization of systems using the signal-to-noise ratio. Also, further research may be conducted on how the new methodology affects the signal-to-noise ratio for operating window. Another possibility for research is to extend the quality loss function thus obtained for LTB characteristics with a finite target to multivariate cases wherein other types (e.g., STB and NTB) are also part of the quality loss function.

The quality loss function takes into account the immediate issues within manufacturing facilities whereas warranty cost occurs during customer use. Therefore, the researcher felt it was necessary to develop a methodology that can predict warranty probability, the probability of customer complaint, on the basis of two independent variables: product performance and customer expectation. The warranty cost models using the methodology have been developed for smaller-the-better, nominal-the-best, and larger-the-better cases. Both customer expectation and product performance were considered to be normally distributed in this methodology. Although the approach is novel, it may be argued that because almost all the characteristics follow a non-negative normal distribution, it is not the best choice for the formulation of the problem. The reliability in terms of mean-time-to-failure (MTTF), mean-time-between-failures (MTBF), etc., of a product is an important characteristic that can be used with the proposed methodology to predict the warranty cost. Therefore, in future research other distributions spreading from zero to infinity should be considered with examples in place of the normal distribution.

The signal-to-noise ratio is used in robust engineering to improve the robustness of a system. However, most processes have more than one quality characteristic or output response. Also, it is very difficult to minimize quality loss and maximize the signal-to-noise ratio at the same time. Therefore, the research also proposes a metric parallel to the signal-to-noise ratio that can be used for multi-response experiments for minimizing quality loss and improving robustness at the same time. The methodology proposed incorporates all three types of characteristics—smaller-the-better, nominal-the-best, and larger-the-better—and gives one number for the metric based on quality loss even if several characteristics are taken into account. The metric thus proposed is called the product of normalized squared-bias and variance and is equally useful for dynamic experiments. The methodology proposed has been demonstrated using a dynamic experiment, but further research may be conducted to use this methodology for static cases as well. Further research is also needed to find out how the weights can be determined for unequal weight case.

APPENDIX A

. SIMULATED RESULTS OF RECIPROCAL OF PERFORMANCE

Simulated Results of Reciprocal of Performance

Sl No.	y	w = 1/y	Blank Column	y	Frequency	1/y	Frequency	y	Frequency	1/y	Frequency	1/y	Frequency
1	104.2281	0.009594		53	1	77	0	95	353	119	236	161	5
2	87.80026	0.011389		54	0	78	1	96	366	120	230	162	1
3	104.6066	0.00956		55	1	79	0	97	314	121	200	163	4
4	102.2189	0.009783		56	0	80	0	98	295	122	212	164	4
5	92.38272	0.010825		57	0	81	1	99	264	123	178	165	1
6	102.3869	0.009767		58	5	82	2	100	226	124	169	166	0
7	95.84815	0.010433		59	6	83	5	101	215	125	164	167	2
8	94.68419	0.010561		60	3	84	13	102	208	126	140	168	1
9	78.68298	0.012709		61	8	85	9	103	152	127	119	169	1
10	85.20883	0.011736		62	6	86	19	104	152	128	116	170	3
11	77.75679	0.012861		63	9	87	17	105	148	129	121	171	2
12	84.64676	0.011814		64	13	88	24	106	124	130	95	172	1
13	93.11375	0.01074		65	16	89	44	107	92	131	89	173	3
14	95.83182	0.010435		66	30	90	41	108	81	132	89	174	0
15	90.51239	0.011048		67	27	91	47	109	66	133	80	175	0
16	86.74301	0.011528		68	32	92	81	110	39	134	58	176	0
17	106.8661	0.009358		69	49	93	112	111	37	135	54	177	0
18	92.7472	0.010782		70	41	94	121	112	35	136	57	178	0
19	90.28347	0.011076		71	73	95	161	113	23	137	39	179	0
20	90.35694	0.011067		72	80	96	162	114	18	138	38	180	1
21	105.9605	0.009437		73	74	97	161	115	12	139	45	181	0
22	77.63778	0.01288		74	106	98	214	116	15	140	40	182	0
23	88.33965	0.01132		75	125	99	218	117	10	141	34	183	0
24	108.0206	0.009257		76	157	100	226	118	4	142	25	184	0
25	89.52362	0.01117		77	165	101	259	119	11	143	19	185	0
26	82.20183	0.012165		78	196	102	277	120	6	144	29	186	0
27	99.13396	0.010087		79	208	103	289	121	2	145	21	187	0
28	74.30185	0.013459		80	250	104	337	122	1	146	12	188	1
29	83.88101	0.011922		81	271	105	333	123	0	147	19	189	0
30	80.41074	0.012436		82	291	106	300	124	1	148	16	190	0
31	87.72026	0.0114		83	310	107	331	125	0	149	15		$\sum 10000$
32	96.91855	0.010318		84	348	108	343	126	0	150	5		
33	86.87019	0.011511		85	362	109	346	127	0	151	15		
34	88.54563	0.011294		86	378	110	326	128	1	152	13		
…	…	…		87	357	111	296	129	0	153	6		
…	…	…		88	417	112	316	130	0	154	3		
…	…	…		89	397	113	322		$\sum 10000$	155	14		
9996	87.70769	0.011402		90	369	114	303			156	3		
9997	96.54907	0.010357		91	406	115	275			157	5		
9998	100.2901	0.009971		92	404	116	279			158	4		
9999	71.6599	0.013955		93	379	117	268			159	3		
10000	68.69654	0.014557		94	359	118	259			160	2		

APPENDIX B

COMPARISON OF RESULTS BETWEEN TAGUCHI AND NEW METHODS

Comparison of Results between Taguchi and New Methods

STB	LTB		LTB	LTB	LTB	LTB	LTB
Taguchi Method	Taguchi Method		New Method				
			α=1.5	α=2	α=2.5	α=4	α=5
y_i	y_i	$1/y_i^2$	y_i	y_i	y_i	y_i	y_i
13.97	13.97	0.00512	13.97	13.97	13.97	13.97	13.97
14.04	14.04	0.00507	14.04	14.04	14.04	14.04	14.04
14.09	14.09	0.00504	14.09	14.09	14.09	14.09	14.09
13.99	13.99	0.00511	13.99	13.99	13.99	13.99	13.99
14.00	14.00	0.00510	14.00	14.00	14.00	14.00	14.00
14.06	14.06	0.00506	14.06	14.06	14.06	14.06	14.06
13.92	13.92	0.00516	13.92	13.92	13.92	13.92	13.92
13.93	13.93	0.00515	13.93	13.93	13.93	13.93	13.93
14.01	14.01	0.00509	14.01	14.01	14.01	14.01	14.01
14.03	14.03	0.00508	14.03	14.03	14.03	14.03	14.03
15.12	15.12	0.00437	15.12	15.12	15.12	15.12	15.12
15.09	15.09	0.00439	15.09	15.09	15.09	15.09	15.09
13.93	13.93	0.00515	13.93	13.93	13.93	13.93	13.93
13.98	13.98	0.00512	13.98	13.98	13.98	13.98	13.98
14.02	14.02	0.00509	14.02	14.02	14.02	14.02	14.02
14.05	14.05	0.00507	14.05	14.05	14.05	14.05	14.05
14.08	14.08	0.00504	14.08	14.08	14.08	14.08	14.08
13.98	13.98	0.00512	13.98	13.98	13.98	13.98	13.98
14.00	14.00	0.00510	14.00	14.00	14.00	14.00	14.00
14.11	14.11	0.00502	14.11	14.11	14.11	14.11	14.11
14.12	← **Mean** →	0.00502	14.12	14.12	14.12	14.12	14.12
0.3323	← **SD** →	0.0002	0.3323	0.3323	0.3323	0.3323	0.3323
2	← **A_0** →	2	2	2	2	2	2
15	← **Δ_0** →	15	15	15	15	15	15
0.008889	← **k** →	450	0.008889	0.008889	0.008889	0.008889	0.008889
199.4848	← **MSD** →	0.0051	49.9540	199.4848	448.7028	1794.4800	3190.1008
1.77	← **QL ($)** →	2.31	0.44	1.77	3.99	15.95	28.36
A*		**B***		**C***			

APPENDIX C

. SIMULATION OF WARRANTY COST FOR STB, NTB, AND LTB CASES

Simulation of Warranty Cost for Brake Rotor, STB Case
(*All complaints averaged to actual, †Random actual complaints)

Sl No.	PP	CE	PP-CE = Y*	$\{(Y^*-0)^2 \mid Y^* > 0\}$	Random #	Actual Complaint (Simulated)
1	65.056	30.506	34.550	1193.674	19120	0.000
2	-5.608	32.503	-38.111	0.000	12663	0.000
3	24.723	28.715	-3.992	0.000	24259	0.000
4	37.085	26.169	10.916	119.151	12550	0.000
5	59.976	28.578	31.399	985.871	24189	0.000
6	0.422	29.725	-29.303	0.000	27490	0.000
:	:	:	:	:	:	:
166	32.950	31.700	1.250	1.561	78	1.561
167	31.880	26.983	4.896	23.975	4104	0.000
168	43.507	27.217	16.290	265.369	7589	0.000
169	27.057	28.136	-1.079	0.000	22795	0.000
170	20.492	28.590	-8.098	0.000	28141	0.000
171	49.122	30.487	18.634	347.237	356	347.237
172	8.572	29.430	-20.858	0.000	10543	0.000
173	42.362	32.447	9.915	98.304	29712	0.000
174	22.042	31.236	-9.194	0.000	17309	0.000
175	29.585	30.622	-1.037	0.000	11362	0.000
176	76.908	36.043	40.866	1670.008	17208	0.000
177	65.644	27.216	38.429	1476.751	25029	0.000
178	23.649	34.763	-11.114	0.000	2351	0.000
179	-2.883	37.198	-40.081	0.000	2442	0.000
180	47.934	24.719	23.215	538.923	27744	0.000
181	33.823	32.419	1.404	1.971	4884	0.000
182	27.478	28.204	-0.726	0.000	20014	0.000
183	39.707	35.055	4.653	21.648	12905	0.000
184	37.307	27.561	9.746	94.985	589	94.985
185	20.093	29.949	-9.856	0.000	29878	0.000
:	:	:	:	:	:	:
29998	35.563	30.208	5.355	28.677	4476	0.000
29999	9.449	28.191	-18.741	0.000	25313	0.000
30000	20.024	34.219	-14.194	0.000	27100	0.000
SUM	921629.234	899516.335	22112.899	5405326.985	$T_p = 636000$	295638.938
MEAN	30.721	29.984	0.737	***WC = $339195.10**	k = 0.0592	**†WC = $371038.70**
ST DEV	18.141	3.337	18.465	***WC_{unit} = $0.53**	ω = 0.05	**†WC_{unit} = $0.58**

Simulation of Warranty Cost for Gear Housing, NTB Case
(*All complaints averaged to actual, †Random actual complaints)

Sl No.	PP	CE	PP-CE = Y*	$\{(-0.002-Y^*)^2 \mid Y^* < -0.002\}$	$\{(Y^*-0.002)^2 \mid Y^* > 0.002\}$	Random #	Actual Complaint (Simulated) $\mid Y^* < -0.002$	Actual Complaint (Simulated) $\mid Y^* > 0.002$
1	0.99867	0.99877	-0.00010	0	0	16380	0	0
2	0.99564	1.00145	-0.00581	1.45162E-05	0	13990	0	0
3	1.00036	0.99967	0.00069	0	0	15546	0	0
4	1.00356	1.00001	0.00355	0	2.38756E-06	16111	0	0
:	:	:	:	:	:	:	:	:
2555	1.00252	0.99981	0.00272	0	5.11428E-07	12478	0	0
2556	1.00761	1.00087	0.00674	0	2.24665E-05	566	0	2.2466E-05
2557	1.00163	0.99927	0.00236	0	1.28146E-07	18721	0	0
2558	0.99885	0.99982	-0.00098	0	0	15311	0	0
2559	0.99714	0.99950	-0.00235	1.25124E-07	0	7451	0	0
2560	0.99868	0.99880	-0.00011	0	0	10096	0	0
2561	1.00086	0.99962	0.00125	0	0	576	0	0
2562	0.99737	0.99932	-0.00195	0	0	17051	0	0
2563	1.00001	1.00006	-0.00005	0	0	4414	0	0
2564	0.99699	1.00159	-0.00460	6.75844E-06	0	350	6.75844E-06	0
2565	1.00148	0.99747	0.00400	0	4.01175E-06	3301	0	0
2566	0.99916	0.99898	0.00018	0	0	17664	0	0
2567	1.00187	0.99937	0.00250	0	2.50198E-07	8440	0	0
2568	1.00193	0.99871	0.00322	0	1.47893E-06	14006	0	0
2569	0.99624	0.99889	-0.00265	4.23629E-07	0	15173	0	0
2570	0.99911	1.00096	-0.00185	0	0	13423	0	0
2571	0.99717	1.00007	-0.00290	8.12398E-07	0	17389	0	0
2572	1.00194	1.00157	0.00037	0	0	7418	0	0
2573	0.99651	0.99836	-0.00185	0	0	7323	0	0
2574	0.99535	1.00027	-0.00492	8.53246E-06	0	317	8.53246E-06	0
2575	1.00218	0.99796	0.00422	0	4.91281E-06	5145	0	0
2576	0.99968	1.00101	-0.00132	0	0	3394	0	0
:	:	:	:	:	:	:	:	:
19997	1.00030	1.00045	-0.00015	0	0	10244	0	0
19998	1.00333	0.99993	0.00339	0	1.93565E-06	4066	0	0
19999	0.99996	0.99965	0.00031	0	0	1520	0	0
20000	1.00016	1.00023	-0.00007	0	0	15943	0	0
SUM	19992	20000	-8.161	0.045444743	0.027552819	T_p = 20000	0.00068	0.00074213
MEAN	0.9996	1.0000	-0.0004	**$^*WC_1$ = \$59078.17**	**$^*WC_2$ = \$107455.99**	k_1 = 65000000, k_2 = 130000000	**$^\dagger WC_1$ = \$44447.77**	**$^\dagger WC_2$ = \$96477.23**
ST DEV	0.00310	0.00101	0.003	**$^*WC = {}^*WC_1 + {}^*WC_2$ = \$166534.16**	**$^*WC_{unit}$ = \$8.33**	$\omega_1 = 0.02$, $\omega_2 = 0.03$	**$^\dagger WC = {}^\dagger WC_1 + {}^\dagger WC_2$ = \$140924.50**	**$^\dagger WC_{unit}$ = \$7.05**

Simulation of Warranty Cost for Gear Housing, LTB Case
(*All complaints averaged to actual, †Random actual complaints)

Sl No.	PP	CE	PP-CE = Y*	$\{(1-Y^*)^2 \mid Y^* < 1\}$	Random #	Actual Complaint (Simulated)
1	68.1476	67.2037	0.9439	0.00315	5629	0.0000
2	65.2838	67.9722	-2.6883	13.60371	9197	0.0000
3	69.7428	65.6838	4.0590	0.00000	2023	0.0000
4	72.7670	68.2041	4.5629	0.00000	4512	0.0000
5	72.5381	67.1048	5.4333	0.00000	4483	0.0000
6	74.1049	66.8201	7.2848	0.00000	6274	0.0000
7	62.6297	66.6550	-4.0253	25.25347	8733	0.0000
8	68.3411	67.7992	0.5419	0.20981	4018	0.0000
9	72.2354	67.1579	5.0775	0.00000	6670	0.0000
10	65.8434	69.1982	-3.3548	18.96442	8701	0.0000
11	67.0050	66.8630	0.1420	0.73611	1140	0.7361
12	64.0746	68.1767	-4.1021	26.03168	7755	0.0000
13	63.6161	68.5441	-4.9280	35.14140	2539	35.1414
14	66.1629	65.8388	0.3241	0.45678	35	0.4568
15	66.7610	68.5886	-1.8277	7.99571	9792	0.0000
16	62.8221	66.4830	-3.6609	21.72378	5415	0.0000
17	67.3633	66.7908	0.5724	0.18280	3950	0.0000
18	67.8434	69.8030	-1.9595	8.75885	6861	0.0000
19	69.4223	67.2397	2.1826	0.00000	3413	0.0000
20	67.9564	67.6089	0.3475	0.42579	398	0.4258
21	68.0692	69.6555	-1.5864	6.68923	7041	0.0000
22	67.9425	67.4387	0.5038	0.24620	6772	0.0000
23	72.9609	66.5631	6.3977	0.00000	6224	0.0000
24	68.7773	68.4988	0.2786	0.52049	4036	0.0000
25	68.4818	67.8018	0.6800	0.10242	785	0.1024
:	:	:	:	:	:	:
9995	73.0117	68.4196	4.5921	0.00000	147	0.0000
9996	67.8471	67.9690	-0.1219	1.25866	1772	1.2587
9997	69.8216	67.5523	2.2693	0.00000	1121	0.0000
9998	72.9537	68.6912	4.2625	0.00000	1138	0.0000
9999	73.8457	68.2230	5.6228	0.00000	9686	0.0000
10000	71.4734	65.6781	5.7953	0.00000	4880	0.0000
SUM	921629.2	899516.3	22112.9	36768.7	$T_p = 636000$	295638.9
MEAN	69.016	67.483	1.533	***WC = $9683931**	k = 0.0592	**†WC = $9843395**
ST DEV	2.928	1.109	3.120	***WC$_{unit}$ = $193.68**	ω = 0.05	**†WC$_{unit}$ = $196.87**

APPENDIX D

. WARRANTY PROBABILITY FOR PP WEIBULL AND CE NORMALLY DISTRIBUTED

Warranty Probability for PP Weibull and CE Normally Distributed

Sl No.	Y*=PP(Weibull) - CE(Normal)	Probability	Cumulative probability	Sl No.	Y*=PP(Weibull) - CE(Normal)	Probability	Cumulative probability
1	-40000	0	0	102	400	0.01259048	0.347230144
2	-39600	4.05259E-10	4.05259E-10	103	800	0.012711984	0.359942128
3	-39200	1.78473E-09	2.18999E-09	104	1200	0.01282067	0.372762798
4	-38800	4.79484E-09	6.98483E-09	105	1600	0.012916332	0.38567913
5	-38400	1.02762E-08	1.7261E-08	**106**	**2000**	**0.012998796**	**0.398677926**
6	-38000	1.93283E-08	3.65894E-08	107	2400	0.01306792	0.411745847
7	-37600	3.33955E-08	6.99849E-08	108	2800	0.013123592	0.424869438
8	-37200	5.4373E-08	1.24358E-07	109	3200	0.013165731	0.438035169
9	-36800	8.47378E-08	2.09096E-07	110	3600	0.013194285	0.451229454
10	-36400	1.27711E-07	3.36807E-07	111	4000	0.013209231	0.464438684
11	-36000	1.87459E-07	5.24266E-07	112	4400	0.013210571	0.477649255
12	-35600	2.69334E-07	7.936E-07	113	4800	0.013198333	0.490847588
13	-35200	3.80177E-07	1.17378E-06	114	5200	0.013172567	0.504020155
14	-34800	5.28666E-07	1.70244E-06	115	5600	0.013133344	0.517153499
15	-34400	7.25745E-07	2.42819E-06	116	6000	0.01308075	0.530234249
16	-34000	9.85124E-07	3.41331E-06	117	6400	0.01301489	0.543249139
17	-33600	1.32386E-06	4.73718E-06	118	6800	0.012935878	0.556185017
18	-33200	1.76306E-06	6.50023E-06	119	7200	0.012843839	0.569028855
19	-32800	2.3286E-06	8.82883E-06	120	7600	0.012738904	0.58176776
20	-32400	3.05209E-06	1.18809E-05	121	8000	0.012621211	0.594388971
21	-32000	3.97179E-06	1.58527E-05	122	8400	0.012490897	0.606879868
22	-31600	5.13378E-06	2.09865E-05	123	8800	0.012348102	0.619227969
23	-31200	6.5931E-06	2.75796E-05	124	9200	0.012192964	0.631420933
24	-30800	8.4151E-06	3.59947E-05	125	9600	0.012025622	0.643446555
25	-30400	1.06768E-05	4.66715E-05	126	10000	0.011846215	0.655292769
26	-30000	1.34685E-05	6.01401E-05	127	10400	0.011654882	0.666947651
27	-29600	1.68951E-05	7.70352E-05	128	10800	0.01145177	0.678399421
28	-29200	2.10776E-05	9.81128E-05	129	11200	0.011237031	0.689636453
29	-28800	2.6155E-05	0.000124268	130	11600	0.011010833	0.700647286
30	-28400	3.22853E-05	0.000156553	131	12000	0.010773362	0.711420648
31	-28000	3.96473E-05	0.0001962	132	12400	0.010524833	0.721945481
32	-27600	4.84415E-05	0.000244642	133	12800	0.010265495	0.732210976
33	-27200	5.88914E-05	0.000303533	134	13200	0.009995643	0.742206619
34	-26800	7.1244E-05	0.000374777	135	13600	0.009715624	0.751922243
35	-26400	8.57705E-05	0.000460548	136	14000	0.00942585	0.761348093
36	-26000	0.000102766	0.000563314	137	14400	0.009126803	0.770474896
37	-25600	0.000122551	0.000685865	138	14800	0.008819049	0.779293945
38	-25200	0.000145467	0.000831332	139	15200	0.00850324	0.787797185
39	-24800	0.000171878	0.00100321	140	15600	0.008180121	0.795977306
40	-24400	0.000202169	0.00120538	141	16000	0.007850536	0.803827842
41	-24000	0.000236743	0.001442123	142	16400	0.007515428	0.81134327
42	-23600	0.000276016	0.001718139	143	16800	0.007175835	0.818519105
43	-23200	0.000320417	0.002038555	144	17200	0.006832888	0.825351993
44	-22800	0.000370382	0.002408937	145	17600	0.006487804	0.831839797
45	-22400	0.000426351	0.002835288	146	18000	0.006141873	0.83798167
46	-22000	0.000488762	0.00332405	147	18400	0.005796446	0.843778116
47	-21600	0.000558048	0.003882097	148	18800	0.005452919	0.849231035
48	-21200	0.000634628	0.004516726	149	19200	0.005112713	0.854343749
49	-20800	0.000718907	0.005235633	150	19600	0.004777254	0.859121003

Warranty Probability for PP Weibull and CE Normally Distributed

Sl No.	Y*=PP - CE	Probability	Cumulative probability	Sl No.	Y*=PP - CE	Probability	Cumulative probability
50	-20400	0.000811266	0.006046899	151	20000	0.004447951	0.863568953
51	-20000	0.000912059	0.006958958	152	20400	0.004126173	0.867695127
52	-19600	0.001021608	0.007980566	153	20800	0.00381323	0.871508357
53	-19200	0.001140197	0.009120763	154	21200	0.003510346	0.875018703
54	-18800	0.001268069	0.010388832	155	21600	0.003218643	0.878237346
55	-18400	0.001405421	0.011794253	156	22000	0.002939123	0.881176469
56	-18000	0.001552402	0.013346655	157	22400	0.002672648	0.883849117
57	-17600	0.001709109	0.015055764	158	22800	0.002419935	0.886269052
58	-17200	0.001875585	0.016931349	159	23200	0.002181544	0.888450595
59	-16800	0.00205182	0.018983169	160	23600	0.001957869	0.890408465
60	-16400	0.002237746	0.021220915	161	24000	0.001749147	0.892157612
61	-16000	0.002433241	0.023654156	162	24400	0.001555451	0.893713064
62	-15600	0.002638127	0.026292283	163	24800	0.001376701	0.895089765
63	-15200	0.002852173	0.029144457	164	25200	0.001212673	0.896302438
64	-14800	0.003075096	0.032219552	165	25600	0.001063009	0.897365446
65	-14400	0.003306561	0.035526113	166	26000	0.000927233	0.89829268
66	-14000	0.003546189	0.039072302	167	26400	0.00080477	0.899097449
67	-13600	0.003793554	0.042865856	168	26800	0.000694955	0.899792405
68	-13200	0.00404819	0.046914046	169	27200	0.00059706	0.900389465
69	-12800	0.004309595	0.051223641	170	27600	0.000510305	0.900899769
70	-12400	0.00457723	0.055800871	171	28000	0.000433877	0.901333646
71	-12000	0.004850526	0.060651397	172	28400	0.000366948	0.901700594
72	-11600	0.005128889	0.065780286	173	28800	0.000308689	0.902009283
73	-11200	0.0054117	0.071191987	174	29200	0.000258281	0.902267563
74	-10800	0.005698321	0.076890308	175	29600	0.000214929	0.902482492
75	-10400	0.005988096	0.082878404	176	30000	0.000177871	0.902660364
76	-10000	0.006280357	0.089158762	177	30400	0.000146387	0.902806751
77	-9600	0.006574425	0.095733187	178	30800	0.000119801	0.902926552
78	-9200	0.006869613	0.102602799	179	31200	9.74886E-05	0.903024041
79	-8800	0.007165229	0.109768028	180	31600	7.88777E-05	0.903102919
80	-8400	0.00746058	0.117228609	181	32000	6.34499E-05	0.903166368
81	-8000	0.007754973	0.124983581	182	32400	5.07397E-05	0.903217108
82	-7600	0.008047715	0.133031297	183	32800	4.03334E-05	0.903257441
83	-7200	0.00833812	0.141369417	184	33200	3.18663E-05	0.903289308
84	-6800	0.008625508	0.149994925	185	33600	2.502E-05	0.903314328
85	-6400	0.008909207	0.158904132	186	34000	1.95189E-05	0.903333847
86	-6000	0.009188553	0.168092685	187	34400	1.51266E-05	0.903348973
87	-5600	0.009462899	0.177555584	188	34800	1.16417E-05	0.903360615
88	-5200	0.009731606	0.18728719	189	35200	8.89444E-06	0.903369509
89	-4800	0.009994054	0.197281243	190	35600	6.74252E-06	0.903376252
90	-4400	0.010249638	0.207530882	191	36000	5.06789E-06	0.90338132
91	-4000	0.010497773	0.218028655	192	36400	3.77326E-06	0.903385093
92	-3600	0.010737892	0.228766547	193	36800	2.7791E-06	0.903387872
93	-3200	0.01096945	0.239735997	194	37200	2.02092E-06	0.903389893
94	-2800	0.011191924	0.250927921	195	37600	1.4468E-06	0.90339134
95	-2400	0.011404817	0.262332738	196	38000	1.01525E-06	0.903392355
96	-2000	0.011607655	0.273940394	197	38400	6.93406E-07	0.903393049
97	-1600	0.011799992	0.285740386	198	38800	4.55383E-07	0.903393504
98	-1200	0.01198141	0.297721796	199	39200	2.80973E-07	0.903393785
99	-800	0.012151517	0.309873313	200	39600	1.54507E-07	0.90339394
100	-400	0.012309956	0.322183268	201	40000	6.39222E-08	0.903394003
101	0	0.012456396	0.334639664				

APPENDIX E

COMPUTATIONS IN SIMULTANEOUS OPTIMIZATION METHODOLOGY FOR DYNAMIC MULTI-RESPONSE EXPERIMENTS

Experimental Setup, Control Factor Levels with Respect to Runs

Run No.	Control factors					
	x_1	x_2	x_3	x_4	x_5	x_6
1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	1	1	2	2	3	3
5	2	2	3	3	1	1
6	3	3	1	1	2	2
7	1	2	1	3	2	3
8	2	3	2	1	3	1
9	3	1	3	2	1	2
10	1	3	3	2	2	1
11	2	1	1	3	3	2
12	3	2	2	1	1	3
13	1	2	3	1	3	2
14	2	3	1	2	1	3
15	3	1	2	3	2	1
16	1	3	2	3	1	2
17	2	1	3	1	2	3
18	3	2	1	2	3	1

Response Types and Their Acceptable Limits

Responses		Limits	M_1	M_2	M_3
Y1	LTB	Minimum	55	110	165
Y2	NTB	Minimum	7	14	21
		Maximum	13	26	39
Y3	STB	Maximum	3	6	9

Reference: Chang (2006)

Data of Three Responses for 18 Experimental Runs, One Signal Factor at Three Levels, and One Noise Factor at Two Levels

LTB Response Y_1											
M_1=10				M_2=20				M_3=30			
N_1	N_2	N_1	N_2	N_1	N_2	N_1	N_2	N_1	N_2	N_1	N_2
61.6	70.8	78.2	57.1	128	137.3	106	160.3	230.6	282.2	226.9	252.5
88.3	72.9	93.6	72.7	175	174	182	145.5	259.7	258.4	304.5	214.6
80.8	77.2	81.1	83.3	154	167.1	157	159	238.1	251.8	237.8	257.9
65.9	83.7	71.3	78.4	179	135.6	152	177	196.1	246.9	221.6	291.8
79.4	67.8	88.6	87.3	122	113.6	152	141.3	248.8	171.5	245.1	244.7
90.5	87.6	87	87.8	162	160.7	169	163.9	286.9	231.4	236.5	288.7
80.9	69.9	74.7	78.7	166	141.7	163	159.1	232.2	260.4	246.4	239.7
92.3	105	71.7	89.4	186	216.1	154	173	233.1	340.6	240.5	308.9
92.8	82.1	59.8	87	131	175.1	142	138.2	257.3	161.4	266	274.4
86	81.9	100	91.2	179	190.8	175	137.8	246.9	293.5	244.9	227.1
76.3	67.1	78.2	76	140	169.5	155	175.4	264.1	239.2	260.1	251.8
91.4	85.1	81.8	63.8	160	123.8	167	166.4	238.5	233.5	197.2	242
87.9	57.4	82.5	78.5	147	91.6	167	182.3	212.4	250.3	222.2	207.3
88.1	81.7	78.1	75.7	157	140.2	170	127.8	239.1	241.4	215.2	211.7
102	80.4	78.2	76.5	168	206.7	181	222.6	240.6	325.1	235.2	285.2
77.4	72	75.4	69.5	172	189.1	159	168.6	201.3	254.3	219.7	237.3
71.4	77	69.2	70.5	145	158.4	153	154	223.8	218.4	218.7	224.1
82.8	85.2	67.8	92	184	154.4	176	157.6	276.1	249.3	254.4	286.1

NTB Response Y_2											
M_1=10				M_2=20				M_3=30			
N_1	N_2	N_1	N_2	N_1	N_2	N_1	N_2	N_1	N_2	N_1	N_2
7.4	9.1	7.2	10.2	16.7	22.8	13.2	17.8	23.7	26.2	24.1	26.7
10	8.7	8.8	9.2	23.4	19.1	22.6	24.1	29.6	31.4	30.3	30.5
11	11	11	10.9	20.1	21.8	20.9	23.7	30.6	30.5	32.4	32.7
7.6	8.1	7.2	7.6	15.3	14.1	14.8	14.7	22.5	21.9	22.2	21.8
12	11	13	11.8	25.6	25.8	25.7	26.2	36.6	39.1	35.7	33.1
10	11	10	11.2	23.7	22.5	21.8	20	32	34.2	32.8	28
12	12	12	11.6	23.3	22	23.3	22.5	33.5	33.8	32.7	34
8.3	8.2	8.4	6.1	16.9	18.5	18.5	15.1	28.1	29.1	27.3	21.1
8.6	8.2	9.7	8.4	17.3	18.6	18.4	18	30.9	31.3	31.5	30.9
6.9	8.3	6.6	7.2	15.8	14.3	16.4	16.6	22.2	23.1	24.9	24.2
9.5	11	9.2	12.2	27.9	23.6	25.4	20.4	32.3	26.8	33.8	29.2
12	12	11	11.5	22.9	22.1	25.9	22.6	36.9	32.5	36.8	31.9
10	11	10	10.1	19.9	20.1	22.2	19.2	27.1	27.8	27.7	24.6
12	11	10	11.8	24.1	23.3	20.2	24.4	28	32.1	35.6	38.9
10	8.7	7.6	7.2	14.9	16.8	19.4	14.1	26.9	26.6	19.5	25.1
10	11	11	10.9	20.7	20.6	22.2	21.4	34.4	30.5	30	31.4
8.8	9.2	8.4	9	19	16.7	13.8	17.4	26.2	27.1	24.1	26.2
11	11	9.2	7.7	19.8	19.3	20.1	22.6	27.3	29.4	31	24.6

STB Response Y_3											
M_1=10				M_2=20				M_3=30			
N_1	N_2	N_1	N_2	N_1	N_2	N_1	N_2	N_1	N_2	N_1	N_2
1.9	2	1.9	2.1	4.6	3.9	3.7	4.8	7.6	4.7	4.6	4.3
1.8	2	2	2.2	4	3.8	2.8	3.5	6	6.7	3.3	6.1
1	1	3.2	2.6	4.4	3.8	5.4	3.9	8.1	8	6.7	2.7
1.7	2	2.2	2.6	4.1	3.6	4.9	3.3	5.9	7.7	6	6.9
2	3	2.1	1.7	2.7	4.1	3.8	3.6	4.8	5.5	5.8	5.2
1.8	1	2.2	2	2.5	4	4.1	4.4	4.9	6.6	5.4	3.1
1.9	2	1.6	1.7	5.2	3.6	5.1	5.1	6.6	5.8	5.3	5.1
2.8	2	1.5	2.2	3.4	4.1	3.4	4.3	6.2	9	3.7	5
2.3	2	1.7	1.2	4.5	4	5.1	4	7.4	7.2	6.5	6.7
1.7	2	1.7	1.8	3.9	3.9	5.8	4.8	5.2	4.5	8.4	8
1	2	1.7	2.3	4.1	5	4	4.6	3.9	6.7	5.6	4.5
2.1	2	1.9	0.9	4	4.2	3.7	4.1	7.4	2.1	6.2	6.9
2	2	2.3	2.1	5	4.5	4.7	4.7	6.7	7.4	7.3	6.2
1.7	2	2.1	2.7	3.8	3.1	3.3	3.8	7.2	4.9	4.9	4.7
1.7	2	2.2	1.9	5.3	4.8	3.6	2.6	5	3.8	7.5	5.5
1.9	2	2.3	1.6	3.1	4.1	4.2	4.8	5.8	5.4	3.3	6.9
1.5	2	2.1	1.8	3.8	3.7	4.4	4.4	6.9	4.6	4.2	4.2
2.5	2	2.2	1.8	3.1	3.9	3.4	4.7	7.7	6	7.4	7.3

Reference: Chang (2006)

Computations of $\overline{y}_{ijk}$ and s_{ijk} Across Noise Levels

2	**Alpha**											**0**					
159.24	**Targets**	**319.13**		**490.94**		**10**		**20**		**30**		**0**		**0**		**0**	
LTB Response Y_1						**NTB Response Y_2**						**STB Response Y_3**					
M_1=10		**M_2=20**		**M_3=30**		**M_1=10**		**M_2=20**		**M_3=30**		**M_1=10**		**M_2=20**		**M_3=30**	
$\overline{y}_{ijk}$	s_{ijk}	$\overline{y}_{ijk}$	s_{ijk}	$\overline{y}_{ijk}$	s_{ijk}	$\overline{y}_{ijk}$	s_{ijk}	$\overline{y}_{ijk}$	s_{ijk}	$\overline{y}_{ijk}$	s_{ijk}	$\overline{y}_{ijk}$	s_{ijk}	$\overline{y}_{ijk}$	s_{ijk}	$\overline{y}_{ijk}$	s_{ijk}
66.9	9.4	132.9	22.5	248.1	25.4	8.5	1.4	17.6	4.0	25.2	1.5	2.0	0.1	4.3	0.5	5.3	1.5
81.9	10.7	169.1	16.0	259.3	36.7	9.2	0.6	22.3	2.2	30.5	0.7	2.0	0.2	3.5	0.5	5.5	1.5
80.6	2.5	159.5	5.5	246.4	10.1	10.7	0.2	21.6	1.5	31.6	1.2	2.1	1.0	4.4	0.7	6.4	2.5
74.8	7.8	160.8	21.0	239.1	40.8	7.6	0.4	14.7	0.5	22.1	0.3	2.2	0.4	4.0	0.7	6.6	0.8
80.8	9.6	132.1	17.4	227.5	37.4	11.7	0.9	25.8	0.3	36.1	2.5	2.1	0.4	3.6	0.6	5.3	0.4
88.2	1.6	164.0	3.9	260.9	31.2	10.6	0.5	22.0	1.5	31.8	2.7	1.8	0.4	3.8	0.9	5.0	1.5
76.1	4.8	157.5	10.9	244.7	12.0	11.8	0.2	22.8	0.6	33.5	0.6	1.8	0.1	4.8	0.8	5.7	0.7
89.6	13.6	182.3	26.0	280.8	52.5	7.8	1.1	17.3	1.6	26.4	3.6	2.1	0.6	3.8	0.5	6.0	2.3
80.4	14.4	146.6	19.6	239.8	52.7	8.7	0.7	18.1	0.6	31.2	0.3	1.9	0.5	4.4	0.5	7.0	0.4
89.8	7.8	170.8	22.9	253.1	28.4	7.3	0.7	15.8	1.0	23.6	1.2	1.8	0.2	4.6	0.9	6.5	2.0
74.4	5.0	159.9	15.9	253.8	11.0	10.4	1.4	24.3	3.2	30.5	3.1	1.8	0.6	4.4	0.5	5.2	1.2
80.5	11.8	154.4	20.6	227.8	20.7	11.4	0.6	23.4	1.7	34.5	2.7	1.6	0.5	4.0	0.2	5.7	2.4
76.6	13.4	146.9	39.6	223.1	19.2	10.3	0.2	20.4	1.3	26.8	1.5	2.2	0.1	4.7	0.2	6.9	0.6
80.9	5.4	148.7	18.5	226.9	15.6	11.3	0.9	23.0	1.9	33.7	4.7	2.1	0.4	3.5	0.4	5.4	1.2
84.2	11.8	194.6	24.6	271.5	42.2	8.4	1.3	16.3	2.4	24.5	3.4	2.0	0.2	4.1	1.2	5.5	1.5
73.6	3.5	172.1	12.6	228.2	22.8	10.8	0.3	21.2	0.7	31.6	2.0	2.0	0.3	4.1	0.7	5.4	1.5
72.0	3.4	152.5	5.6	221.3	3.1	8.9	0.3	16.7	2.2	25.9	1.3	1.8	0.3	4.1	0.4	5.0	1.3
82.0	10.2	167.8	14.1	266.5	17.5	9.7	1.6	20.5	1.5	28.1	2.8	2.0	0.4	3.8	0.7	7.1	0.8
79.6	**Means**	**159.6**		**245.5**		**9.7**		**20.2**		**29.3**		**1.9**		**4.1**		**5.9**	

Squared Biases, Variances and Maxima Thereof

LTB Response Y_1						NTB Response Y_2						STB Response Y_3					
M_1=10		M_2=20		M_3=30		M_1=10		M_2=20		M_3=30		M_1=10		M_2=20		M_3=30	
$\bar{y}_{ijk}^2(1-2)^2$	s_{ijk}^2	$\bar{y}_{ijk}^2(1-2)^2$	s_{ijk}^2	$\bar{y}_{ijk}^2(1-2)^2$	s_{ijk}^2	$(\bar{y}_{ijk}-m_{jk})^2$	s_{ijk}^2	$(\bar{y}_{ijk}-m_{jk})^2$	s_{ijk}^2	$(\bar{y}_{ijk}-m_{jk})^2$	s_{ijk}^2	$\bar{y}_{ijk}^2$	s_{ijk}^2	$\bar{y}_{ijk}^2$	s_{ijk}^2	$\bar{y}_{ijk}^2$	s_{ijk}^2
8522.9	89.009	34683	505.91	58996	645.95	2.3256	2.0492	5.64063	15.749	23.281	2.2358	3.9006	0.0092	18.063	0.2833	28.09	2.38
5986	114.5	22525	257.31	53658	1347.4	0.64	0.4067	5.29	4.9267	0.2025	0.55	3.9006	0.0292	12.426	0.2758	30.526	2.2958
6184.9	6.38	25499	29.817	59801	101.42	0.49	0.0333	2.64063	2.3958	2.4025	1.35	4.2025	1.05	19.141	0.5358	40.641	6.4092
7126.6	61.209	25062	440.84	63424	1664.5	5.6406	0.1358	27.8256	0.2425	62.41	0.1	4.6225	0.1367	15.801	0.4892	43.891	0.7158
6157.5	91.349	34981	303.73	69388	1398.4	2.9756	0.8292	33.9306	0.0692	37.516	6.1358	4.41	0.14	12.603	0.3633	28.356	0.1825
5043.8	2.4158	24082	14.963	52931	971.48	0.3306	0.2292	4	2.3933	3.0625	7.0767	3.24	0.1867	14.063	0.7233	25	2.1133
6921.3	23.397	26117	119.19	60647	143.54	3.0625	0.0567	7.70062	0.4092	12.25	0.3267	3.1506	0.0225	22.563	0.59	32.49	0.4467
4857.3	186.26	18730	674.83	44170	2755	5.0625	1.2167	7.5625	2.6233	12.96	13.027	4.3056	0.3158	14.44	0.22	35.701	5.1092
6212.5	208.19	29785	384.99	63085	2778.8	1.6256	0.4492	3.70562	0.3292	1.3225	0.09	3.4225	0.2567	19.36	0.2733	48.303	0.1767
4826	60.949	22010	526.35	56569	804.61	7.5625	0.55	17.8506	1.0825	40.96	1.42	3.3306	0.0358	21.16	0.82	42.576	3.8492
7198.6	24.633	25355	251.82	56236	120.98	0.1806	1.8825	18.7056	9.9558	0.2756	9.8358	3.1506	0.3292	19.581	0.2158	26.781	1.5292
6196.8	140.18	27145	425.56	69244	428.33	1.8225	0.3367	11.3906	2.9425	20.476	7.2692	2.6406	0.2758	16	0.0467	31.923	5.8433
6834.2	178.25	29682	1571.8	71766	368.26	0.0756	0.0558	0.1225	1.67	10.24	2.2467	4.6225	0.0167	22.326	0.0425	47.61	0.3133
6137.9	29.12	29048	343.67	69744	242.34	1.69	0.86	9	3.7	13.323	21.897	4.3056	0.2025	12.25	0.1267	29.431	1.4092
5627.9	139.83	15509	604.18	48144	1777.7	2.4806	1.8025	13.69	5.5533	29.976	11.843	4	0.06	16.606	1.4758	29.703	2.3767
7339.3	12.349	21619	157.67	69059	519.96	0.5625	0.0967	1.50063	0.5492	2.4806	3.8825	3.9006	0.0892	16.403	0.4967	28.623	2.27
7607.2	11.816	27775	31.103	72734	9.75	1.3225	0.1167	10.7256	4.7292	16.81	1.62	3.0625	0.07	16.606	0.1425	24.751	1.6825
5974.4	104.17	22902	198.57	50385	306.19	0.0756	2.6025	0.2025	2.1633	3.7056	7.6625	4.1006	0.1625	14.251	0.4892	50.41	0.5667
8522.9	208.19	34981	1571.8	72734	2778.8	7.5625	2.6025	33.9306	15.749	62.41	21.897	4.6225	1.05	22.563	1.4758	50.41	6.4092

Last row contains maxima, i.e., $\max_{i=1}^{18}\left(\bar{y}_{ijk}^2(1-2)^2\right)$, $\max_{i=1}^{18}\left((\bar{y}_{ijk}-m_{jk})^2\right)$, $\max_{i=1}^{18}\left(\bar{y}_{ijk}^2\right)$, $\max_{i=1}^{18}\left(s_{ijk}^2\right)$

Normalized Biases b_{ijk} and Normalized Variances v_{ijk} and Given Equal Weights

0.2500	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**
LTB Response Y_1						**NTB Response Y_2**						**STB Response Y_3**					
0.5			**Weight**			**0.5**			**Weight**			**0.5**		***1.5***	**Weight**		
M_1=10		**M_2=20**		**M_3=30**		**M_1=10**		**M_2=20**		**M_3=30**		**M_1=10**		**M_2=20**		**M_3=30**	
b_{i11}	v_{i11}	b_{i12}	v_{i12}	b_{i13}	v_{i13}	b_{i21}	v_{i21}	b_{i22}	v_{i22}	b_{i23}	v_{i23}	b_{i31}	v_{i31}	b_{i32}	v_{i32}	b_{i33}	v_{i33}
1	0.4275	0.9915	0.3219	0.8111	0.2325	0.3075	0.7874	0.1662	1	0.373	0.1021	0.8438	0.0087	0.8006	0.192	0.5572	0.3713
0.7023	0.55	0.6439	0.1637	0.7377	0.4849	0.0846	0.1563	0.1559	0.3128	0.0032	0.0251	0.8438	0.0278	0.5507	0.1869	0.6055	0.3582
0.7257	0.0306	0.7289	0.019	0.8222	0.0365	0.0648	0.0128	0.0778	0.1521	0.0385	0.0617	0.9091	1	0.8483	0.3631	0.8062	1
0.8362	0.294	0.7164	0.2805	0.872	0.599	0.7459	0.0522	0.8201	0.0154	1	0.0046	1	0.1302	0.7003	0.3315	0.8707	0.1117
0.7225	0.4388	1	0.1932	0.954	0.5032	0.3935	0.3186	1	0.0044	0.6011	0.2802	0.954	0.1333	0.5586	0.2462	0.5625	0.0285
0.5918	0.0116	0.6884	0.0095	0.7277	0.3496	0.0437	0.0881	0.1179	0.152	0.0491	0.3232	0.7009	0.1778	0.6233	0.4901	0.4959	0.3297
0.8121	0.1124	0.7466	0.0758	0.8338	0.0517	0.405	0.0218	0.227	0.026	0.1963	0.0149	0.6816	0.0214	1	0.3998	0.6445	0.0697
0.5699	0.8947	0.5354	0.4293	0.6073	0.9914	0.6694	0.4675	0.2229	0.1666	0.2077	0.5949	0.9314	0.3008	0.64	0.1491	0.7082	0.7972
0.7289	1	0.8515	0.2449	0.8673	1	0.215	0.1726	0.1092	0.0209	0.0212	0.0041	0.7404	0.2444	0.8581	0.1852	0.9582	0.0276
0.5662	0.2928	0.6292	0.3349	0.7778	0.2896	1	0.2113	0.5261	0.0687	0.6563	0.0649	0.7205	0.0341	0.9378	0.5556	0.8446	0.6006
0.8446	0.1183	0.7248	0.1602	0.7732	0.0435	0.0239	0.7233	0.5513	0.6321	0.0044	0.4492	0.6816	0.3135	0.8678	0.1462	0.5313	0.2386
0.7271	0.6733	0.776	0.2707	0.952	0.1541	0.241	0.1294	0.3357	0.1868	0.3281	0.332	0.5713	0.2627	0.7091	0.0316	0.6333	0.9117
0.8019	0.8562	0.8485	1	0.9867	0.1325	0.01	0.0215	0.0036	0.106	0.1641	0.1026	1	0.0159	0.9895	0.0288	0.9445	0.0489
0.7202	0.1399	0.8304	0.2186	0.9589	0.0872	0.2235	0.3305	0.2652	0.2349	0.2135	1	0.9314	0.1929	0.5429	0.0858	0.5838	0.2199
0.6603	0.6716	0.4433	0.3844	0.6619	0.6397	0.328	0.6926	0.4035	0.3526	0.4803	0.5408	0.8653	0.0571	0.736	1	0.5892	0.3708
0.8611	0.0593	0.618	0.1003	0.9495	0.1871	0.0744	0.0371	0.0442	0.0349	0.0397	0.1773	0.8438	0.0849	0.727	0.3365	0.5678	0.3542
0.8926	0.0568	0.794	0.0198	1	0.0035	0.1749	0.0448	0.3161	0.3003	0.2693	0.074	0.6625	0.0667	0.736	0.0966	0.491	0.2625
0.701	0.5004	0.6547	0.1263	0.6927	0.1102	0.01	1	0.006	0.1374	0.0594	0.3499	0.8871	0.1548	0.6316	0.3315	1	0.0884

Weighted Normalized Biases $b_{ijk}^{1-u_j}$ and Weighted Normalized Variances $v_{ijk}^{1-w_j}$ Considering Equal Weights

LTB Response Y_1						NTB Response Y_2						STB Response Y_3					
M_1=10		M_2=20		M_3=30		M_1=10		M_2=20		M_3=30		M_1=10		M_2=20		M_3=30	
$b_{i11}^{1-u_1}$	$v_{i11}^{1-u_1}$	$b_{i12}^{1-u_1}$	$v_{i12}^{1-u_1}$	$b_{i13}^{1-u_1}$	$v_{i13}^{1-u_1}$	$b_{i21}^{1-u_2}$	$v_{i21}^{1-u_2}$	$b_{i22}^{1-u_2}$	$v_{i22}^{1-u_2}$	$b_{i23}^{1-u_2}$	$v_{i23}^{1-u_2}$	$b_{i31}^{1-u_3}$	$v_{i31}^{1-u_3}$	$b_{i32}^{1-u_3}$	$v_{i32}^{1-u_3}$	$b_{i33}^{1-u_3}$	$v_{i33}^{1-u_3}$
1.0000	0.5287	0.9936	0.4273	0.8547	0.3348	0.4130	0.8359	0.2603	1.0000	0.4773	0.1806	0.8804	0.0286	0.8463	0.2900	0.6450	0.4757
0.7672	0.6386	0.7188	0.2574	0.7960	0.5811	0.1569	0.2485	0.2481	0.4183	0.0136	0.0631	0.8804	0.0680	0.6393	0.2843	0.6865	0.4630
0.7863	0.0732	0.7889	0.0511	0.8634	0.0835	0.1284	0.0381	0.1473	0.2436	0.0869	0.1237	0.9311	1.0000	0.8839	0.4677	0.8508	1.0000
0.8744	0.3993	0.7787	0.3854	0.9024	0.6809	0.8026	0.1092	0.8618	0.0437	1.0000	0.0176	1.0000	0.2167	0.7655	0.4368	0.9013	0.1932
0.7836	0.5391	1.0000	0.2914	0.9653	0.5975	0.4968	0.4241	1.0000	0.0171	0.6827	0.3851	0.9653	0.2207	0.6461	0.3495	0.6495	0.0693
0.6747	0.0354	0.7558	0.0305	0.7879	0.4547	0.0956	0.1616	0.2012	0.2434	0.1043	0.4286	0.7660	0.2738	0.7015	0.5858	0.5910	0.4351
0.8555	0.1941	0.8032	0.1445	0.8726	0.1084	0.5076	0.0567	0.3288	0.0647	0.2949	0.0427	0.7501	0.0560	1.0000	0.5028	0.7193	0.1356
0.6559	0.9199	0.6259	0.5304	0.6879	0.9936	0.7401	0.5654	0.3244	0.2607	0.3076	0.6774	0.9481	0.4062	0.7155	0.2399	0.7720	0.8436
0.7889	1.0000	0.8864	0.3482	0.8988	1.0000	0.3157	0.2678	0.1900	0.0550	0.0555	0.0162	0.7982	0.3476	0.8915	0.2823	0.9685	0.0676
0.6528	0.3980	0.7065	0.4402	0.8282	0.3947	1.0000	0.3117	0.6177	0.1342	0.7292	0.1285	0.7821	0.0794	0.9530	0.6436	0.8810	0.6822
0.8810	0.2017	0.7855	0.2532	0.8245	0.0953	0.0608	0.7843	0.6398	0.7089	0.0171	0.5487	0.7501	0.4190	0.8991	0.2365	0.6223	0.3414
0.7874	0.7433	0.8268	0.3753	0.9638	0.2460	0.3440	0.2157	0.4410	0.2842	0.4335	0.4374	0.6571	0.3669	0.7728	0.0750	0.7099	0.9330
0.8474	0.8901	0.8841	1.0000	0.9900	0.2196	0.0316	0.0561	0.0147	0.1858	0.2578	0.1813	1.0000	0.0447	0.9921	0.0699	0.9580	0.1040
0.7818	0.2287	0.8699	0.3198	0.9690	0.1605	0.3250	0.4358	0.3696	0.3374	0.3140	1.0000	0.9481	0.2910	0.6325	0.1586	0.6679	0.3211
0.7325	0.7419	0.5433	0.4882	0.7338	0.7153	0.4334	0.7592	0.5062	0.4576	0.5769	0.6307	0.8972	0.1169	0.7946	1.0000	0.6725	0.4752
0.8939	0.1202	0.6970	0.1782	0.9619	0.2845	0.1424	0.0846	0.0964	0.0807	0.0890	0.2732	0.8804	0.1573	0.7873	0.4418	0.6541	0.4591
0.9183	0.1163	0.8411	0.0528	1.0000	0.0144	0.2704	0.0974	0.4216	0.4056	0.3739	0.1419	0.7343	0.1312	0.7946	0.1732	0.5865	0.3667
0.7661	0.5949	0.7278	0.2119	0.7593	0.1912	0.0316	1.0000	0.0215	0.2256	0.1203	0.4550	0.9141	0.2467	0.7085	0.4368	1.0000	0.1621

Products of Weighted Normalized Biases and Weighted Normalized Variances, $b_{ijk}^{1-u_j} v_{ijk}^{1-w_j}$ Considering Equal Weights

LTB Response Y_1			NTB Response Y_2			STB Response Y_3			Y_1	Y_2	Y_3	$Y_1Y_2Y_3$	
M_1	M_2	M_3	M_2	M_1	M_3	M_1	M_2	M_3				= †	
$b_{i11}^{1-u_1} v_{i11}^{1-w_1}$	$b_{i12}^{1-u_1} v_{i12}^{1-w_1}$	$b_{i13}^{1-u_1} v_{i13}^{1-w_1}$	$b_{i21}^{1-u_2} v_{i21}^{1-w_2}$	$b_{i22}^{1-u_2} v_{i22}^{1-w_2}$	$b_{i23}^{1-u_2} v_{i23}^{1-w_2}$	$b_{i31}^{1-u_3} v_{i31}^{1-w_3}$	$b_{i32}^{1-u_3} v_{i32}^{1-w_3}$	$b_{i33}^{1-u_3} v_{i33}^{1-w_3}$	$\prod_{k=1}^{3} b_{i1k}^{1-u_1} v_{i1k}^{1-w_1}$	$\prod_{k=1}^{3} b_{i2k}^{1-u_2} v_{i2k}^{1-w_2}$	$\prod_{k=1}^{3} b_{i3k}^{1-u_3} v_{i3k}^{1-w_3}$	$\prod_{j=1}^{3}\prod_{k=1}^{3} b_{ijk}^{1-u_j} v_{ijk}^{1-w_j}$	-log10 †
0.5287	0.4246	0.2861	0.3452	0.2603	0.0862	0.0251	0.2455	0.3068	0.0642	0.0077	0.0019	9.425E-07	6.02572
0.49	0.185	0.4625	0.039	0.1038	0.0009	0.0599	0.1817	0.3178	0.0419	3E-06	0.0035	5.036E-10	9.29793
0.0576	0.0403	0.0721	0.0049	0.0359	0.0108	0.9311	0.4134	0.8508	0.0002	2E-06	0.3275	1.035E-10	9.98518
0.3491	0.3001	0.6144	0.0876	0.0377	0.0176	0.2167	0.3344	0.1741	0.0644	6E-05	0.0126	4.712E-08	7.32683
0.4225	0.2914	0.5768	0.2107	0.0171	0.2629	0.213	0.2258	0.045	0.071	0.0009	0.0022	1.453E-07	6.83763
0.0239	0.023	0.3582	0.0155	0.049	0.0447	0.2097	0.4109	0.2572	0.0002	3E-05	0.0222	1.475E-10	9.83112
0.166	0.1161	0.0945	0.0288	0.0213	0.0126	0.042	0.5028	0.0976	0.0018	8E-06	0.0021	2.894E-11	10.5385
0.6034	0.332	0.6835	0.4184	0.0846	0.2084	0.3851	0.1717	0.6513	0.1369	0.0074	0.0431	4.347E-05	4.36179
0.7889	0.3086	0.8988	0.0845	0.0104	0.0009	0.2775	0.2517	0.0655	0.2188	8E-07	0.0046	7.969E-10	9.09862
0.2598	0.311	0.3269	0.3117	0.0829	0.0937	0.0621	0.6133	0.601	0.0264	0.0024	0.0229	1.464E-06	5.83438
0.1777	0.1989	0.0786	0.0477	0.4536	0.0094	0.3143	0.2126	0.2124	0.0028	0.0002	0.0142	8.014E-09	8.09613
0.5853	0.3103	0.2371	0.0742	0.1253	0.1896	0.2411	0.0579	0.6623	0.0431	0.0018	0.0093	7.025E-07	6.15334
0.7542	0.8841	0.2174	0.0018	0.0027	0.0467	0.0447	0.0694	0.0996	0.145	2E-07	0.0003	1.016E-11	10.9933
0.1788	0.2781	0.1555	0.1417	0.1247	0.314	0.2759	0.1003	0.2145	0.0077	0.0055	0.0059	2.547E-07	6.59401
0.5435	0.2652	0.5249	0.3291	0.2316	0.3639	0.1049	0.7946	0.3196	0.0757	0.0277	0.0266	5.589E-05	4.2527
0.1074	0.1242	0.2737	0.0121	0.0078	0.0243	0.1385	0.3479	0.3003	0.0037	2E-06	0.0145	1.206E-10	9.91879
0.1068	0.0444	0.0144	0.0263	0.171	0.053	0.0963	0.1376	0.2151	7E-05	0.0002	0.0029	4.656E-11	10.3319
0.4558	0.1542	0.1452	0.0316	0.0048	0.0547	0.2255	0.3095	0.1621	0.0102	8E-06	0.0113	9.687E-10	9.01383

Also, Their Products Across Signal Factor Levels $\prod_{k=1}^{3} b_{ijk}^{1-u_j} v_{ijk}^{1-w_j}$, and Products Across Responses $\prod_{j=1}^{3}\prod_{k=1}^{3} b_{ijk}^{1-u_j} v_{ijk}^{1-w_j}$

Last column contains negative log to base of ten of $\prod_{j=1}^{3}\prod_{k=1}^{3} b_{ijk}^{1-u_j} v_{ijk}^{1-w_j}$

Normalized Biases b_{ijk} and Normalized Variances v_{ijk} and Given Unequal Weights

0.18	**0.07**	**0.18**	**0.07**	**0.18**	**0.07**	**0.10**	**0.65**	**0.10**	**0.65**	**0.10**	**0.65**	**0.50**	**0.00**	**0.50**	**0.00**	**0.50**	**0.00**
LTB Response Y_1						**NTB Response Y_2**						**STB Response Y_3**					
0.25			**Weight**			**0.75**			**Weight**			**0.5**		**1.5**	**Weight**		
M_1=10		**M_2=20**		**M_3=30**		**M_1=10**		**M_2=20**		**M_3=30**		**M_1=10**		**M_2=20**		**M_3=30**	
b_{i11}	v_{i11}	b_{i12}	v_{i12}	b_{i13}	v_{i13}	b_{i21}	v_{i21}	b_{i22}	v_{i22}	b_{i23}	v_{i23}	b_{i31}	v_{i31}	b_{i32}	v_{i32}	b_{i33}	v_{i33}
1	0.4275	0.9915	0.3219	0.8111	0.2325	0.3075	0.7874	0.1662	1	0.373	0.1021	0.8438	0.00873	0.8006	0.192	0.5572	0.3713
0.7023	0.55	0.6439	0.1637	0.7377	0.4849	0.0846	0.1563	0.1559	0.3128	0.0032	0.0251	0.8438	0.02778	0.5507	0.1869	0.6055	0.3582
0.7257	0.0306	0.7289	0.019	0.8222	0.0365	0.0648	0.0128	0.0778	0.1521	0.0385	0.0617	0.9091	1	0.8483	0.3631	0.8062	1
0.8362	0.294	0.7164	0.2805	0.872	0.599	0.7459	0.0522	0.8201	0.0154	1	0.0046	1	0.13016	0.7003	0.3315	0.8707	0.1117
0.7225	0.4388	1	0.1932	0.954	0.5032	0.3935	0.3186	1	0.0044	0.6011	0.2802	0.954	0.13333	0.5586	0.2462	0.5625	0.0285
0.5918	0.0116	0.6884	0.0095	0.7277	0.3496	0.0437	0.0881	0.1179	0.152	0.0491	0.3232	0.7009	0.17778	0.6233	0.4901	0.4959	0.3297
0.8121	0.1124	0.7466	0.0758	0.8338	0.0517	0.405	0.0218	0.227	0.026	0.1963	0.0149	0.6816	0.02143	1	0.3998	0.6445	0.0697
0.5699	0.8947	0.5354	0.4293	0.6073	0.9914	0.6694	0.4675	0.2229	0.1666	0.2077	0.5949	0.9314	0.30079	0.64	0.1491	0.7082	0.7972
0.7289	1	0.8515	0.2449	0.8673	1	0.215	0.1726	0.1092	0.0209	0.0212	0.0041	0.7404	0.24444	0.8581	0.1852	0.9582	0.0276
0.5662	0.2928	0.6292	0.3349	0.7778	0.2896	1	0.2113	0.5261	0.0687	0.6563	0.0649	0.7205	0.03413	0.9378	0.5556	0.8446	0.6006
0.8446	0.1183	0.7248	0.1602	0.7732	0.0435	0.0239	0.7233	0.5513	0.6321	0.0044	0.4492	0.6816	0.31349	0.8678	0.1462	0.5313	0.2386
0.7271	0.6733	0.776	0.2707	0.952	0.1541	0.241	0.1294	0.3357	0.1868	0.3281	0.332	0.5713	0.2627	0.7091	0.0316	0.6333	0.9117
0.8019	0.8562	0.8485	1	0.9867	0.1325	0.01	0.0215	0.0036	0.106	0.1641	0.1026	1	0.01587	0.9895	0.0288	0.9445	0.0489
0.7202	0.1399	0.8304	0.2186	0.9589	0.0872	0.2235	0.3305	0.2652	0.2349	0.2135	1	0.9314	0.19286	0.5429	0.0858	0.5838	0.2199
0.6603	0.6716	0.4433	0.3844	0.6619	0.6397	0.328	0.6926	0.4035	0.3526	0.4803	0.5408	0.8653	0.05714	0.736	1	0.5892	0.3708
0.8611	0.0593	0.618	0.1003	0.9495	0.1871	0.0744	0.0371	0.0442	0.0349	0.0397	0.1773	0.8438	0.08492	0.727	0.3365	0.5678	0.3542
0.8926	0.0568	0.794	0.0198	1	0.0035	0.1749	0.0448	0.3161	0.3003	0.2693	0.074	0.6625	0.06667	0.736	0.0966	0.491	0.2625
0.701	0.5004	0.6547	0.1263	0.6927	0.1102	0.01	1	0.006	0.1374	0.0594	0.3499	0.8871	0.15476	0.6316	0.3315	1	0.0884

Weighted Normalized Biases $b_{ijk}^{1-u_j}$ and Weighted Normalized Variances $v_{ijk}^{1-w_j}$ Considering Unequal Weights

LTB Response Y_1						NTB Response Y_2						STB Response Y_3					
M_1=10		M_2=20		M_3=30		M_1=10		M_2=20		M_3=30		M_1=10		M_2=20		M_3=30	
$b_{i11}^{1-u_1}$	$v_{i11}^{1-u_1}$	$b_{i12}^{1-u_1}$	$v_{i12}^{1-u_1}$	$b_{i13}^{1-u_1}$	$v_{i13}^{1-u_1}$	$b_{i21}^{1-u_2}$	$v_{i21}^{1-u_2}$	$b_{i22}^{1-u_2}$	$v_{i22}^{1-u_2}$	$b_{i23}^{1-u_2}$	$v_{i23}^{1-u_2}$	$b_{i31}^{1-u_3}$	$v_{i31}^{1-u_3}$	$b_{i32}^{1-u_3}$	$v_{i32}^{1-u_3}$	$b_{i33}^{1-u_3}$	$v_{i33}^{1-u_3}$
1.0000	0.4537	0.9930	0.3484	0.8423	0.2575	0.3460	0.9197	0.1989	1.0000	0.4117	0.4500	0.9186	0.0087	0.8947	0.1920	0.7465	0.3713
0.7485	0.5735	0.6970	0.1858	0.7792	0.5101	0.1083	0.5222	0.1877	0.6658	0.0058	0.2754	0.9186	0.0278	0.7421	0.1869	0.7782	0.3582
0.7688	0.0391	0.7716	0.0250	0.8517	0.0460	0.0852	0.2176	0.1005	0.5173	0.0533	0.3771	0.9535	1.0000	0.9211	0.3631	0.8979	1.0000
0.8635	0.3203	0.7607	0.3066	0.8938	0.6209	0.7681	0.3558	0.8365	0.2321	1.0000	0.1517	1.0000	0.1302	0.8368	0.3315	0.9331	0.1117
0.7660	0.4648	1.0000	0.2168	0.9621	0.5280	0.4319	0.6701	1.0000	0.1496	0.6325	0.6407	0.9767	0.1333	0.7474	0.2462	0.7500	0.0285
0.6504	0.0159	0.7363	0.0132	0.7706	0.3763	0.0598	0.4272	0.1460	0.5171	0.0663	0.6735	0.8372	0.1778	0.7895	0.4901	0.7042	0.3297
0.8431	0.1310	0.7869	0.0908	0.8616	0.0636	0.4433	0.2620	0.2632	0.2787	0.2310	0.2295	0.8256	0.0214	1.0000	0.3998	0.8028	0.0697
0.6306	0.9016	0.5992	0.4555	0.6643	0.9920	0.6968	0.7663	0.2590	0.5340	0.2430	0.8338	0.9651	0.3008	0.8000	0.1491	0.8415	0.7972
0.7716	1.0000	0.8765	0.2703	0.8898	1.0000	0.2507	0.5407	0.1363	0.2583	0.0312	0.1462	0.8605	0.2444	0.9263	0.1852	0.9789	0.0276
0.6273	0.3190	0.6839	0.3615	0.8137	0.3158	1.0000	0.5804	0.5610	0.3918	0.6845	0.3839	0.8488	0.0341	0.9684	0.5556	0.9190	0.6006
0.8707	0.1374	0.7680	0.1821	0.8098	0.0542	0.0347	0.8928	0.5851	0.8517	0.0076	0.7557	0.8256	0.3135	0.9316	0.1462	0.7289	0.2386
0.7700	0.6922	0.8122	0.2967	0.9605	0.1757	0.2778	0.4888	0.3744	0.5559	0.3668	0.6798	0.7558	0.2627	0.8421	0.0316	0.7958	0.9117
0.8344	0.8655	0.8740	1.0000	0.9891	0.1527	0.0158	0.2606	0.0063	0.4559	0.1966	0.4507	1.0000	0.0159	0.9947	0.0288	0.9718	0.0489
0.7640	0.1605	0.8586	0.2432	0.9662	0.1034	0.2596	0.6787	0.3029	0.6023	0.2491	1.0000	0.9651	0.1929	0.7368	0.0858	0.7641	0.2199
0.7115	0.6906	0.5132	0.4110	0.7130	0.6601	0.3667	0.8794	0.4418	0.6943	0.5168	0.8064	0.9302	0.0571	0.8579	1.0000	0.7676	0.3708
0.8846	0.0723	0.6739	0.1178	0.9584	0.2104	0.0965	0.3158	0.0604	0.3089	0.0549	0.5458	0.9186	0.0849	0.8526	0.3365	0.7535	0.3542
0.9110	0.0694	0.8277	0.0260	1.0000	0.0052	0.2082	0.3373	0.3547	0.6563	0.3071	0.4020	0.8140	0.0667	0.8579	0.0966	0.7007	0.2625
0.7473	0.5252	0.7066	0.1460	0.7401	0.1286	0.0158	1.0000	0.0100	0.4992	0.0787	0.6925	0.9419	0.1548	0.7947	0.3315	1.0000	0.0884

Products of Weighted Normalized Biases and Weighted Normalized Variances, $b_{ijk}^{1-u_j} v_{ijk}^{1-w_j}$ Considering Equal Weights

LTB Response Y_1			NTB Response Y_2			STB Response Y_3			Y_1	Y_2	Y_3	$Y_1Y_2Y_3$	
M_1	M_2	M_3	M_2	M_1	M_3	M_1	M_2	M_3				= †	
$b_{i11}^{1-u_1} v_{i11}^{1-w_1}$	$b_{i12}^{1-u_1} v_{i12}^{1-w_1}$	$b_{i13}^{1-u_1} v_{i13}^{1-w_1}$	$b_{i21}^{1-u_2} v_{i21}^{1-w_2}$	$b_{i22}^{1-u_2} v_{i22}^{1-w_2}$	$b_{i23}^{1-u_2} v_{i23}^{1-w_2}$	$b_{i31}^{1-u_3} v_{i31}^{1-w_3}$	$b_{i32}^{1-u_3} v_{i32}^{1-w_3}$	$b_{i33}^{1-u_3} v_{i33}^{1-w_3}$	$\prod_{k=1}^{3} b_{i1k}^{1-u_1} v_{i1k}^{1-w_1}$	$\prod_{k=1}^{3} b_{i2k}^{1-u_2} v_{i2k}^{1-w_2}$	$\prod_{k=1}^{3} b_{i3k}^{1-u_3} v_{i3k}^{1-w_3}$	$\prod_{j=1}^{3}\prod_{k=1}^{3} b_{ijk}^{1-u_j} v_{ijk}^{1-w_j}$	-log10 †
0.4537	0.346	0.2168	0.3182	0.1989	0.1852	0.008	0.1718	0.2772	0.034	0.0117	0.0004	1.524E-07	6.816902
0.4292	0.1295	0.3975	0.0566	0.125	0.0016	0.0255	0.1387	0.2787	0.0221	1E-05	0.001	2.443E-10	9.611992
0.0301	0.0193	0.0392	0.0185	0.052	0.0201	0.9535	0.3344	0.8979	2E-05	2E-05	0.2863	1.263E-10	9.898751
0.2766	0.2332	0.5549	0.2733	0.1941	0.1517	0.1302	0.2774	0.1042	0.0358	0.008	0.0038	1.084E-06	5.965167
0.3561	0.2168	0.508	0.2894	0.1496	0.4052	0.1302	0.184	0.0214	0.0392	0.0175	0.0005	3.521E-07	6.45332
0.0103	0.0097	0.29	0.0255	0.0755	0.0447	0.1488	0.3869	0.2322	3E-05	9E-05	0.0134	3.344E-11	10.47575
0.1104	0.0715	0.0548	0.1161	0.0734	0.053	0.0177	0.3998	0.0559	0.0004	0.0005	0.0004	7.724E-11	10.11213
0.5686	0.2729	0.659	0.534	0.1383	0.2026	0.2903	0.1193	0.6709	0.1023	0.015	0.0232	3.554E-05	4.449247
0.7716	0.2369	0.8898	0.1355	0.0352	0.0046	0.2103	0.1716	0.027	0.1627	2E-05	0.001	3.441E-09	8.463378
0.2001	0.2473	0.257	0.5804	0.2198	0.2628	0.029	0.5381	0.5519	0.0127	0.0335	0.0086	3.667E-06	5.435729
0.1196	0.1399	0.0439	0.031	0.4983	0.0057	0.2588	0.1362	0.1739	0.0007	9E-05	0.0061	3.992E-10	9.398806
0.533	0.241	0.1688	0.1358	0.2081	0.2493	0.1986	0.0266	0.7255	0.0217	0.007	0.0038	5.86E-07	6.232092
0.7222	0.874	0.151	0.0041	0.0029	0.0886	0.0159	0.0286	0.0475	0.0953	1E-06	2E-05	2.177E-12	11.6622
0.1226	0.2088	0.0999	0.1762	0.1824	0.2491	0.1861	0.0632	0.168	0.0026	0.008	0.002	4.053E-08	7.392212
0.4914	0.2109	0.4706	0.3225	0.3067	0.4168	0.0532	0.8579	0.2846	0.0488	0.0412	0.013	2.611E-05	4.583269
0.0639	0.0794	0.2017	0.0305	0.0187	0.03	0.078	0.2869	0.2669	0.001	2E-05	0.006	1.042E-10	9.982284
0.0632	0.0216	0.0052	0.0702	0.2328	0.1234	0.0543	0.0828	0.1839	7E-06	0.002	0.0008	1.185E-11	10.92639
0.3925	0.1032	0.0952	0.0158	0.005	0.0545	0.1458	0.2634	0.0884	0.0039	4E-06	0.0034	5.621E-11	10.25022

Also, Their Products Across Signal Factor Levels $\prod_{k=1}^{3} b_{ijk}^{1-u_j} v_{ijk}^{1-w_j}$, and Products Across Responses $\prod_{j=1}^{3}\prod_{k=1}^{3} b_{ijk}^{1-u_j} v_{ijk}^{1-w_j}$

Last column contains negative log to base of ten of $\prod_{j=1}^{3}\prod_{k=1}^{3} b_{ijk}^{1-u_j} v_{ijk}^{1-w_j}$

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