Development of New Stiffness and Damping Expressions for Footing Vibrations

H. V. P. Truong
VicRoads, Australia

P. J. Moore
The University of Melbourne, Australia

Proceedings: Third International Conference on Recent Advances in Geotechnical Earthquake Engineering and Soil Dynamics, April 2-7, 1995, Volume II, St. Louis, Missouri

Development of New Stiffness and Damping Expressions for Footing Vibrations

Paper No. 12.04

H.V.P. Truong
Civil Engineer, VicRoads, Australia

P.J. Moore
Former Reader in Civil Engineering, The University of Melbourne, Australia

SYNOPSIS New expressions for stiffness and radiation damping, which have been developed for a new forcing function, are based on the simple equation of wave propagation in a perfectly elastic half space for different modes of vibrations, particularly vertical and horizontal vibrations. A differential equation including the effect of foundation mass is presented and the results of the amplitudes of vibration obtained from this differential equation are compared with those in the standard differential equation in soil dynamics textbooks. Furthermore, added soil masses for vertical and horizontal vibrations are also derived based on the equation of wave propagation and discussed with other findings.

Finally, this paper also compared different ways of using the total damping, which is composed of radiation damping and internal damping.

INTRODUCTION

The periodic function of $P_o e^{i\omega t}$ has been used extensively in deriving old lumped parameters, and predicting resonant frequency and amplitude of vibration of foundations (Arnold et al. (1955), Bycroft (1956 and 1959), Hsieh (1962), Hall (1967) and Dobry and Gazetas (1985 and 1986)). In this paper, a new periodic forcing function, $P(t) = P_o T + P_o \sin\omega t$, has been introduced. Using the treatment of Lysmer and Richart (1966) based on a simple equation of wave propagation with an appropriate correction of pressure distribution, new expressions have been generated for stiffness and damping for use in calculations of permanent displacements of different modes of vibrations and to confirm the findings by Truong (1991a). Note that expressions of stiffness and damping are completely dependent on the type of the forcing function (Truong, 1992).

For the new forcing function, Truong (1991a), using the method by Hsieh (1962), derived new dynamic stiffness and damping values, which are dependent on the frequency of the dynamic load. The new dynamic damping value, which is inversely proportional to the circular frequency, has the same form as material damping, while the value of stiffness increases with increase in the circular frequency. Similar methods can be used to obtain the new dynamic spring and damping values for other modes of vibrations such as vertical, rocking and torsional (Truong, 1992). The new expressions of stiffness and damping have been introduced into calculations of horizontal permanent displacement (Truong, 1991b, c and d and Truong 1992).

The effect of mass of footing system on new expression of stiffness and damping for the new forcing function, which was not covered in Truong (1991a), are investigated by using the treatment by Hsieh (1962). This process leads to a new differential equation in soil dynamics.

NEW HORIZONTAL STIFFNESS AND ADDED SOIL MASS

If a horizontal force P(t) applied to the footing has the following form:

$$P(t) = P_o T + P_o \sin\omega t \qquad (1)$$

where T and P_o are constants, $P_o T$ is a constant load, ω is the circular frequency of the forcing function and t is the time variable.

Using the symbol Re to mean "the real part of", then

$$P(t) = P_o T + Re(-iP_o e^{i\omega t}) \qquad (2)$$

Since expressions of stiffness and damping are related to the values of the oscillatory part of Eq. (1), and if T = 0, then

$$P(t) = Re(-iP_o e^{i\omega t}) = P_o \sin\omega t \qquad (3)$$

The way to find a new horizontal stiffness for the new forcing function (Eq.1) is to consider the simple equation of wave propagation of horizontal vibration in a perfectly elastic half space. This method, presented by Lysmer and Richart (1966) for vertical vibration, will also be repeated in determining the new vertical stiffness in the next section.

The elastic half space is excited by a horizontal load, $P_o e^{i\omega t}$, and the horizontal displacement u of the free surface is to be evaluated. Assuming linear elasticity of the material, the shear stress can be given by

$$\tau = G\frac{du}{dt} \qquad (4)$$

in which G is the shear modulus and du/dt stands for the shear strain.

The horizontal displacement u on a horizontal plane is

$$u(z) = A_h e^{i\omega(t - \frac{z}{V_s})} \qquad (5)$$

which is the general expression for a sinusoidal wave with amplitude A_h propagated horizontally with the constant shear velocity V_s.

Taking the derivative of Eq. (5),

$$\frac{du}{dz} = -\frac{i\omega}{V_s}A_k e^{i\omega(t-\frac{z}{V_s})} \tag{6}$$

The shear stress on a horizontal plane is

$$\tau(z) = G\frac{du}{dz} = -\frac{i\omega}{V_s}GA_k e^{i\omega(t-\frac{z}{V_s})} \tag{7}$$

and the stress boundary condition at the surface (z=0) yields, consequently,

$$\tau(0) = -\frac{P_o e^{i\omega t}}{A} = -\frac{i\omega}{V_s}GA_k e^{i\omega t} \tag{8}$$

in which A is the area of the footing.
Then

$$A_k = -i\frac{P_o V_s}{A\omega G} \tag{9}$$

The horizontal displacement u(0) at the surface is

$$u(0) = \frac{V_s}{A\omega G}(-iP_o e^{i\omega t}) \tag{10}$$

Eq. (10) has shown that the new stiffness, K'_x of the new forcing function (Eq.1) can be expressed as follows:

$$K'_x = \frac{A\omega G}{V_s} = A\omega\sqrt{\rho G} \tag{11}$$

which is independent of the Poisson's ratio, μ.

For a circular footing with uniform pressure distribution,

$$K'_x = \pi r_o^2\omega\sqrt{\rho G} \tag{12}$$

For a rectangular footing,

$$K'_x = BL\omega\sqrt{\rho G} \tag{13}$$

For a circular footing with a rigid base pressure distribution, the stiffness increases with the change in pressure distribution from uniform to rigid base as found by Nagendra et al. (1982), so the new stiffness with their central displacement condition is as follows:

$$K'_x = \omega r_o^2\sqrt{\rho G}(\pi\frac{8}{2\pi}) = 4\omega r_o^2\sqrt{\rho G} \tag{14}$$

First derivative of Eq. (10),

$$\frac{du(0)}{dt} = \frac{1}{A\sqrt{\rho G}}P_o e^{i\omega t} \tag{15}$$

Eq. (15) shows the old value of damping for the old function, $P_o e^{i\omega t}$, is proportional to $(\rho G)^{1/2}$ (Richart et al., 1970).

Taking the second derivative of the horizontal displacement u(0),

$$\frac{d^2u}{dt^2} = -\frac{\omega}{A\sqrt{\rho G}}(-iP_o e^{i\omega t}) \tag{16}$$

This is also the equation of motion for the simple mass, which has been considered as the added soil mass for the old function. This mass, m_x' - increasing with the area of the footing, shear modulus and density, and decreasing with the circular frequency - is presented as follows:

$$m_x' = \frac{A\sqrt{\rho G}}{\omega} \tag{17}$$

The new value of damping, C'_x, to be used with the new horizontal stiffness (Eq. 14), is given by Eq. (18) as follows:

$$C'_x = \frac{8}{(2-\mu)}\frac{Gr_o}{\omega} \tag{18}$$

The derivation for the equation (18) can be found in Truong (1991a or 1992).

THE EFFECT OF MASS OF THE FOOTING SYSTEM

Lysmer and Richart (1966) extended the work by Hsieh (1962) by including the effect of the mass m of the footing. The displacement, x of the mass m can be written in the form

where F^* is the displacement function and k is a quantity having the

$$x = \frac{P_o F^* e^{i\omega t}}{k} \tag{19}$$

dimension force/length.

The quantity k, referred to as the "spring constant", is usually equal to the static spring constant (Lysmer et al., 1966). The displacement function can be expressed as

$$F^* = \frac{(F_1^* + iF_2^*)}{[1-(\frac{m\omega^2}{k})(F_1^* + iF_2^*)]} \tag{20}$$

where F_1^* and F_2^* are functions of the dimensionless constant **a** and Poisson's ratio μ.

$$F^* = \frac{(a_4 + ia_5)}{(a_2^2 + a_3^2)} \tag{21}$$

$$a_2 = \frac{m\omega^2 F_2^*}{k} \tag{22}$$

$$a_3 = (1 - \frac{m\omega^2 F_1^*}{k}) \tag{23}$$

$$a_4 = a_3 F_1^* - a_2 F_2^* \tag{24}$$

$$a_5 = a_2 F_1^* + a_3 F_2^* \tag{25}$$

Different expressions for stiffness and damping for two different forcing functions have been derived in the following section.

(a). The forcing function in Eq. (19) is $P(t) = P_o e^{i\omega t}$ and the relationship between displacement, velocity and force is as follows:

The general expressions for stiffness K and damping C for different modes have been derived (Lysmer et al., 1966) as follows:

$$a_4 \omega x - a_5 \frac{dx}{dt} = \frac{P_o e^{i\omega t} \omega}{k} \frac{(a_4^2 + a_5^2)}{(a_3^2 + a_2^2)} \tag{26}$$

$$K = \frac{F_1^* k}{(F_1^{*2} + F_2^{*2})} - m\omega^2 \tag{27}$$

$$C = \frac{F_2^* k}{\omega (F_1^{*2} + F_2^{*2})} \tag{28}$$

K and C will be K_x, C_x, K_z, C_z, K_θ and C_θ, if k is used as the static constant of horizontal, vertical and rocking vibration, respectively.

(b). If the forcing function is given by Eq. (1), then Eq. (25) becomes

$$a_5 \omega x + a_4 \frac{dx}{dt} = -\frac{\omega}{k} \frac{(a_4^2 + a_5^2)}{(a_3^2 + a_2^2)} P_o \sin\omega t \tag{29}$$

The new general values of stiffness K' and damping C' are

$$K' = \frac{F_2^* k}{(F_1^{*2} + F_2^{*2})} \tag{30}$$

$$C' = \frac{F_1^* k}{\omega (F_1^{*2} + F_2^{*2})} - m\omega \tag{31}$$

For vertical vibration, K' and C' will become K'_z and C'_z, respectively. The undamped natural frequency, ω_o, is the same value (= $(k/m)^{1/2}$) for a simplified analog, when C' = 0. It seems much more logical than letting K = 0 (Eq.27). The static vertical constant k (Timoshenko & Goodier, 1951) is

Then Lysmer and Richart (1966) also found that the variation of F^* with μ is insignificant for practical calculations. For vertical vibration,

$$k = \frac{4 G r_o}{(1 - \mu)} \tag{32}$$

$$F^* = \frac{4 (f_1 + i f_2)}{(1 - \mu)} \tag{33}$$

where f_1 and f_2 are functions (first introduced by Reissner (1936) which depend upon the Poisson's ratio μ of the medium and a dimensionless quantity **a** described by

$$a = \omega r_o \sqrt{\frac{\rho}{G}} \tag{34}$$

NEW VERTICAL STIFFNESS BY THE EQUATION OF WAVE PROPAGATION

The method, presented by Lysmer and Richart (1966) for vertical vibration, is repeated to determine the new vertical stiffness in the following section.

The elastic half space is excited by a vertical load, $P_o e^{i\omega t}$, and the displacement w of the free surface is to be evaluated. Assuming elasticity of the material, the tensile stress can be given by

$$\sigma(z) = E_p \frac{dw}{dz} \tag{35}$$

where E_p, which is the constrained elastic modulus with zero lateral displacement, is as follows:

$$E_p = \frac{G}{s^2} \tag{36}$$

in which

$$s^2 = \frac{1 - 2\mu}{2(1 - \mu)} \tag{37}$$

The velocity of the push wave (P-waves) propagated downwards is

$$V_p = \sqrt{\frac{E_p}{\rho}} = \frac{1}{s}\sqrt{\frac{G}{\rho}} = \frac{V_s}{s} \tag{38}$$

The vertical displacement, w, on a horizontal plane with no reflected wave due to infinite half space is

$$w(z) = A_v e^{i\omega\left(t - \frac{z}{V_p}\right)} \tag{39}$$

which is the general expression for a sinusoidal wave with amplitude A_v propagated downwards with the constant velocity V_p. Taking the derivative of Eq.(39),

$$\frac{dw}{dz} = -\frac{i\omega}{V_p}A_v e^{i\omega(t-\frac{z}{V_p})} \tag{40}$$

The tensile stress on a horizontal plane is

$$\sigma(z) = E_p\frac{dw}{dz} = -\frac{i\omega}{V_p}E_p A_v e^{i\omega(t-\frac{z}{V_p})} \tag{41}$$

and the stress boundary condition at the surface (z=0) yields, consequently,

$$\sigma(0) = -\frac{P_o e^{i\omega t}}{A} = -\frac{i\omega}{V_p}E_p A_v e^{i\omega(t-\frac{z}{V_p})} \tag{42}$$

then

$$A_v = -i\frac{P_o V_p}{A\,\omega E_p} = \frac{-isP_o}{A\,\omega\sqrt{\rho G}} \tag{43}$$

The vertical displacement w(0) at the surface is

$$w(0) = \frac{s}{A\,\omega\sqrt{\rho G}}(-iP_o e^{i\omega t}) \tag{44}$$

Eq.(44) has shown that the new stiffness of the new forcing function can be expressed as follows:

$$K_z' = \frac{A\,\omega\sqrt{\rho G}}{s} \tag{45}$$

For circular footing with uniform pressure distribution,

$$K_z' = \frac{\pi}{s}r_o^2\omega\sqrt{\rho G} \tag{46}$$

For circular footing with rigid base pressure distribution, the stiffness increases with the change in pressure distribution from uniform to rigid base (Nagendra et al., 1981), so the new stiffness can be expressed as follows:

$$K_z' = \frac{32}{3s\pi}r_o^2\omega\sqrt{\rho G} \tag{47}$$

For $\mu = 1/3$, the ratio of the coefficient of new stiffness and that of new stiffness is 0.88 compared with 0.85 by Lysmer et al. (1966) for $0.3 < a < 0.8$.

First derivative of Eq.(44),

$$\frac{dw}{dt} = \frac{s}{A\sqrt{\rho G}}P_o e^{i\omega t} \tag{48}$$

Eq.(48) shows the that old value of damping for the old function, $P_o\, e^{i\omega t}$, is proportional to $(\mu\ G)^{1/2}$ (Richart et al. 1970).

Taking second derivative of the vertical displacement w(0),

$$\frac{d^2w}{dt^2} = -\frac{\omega s}{A\sqrt{\rho G}}(-iP_o e^{i\omega t}) \tag{49}$$

which is also the equation of motion for the simple mass, which has been considered as the added soil mass for the old function. This mass, m_z' - increasing with the area of the footing, shear modulus and density, and decreasing with the circular frequency - is presented as follows:

$$m_z' = \frac{A\sqrt{\rho G}}{\omega s} \tag{50}$$

The horizontal added soil mass is smaller than that of the vertical vibration by a factor, s. The new value of damping to be used with the new horizontal stiffness, Eq.(47), is as follows (Truong, 1991a or 1992)

$$C_z' = \frac{4}{1-\mu}\left(\frac{Gr_o}{\omega}\right) \tag{51}$$

NEW DIFFERENTIAL EQUATION

The old lumped parameters (Hsieh, 1962 and Richart, et al. 1970) are derived based on the forcing function, $P_o\, e^{i\omega t}$, so the initial value of stiffness ($\omega t = 0$) is the static stiffness, which is mainly deduced using the theory of elasticity (Nagendra and Sridharan, 1982). The static stiffness, consisting of the initial (or maximum) shear modulus, is the maximum value of stiffness for any footing system under vibration. The stiffness for this case will reduce with the increase in the value of ωt from 0 to $\pi/2$. The old lumped parameters are useful in many cases involving with elastic displacement, e.g. for predicting the maximum amplitude of vibration and resonant frequency. For most cases, the value of damping is very small and can be considered as zero (Barkan, 1962).

The new forcing function, $P_o \sin\omega t$, brings in the new stiffness, which becomes 0 when the term, ωt, is equal to 0. While the value of damping becomes very large, as ω approaches to 0. When $\omega t = \pi/2$ or at high frequencies, the value of stiffness is equal to the static stiffness. The new lumped parameters should be used for cases related to plastic displacement, e.g. in the calculation of horizontal and vertical permanent displacements. Initial static shear modulus (or Young's modulus) has to be used for cases with low frequencies, e.g. smaller than 1 Hz.

There are some cases, especially with high frequencies, either one of the two methods will end up with the same results in predicting permanent displacements or resonant frequency. Actually, the amplitude of vibration and resonant frequency, obtained from the following differential equation for the new forcing function (Eq.1) based on Eq.31 for each individual mode of vibration, actually are the same values.

$$(\frac{k_x}{\omega} - m\omega)\frac{dx}{dt} + K'_x x = P_o T + P_o \sin\omega t \qquad (52)$$

where k_x and K'_x are the values of static and new dynamic stiffness for the horizontal vibration, respectively.

If T = 0, the following equation of the amplitude of vibration, obtained after solving the above equation (Truong, 1992), can be found in standard textbook, e.g. Richart et al. (1970)

$$A = \frac{P_o}{\sqrt{(k_x - m\omega^2)^2 + K'^2_x}} \qquad (53)$$

where the square of the new dynamic stiffness K'_x based on Veletsos and Wei (1971) (Truong, 1991a or 1992) is

$$K'^2_x = [\frac{4.6 r_o^2 \sqrt{\rho G}}{(2-\mu)}]^2 \omega^2 \qquad (54)$$

The phase angle between the new forcing function (Eq.1) and displacement, which is different from the old one in text books, e.g. Richart et al. (1970) about 90 degrees, is

$$\tan\phi = \frac{k_x - m\omega^2}{K'_x} = \frac{1 - (\frac{\omega}{\omega_o})^2}{\frac{K'_x}{\omega_o^2}} \qquad (55)$$

where ω_o is the undamped natural frequency as derived from Eq.31 (= $(k/m)^{1/2}$). The frequency at which the maximum amplitude occurs is not the undamped natural circular frequency ω_o, but a frequency slightly less than ω_o. Note that the damping ratio D, can be derived from the denominator of Eq. (55).

The coefficients of old and new values of stiffness and damping, derived in the above sections, are not very important compared with the decision to choose and use the appropriate value of dynamic shear modulus. Other coefficients for stiffness and damping can be found in Nagendra et al. (1982 and 1984) and Gazetas (1987b).

ADDED SOIL MASSES

The above added soil mass, sometimes called an apparent mass or the in-phase mass of soil, which vibrates in sympathy with an oscillating footing. The added soil masses for horizontal and vertical vibrations, which are linearly proportional to the area of the footing and inversely proportional to the circular frequency of the footing, increase with the square root of the density and the shear modulus of the soil (Eqs. 17 and 50). This findings have relatively combined all of those of others, e.g. Pauw (1953) found that the vertical added soil mass is linearly proportional to unit weight and inversely proportional to the gravity. While Barkan (1962) introduced the added soil mass in terms of a coefficient of mass increase, the latter increases as the mass ratio decreases. Barkan and Ilyichev (1977) have expressed the view that this mass, which cannot be ignored in engineering calculations, depends on the sizes of foundation, its embedment, geological formation of a construction site and soil properties and does not depend on the foundation mass. They calculated values of the coefficient of mass increase ranging from 1.5 to 4.1. Finally, Golubtsova (1986) also found that the vertical added soil mass are dependent on the circular frequency, but in a rather complicated form than the ones above.

TOTAL DAMPING

The damping ratio normally represented by viscous dashpot in the mass-spring-dashpot model represents the total damping which is a combination of radiation damping and internal damping. For horizontal vibration, Sankaran et al. (1977) presented that the internal damping is also inversely proportional to the circular frequency as the new expressions of damping. Dobry et al. (1985) have suggested the values of stiffness and damping corrected with the internal damping constant as follows:

$$K^*_x = K_x - \frac{\omega C_x b}{2} \qquad (56)$$

$$C^*_x = C_x + \frac{b K_x}{\omega} \qquad (57)$$

where b is the material damping constant.

CONCLUSIONS

1. Expressions of stiffness and damping are dependent on the forcing function, but the resonant frequency and the maximum amplitude of vibration did not change, and the exceptions are the permanent displacements.

2. Added soil masses are linearly proportional to the area of the footing and the square root of the shear modulus and soil density. The horizontal added soil mass is smaller than that of the vertical vibration. The expressions of horizontal and vertical added soil masses are very simple to use compared with others.

ACKNOWLEDGEMENT

This paper is presented with the permission of Mr. Colin Jordan, Chief Executive Officer, Roads Corporation (Victoria). The views expressed are those of the authors and do not necessarily represent those of parent organisations.

REFERENCES

Arnold, R.N., Bycroft, G.N. and Warburton, G.B. (1955) "Forced Vibrations of a Body on an Infinite Elastic Solid" J. App. Mech. Trans., ASEM, Vol. 22, pp. 391-401.

Barkan, D.D. (1962) Dynamics of Bases and Foundations. Translated from Russian by L. Drashevska, McGraw-Hill Book Co., New York, 434p.

Barkan, D.D. and Shaevich, V.M. (1976) "Effect of an Added Soil Mass and its Nonlinear Properties on Foundation Vibration" Scientific-Research Institute of Foundations and Underground Structures. Translated from Osnovaniya Fundamenty i Mechanika Gruntov, No. 5, pp. 11-14, Sep-Oct, 1976.

Barkan, D.D. and Ilyichev, V.A. (1977), "Dynamics of Bases and Foundations", Proc. Soil Dynamics and Its Application to Foundation Engineering, Proc. 9th ICSMFE, pp. 630-637.

Bycroft, G.N. (1956) "Forced Vibrations of a Rigid Circular Plate on a Semi-Infinite Elastic Space and on an Elastic Stratum, Philosophical Trans., Royal Soc., London, Ser. A, Vol. 248, pp. 327-368.

Bycroft, G.N. (1959) "Machine Foundation Vibration", Int. of M.E., Vol.173 (18), pp. 469-473.

Dobry, R. and Gazetas, G. (1985), "Dynamic Stiffness and Damping of Foundations by Simple Methods" Vibration Problems in Geotechnical Engineering, Proc. of Symposium Sponsored by the Geotechnical Eng. Div. in Conjunction with the ASCE Convention in Detroid, Michigan Oct.22, 1985, pp. 75-107.

Dobry, R. and Gazetas, G. (1986), "Dynamic Response of Arbitrarily Shaped Foundations", J. Geot. Eng., pp.109-135.

Gazetas, G. (1987b), Chapter 2 "Simple Physical Methods For Foundation Impedances", Dynamic Behaviour of Foundation and Buried Structure" Edited by P.K. Banerjee and R. Butterfield, pp. 45-94.

Golubtsova, M.N. (1986) "Allowance For Adjoining Soil Mass in Calculating the Vertical Oscillations of massive Foundations" Scientific-Research Institute of Bases and Underground Structures. Translated From Osnovaniya, Fundamenty i Mekhanika Gruntov, No.1, pp. 7-10, January-February, 1986.

Hall, J.R. (1967), "Coupled Rocking and Sliding Oscillations of Rigid Circular Footings", Proc. Inter. Symp. Materials, Albuquerque, N.M., pp. 139-147.

Hsieh, T.K. (1962) "Foundation Vibrations" Proceedings Institution of Civil Engineers, London, England, Vol. 22, 1962, pp. 211-226.

Lysmer, J. & Richart, F.E. Jr. (1966) "Dynamic Response of Footings to Vertical Loading" Journal of The Soil Mechanics and Foundations Division, ASCE, Vol. 92, No. SM1, Jan. 1966, pp. 65-91.

Nagendra, M.V. and Sridharan, A. (1981), "Response of Circular Footings to Vertical Vibrations", J.Geot.Eng.Div., ASCE, Vol.107, pp. 989-995.

Nagendra, M.V. & Sridharan, A. (1982) "Stiffness Coefficients of Elastic Medium" Journal of The Geotechnical Engineering Division, ASCE, Vol. 108, No. GT4, April 1982, pp. 661-668.

Nagendra, M.V. & Sridharan, A. (1984) "Footing Response to Horizontal Vibration" J. of Eng. Mechanics, Vol.110, pp. 648-654.

Pauw, A. (1953) "A Dynamic Analogy For Foundation-Soil Systems" ASTM Symposium on Dynamic Testing of Soils, 1953, STP No. 156, p.90.

Reissner, E. (1936), "Stationare Axialsymmetrische, durch eine schuttelnde Masse erregte schwingungen eines homogenen elastischen Halbraumes, Ing. Archiv., Vol. 7, Part 6, pp. 381-396.

Richart, F.E. Jr., Woods, R.D. and Hall, J.R. (1970), Vibrations of Soils and Foundations. Prentice-Hall, 414 p.

Sankaran K.S.; Subrahmanyam, M.S. & Sastry, K.R. (1977) "Horizontal Vibrations-New Lumped Parameter Method" Proc. of The Ninth Inter. Conf. on Soil Mech. & Found. Engrg. Vol.2 (1977), pp. 365-368.

Timoshenko, S.P. and Goodier, J.N. (1951) .Theory of Elasticity. McGraw Hill Book Co., 506 p.

Truong, H.V.P. (1991a), "Lumped Parameters in Permanent Displacement", Proc. Second Int. Conf. on Recent Advances in Geot. Earthquake Eng. and Soil Dynamics, St. Louis, Missouri (USA), pp. 13-16.

Truong, H.V.P. (1991b), "Parametric Study of Horizontal Permanent Displacement on Sand" Proc. Second Int. Conf. on Recent Advances in Geot. Earthquake Eng. and Soil Dynamics, St. Louis, Missouri (USA), pp. 675-678.

Truong, H.V.P. (1991c), "Lotus 1-2-3 in Permanent Displacement", Proc. 7th Conf. of the Inter. Association For Computer Methods and Advances in Geomechanics, Cairns, Queenslands, Australia, pp. 93-100.

Truong, H.V.P. (1991d), "Movement of Bridge Abutment Due To Cyclic Loading", Proc. 9th Asian Regional Conference on Soil Mechanics & Found. Eng., pp. 297-300.

Truong, H.V.P. (1992) "Behaviour of Small Footings Subjected to Horizontal Vibratory Loading", Ph.D. Thesis, Department of Civil & Environmental Engineering, The University of Melbourne, Australia.