1968

Inferences on the parameters of the Weibull distribution

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INFERENCES ON THE PARAMETERS OF THE WEIBULL DISTRIBUTION

BY

DARREL RAY THOMAN, 1938

A DISSERTATION

Presented to the Faculty of the Graduate School of the

UNIVERSITY OF MISSOURI AT ROLLA

In Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

1968

ABSTRACT

For the most part, solutions to the problems of making inferences about the parameters in the Weibull distribution have been limited to providing simple estimators of the parameters. Little has been known about the properties of the estimators. In this paper the small and moderate sample size properties of the maximum likelihood estimators are studied and their superiority is established. The problem of making further inferences which are based on the maximum likelihood estimates of the parameters is then considered.

The inferences that are presented can be divided into those based on a single sample and those based on two independent samples from Weibull distributions and include solutions to the standard problems of interval estimation and hypothesis testing. In addition tolerance limits and confidence limits on the reliability are given. These procedures are accomplished by the discovery of certain pivotal functions whose distributions can be obtained by Monte Carlo methods. Although the distributions are only tabulated for complete samples the procedures which are presented can be extended to the case of censored sampling since for this type of sampling the basic functions remain pivotal.

ACKNOWLEDGEMENTS

The author is especially indebted to Dr. Lee J. Bain and Dr. Charles E. Antle for their guidance and encouragement in the preparation of this dissertation. He also wants to acknowledge the valuable assistance in processing the data given by the staff of the University of Missouri at Rolla Computing Center.

The author also expresses his gratitude to his wife and family for their patience through the research and the graduate studies which preceded it.

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I. INTRODUCTION

A. The Weibull Distribution

In 1951, the Swedish Engineer W. Weibull advanced a statistical distribution which had been found to provide a good model for a variety of fatigue studies, [1]. The Weibull cumulative distribution function is given by

$$W(x;G,b,c) = \begin{cases} 1 - \text{Exp}[-(x-G)^c/b^c] & \text{for } x \geq G,\ b \geq 0,\ c \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Here G will be referred to as the location parameter, b as the scale parameter and c as the shape parameter.

When c=1, the Weibull distribution reduces to the exponential distribution which has enjoyed wide use as a model in many failure studies. To some extent this popularity is due more to its simplicity than to its appropriateness as a model since the exponential distribution has the property that the probability of failure of a component for any given interval is independent of its age at the beginning of the interval. This property can be expressed by saying that the failure rate is constant. On the other hand, the Weibull distribution has the property that for $c > 1$ its failure rate is an increasing function of its age and for $c < 1$ its failure rate is a decreasing function. This flexibility along with its success as a model in empirical studies by such men as Weibull [2], Freudenthal and Gumbel [3], and Lieblein and Zelen [4] has brought it into wide use as a model for most failure distributions and a wide variety of other applications.

B. Objectives

In what follows it will be assumed in equation (1) that the location parameter is known but that the scale and shape parameters are both unknown. In this case it can be assumed that G=0 so that from (1) the Weibull density function may be written as

$$w(x;b,c) = cb^{-c}x^{c-1}\mathrm{Exp}[-(x/b)^{c}] , \quad x > 0. \tag{2}$$

The problem of making inferences about the population becomes then a problem of making inferences about the unknown parameters b and c. In all cases these inferences will be based on the maximum likelihood estimators, $\hat{b}$ and $\hat{c}$, which satisfy (see, for example, Leone et al [5]) the equations

$$\frac{n}{\hat{c}} - n \frac{\Sigma\, x_i^{\hat{c}} \ln(x_i)}{\Sigma\, x_i^{\hat{c}}} + \Sigma \ln(x_i) = 0 \tag{3}$$

and

$$\hat{b} = (\Sigma x_i{}^{\hat{c}}/n)^{1/\hat{c}} \tag{4}$$

where x_i, i=1, 2, ..., n, represent a sample from a Weibull distribution.

The inferences which are presented can be divided into those based on a single sample and those based on two independent samples from Weibull distributions. In the case of the single sample the maximum likelihood estimators are compared with other estimators available and unbiasing factors for the estimate of the shape parameter are given. Confidence intervals for each parameter with both parameters assumed unknown are presented. From these, tests of hypothesis are easily obtained. In the case of the test of $c=c_o$ against $c=c_A$ the power is given as a function of c_A/c_o and n. The power of the test of $b=b_o$ against $b=b_A$ is given as a function of $(b_A/b_o)^c$ and n. In addition, the distribution of the maximum likelihood estimator of the reliability is studied. Exact lower confidence limits are given and are compared with those given by Johns and Lieberman [6]. γ probability tolerance limits for proportion β are also derived and tabled as a function of n, γ, and β.

In the case of two independent samples, a test of the equality of the shape parameters in two Weibull distributions with the scale parameters unknown is given. Tests for the equality of the scale parameters are also presented along with a procedure for selecting the Weibull process with the larger mean life.

In each case the inferences are made possible by the use of Monte-Carlo methods to generate the distributions of certain pivotal functions. Tables containing the percentage points of the generated distributions are given in Appendix A. A discussion of the numerical methods and the accuracy of the results is included.

C. Review of the Literature

Since the maximum likelihood estimators have not been obtained in closed form, most of the published work on the

Weibull distribution has been concerned with presenting simple point estimators. Among them are estimators given by Gumbel [7], Menon [8], Miller and Freund [9], and Antle and Bain [10]. A comparison of these estimators is also made in [10]. The maximum likelihood estimators have been obtained by Leone et al [5] and also by Cohen [11] and Harter and Moore [12]. However, no extensive comparison has been made between the maximum likelihood estimators and the others. The distributions of these estimators has not been obtained and little has been given on their properties for small and moderate sample sizes.

Very little has been done with regard to confidence intervals for the parameters or tests of hypotheses. Bain and Weeks [13] have provided confidence intervals for each parameter with the other parameter known based on a single order statistic, and confidence intervals for b based on the maximum likelihood estimator of b with c known. Harter and Moore [12] give confidence intervals for b based on the maximum likelihood estimator of b with c known for censored samples.

Johns and Lieberman [6] have given exact confidence limits on the reliability which are asymptotically efficient. The procedure is valid for censored sampling.

The only work on the two sample problems in the Weibull distribution is due to Qureishi [14] and Qureishi, Nabavian and Alanen [15]. These papers give procedures for selecting the Weibull process with the larger mean life when the shape parameters are equal.

II. INFERENCES BASED ON A SINGLE SAMPLE

A. Estimation of c (b unknown)

1. Confidence Intervals for c

In what follows, $\hat{c}_{11}$ is used to denote the maximum likelihood estimator of c when in fact the sampling is from a Weibull distribution with b=1 and c=1, i.e. a standard exponential distribution. The following theorem which was noted in [10] will be useful.

Theorem A: $\hat{c}/c$ is distributed independently of b and c and has the same distribution as $\hat{c}_{11}$.

Proof: Let y_i, i=1,..., n, be a random sample of size n from a standard exponential distribution and x_i, i=1,...,n, the random sample from a Weibull generated by taking $x_i = b(y_i)^{1/c}$. Now $\hat{c}$, the maximum likelihood estimate based on the x_i's, satisfies (3). But if (3) is expressed in terms of the y_i's it becomes

$$\frac{n}{(\hat{c}/c)} - n \frac{\Sigma\, y_i^{\hat{c}/c} \ln(y_i)}{\Sigma\, y_i^{\hat{c}/c}} + \Sigma \ln(y_i) = 0. \tag{5}$$

But the solution of (5) for $\hat{c}/c$ is the same as the solution of

$$\frac{n}{\hat{c}_{11}} - n \frac{\Sigma\, y_i^{\hat{c}_{11}} \ln(y_i)}{\Sigma\, y_i^{\hat{c}_{11}}} + \Sigma \ln(y_i) = 0, \tag{6}$$

for $\hat{c}_{11}$. Thus $\hat{c}/c = \hat{c}_{11}$ whenever $\hat{c}$ and $\hat{c}_{11}$ are based on samples related in the manner described above, and it follows that $\hat{c}/c$ has the same distribution as $\hat{c}_{11}$.

The distribution of $\hat{c}_{11}$ was obtained by Monte Carlo methods. Table A1 contains percentage points of the distribution of $\hat{c}_{11}$ which can then be used to construct confidence intervals for c with b unknown. 100(1-γ) percent confidence intervals will be of the form $(\hat{c}/\ell_2, \hat{c}/\ell_1)$ where ℓ_1 and ℓ_2, from Table A1, are such that

$$P[\ell_1 < \hat{c}_{11} < \ell_2] = 1-\gamma.$$

2. Unbiased Maximum Likelihood Estimator of c

Theorem A confirms the feeling expressed by Leone et al [5] that the percent of bias in $\hat{c}$ is independent of the true value of c and b. The generated distribution of $\hat{c}_{11}$ provides the factors B(n) such that $E[B(n)\hat{c}] = c$. These unbiasing factors are given in Table 1.

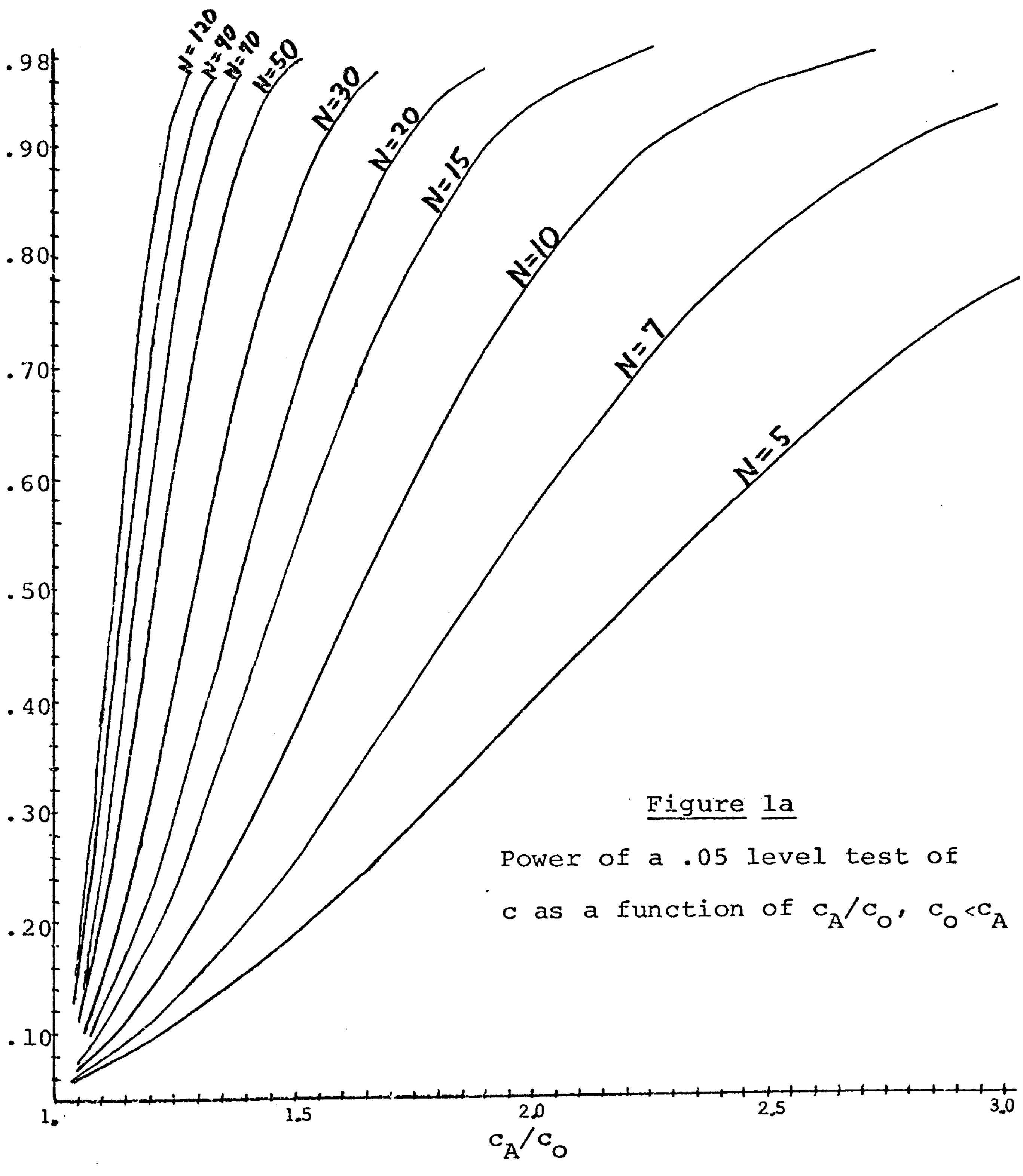
Table 1

Unbiasing Factors for the M.L.E. of c

n	5	6	7	8	9	10	11	12	13
B(n)	.669	.752	.792	.820	.842	.859	.872	.883	.893
n	14	15	16	18	20	22	24	26	28
B(n)	.901	.908	.914	.923	.931	.938	.943	.947	.951
n	30	32	34	36	38	40	42	44	46
B(n)	.955	.958	.960	.962	.964	.966	.968	.970	.971
n	48	50	52	54	56	58	60	62	64
B(n)	.972	.973	.974	.975	.976	.977	.978	.979	.980
n	66	68	72	76	80	85	90	100	120
B(n)	.980	.981	.982	.983	.984	.985	.986	.987	.990

3. Tests of Hypotheses of c and the Power of the Tests

Consider the test of H_o: $c=c_o$ against H_A: $c=c_A$ where $c_A > c_o$. Clearly, the γ significance level test based on the function $\hat{c}/c_o$ yields the critical region $(c_o \ell_{1-\gamma}, \infty)$. The power of this test is $P[\ \hat{c} > c_o \ell_{1-\gamma} | H_A\ ]$ or, equivalently, $P[\ \hat{c}_{11} > (c_o/c_A) \ell_{1-\gamma}\ ]$. It is independent of b and depends only on c_o/c_A, γ and n. Figures 1a and 1b give the power of the .05 and .10 level tests as a function of c_A/c_o, where $c_A/c_o > 1$ for n=5, 7, 10, 15, 20, 30, 50, 70, 90 and 120. Similarly, the power of the .10 level test with $c_A/c_o < 1$ is given in Figure 2.



<u>Figure 1a</u>

Power of a .05 level test of c as a function of c_A/c_o, $c_o < c_A$

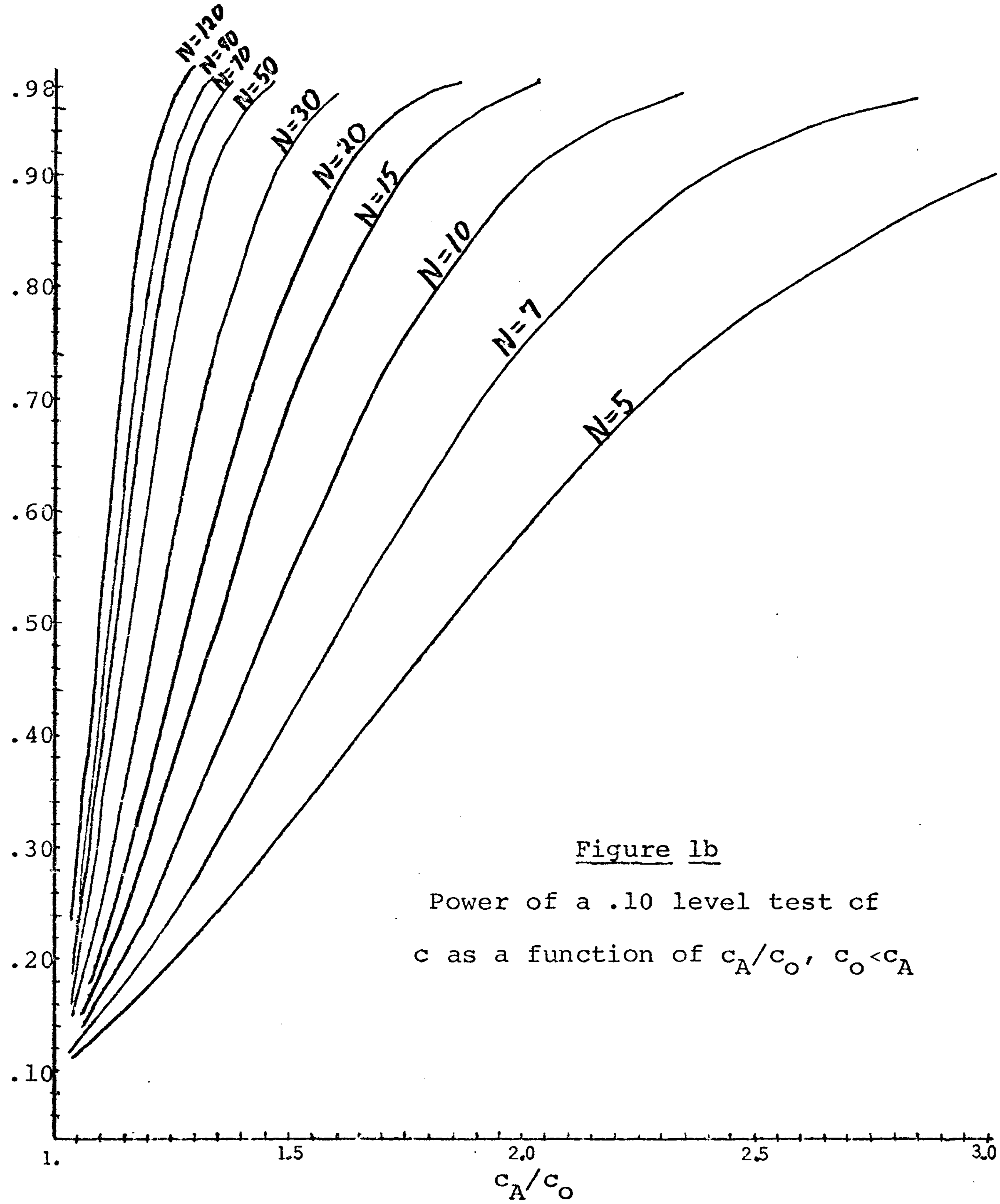


Figure 1b

Power of a .10 level test cf c as a function of c_A/c_o, $c_o < c_A$

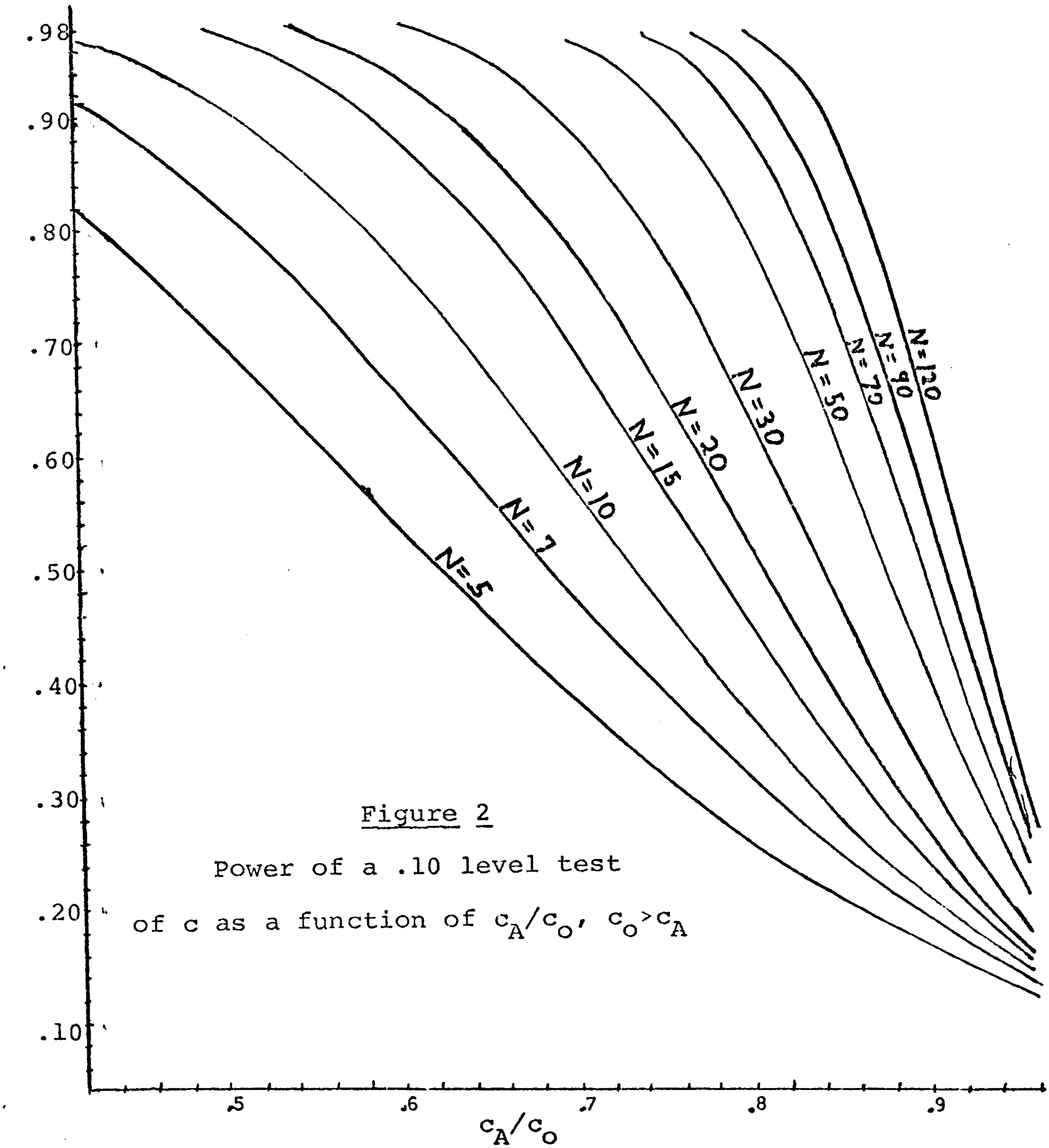


Figure 2

Power of a .10 level test of c as a function of c_A/c_o, $c_o > c_A$

4. Asymptotic Convergence of the Distribution of $\hat{c}$

Although our immediate concern is with $\hat{c}$, for future reference the asymptotic covariance matrix of $\hat{b}$ and $\hat{c}$ will be derived. The asymptotic covariance matrix is given by

$$1/n \begin{bmatrix} -E[\frac{\partial^2 \ln L}{\partial b^2}] & -E[\frac{\partial \ln L}{\partial b}\frac{\partial \ln L}{\partial c}] \\ -E[\frac{\partial \ln L}{\partial b}\frac{\partial \ln L}{\partial c}] & -E[\frac{\partial^2 \ln L}{\partial c^2}] \end{bmatrix}^{-1}$$

where L denotes the likelihood function, $w(x_i; b, c)$. Differentiation of ln L yields

$$\frac{\partial^2 \ln L}{\partial b^2} = c/b^2 - c(c+1)x^c/b^{c+2}$$

$$\frac{\partial^2 \ln L}{\partial b\,\partial c} = -1/b + (c/b)x/b)^c \ln(x/b) + (x/b)^c/b$$

$$\frac{\partial^2 \ln L}{\partial c^2} = -1/c^2 - (x/b)^c \ln^2(x/b) .$$

Now $E(x^c) = b^c$, $E[(x/b)^c \ln(x/b)]=(1/c)\int_0^\infty t\ln(t)e^{-t}dt=[1-\Gamma'(1)]/c$,

and $E[(x/b)^c \ln^2(x/b)] = (1/c^2)\int_0^\infty t\ln(t)e^{-t}dt=[2\Gamma'(1)+\Gamma''(1)]/c^2$.

Using the results given in [8] that $\Gamma'(1) = -.5772$ and $\Gamma''(1) = 1.9781$, we have

$$E[(x/b)^c \ln(x/b)] = .4228/c$$

and

$$E[(x/b)^c \ln^2(x/b)] = .8238/c^2.$$

Thus, $E[\frac{\partial^2 \ln L}{\partial b^2}]=c^2/b^2$, $E[\frac{\partial^2 \ln L}{\partial b\,\partial c}]=-.4228/b^2$ and $E[\frac{\partial^2 \ln L}{\partial c^2}]=\frac{1.828}{c^2}$

and therefore the asymptotic covariance matrix is

$$1/n \begin{bmatrix} 1.109\ b^2/c^2 & .257\ b \\ .257\ b & .608\ c^2 \end{bmatrix} \qquad (7)$$

It is seen then that $\hat{c}/c$ is asymptotically normal with mean 1 and variance .608/n. Reference can be made to curves (2) and (3) of Figure 4 in section II.C and Table 1 for an idea as to the rate of convergence of the distribution of $\hat{c}/c$ to its asymptotic distribution. It will, however, be of more interest to consider directly the difference between the confidence limits obtained from the tabulated and asymptotic distributions. In the case of a $100(1-\gamma)$ percent lower confidence limit this difference is

$$D = \left[\frac{1}{\ell_{1-\gamma}} - \frac{1}{1 + \sqrt{(.608/n)}\,\ell^*_{1-\gamma}} \right] \hat{c}$$

where $\ell^*_{1-\gamma}$ is the $100(1-\gamma)$ percentage point from the standard normal distribution. For a $100(1-\gamma)$ percent upper confidence limit, D is obtained by replacing $1-\gamma$ by γ. Table 2a gives approximate values of n at which the absolute difference relative to $\hat{c}$, $|D|/\hat{c}$, becomes less than .1, .05, .02 for $\gamma = .02, .05, .10$.

Table 2a

Sample Sizes at which the Absolute Difference in Exact and Asymptotic Confidence Limits Relative to $\hat{c}$ Become Less than $|D|/\hat{c}$

	Lower Limits			Upper Limits		
$\|D\|/\hat{c}$ γ	.02	.05	.10	.02	.05	.10
.1	22	17	14	40	27	20
.05	48	38	30	66	49	37
.02	>130	100	80	>130	115	90

The convergence rate can be increased if the asymptotic distribution of the unbiased estimator is used. Since the unbiased estimator of c, $B(n)\hat{c}$, is asymptotically normal with mean, c, and variance, $[B(n)]^2(.608/n)c^2$, the difference in lower confidence limits now becomes

$$D = \left[\frac{1}{\ell_{1-\gamma}} - \frac{B(n)}{1 + B(n)\sqrt{.608/n}\ \ell^*_{1-\gamma}}\right]\hat{c}\ .$$

Table 2b gives the required sample sizes for this case and it is seen that there is a substantial decrease in the sample size needed to achieve a given amount of accuracy.

Table 2b

Sample Sizes at which the Absolute Difference in Exact and Asymptotic Confidence Limits Based on the Unbiased Estimator of c Become Less than $|D|/\hat{c}$

	Lower Limits			Upper Limits		
$\|D\|/\hat{c}$ γ	.02	.05	.10	.02	.05	.10
.05	27	18	10	22	16	12
.02	80	52	27	76	35	19
.01	>130	120	64	>130	54	28

B. Estimation of b (c unknown)

1. Confidence Intervals for b

The following theorem will enable us to establish a pivotal function of b only, whose distribution is independent of both parameters.

Theorem B: $\ln(\hat{b}_s) = c \ln(\hat{b}/b)$ and $\hat{c}_s = \hat{c}/c$ have a joint distribution which does not depend on b and c.

Proof: If b_o and c_o represent the true values of b and c then $z = (x/b_o)^{c_o}$ has the standard exponential distribution. From the definition of the maximum likelihood estimators of b and c,

$$\hat{c}^n \mathrm{Exp}[-\Sigma(x_i/\hat{b})^{\hat{c}}] \quad (x_i/\hat{b})^{\hat{c}} = \mathrm{Max}\{ c^n \mathrm{Exp}[-\Sigma(x_i/b)^c] \quad (x_i/b)^c \}$$

This is the same as

$$c_o^n(\hat{c}/c_o)^n \mathrm{Exp}\{- \Sigma[(x_i/b_o)^{c_o}(b_o/\hat{b})^{c_o}]^{\hat{c}/c_o}\} \quad [(x_i/b_o)^c(b_o/\hat{b})^c]^{\frac{\hat{c}}{c_o}}$$

$$= \mathrm{Max}\{c_o^n(\frac{c}{c_o})^n \mathrm{Exp}\{- \Sigma[(\frac{x_i}{b_o})^{c_o}(\frac{b_o}{b})^{c_o}]^{c/c_o}\} \quad [(x_i/b_o)^{c_o}(b_o/b)^{c_o}]^{\frac{c}{c_o}}\}$$

or

$$\hat{c}_s^n \mathrm{Exp}[-\Sigma(z_i/\hat{b}_s)^{\hat{c}_s}] \quad (z_i/\hat{b}_s)^{\hat{c}_s} = \mathrm{Max}\{c_s^n \mathrm{Exp}[-\Sigma(z_i/b_s)^{c_s}] \quad (z_i/b_s)^{c_s}\}$$

where $c_s = c/c_o$ and $b_s = (b/b_o)^{c_o}$. Therefore $\hat{c}_s$ and $\hat{b}_s$ correspond to the maximum likelihood estimators of b and c when the sampling is actually on a standard exponential variate z. Thus the joint distribution of $\ln(\hat{b}_s) = c \ln(\hat{b}/b)$ and $\hat{c}_s = \hat{c}/c$ is independent of b and c.

Since the joint density of $c \ln(\hat{b}/b)$ and $\hat{c}/c$ does not depend on b and c, neither does the distribution of $\hat{c} \ln(\hat{b}/b)$. In particular, $\hat{c} \ln(\hat{b}/b)$ will have the same distribution as

$\hat{c}_{11}\ln(\hat{b}_{11})$ where, as before, $\hat{b}_{11}$ will denote the maximum likelihood estimator of b when in fact the sampling is from a Weibull distribution with b=1 and c=1.

Clearly 100(1-γ) percent confidence intervals for b can now be constructed and will be of the form

$$(\hat{b}e^{-t_2/\hat{c}} \quad , \quad \hat{b}e^{-t_1/\hat{c}}) \tag{8}$$

where t_1 and t_2, from Table A2e are such that $G_1(t_2) - G_1(t_1) = 1 - \gamma$.

2. Asymptotic Convergence

The asymptotic distribution of $\hat{c}\ln(\hat{b}/b)$ can be found from (7) and the following theorem on functions of asymptotic normal variables [16].

Theorem: If $f(T_1, \ldots, T_k)$ is a continuous function with continuous first partials and if $\sqrt{n}(\vec{T} - \vec{\theta}) \underset{\infty}{\sim} N(\Phi, \Sigma)$ then $\sqrt{n}[f(T_1,\ldots,T_k)-f(\theta_1,\ldots,\theta_k)] \underset{\infty}{\sim} N(0, \Sigma\Sigma\sigma_{ij}\frac{\partial f}{\partial T_i}\Big|_{\vec{\theta}}\frac{\partial f}{\partial T_j}\Big|_{\vec{\theta}})$.

For the function $f(\hat{c}, \hat{b}) = \hat{c}\ln(\hat{b}/b)$, $\frac{\partial f}{\partial \hat{c}}\Big|_{b,c} = 0$ and $\frac{\partial f}{\partial \hat{b}}\Big|_{b,c} = c/b$. Therefore, from the above theorem and (7) we have that $\sqrt{n}\,\hat{c}\ln(\hat{b}/b) \underset{\infty}{\sim} N(0, 1.109)$.

It would again be useful to determine the sample size needed so that the normal approximation can be used. The difference between the approximate and exact 100(1-γ) percent lower (or uppwer) confidence limits for b is:

$$D = [e^{-\ell/\hat{c}} - e^{-\ell^*(\sqrt{\frac{1.109}{n}})/\hat{c}}\ \hat{b}] \tag{9}$$

where ℓ is the 100(1-γ) (or 100γ) percentage point from Table A2e and ℓ^* is the corresponding percentage point from the standard normal. Sample sizes as a function of $|D|/\hat{b}$ and γ are given in Table 3 for $\hat{c}$ = .6, 1. and 1.6. It may be noted from (9) and Table A2e that for $\hat{c}$ < .6 and fixed $|D|/\hat{b}$ < .02 the sample size is a decreasing function of $\hat{c}$. Thus the sample sizes for $\hat{c}$ = .6 are conservative estimates whenever $\hat{c}$ > .6. The values of n for $\hat{c}$ = 1 and 1.6 indicate the amount of conservativeness.

Table 3

Sample Sizes at which the Absolute Difference in Exact and Asymptotic Confidence Limits Relative to $\hat{b}$ Become Less than $|D|/\hat{b}$

$\hat{c}$	$|D|/\hat{b}$ \ γ	Lower Limits .02	.05	.10	Upper Limits .02	.05	.10
.6	.02	62	29	17	85	56	45
	.01	130	50	31	>130	80	63
	.005	>130	76	52	>130	105	79
1.0	.02	40	22	14	56	39	32
	.01	78	40	25	70	55	48
	.005	>130	60	41	100	77	68
1.6	.02	31	18	12	35	27	22
	.01	60	28	18	55	46	35
	.005	110	50	32	80	72	52

3. Tests of Hypotheses of b and the Power of the Tests

A test of the hypothesis H_o: $b=b_o$ against H_A: $b=b_A$ can be based on the function $\hat{c}\ \ln(\hat{b}/b)$. If $b_o < b_A$ then the critical region corresponding to a test at the γ significance level is

$$(b_o e^{\ell/\hat{c}}\ ,\ \infty)$$

where ℓ, from Table A2e, is such that $G_1(\ell) = 1-\gamma$.

In order to obtain the power of the above test it is useful to generalize the result given in Theorem B as follows.

Theorem C: For any positive constant K,

$\hat{c}[\ln(\hat{b}/b) - (1/c)\ln(K)]$ has the same distribution as

$\hat{c}_{11}[\ln(\hat{b}_{11}) - \ln(K)]$.

Proof: From equation (3)

$$\hat{c}[\ln(\hat{b}) - (1/c)\ln(K)] = \ln(\Sigma\ x_i^{\hat{c}}/n) - (\hat{c}/c)\ln(K).$$

Expressing this in terms of the y_i's, where $y_i=(x_i/b)^c$, we have

$$\hat{c}[\ln(\hat{b}/b) - (1/c)\ln(K)] = \ln(\Sigma\ y_i^{\hat{c}/c}/n) - (\hat{c}/c)\ln(K).$$

But direct use of (3) and (4) to obtain the maximum likelihood estimate of $c[\ln(b) - \ln(K)]$ gives

$$\hat{c}_{11}[\ln(\hat{b}_{11}) - \ln(K)] = \ln(\Sigma\ y_i^{\hat{c}_{11}}/n) - \hat{c}_{11}\ln(K)$$

and the theorem follows from Theorem A.

This result reduces, when K=1, to Theorem B. Generalizing the notation of section II.B-1, let G_K denote the common cumulative distribution of $\hat{c}_{11}[\ln(\hat{b}_{11}) - \ln(K)]$ and $\hat{c}[\ln(\hat{b}/b) - (1/c)\ln(K)]$.

The distribution, G_K, was obtained empirically for several values of K and percentage points are given in Tables A2a, b, c, d, e, f, g, h, i, as a function of N for K = .51083, .69315, .80, .90, 1., 1.05, 1.10, 1.15 and 2.0. Additional related tables needed in section II.E to derive tolerance limits are given. Tables A3a, b, c, d, and Tables A4a, b, c, d, give γ percentage points of G_K as a function of N and K for γ = .02, .05, .10, .25, .80, .90, .95 and .98.

The power of the test with $b_o < b_A$ based on $\hat{c}\ \ln(\hat{b}/b)$ is

$$P[b_o e^{\ell/\hat{c}} < \hat{b} \mid H_A\!: b=b_A] = P[\ell < \hat{c}\ \ln(\hat{b}/b_o) \mid H_A]$$

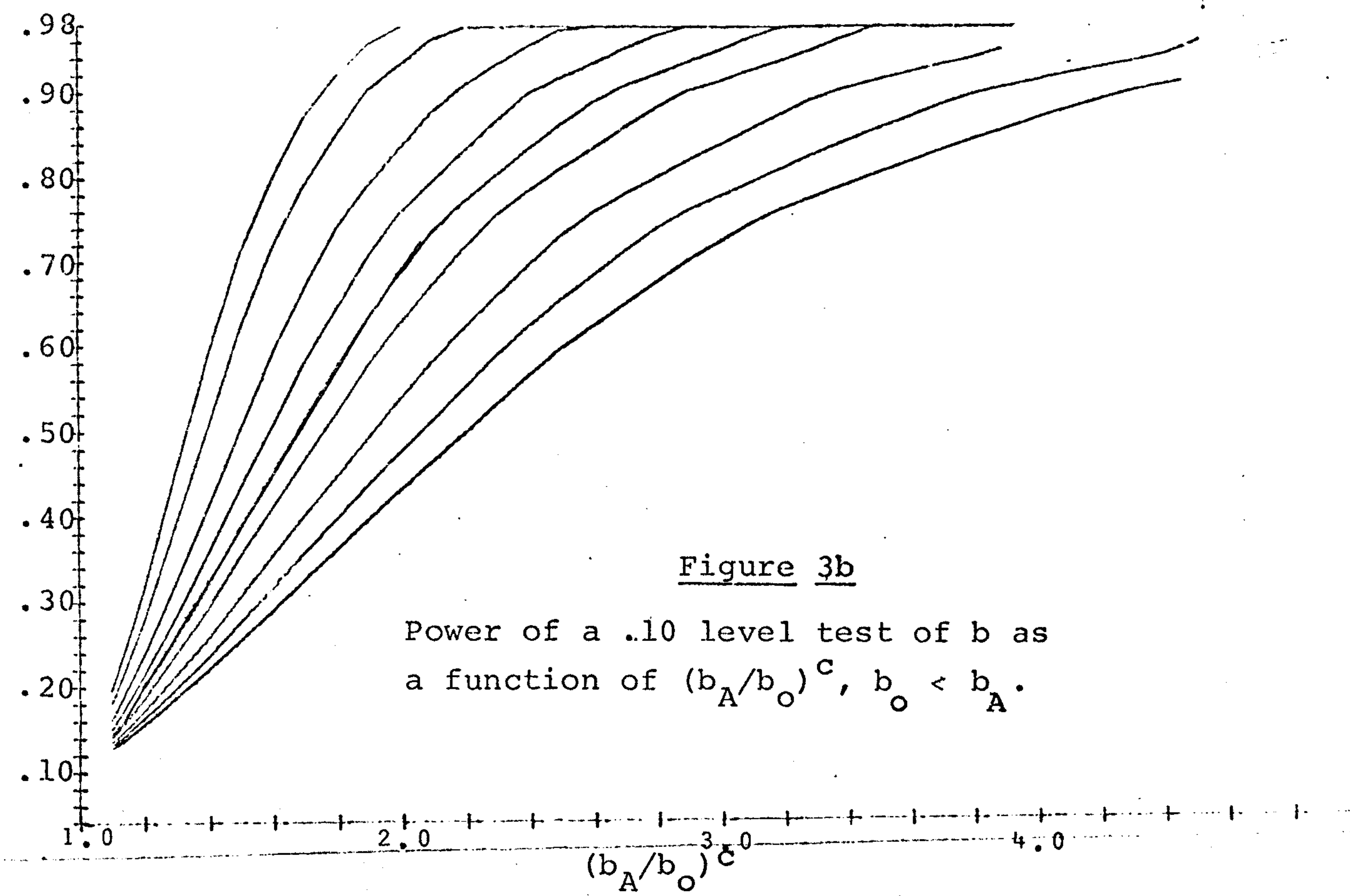
$$= P\{\ell < \hat{c}[\ln(\hat{b}/b_A) - \ln(b_o/b_A)] \mid H_A\}$$

$$= 1 - G_K(\ell) \quad \text{where} \quad K = (b_o/b_A)^c.$$

The power of the test, then, is a function of $(b_o/b_A)^c$, γ, and N. Figures 3a and 3b give the power of the .05 and .10 level tests as a function of $(b_A/b_o)^c$ for N = 10, 12, 15, 20, 24, 30, 40, 60 and 80 with $(b_A/b_o)^c > 1$.

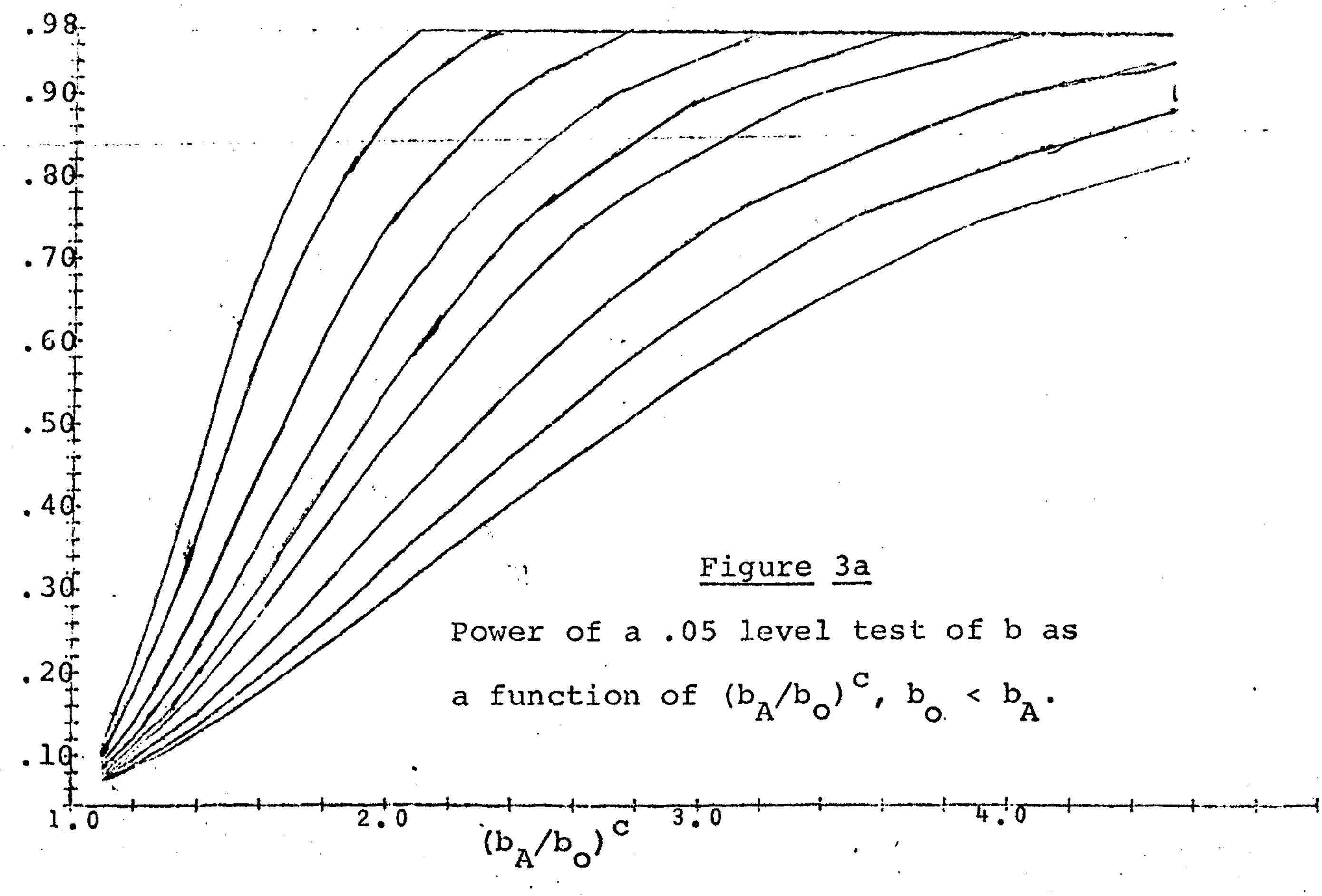
For large samples the asymptotic normal distribution of G_K may be used. The asymptotic distribution can be found by applying the Theorem in section II.A-4 to the function $f(\hat{b},\hat{c}) = \hat{c}[\ln(\hat{b}/b) - (1/c)\ln(K)]$. In this case, $f(b,c) = -\ln K$, $\frac{\partial f}{\partial \hat{b}}\Big|_{b,c} = c/b$ and $\frac{\partial f}{\partial \hat{c}}\Big|_{b,c} = -(1/c)\ln(K)$. Therefore from (7)

$$\sqrt{n}\ \hat{c}[\ln(\hat{b}/b)-(1/c)\ln(K)] \underset{\infty}{\sim} N[-\ln(K),\ .608(\ln K)^2 - .514\ln K + 1.109].$$



<u>Figure 3b</u>

Power of a .10 level test of b as a function of $(b_A/b_o)^c$, $b_o < b_A$.



<u>Figure 3a</u>

Power of a .05 level test of b as a function of $(b_A/b_o)^c$, $b_o < b_A$.

C. Comparison of the Estimators of b and c

The properties stated in Theorems A and B also hold for Menon's estimators [10]. All of the work represented so far was carried out simultaneously for the maximum likelihood and Menon's estimators and comparisons will be primarily limited to a comparison of these. A comparison of the maximum likelihood estimators with others available can be achieved through the comparison with Menon's and by reference to [10].

The biases of the two estimators of c are nearly equal. Both are highly biased for small n. The bias in the maximum likelihood estimator is slightly less than that of Menon's for $n > 20$.

The variances of both estimators of c as well as the asymptotic variance of the maximum likelihood estimator are included in Figure 4 for $n > 8$. Except for $n=5$, the variance of the maximum likelihood estimator is less than that of Menon's. The ratio of the variances approaches .55, the asymptotic efficiency of Menon's estimator.

Fortunately, as seen in section II.A-2, the estimators of c can be unbiased. The variances of the unbiased estimators are given in Table 4. The unbiased maximum likelihood estimator is clearly superior for even small values of n.

The variance of $\hat{c}\ \ln(\hat{b}/b)$ based on Menon's and the maximum likelihood estimators of b and c as well as its asymptotic variance when it is based on the maximum likelihood estimators is given in Table 5. The variance of

$\hat{c} \ln(\hat{b}/b)$ when it is based on the maximum likelihood estimators is smaller for $n > 10$; however, the difference is small. The ratio of the variance approaches .95, the asymptotic efficiency of $\hat{c} \ln(\hat{b}/b)$ based on Menon's estimators.

<u>Figure 4</u>

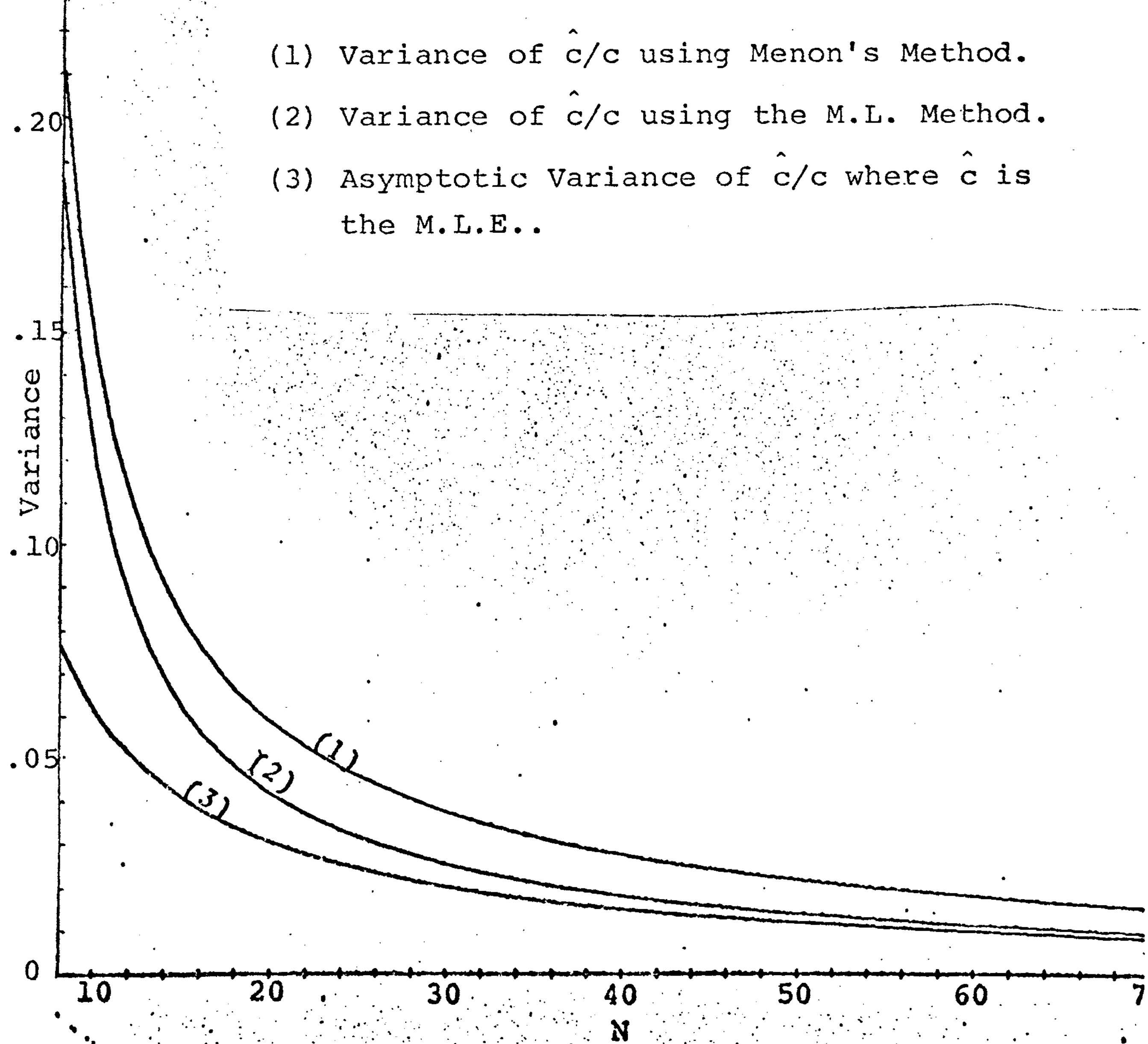


Table 4

Variance of Menon's and the Maximum Likelihood Estimators of c

N	5	6	8	10	12	14	16
Menon's	.334	.236	.147	.108	.086	.073	.063
M.L.E.	.320	.215	.124	.087	.067	.055	.047
N	18	20	25	30	35	40	45
Menon's	.056	.050	.040	.034	.029	.026	.023
M.L.E.	.041	.036	.028	.023	.020	.017	.015
N	50	60	70	80	100	120	
Menon's	.021	.017	.015	.013	.011	.009	
M.L.E.	.014	.011	.010	.008	.006	.005	

Table 5

Variance of $\hat{c}\ \ln(\hat{b}/b)$ Using Menon's and the Maximum Likelihood Estimators of b and c and its Asymptotic Variance Based on the Maximum Likelihood Estimators

N	5	6	8	10	12	15	20
Menon's	.604	.387	.233	.169	.128	.097	.070
M.L.E.	.642	.406	.234	.168	.125	.094	.067
Asymptotic	.222	.185	.139	.111	.092	.074	.055
N	25	30	40	50	75	100	
Menon's	.055	.045	.032	.0253	.0163	.0119	
M.L.E.	.052	.042	.030	.0240	.0154	.0114	
Asymptotic	.044	.037	.028	.0222	.0148	.0111	

Both estimators have the disadvantage of not being applicable to censored sampling. It may be noted that the maximum likelihood estimators corresponding to the censored sampling, [11], possess the same important properties stated in Theorems A and B. However, the necessity of tabulating the distribution for each possible point of censoring greatly enlarges the task.

Even though for complete samples the maximum likelihood estimators appear to be superior to the other estimators they have not, in the past, received as much attention as they might have if they were of a simpler type. However, it has been found that if a computer is available the maximum likelihood estimates can be readily and accurately obtained from a routine such as the one given in Appendix B.

D. Conservative Confidence Limits on the Mean

The mean of the Weibull distribution is given by $b\Gamma(1+1/c)$. However, $\Gamma(1+1/c) \geq .886$ for all c and assumes its minimum value at c=2.16. Hence, if $\ell_{1-\gamma}$, from Table A2e, is chosen such that

$$P[\ \hat{c}_{11}\ln(\hat{b}_{11}) < \ell_{1-\gamma}\ ] = 1 - \gamma,$$

then $(.886\hat{b}\ e^{-\ell_{1-\gamma}/\hat{c}},\ \infty)$ is a conservative (1-γ)100 percent upper confidence intervals for the mean. The true confidence is

$$P[\ \mu > .886\hat{b}\ e^{-\ell_{1-\gamma}/\hat{c}}\ ] = P[\ b\Gamma(1+1/c) > .886\hat{b}\ e^{-\ell_{1-\gamma}/\hat{c}}\ ]$$

$$= P\{\ \ell_{1-\gamma} > \hat{c}[\ln(\hat{b}/b) - \ln(\frac{\Gamma(1+1/c)}{.886})]\}$$

$$= G_K(\ell_{1-\gamma}) \quad \text{where} \quad K = [\ \frac{\Gamma(1+1/c)}{.886}\ ]^c.$$

The conservativeness follows from the fact that $K \geq 1$ for all c and that for $K \geq 1$, $G_k(\ell_{1-\gamma}) \geq G_1(\ell_{1-\gamma}) = 1 - \gamma$.

The true confidence can be computed from Tables A2e, f, g, h, for any given value of c and is given as a function of c in Figure 5a for γ=.05 and Figure 5b for γ=.10. The conservativeness is relatively insensitive to the value of c, especially when the sample size is small. For example, when n=10 and γ=.05 the true confidence is between .95 and .96 for all values of c between 1.3 and 3.4. When n=30, the true confidence is between .95 and .96 for all values of c between 1.5 and 3. It appears that the procedure in section II.A. for

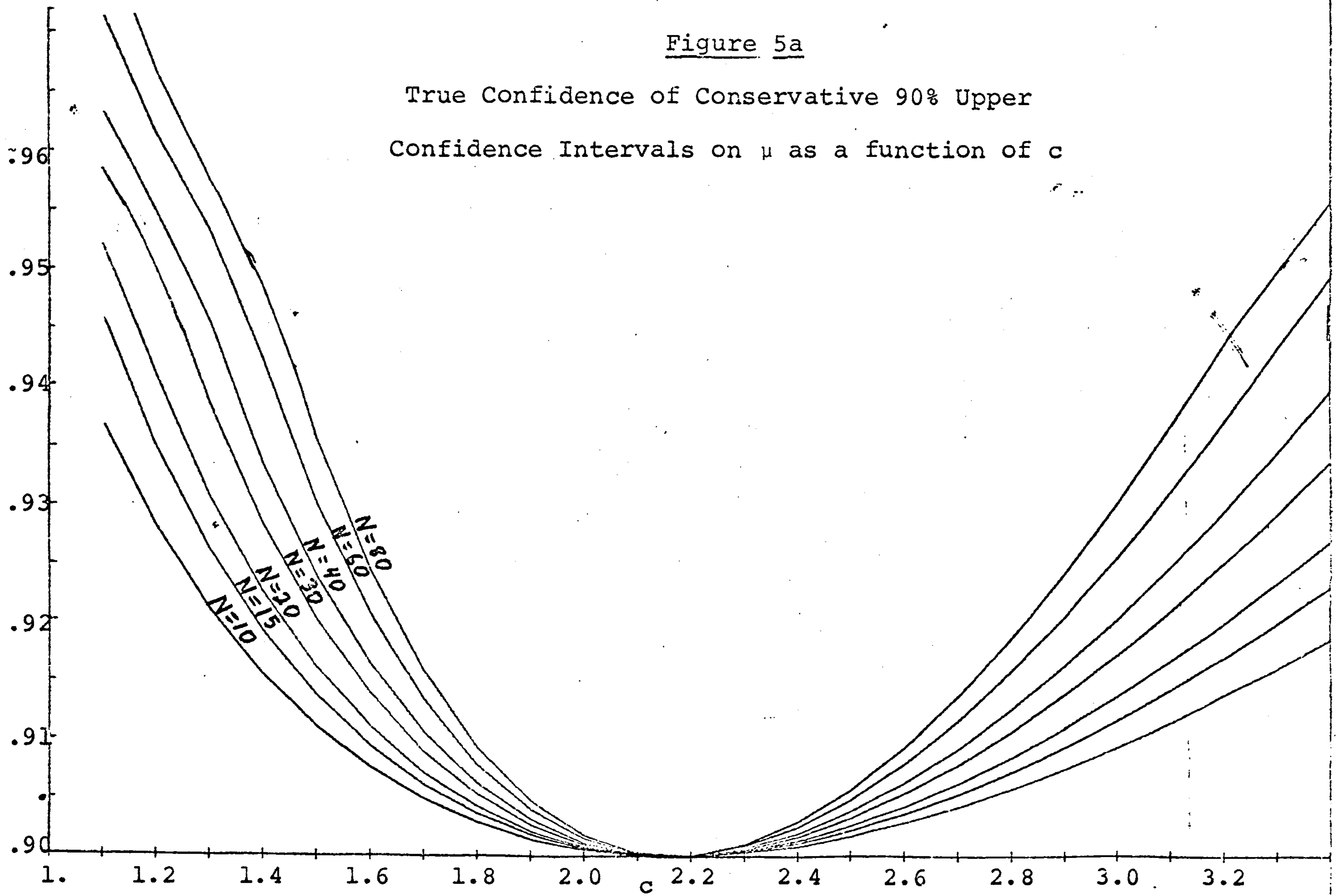


Figure 5a

True Confidence of Conservative 90% Upper Confidence Intervals on μ as a function of c

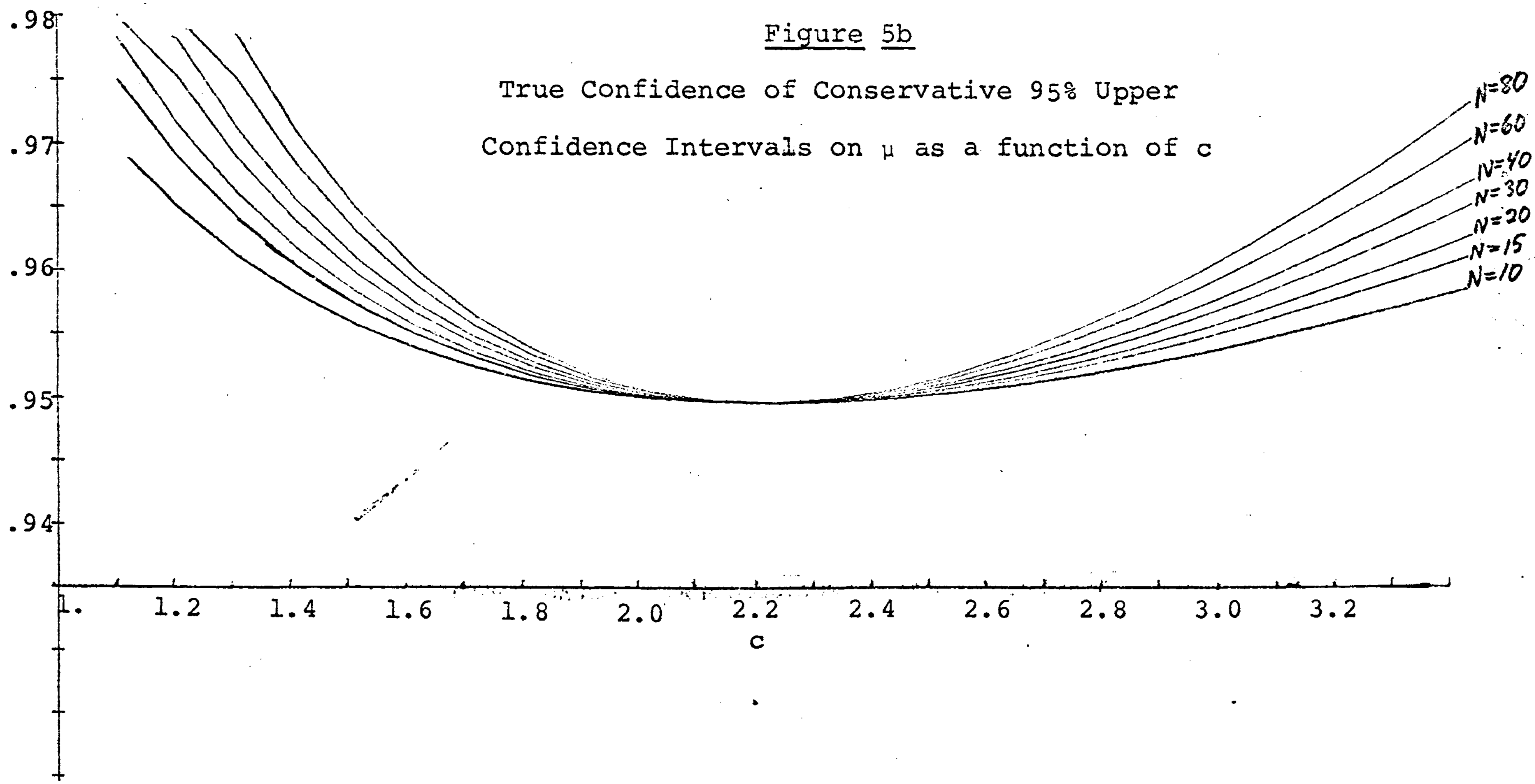


Figure 5b

True Confidence of Conservative 95% Upper

Confidence Intervals on μ as a function of c testing the value of c could be used in conjunction with the above procedure to make useful inferences about the mean.

Conservative upper confidence limits can also be obtained for $c \geq 1$ since $\mu \leq b$ for all $c \geq 1$. Thus, the upper confidence limit on b developed in section II.B-1 can serve as a conservative upper confidence limit on μ when $c \geq 1$. The true confidence can be seen to be $1 - G_K(\ell_\gamma)$ where $K = \Gamma(1+1/c)^c$. Unfortunately, it is more sensitive to the true value of c. For $\gamma = .05$ the true confidence exceeds .98 for all $c > 1.2$.

E. Tolerance Limits

$L(x_1, \ldots, x_n)$ is said to be a lower probability tolerance limit of proportion if

$$P\left[\int_{L(X_1,\ldots,X_n)}^{\infty} f(x;\ \Theta)\,dx > \beta\right] = \gamma.$$

For the Weibull distribution this reduces to

$$P[\ L(X_1,\ldots,\ X_n) < b(-\ln(\beta))^{1/c}\ ] = \gamma. \qquad (11)$$

That is, the problem of finding a lower tolerance limit reduces to a problems of finding a 100γ percent lower confidence limit for $b(-\ln(\beta))^{1/c}$, the $(1-\gamma)$ percentile point in the Weibull.

If ℓ_γ is chosen such that $G_K(\ell_\gamma) = \gamma$ with $K = -\ln(\beta)$ we see that this reduces to

$$P[\ \hat{c}\ \ln(\hat{b}/b) - (\hat{c}/c)\ln(-\ln(\beta)) < \ell_\gamma] = \gamma$$

or

$$P[\ \ln(\hat{b}) - \ell_\gamma/\hat{c} < \ln(b) + (1/c)\ln(-\ln(\beta))] = \gamma$$

and finally

$$P[\ \hat{b}e^{-\ell_\gamma/\hat{c}} < b(-\ln(\beta))^{1/c}\ ] = \gamma.$$

Thus, from (11), $\hat{b}e^{-\ell_\gamma/\hat{c}}$ is the desired γ lower probability tolerance limit for proportion β.

For a given value of β, the tabulated distributions, G_K, can be used to find the desired tolerance limits. Tables A3a, b, c, d, give the values of ℓ_γ as a function of β with γ = .80, .90, .95, .98. Tables A4a, b, c, d, give ℓ_γ as a function of β with γ = .02, .05, .10 and .25. These can be used to find upper γ tolerance limits of proportion β from the fact that they are equivalent to $(1-\gamma)$ lower tolerance limits of proportion $1-\beta$.

F. Estimation of the Reliability

1. Introduction

In the application of the Weibull to the distribution of the time to failure of a component most questions that arise involve the concept of the reliability of the component. The reliability for time t is given by

$$R(t) = P[\ X > t\ ] = \text{Exp}[-(t/b)^c].$$

Although the maximum likelihood estimators, $\hat{b}$ and $\hat{c}$, are computationally tedious to calculate, it has been shown that they are usually better than other more convenient estimators of b and c. Thus the maximum likelihood estimator, $\hat{R}(t)$, of the reliability, R(t) might be expected to have good properties. It is shown in this section that $\hat{R}(t)$ is nearly a minimum variance unbiased estimator of R(t). It is also shown that the density of $\hat{R}(t)$ depends only on the parameter R(t). This makes it possible to use the general method (see, for example [17]) for obtaining confidence intervals for R(t) based on $\hat{R}(t)$ or for testing hypotheses concerning R(t). These confidence intervals or tests based on $\hat{R}(t)$ should be expected to have good properties.

The distribution of $\hat{R}(t)$ was determined by Monte Carlo methods and the results were used to form Table A5. For an observed $\hat{R}(t)$, the lower confidence limit for R(t) can be read directly from Table A5 for confidence levels $\gamma = .75$, .80, .85, .90, .95, .98 and sample sizes n = 8, 9, 10, 12, 15, 18, 20, 25, 30, 40, 50, 75, 100. Thus the lower confidence limit for R(t) is very easy to determine when the maximum likelihood estimates are available. A comparison of the exact confidence limits obtained from the distribution of $\hat{R}(t)$ with the approximate confidence limits obtained by means of a normal approximation shows that the normal approximation is quite adequate for sample sizes as large as 50.

Johns and Lieberman [6] have also presented a method for obtaining confidence limits for the reliability in the case of the Weibull distribution. They provide necessary tables for sample sizes of 10, 15, 20, 30, 50 and 100 and for various censoring fractions. Their method is asymptotically efficient but no evaluation of it has been reported for small samples. A preliminary comparison indicates that these two methods give almost identical lower limits for any given sample, which is a very interesting result. Thus, if for some reason a lower confidence limit is desired when the maximum likelihood estimates are not readily available, it would probably be more convenient to use the tables given by Johns and Lieberman.

The above results might also indicate that limits based on maximum likelihood estimators from censored samples would be about the same as those given by Johns and Lieberman.

2. The Distribution of $\hat{R}(t)$

The distribution of $\hat{R}(t)$ based on a sample of size n will now be considered. Let $\hat{b}_s = (\hat{b}/b)^c$ and $\hat{c}_s = \hat{c}/c$. It essentially follows from Theorems A and B that the joint distribution of $\hat{b}_s$ and $\hat{c}_s$ is independent of both parameters. It will now be shown that the distribution of $\hat{R}(t)$ depends only on R(t).

$$\hat{R}(t) = \text{Exp}[-(t/\hat{b})^{\hat{c}}],$$

so that

$$\begin{aligned}\ln[-\ln(\hat{R}(t))] &= \hat{c}\ \ln(t/\hat{b}) \\ &= (\hat{c}/c)\ln[(t/b)^c(\hat{b}/b)^{-c}] \\ &= \hat{c}_s\ \ln[-\hat{b}_s^{-1}\ln(R(t))].\end{aligned}$$

Thus the distribution of $\hat{R}(t)$ depends on b, c and t only through R(t).

This result makes it feasible to study the distribution of $\hat{R}(t)$ empirically. It also now is possible to give confidence intervals for R(t) based on $\hat{R}(t)$ with both b and c unknown.

3. Point Estimation of R(t)

The properties of $\hat{R}(t)$ as a point estimator are considered first. Table 6 gives the bias of $\hat{R}(t)$, $E[\hat{R}(t)]-R(t)$, for R(t) = .50, .60, .70, .75, .80, .85, .90, .925, .95, and .98 and n = 8, 10, 12, 15, 18, 20, 25, 30, 40, 50, 75, and 100. As indicated in Table 6 the bias is quite small and it does not seem worth an attempt to eliminate it.

Table 6

Bias in $\hat{R}(t)$

R(t)	n = 8	10	12	15	20	25	30	40	50	70	100
.50	.005	.003	.003	.002	.002	.002	.001	.001	.001	.001	.001
.60	.012	.009	.008	.006	.005	.004	.003	.002	.002	.002	.001
.70	.015	.011	.010	.008	.006	.005	.004	.003	.003	.002	.001
.75	.014	.011	.010	.008	.006	.005	.004	.003	.002	.002	.001
.80	.013	.010	.008	.006	.005	.004	.003	.002	.002	.002	.001
.85	.010	.007	.006	.005	.004	.003	.003	.002	.002	.001	.001
.90	.006	.004	.004	.002	.002	.002	.001	.001	.001	.001	.000
.925	.003	.002	.002	.001	.001	.001	.001	.000	.000	.000	.000
.95	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
.98	-.002	.002	.001	.001	.001	.001	.001	.000	.000	.000	.000

The variance of $\hat{R}(t)$ is given in Table 7 for the same values of R(t) and n. It is of interest to compare the variance of $\hat{R}(t)$ with the Cramer-Rao Lower Bound, CRLB, for a regular unbiased estimator of R(t).

The CRBL may be computed directly. However it is equal to the asymptotic variance of $\hat{R}(t)=\text{Exp}[-(t/\hat{b})^{\hat{c}}]$ so that the Theorem given in section II.A-4 may be used. In this case $\left.\frac{\partial R}{\partial \hat{c}}\right|_{b,c} = -R\ \ln(R)\ \ln(-\ln(R))$ and $\left.\frac{\partial R}{\partial \hat{b}}\right|_{b,c} = -(c/b)R(-\ln(R))$.

Therefore, using (7),

$$n(\hat{R}(t) - R(t)) \underset{\infty}{\sim} N[0,\ R^2(\ln R)^2[1.108665 - .514044\ \ln(-\ln R) + .607927(\ln[-\ln R])^2]].$$

The difference between the variance of $\hat{R}(t)$ and the CRLB is given in Table 8 for certain values of R(t) and n. The maximum difference occurred for R(t) = .5. As indicated in the table the variance of $\hat{R}(t)$ is approximately equal to the CRLB, especially for the values of reliability of interest.

Table 7

Variance of $\hat{R}(t)x10^4$

R(t)	n: 8	10	12	15	20	25	30	40	50	75	100
.50	266	200	167	124	090	072	059	043	034	023	017
.60	242	187	154	118	086	068	057	042	033	022	016
.70	194	153	126	099	072	058	048	036	029	019	014
.75	163	130	107	086	062	050	042	031	025	017	012
.80	130	103	086	070	051	041	034	026	020	014	010
.85	095	076	063	051	037	030	025	019	015	010	008
.90	059	047	039	032	023	019	016	012	009	006	005
.925	041	033	027	022	016	013	011	008	007	004	003
.95	025	019	016	013	009	007	006	005	004	003	002
.98	006	005	004	003	002	002	001	001	001	001	001

Table 8

Variance$[\hat{R}(t)]$ - Cramer-Rao Lower Bound

R(t)	n: 10	12	15	20	25	30	40	50	75
.50	.0034	.0029	.0014	.0007	.0006	.0004	.0002	.0001	.0001
.75	.0005	.0003	.0002	.0000	.0000	.0000	.0000	.0000	.0000
.95	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

4. Exact Confidence Limits for R(t)

Since the distribution of $\hat{R}(t)$ depends only on R(t) confidence limits for R(t) based on $\hat{R}(t)$ can be determined. Monte Carlo methods were used to obtain the distribution of $\hat{R}(t)$ for a given R(t) and the general method for constructing confidence limits was applied to determine the lower confidence limit for R(t). Thus for a confidence level γ, sample size n, and observed value of $\hat{R}(t)$, the lower 100γ percent confidence limit can be read directly from Table A5 for γ = .75, .80, .85, .90, .95, .98, n = 8, 10, 12, 15, 18, 20, 25, 30, 40, 50, 75, 100 and $\hat{R}(t)$ = .50(.02).98. The tables were obtained by generating 10,000 samples for each of the above sample sizes.

5. Approximate Confidence Limits for Large n

The standard procedure for obtaining confidence limits for R(t) when n is large is to assume that $\hat{R}(t)$ is normally distributed with mean R and variance V(R), where

$$V(R) = R^2(\ln R)^2\{1.108665 - .514044 \ln(-\ln R) + .607927[\ln(-\ln R)]^2\}/n.$$

The true reliability, R, could be replaced by $\hat{R}$ in the expression for the variance and an approximate lower γ confidence limit would be

$$L_1 = \hat{R} - z_\gamma [V(\hat{R})]^{1/2}$$

where z_γ is the γ percentage point of the normal distribution.

The limit L_1 will be called the direct approximation to the exact lower confidence limit L. It was found that L_1 differs from L by less than .005 for n=100. Also L_1 is usually too large.

The direct approximation can be improved considerably by using an iterative procedure. Let

$$L_i = \hat{R} - z_\gamma [V(L_{i-1})]^{1/2}, \quad i=2,\ 3,\ \ldots\ .$$

It was observed that after 4 or 5 iterations the changes in L_i were less than .00005, and the values of L_i were in much better agreement with the exact values. The maximum difference between the exact limits and the lower limits obtained from the iterative approximation was .005 for $n \geq 40$. The maximum difference was .002 for n=100. Thus it appears that the iterative approximation methods should be used if the appropriate table is not available. Perhaps it should be noted that the iterative procedure results from applying the general method for obtaining confidence intervals to the normal approximation of the density.

G. Example

Lieblein and Zelen [4] give the results of tests of the endurance of nearly 5000 deep-groove ball bearings. The graphical estimates of c over all lots tested appear to have an average value of about 1.6. Consider the following sample given on page 286, [4].

The results of the tests, in millions of revolutions, of 23 ball bearings were: 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.48, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

The maximum likelihood estimate of c is 2.102. The unbiasing factor (Table 1) is .940 so that the unbiased estimate of c is 1.976. The estimate of b from equation (4) is 81.99. From Table A1, a 90 percent confidence interval for c is (1.50, 2.62) and from (8) and Table A2e a 90 percent confidence interval for b is (68.04, 98.75). (The estimates of b and c given in [4] were 80 and 2.23, respectively.)

If we had wished to test at the .10 level the hypothesis H_o: c=1 against H_A: c > 1.6, the power of the test from Figure 1b would have exceeded .89 and based on the above sample the test would have led to the rejection of the null hypothesis.

From section II.D, a conservative 90 percent lower confidence limit on μ is given by 71.26. For values of c between 1.5 and 2.6 the true confidence, from Figure 5b, is between .90 and .917.

From section II.E, the 90 percent lower tolerance limit for proportion .90 is 18.85. The maximum likelihood estimate of the reliability for time t=40 is .802 and the .90 percent lower confidence limit on R_{40} is, from Table A5, .694.

III. INFERENCES BASED ON TWO INDEPENDENT SAMPLES

A. Introduction

In the work to follow it will be assumed that independent random samples of equal size have been drawn from Weibull distributions $w(x;\ b_1,\ c_1)$ and $w(x;\ b_2,\ c_2)$ where

$$w(x;\ b_i,\ c_i) = c_i (b_i)^{-c_i} (x)^{c_i - 1} \operatorname{Exp}[-(x/b_i)^{c_i}],\ i=1,\ 2.$$

The problems to be considered are those of testing $c_1=c_2$ and $b_1=b_2$. The procedures for performing these tests will be based on certain functions of the maximum likelihood estimators of the parameters whose distributions are parameter free. In addition to providing solutions to the above problems the functions lead to the construction of a procedure for selecting the Weibull process with the larger average life time; a problem considered by Qureishi et al [14], [15].

The assumption of equal sample sizes is not inherently required by the test procedures presented but was deemed necessary in order to simplify the task of obtaining the distributions by Monte Carlo methods.

B. Testing the Equality of the Shape Parameters (b unknown)

In order to test $c_1=c_2$ we recall from section II.A-1 that the maximum likelihood estimator, $\hat{c}$, of c has the property that $\hat{c}/c$ has the same distribution as $\hat{c}^*$ where $\hat{c}^*$ is the maximum likelihood estimator of c based on a sample from the standard exponential distribution. It then follows that $(\hat{c}_1/c_1)/(\hat{c}_2/c_2)$ has the same distribution as that of $\hat{c}_1^*/\hat{c}_2^*$ where, again, $\hat{c}_1^*$ and $\hat{c}_2^*$ are the maximum likelihood estimators of c_1 and c_2 based on independent random samples which are in fact from standard exponential distributions.

The distribution of $\hat{c}_1^*/\hat{c}_2^*$ was obtained by Monte Carlo methods and percentage points ℓ_γ such that $P[\hat{c}_1^*/\hat{c}_2^* < \ell_\gamma] = \gamma$ are given in Table A6 as a function of γ and the common sample size n. Pôints, ℓ_γ, for $\gamma < .50$ can be found by using the fact that $\ell_\gamma = 1/\ell_{1-\gamma}$.

A test of H_o: $c_1=c_2$ against H_A: $c_1 = kc_2$, $k > 1$, can now be made by using the fact that, under H_o, $\hat{c}_1/\hat{c}_2$ has the same distribution as $\hat{c}_1^*/\hat{c}_2^*$. That is, $P[\hat{c}_1/\hat{c}_2 > \ell_{1-\gamma} \mid H_o] = \gamma$ and a size γ test is given by rejecting if $\hat{c}_1/\hat{c}_2 > \ell_{1-\gamma}$ where $\ell_{1-\gamma}$ is obtained from Table A6.

The power of this test is

$$P[\hat{c}_1/\hat{c}_2 > \ell_{1-\gamma} \mid H_A: c_1 = kc_2] = P[\hat{c}_1^*/\hat{c}_2^* > (1/k)\ell_{1-\gamma}]$$

which can also be obtained from Table A6. The power as a function of $k > 1$ is given in Figures 6a and 6b for certain values of n and with $\gamma = .05$ and $.10$.

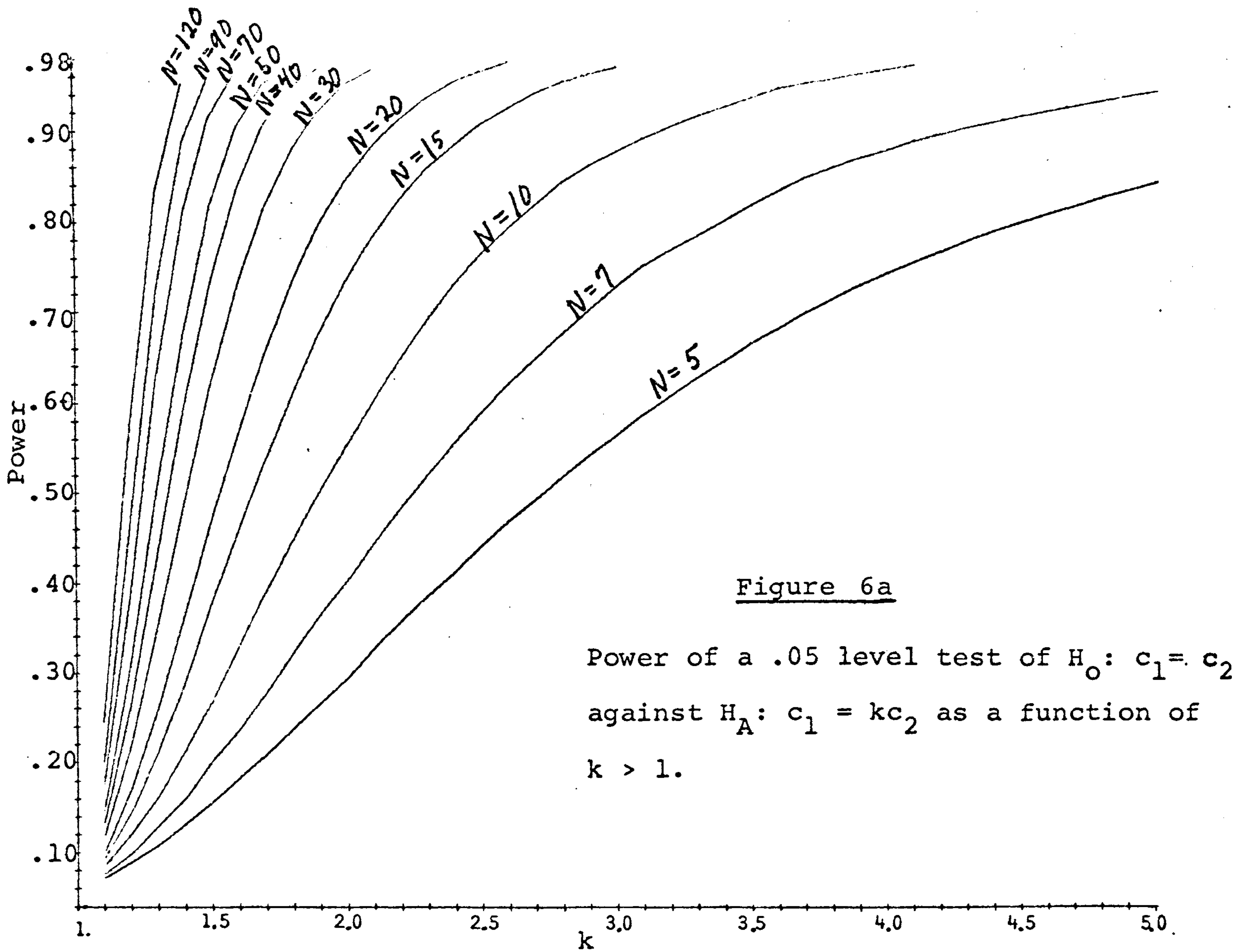


Figure 6a

Power of a .05 level test of H_o: $c_1 = c_2$ against H_A: $c_1 = kc_2$ as a function of $k > 1$.

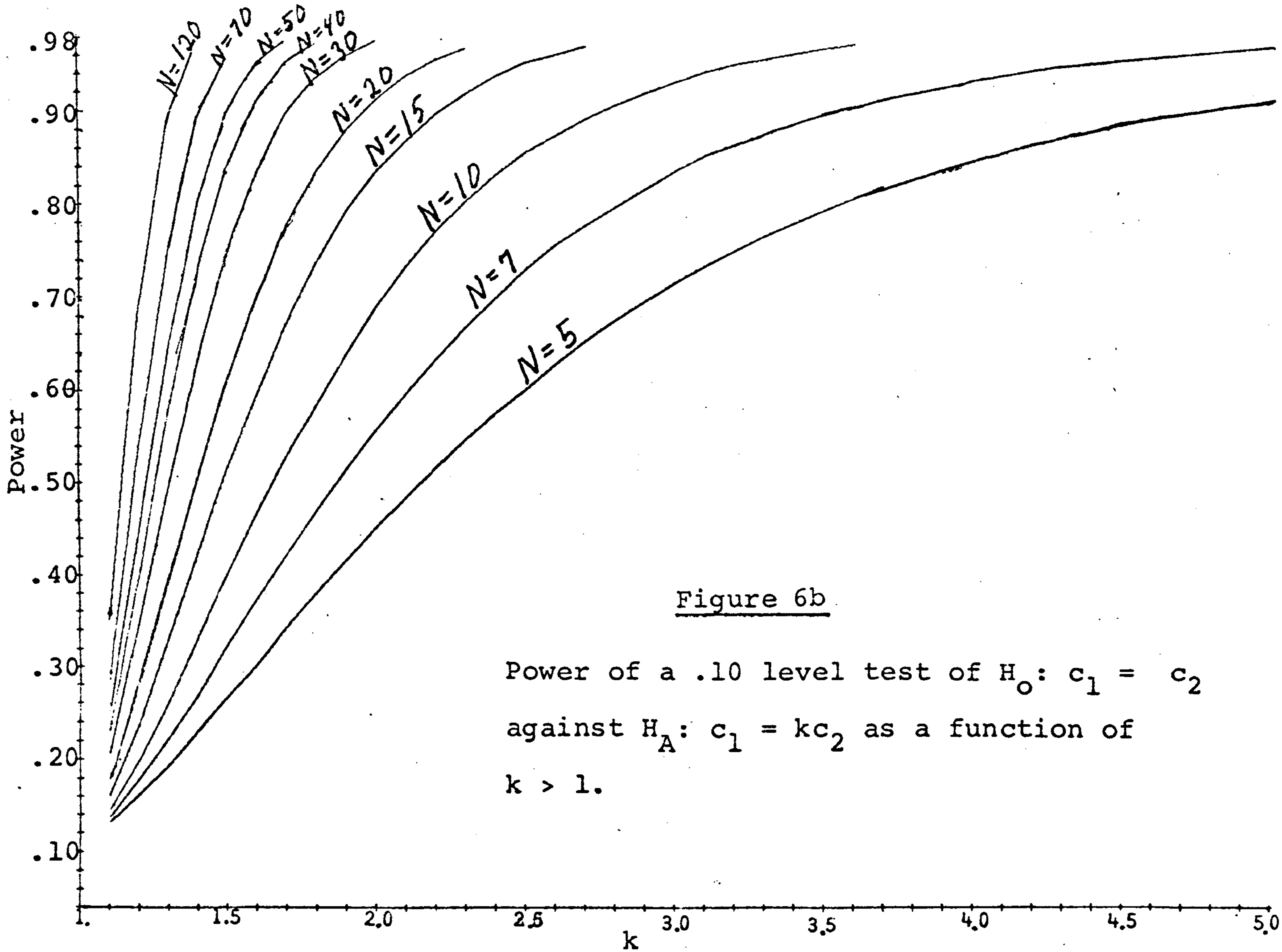


Figure 6b

Power of a .10 level test of H_o: $c_1 = c_2$ against H_A: $c_1 = kc_2$ as a function of $k > 1$.

The above procedure can, of course, be generalized to a test of H_o: $c_1 = kc_2$ against H_A: $c_1 = k'c_2$. For the case when $k < k'$ the rejection region becomes

$$\{(\hat{c}_1, \hat{c}_2) \mid \hat{c}_1/\hat{c}_2 > k\, \ell_{1-\gamma}\}$$

and the power of the test is

$$P[\hat{c}_1^*/\hat{c}_2^* > (k/k')\ell_{1-\gamma}].$$

C. Testing the Equality of the Scale Parameters

1. Tests with $c_1 = c_2$

In the development of the one sample test of $b=b_o$ in II.B it was observed that $\hat{c}\ \ln(\hat{b}/b)$ has the same distribution as $\hat{c}^*\ln(\hat{b}^*)$. For the case of two independent samples it follows from Theorem B that $\hat{c}_1/c_1$, $\hat{c}_2/c_2$, $c_1\ln(\hat{b}_1/b_1)$ and $c_2\ln(\hat{b}_2/b_2)$ have a joint distribution which is independent of the parameters c_1, c_2, b_1, b_2. Therefore, if $c_1 = c_2 = c$,

$$z(M) = \frac{\hat{c}_1 + \hat{c}_2}{2}[\ln(\hat{b}_1/b_1) - \ln(\hat{b}_2/b_2) - (1/c)\ln(M)], \tag{12}$$

where M is any positive constant, has a distribution which is independent of the parameters. In particular it will have the same distribution as

$$z^*(M) = \frac{\hat{c}_1^* + \hat{c}_2^*}{2}[\ln(\hat{b}_1^*) - \ln(\hat{b}_2^*) - \ln(M)]. \tag{13}$$

Let H_M denote the common cumulative distribution function of $z(M)$ and $z^*(M)$. For simplicity, $z(M)$ will be denoted by z when $M=1$.

A test of H_o: $b_1=b_2$, $c_1=c_2$ against H_A: $b_1= kb_2$, $c_1=c_2$ can now be made by using the fact that

$$P\{ \frac{\hat{c}_1+\hat{c}_2}{2}[\ln(\hat{b}_1) - \ln(\hat{b}_2)] < t | H_o \} = H_1(t).$$

Thus, a 100(1-γ) percent critical region for making this test with k > 1 is $\{z| z > z_{1-\gamma}\}$, where $z_{1-\gamma}$ is such that $H_1(z_{1-\gamma}) = 1-\gamma$.

The power of this test can also be expressed in terms of H_M since

$$P\{ \frac{\hat{c}_1+ \hat{c}_2}{2}[\ln(\hat{b}_1) - \ln(\hat{b}_2)] > z_{1-\gamma} | H_A: b_1=kb_2, k>1, c_1=c_2\}$$

$$= 1 - H_{K^c}(z_{1-\gamma}).$$

The distributions, H_M, were obtained by Monte Carlo methods for various values of M. The percentage points of G_1, needed to make the above test, are given in Table A7. The power of the test, as seen above, is a function of K^c and is given in Figure 7 for N = 7, 10, 15, 20, 30, 40, 60 and 80 with $\gamma = .10$.

A test of H_o with k < 1 in the alternative can be constructed in a similar fashion. The critical points z_γ, needed to make the test can be obtained from Table A7 by using the fact that $z_\gamma = - z_{1-\gamma}$.

It should be noted that the test of this section on b_1 and b_2 with c_1 and c_2 assumed equal is equivalent to a test on the means of the two Weibull distributions since $E(x) = c\ (1+1/b)$. In section III.D the above procedure will

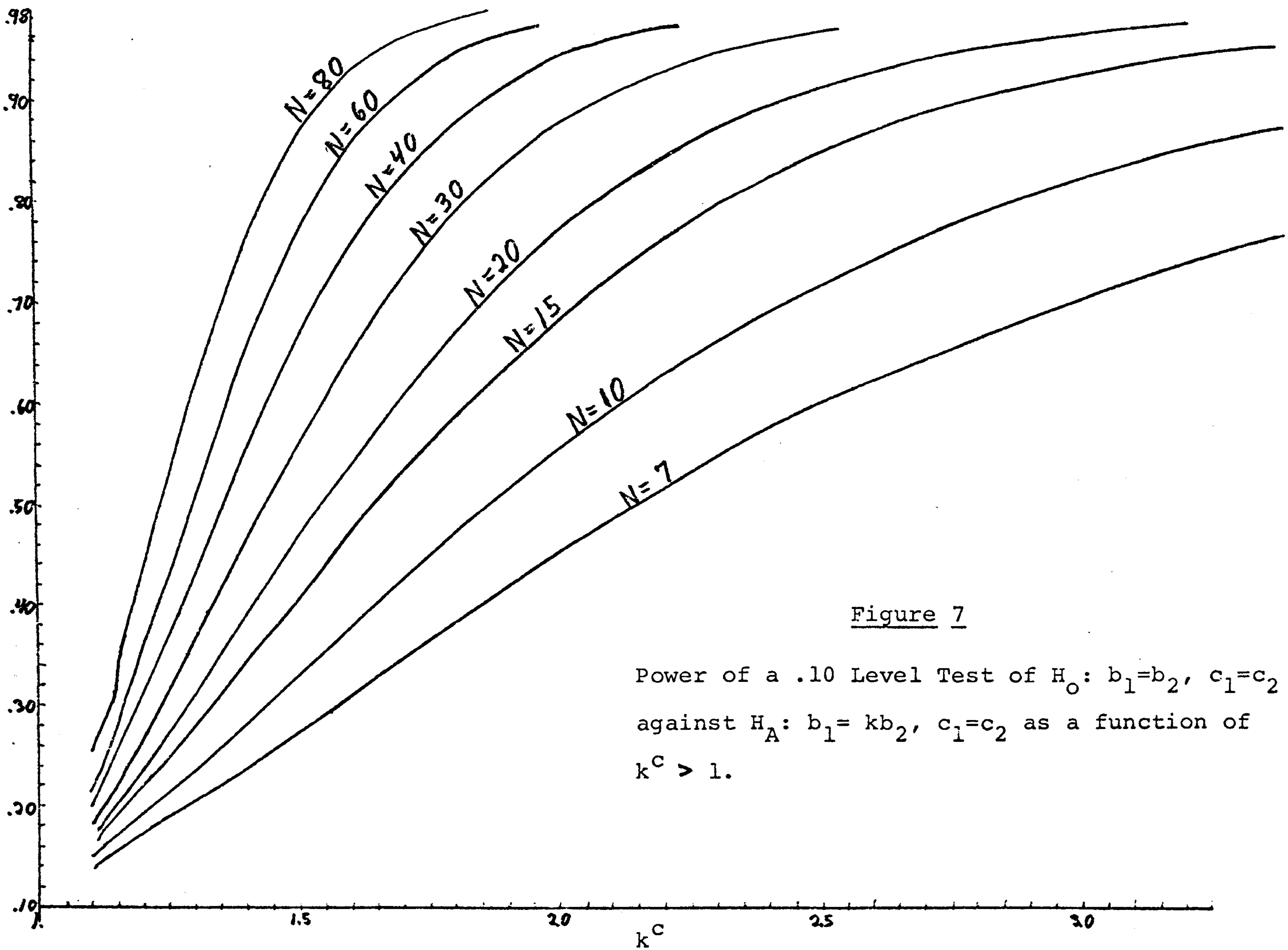


Figure 7

Power of a .10 Level Test of H_o: $b_1=b_2$, $c_1=c_2$ against H_A: $b_1= kb_2$, $c_1=c_2$ as a function of $k^c > 1$.

be used to solve the particular problem of choosing the Weibull process with the larger mean life with $c_1=c_2$ and the procedure will then be compared with procedures that already exist for handling this special problem.

2. Tests with $c_1 \neq c_2$

Consider the test of the one-sided hypothesis H_o: $b_1 > b_2$ against H_A: $b_1 < b_2$. In the special case where $c_1 \leq c_2$ the test defined by the procedure: reject the hypothesis if

$\frac{\hat{c}_1 + \hat{c}_2}{2} \ln(\hat{b}_1/\hat{b}_2) < z_\gamma$ where z_γ from Table A7 is such that $H(z_\gamma) = \gamma$, can be shown to be conservative in the sense that the probability of a type 1 error will not exceed γ. This follows since, under H_o,

$$P[\frac{\hat{c}_1+\hat{c}_2}{2}\ln(\hat{b}_1/\hat{b}_2) < z_\gamma] = P\{\frac{\hat{c}_1+\hat{c}_2}{2}[\ln(\hat{b}_1/b_1) - \ln(\hat{b}_2/b_2)]<z_\gamma\}$$

$$\leq P\{\frac{\hat{c}_1/c_1+\hat{c}_2/c_2}{2}[c_1 \ln(\hat{b}_1/b_1) - c_2 \ln(\hat{b}_2/b_2)] <z_\gamma\}=H_1(z_\gamma)=\gamma.$$

No extensive work has been done to investigate the conservativeness of this test however preliminary investigations for $c_2/c_1 = 1.2$ seem to indicate that the amount of conservativeness when $\gamma = .10$ is quite small, about .01.

In a similar manner it can be seen that if $c_1 \geq c_2$ in the above test then the power of H_o: $b_1 > b_2$ against H_A: $b_1 = kb_2$ with $k < 1$ will be at least the power of the corresponding test in section III.C-1 with $c_1 = c_2$, i.e. $H_{(1/k)^{c_1}}(z_\gamma)$.

D. Discrimination Between Two Weibull Processes

Consider two Weibull processes whose distributions have the same unknown shape parameter, c, but different and unknown scale parameters, b_1 and b_2. Procedures for detecting the process with the larger scale parameter, or, equivalently, the process with the larger average life time, have been given by Qureishi et al in [14] and [15]. For example, procedure R_1 given in [14] is to choose, as the process with the smaller average life time, the one which first produces a predetermined number, R, of failures from samples each of size N_1. The probability of correct selection when $b_1/b_2 = \alpha$ is greater than one is given by equation 12, [14]. The probability of a correct selection depends on c through α^c and therefore the evaluation of any particular procedure requires a knowledge or at least a good estimate of c. Another specification of the test procedure is α_s, the smallest value of α that is worth detecting. The probability of correct selection is tabulated in [15] as a function of α_s^c for a few values of R and N_1.

When the procedure of the previous section is applied to this problem it leads to the following procedure: compute $\hat{b}_1/\hat{b}_2$, where $\hat{b}_1$ and $\hat{b}_2$ are the maximum likelihood estimates based on the life times of units in samples from each of the two processes, and choose process 1 if $\hat{b}_1/\hat{b}_2 > 1$ and process 2 if $\hat{b}_1/\hat{b}_2 < 1$.

If the samples are of equal size, say N, then the distribution G_M, defined in section III.C, can be used to determine the probability of correct selection for the above procedure. If $b_1/b_2 = \alpha$ and the common shape parameter is c then the probability of correct selection is

$$P[\hat{b}_1/\hat{b}_2 > 1 \mid b_1/b_2 = \alpha > 1]$$

$$= P\{\frac{\hat{c}_1 + \hat{c}_2}{2}[\ln(\hat{b}_1/b_1) - \ln(\hat{b}_2/b_2) - \ln(b_2/b_1)] > 0 \mid b_1/b_2 = \alpha\}$$

$$= 1 - H_{\alpha^{-c}}(0) \quad \text{from equation (12), section III.C-1.}$$

For convenience we will denote the probability of a correct selection by $P(\alpha^c)$. Again considering α_s ($\alpha_s > 1$) as the smallest value of b_1/b_2 worth detecting, it follows that $P(\alpha_s^c) < P(\alpha^c)$ for all $\alpha > \alpha_s$. Values of $P(\alpha_s^c)$ are given in Table A8 as a function of N and α_s^c. It should be noted that if a lower confidence bound, c_L, is obtained for c, as in section II.A-1, the $P(\alpha_s^{c_L})$ will serve as a lower bound for $P(\alpha^c)$ for all $c > c_L$.

In order to compare this procedure with R_1, the cost of destructive testing will be set equal by choosing R in procedure R_1 to be equal to N, the number tested in the procedure presented in this section. If N_1 is chosen so that both procedures have the same probability of correct selection then N_1/N reflects the increased number of items to be put on test in procedure R_1. Using 12, [14], and Table A8 it can be seen that for $\alpha_s^c = 1.4$, the value of N_1/N is about 134% for N = 7 and increases to about 140% at N = 20.

Weighted against the cost of extra units being placed on test in procedure R_1 is its reduced experiment time. The expected duration of the experiment for procedure R_1 was given by equation 15, [14], but should be corrected to read:

$$E_1(T) = \Gamma(1+1/c)b_2R^2 \binom{N}{R}^2 \sum_{j=1}^{R} \sum_{i=1}^{R} (-1)^{i+j} \binom{R-1}{i-1}\binom{R-1}{j-1}$$

$$\left\{\frac{1}{\alpha^c(N-R+j)[\alpha^{-c}(N-R+i) + (N-R+j)]^{1+1/c}}\right.$$

$$\left. + \frac{1}{(N-R+j)[N-R+i) + \alpha^{-c}(N-R+j)]^{1+1/c}}\right\}.$$

The expected duration in the case of the procedure of this section can be found in a similar way to be

$$E_2(T) = b_2\Gamma(1+1/c)N \sum_{j=0}^{N} \sum_{i=1}^{N} (-1)^{i+j} \binom{N}{j}\binom{N-1}{i-1}$$

$$\left[\frac{1}{\alpha^c(i\alpha^{-c} + j)^{1+1/c}} + \frac{1}{(i + j\alpha^{-c})^{1+1/c}}\right].$$

$E_2(T)/b_2$ is given in Table 9 for a few values of c and N with $\alpha^c = 1.4$. It should be noted that contrary to a statement in [14] both $E_1(T)/b_2$ and $E_2(T)/b_2$ depend not only on α^c but also on c. In fact, as seen in Table 9, this dependence is quite heavy.

An idea of the time saved in procedure R_1 can be obtained from $E_1(T)/E_2(T)$ where, as before, R = N and N_1 is chosen so that the probability of correct selection is the same for both procedures. The value of $E_1(T)/E_2(T)$ was checked for small values of N and was found to be about .38 for c = 1.4 and about .42 for c = 1.6.

It can be noted that the parameter free properties in section III.C are valid for censored samples and thus so is the above procedure. The procedure based on the maximum likelihood estimators is also expected to be better than R_1 for equally censored samples. However, since the existing tables are valid only for complete samples the truncated nature of R_1 leads to a considerable saving in time.

Table 9

Expected Duration, $E_2(t)$, Relative to b_2
with $\alpha^c = 1.4$

N \ c	1.0	1.2	1.4	1.6	1.8	2.0
10	4.44	3.44	2.86	2.50	2.25	2.08
15	4.95	3.77	3.10	2.69	2.40	2.20
20	5.32	4.00	3.27	2.81	2.50	2.28
25	5.61	4.18	3.41	2.91	2.60	2.32

E. Example

As an example consider the following samples of size 30 from Weibull distributions with common shape parameter equal to 2.0 and scale parameters equal to 50 and 60, respectively.

Sample 1: 18.02, 18.03, 19.84, 19.86, 21.31, 25.95, 29.10 29.21, 31.34, 32.99, 34.22, 35.02, 36.70, 38.62, 41.28, 41.32, 42.05, 43.79, 44.72, 45.02, 45.71, 48.08, 58.18, 61.27, 64.90, 71.35, 72.78, 76.52, 90.91, 91.40.

Sample 2: 13.54, 14.47, 19.83, 20.17, 33.15, 34.40, 36.69 39.42, 40.29, 40.81, 43.94, 45.78, 50.49, 52.59, 54.29, 54.92, 55.76, 58.88, 63.15, 63.93, 65.48, 68.34, 75.46, 81.11, 87.35, 86.27, 88.93, 92.04, 99.48, 105.58.

Using the routine given in Appendix B we find that $\hat{b}_1=50.22$, $\hat{b}_2=63.44$, the unbiased estimates of c are 2.23 and 2.33, respectively, and the maximum likelihood estimates of the reliability for t=30 are .741 and .851.

The acceptance region for testing at the .10 level the hypothesis $c_1=c_2$ against the alternative $c_1\neq c_2$ based on $\hat{c}_1/\hat{c}_2$ is, from section III.B and Table A6, (.710, 1.409). Thus, based on the above samples, the null hypothesis would not have been rejected.

The critical region for $z=\frac{\hat{c}_1+\hat{c}_2}{2}[\ln(\hat{b}_1) - \ln(\hat{b}_2)]$ for testing $b_1=b_2$ against the alternative $b_1< b_2$ is $(-\infty, -.366)$ from section III.C and Table A7. The value of z based on the above samples is -.550 and thus the hypothesis is rejected.

Also, process 1 is correctly picked as the process with the larger average life time. From Table A8, the probability of correct selection for n=30 and $(\alpha_s)^c=1.4$ is .891.

IV. DISCUSSION OF NUMERICAL METHODS AND ACCURACY OF RESULTS

The maximum likelihood estimates of c were obtained from equation (3) by the Newton-Raphson iterative procedure [17]. When this method is applied to equation (3) we obtain the following relation between $\hat{c}_{(k)}$, the k<u>th</u> approximation to $\hat{c}$, and $\hat{c}_{(k+1)}$, the (k+1)<u>st</u> approximation:

$$\hat{c}_{(k+1)} = \hat{c}_{(k)} + \frac{\dfrac{1}{\hat{c}_{(k)}} + \dfrac{S_1}{n} - \dfrac{S_3^{(k)}}{S_2^{(k)}}}{\dfrac{1}{\hat{c}^{\,2}_{(k)}} + \dfrac{S_2^{(k)} S_4^{(k)} - (S_3^{(k)})^2}{(S_2^{(k)})^2}}$$

where

$S_1 = \Sigma \ln x_i$, $S_2^{(k)} = \Sigma\, x_i^{\hat{c}_{(k)}}$, $S_3^{(k)} = \Sigma\, (\ln x_i) x_i^{\hat{c}_{(k)}}$, and

$S_4^{(k)} = \Sigma\, (\ln x_i)^2\, x_i^{\hat{c}_{(k)}}$. The convergence of the iterates is, in general, very fast. If, for example, Menon's estimate is used as the initial approximation of $\hat{c}$, then the average number of iterations required to obtain four place accuracy when sampling from a standard exponential is about 3.5. As further evidence of the speed of convergence and the capacity of modern computers it was noted that the time required for the IBM 360, model 40, to generate 100 samples of size 20 and solve equation (3) and (4) for all samples was 35 seconds.

The distributions of the pivotal functions discussed in the preceding sections were based on the results of 20,000 "random" samples of size 5, 10,000 samples of size 6, 8, 10, 12, 15, 20, 30, 40, 50 and 75, and 6,000 samples of size 100 which were generated from an exponential distribution. The empirical distributions of the generated values of the pivotal functions was tabulated and the percentage points, $y_\gamma(n)$, were obtained for each sample size and various percentages. Interpolation on the sample size was accomplished by fitting, according to the criteria of least squares, the quadratic $y_\gamma(n) = a_o + a_1x + a_2x^2$ where $x = 1/(n-d)^p$. The work in sections II.B-3, II.D, II.E, III.C, and III.D required interpolation on K in the tabulated distributions, G_K and H_K. For this purpose the model $y_{\gamma,n}(K) = b_o + b_1x + b_2x^2$ where $x = \ln(dK + e)$ was used for each value of γ and n.

As an aid in evaluating the accuracy of the results, the distribution of the means of the samples generated during the process was obtained and smoothed in the same manner as above and the resulting points were compared with the known values. Except for the .98 percentage points, the procedure led to percentage points that were within .005 of the true values. The difference attained a value of .010 for a few of the .98 percentage points but the maximum relative error was only .006. Most of the errors occurred for the values of n from 5 to 15. The average absolute error in this range of n was .0023 and the average relative error was .0015. The first four sample moments of the generated exponential random variables were also in close agreement with the population moments.

It is difficult to make exact statements concerning the accuracy of the Monte Carlo results but in view of the studies made it is felt that the accuracy exhibited by the empirical distribution of the sample mean is typical of the accuracy in the tables given in Appendix A.

V. SUMMARY, CONCLUSIONS AND FURTHER PROBLEMS

As noted in section II.C, up to this point progress on providing solutions to the problems of making inferences in the Weibull distribution has primarily been limited to the advancing of simple estimators of the parameters. Little has been known even about the properties of these estimators except in the asymptotic sense. Except for the significant results by Johns and Lieberman, [6], giving exact confidence limits on the reliability, contributions to this area have had to resort to asymptotic theory to obtain, for example, approximate solutions to the problems of interval estimation and hypothesis testing.

In this paper the superiority of the maximum likelihood estimators has been established and their small and moderate sample size properties have been studied. But the most significant results have been the solution, through the discovery of certain pivotal functions, of the standard problems of estimation and hypothesis testing in the Weibull distribution.

Areas which warrant further investigations include a search for good approximations to the distributions of the pivotal functions for moderate samples. The conservativeness of the procedure in section III.C-2 for testing $b_1=b_2$ could be investigated and a study made into how the tests on c_1 and c_2 in section III.B could be used to determine the appropriateness of the assumption of $c_1=c_2$ in sections III.C-1 and III.D and $c_1 < c_2$ in section III.C-2.

Another area of consideration arises immediately from the fact that the pivotal functions remain pivotal even for type II censored sampling. The results in this paper can readily be extended to this case although the necessity of considering various points of censoring greatly enlarges the amount of simulation required to generate the distributions.

In addition, the results can be applied to the important three parameter Weibull given by equation (1). If c is known it can be observed through the change of variables, $y=e^x$, and a reparametrization that $\hat{b}/b$, $(\hat{G} - G)/b$ and $(\hat{G} - G)/\hat{b}$ have distributions that are independent of the parameters. The generation of these distributions would yield inferences concerning b and G with c known.

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APPENDIX A

TABLES

Table A1

Percentage Points, ℓ_γ, such that $P[\hat{c}/c < \ell_\gamma] = \gamma$

N \ γ	.02	.05	.10	.25	.40	.50	.60
5	.604	.683	.766	.951	1.116	1.238	1.378
6	.623	.697	.778	.937	1.080	1.188	1.304
7	.639	.709	.785	.930	1.059	1.155	1.256
8	.653	.720	.792	.926	1.045	1.131	1.223
9	.665	.729	.797	.925	1.035	1.114	1.198
10	.676	.738	.802	.924	1.028	1.101	1.179
11	.686	.745	.807	.924	1.022	1.090	1.163
12	.695	.752	.811	.924	1.017	1.082	1.151
13	.703	.759	.815	.924	1.014	1.075	1.140
14	.710	.764	.819	.925	1.011	1.069	1.132
15	.716	.770	.823	.925	1.008	1.064	1.124
16	.723	.775	.826	.926	1.006	1.059	1.117
17	.728	.779	.829	.927	1.004	1.056	1.111
18	.734	.784	.832	.927	1.003	1.052	1.106
19	.739	.788	.835	.928	1.001	1.049	1.101
20	.743	.791	.838	.929	1.000	1.047	1.097
22	.752	.798	.843	.930	0.998	1.042	1.090
24	.759	.805	.848	.932	0.997	1.038	1.084
26	.766	.810	.852	.933	0.995	1.035	1.079
28	.772	.815	.856	.934	0.994	1.033	1.074
30	.778	.820	.860	.935	0.993	1.030	1.070
32	.783	.824	.863	.937	0.993	1.028	1.067
34	.788	.828	.866	.938	0.992	1.027	1.064
36	.793	.832	.869	.939	0.992	1.025	1.061
38	.797	.835	.872	.940	0.991	1.024	1.059
40	.801	.839	.875	.940	0.991	1.023	1.056
42	.804	.842	.877	.941	0.990	1.022	1.054
44	.808	.845	.880	.942	0.990	1.021	1.052
46	.811	.847	.882	.943	0.990	1.020	1.051
48	.814	.850	.884	.944	0.990	1.019	1.049
50	.817	.852	.886	.944	0.989	1.018	1.048
52	.820	.854	.888	.945	0.989	1.017	1.046
54	.822	.857	.890	.946	0.989	1.017	1.045
56	.825	.859	.891	.946	0.989	1.016	1.044
58	.827	.861	.893	.947	0.989	1.015	1.043
60	.830	.863	.894	.948	0.989	1.015	1.041
62	.832	.864	.896	.948	0.989	1.014	1.040
64	.834	.866	.897	.949	0.989	1.014	1.040
66	.836	.868	.899	.949	0.988	1.014	1.039
68	.838	.869	.900	.950	0.988	1.013	1.038
70	.840	.871	.901	.950	0.988	1.013	1.037
72	.841	.872	.903	.951	0.988	1.012	1.036
74	.843	.874	.904	.951	0.988	1.012	1.036
76	.845	.875	.905	.952	0.988	1.012	1.035
80	.848	.878	.907	.952	0.988	1.011	1.034
85	.852	.881	.910	.953	0.988	1.011	1.032
90	.855	.883	.912	.954	0.988	1.010	1.031
100	.861	.888	.916	.956	0.988	1.009	1.029
120	.871	.897	.923	.959	0.988	1.007	1.025

Table A1 (cont.)

Percentage Points, ℓ_γ, such that $P[\hat{c}/c < \ell_\gamma]=\gamma$

N \ γ	.70	.75	.80	.85	.90	.95	.98
5	1.557	1.671	1.812	2.001	2.277	2.779	3.518
6	1.453	1.543	1.662	1.812	2.030	2.436	3.067
7	1.386	1.461	1.561	1.688	1.861	2.183	2.640
8	1.338	1.404	1.491	1.602	1.747	2.015	2.377
9	1.303	1.361	1.439	1.538	1.665	1.896	2.199
10	1.275	1.328	1.399	1.489	1.602	1.807	2.070
11	1.253	1.302	1.367	1.450	1.553	1.738	1.972
12	1.234	1.281	1.341	1.418	1.513	1.682	1.894
13	1.219	1.263	1.319	1.391	1.480	1.636	1.830
14	1.206	1.248	1.300	1.369	1.452	1.597	1.777
15	1.195	1.234	1.284	1.349	1.427	1.564	1.732
16	1.185	1.223	1.270	1.332	1.406	1.535	1.693
17	1.176	1.213	1.258	1.317	1.388	1.510	1.660
18	1.168	1.204	1.247	1.303	1.371	1.487	1.630
19	1.162	1.196	1.237	1.291	1.356	1.467	1.603
20	1.155	1.188	1.228	1.281	1.343	1.449	1.579
22	1.144	1.176	1.213	1.262	1.320	1.418	1.538
24	1.135	1.165	1.200	1.246	1.301	1.392	1.504
26	1.128	1.156	1.189	1.232	1.284	1.370	1.475
28	1.121	1.148	1.180	1.220	1.269	1.351	1.450
30	1.115	1.141	1.171	1.210	1.257	1.334	1.429
32	1.110	1.135	1.164	1.201	1.246	1.319	1.409
34	1.105	1.129	1.157	1.193	1.236	1.306	1.392
36	1.101	1.125	1.151	1.186	1.227	1.294	1.377
38	1.097	1.120	1.146	1.179	1.219	1.283	1.363
40	1.094	1.116	1.141	1.173	1.211	1.273	1.351
42	1.091	1.112	1.137	1.167	1.204	1.265	1.339
44	1.088	1.109	1.132	1.162	1.198	1.256	1.329
46	1.085	1.106	1.129	1.158	1.192	1.249	1.319
48	1.083	1.103	1.125	1.153	1.187	1.242	1.310
50	1.081	1.100	1.122	1.149	1.182	1.235	1.301
52	1.078	1.098	1.119	1.145	1.177	1.229	1.294
54	1.076	1.095	1.116	1.142	1.173	1.224	1.286
56	1.075	1.093	1.113	1.139	1.169	1.218	1.280
58	1.073	1.091	1.111	1.135	1.165	1.213	1.273
60	1.071	1.089	1.108	1.133	1.162	1.208	1.267
62	1.070	1.087	1.106	1.130	1.158	1.204	1.262
64	1.068	1.086	1.104	1.127	1.155	1.200	1.256
66	1.067	1.084	1.102	1.125	1.152	1.196	1.251
68	1.066	1.083	1.100	1.122	1.149	1.192	1.246
70	1.064	1.081	1.098	1.120	1.146	1.188	1.242
72	1.063	1.080	1.097	1.118	1.144	1.185	1.237
74	1.062	1.078	1.095	1.116	1.141	1.182	1.233
76	1.061	1.077	1.093	1.114	1.139	1.179	1.229
80	1.059	1.075	1.090	1.110	1.134	1.173	1.222
85	1.057	1.072	1.087	1.106	1.129	1.166	1.213
90	1.055	1.069	1.084	1.102	1.124	1.160	1.206
100	1.051	1.065	1.079	1.096	1.116	1.150	1.192
120	1.046	1.058	1.070	1.086	1.104	1.133	1.171

Table A2a; Percentage Points for $\hat{c}[\ln(\hat{b}/b) - \ln(K)]$ with K=.511, $\beta=e^{-K}=.60$

N	γ .02	.05	.10	.25	.40	.50	.60	.70	.75	.80	.85	.90	.95	.98
7	-.312	-.090	.086	.379	.584	.723	.871	1.053	1.158	1.284	1.438	1.657	2.009	2.505
8	-.228	-.037	.125	.397	.588	.715	.850	1.015	1.110	1.224	1.357	1.550	1.863	2.272
10	-.115	.042	.183	.424	.596	.705	.823	.963	1.044	1.140	1.249	1.408	1.669	1.983
12	-.040	.099	.224	.444	.601	.700	.805	.928	1.000	1.083	1.178	1.316	1.543	1.806
14	.015	.142	.256	.459	.605	.696	.793	.903	.968	1.042	1.128	1.250	1.454	1.683
16	.058	.176	.282	.472	.609	.694	.783	.885	.944	1.011	1.090	1.201	1.386	1.592
18	.093	.204	.303	.482	.612	.692	.776	.869	.925	.987	1.059	1.162	1.332	1.521
20	.122	.228	.321	.491	.614	.690	.769	.857	.909	.966	1.035	1.130	1.288	1.464
22	.147	.248	.337	.499	.617	.689	.764	.847	.896	.950	1.014	1.104	1.251	1.416
24	.169	.266	.351	.506	.619	.688	.759	.838	.884	.935	.997	1.082	1.220	1.376
26	.188	.281	.363	.512	.620	.687	.755	.830	.875	.923	.982	1.062	1.193	1.341
28	.205	.295	.374	.518	.622	.686	.752	.824	.866	.912	.969	1.045	1.169	1.311
30	.220	.307	.383	.523	.623	.685	.749	.818	.858	.903	.957	1.030	1.148	1.284
32	.234	.318	.392	.527	.624	.685	.746	.812	.852	.895	.947	1.017	1.129	1.260
34	.246	.328	.400	.532	.626	.684	.743	.808	.846	.887	.937	1.005	1.112	1.239
36	.258	.338	.408	.535	.627	.683	.741	.804	.840	.880	.929	.994	1.097	1.219
38	.269	.346	.415	.539	.628	.683	.739	.800	.835	.874	.921	.984	1.083	1.202
40	.278	.354	.421	.542	.628	.683	.737	.796	.830	.868	.914	.975	1.071	1.185
42	.288	.362	.427	.545	.620	.682	.735	.793	.826	.863	.908	.967	1.059	1.171
44	.296	.368	.432	.548	.630	.682	.733	.790	.822	.859	.902	.959	1.048	1.157
46	.304	.375	.438	.551	.631	.682	.732	.787	.818	.854	.896	.952	1.038	1.144
48	.311	.381	.442	.553	.632	.681	.730	.784	.815	.850	.891	.946	1.029	1.133
50	.318	.386	.447	.555	.632	.681	.729	.782	.812	.846	.886	.939	1.020	1.122
52	.325	.392	.451	.558	.633	.681	.728	.779	.809	.843	.882	.934	1.012	1.111
54	.331	.397	.455	.560	.633	.680	.726	.777	.906	.839	.877	.928	1.004	1.102
56	.337	.401	.459	.562	.634	.680	.725	.775	.803	.836	.873	.923	.997	1.092
58	.343	.406	.463	.564	.635	.680	.724	.773	.801	.833	.869	.918	.990	1.084
60	.348	.410	.466	.565	.635	.680	.723	.771	.799	.830	.866	.914	.984	1.076
64	.358	.418	.479	.569	.636	.679	.721	.768	.794	.825	.859	.905	.972	1.061
68	.367	.426	.479	.572	.637	.679	.719	.765	.790	.821	.853	.898	.961	1.047
72	.376	.432	.484	.575	.638	.679	.718	.762	.787	.817	.848	.891	.951	1.035
76	.383	.438	.489	.577	.638	.678	.716	.759	.783	.813	.843	.885	.943	1.024
80	.390	.444	.494	.580	.639	.678	.715	.757	.780	.809	.838	.879	.934	1.013

Table A2b: Percentage Points for $\hat{c}[\ln(\hat{b}/b) - \ln(K)]$ with K=.693, $\beta=e^{-K}=.50$

N \ γ	.02	.05	.10	.25	.40	.50	.60	.70	.75	.80	.85	.90	.95	.98
7	-.665	-.423	-.229	.058	.255	.376	.500	.656	.747	.854	.981	1.157	1.448	1.827
8	-.563	-.357	-.183	.081	.263	.375	.487	.630	.712	.808	.920	1.080	1.336	1.657
10	-.430	-.263	-.117	.115	.274	.373	.472	.594	.663	.745	.837	.975	1.186	1.441
12	-.343	-.199	-.072	.138	.283	.372	.462	.570	.631	.402	.784	.904	1.088	1.305
14	-.281	-.152	-.038	.156	.289	.372	.455	.553	.607	.671	.745	.852	1.017	1.209
16	-.233	-.115	-.010	.170	.295	.371	.449	.539	.589	.647	.715	.813	.963	1.137
18	-.195	-.085	.012	.182	.299	.371	.445	.528	.575	.628	.692	.781	.920	1.080
20	-.163	-.061	.031	.192	.303	.371	.441	.519	.563	.613	.672	.755	.884	1.034
22	-.136	-.040	.047	.200	.306	.371	.438	.511	.552	.600	.656	.734	.855	.996
24	-.113	-.021	.061	.207	.309	.370	.435	.504	.544	.588	.642	.715	.829	.963
26	-.093	-.006	.073	.214	.311	.370	.442	.499	.536	.579	.630	.699	.807	.935
28	-.076	.008	.084	.219	.313	.370	.430	.493	.529	.570	.619	.685	.788	.910
30	-.060	.021	.094	.225	.315	.370	.427	.489	.523	.563	.610	.673	.771	.888
32	-.046	.032	.103	.229	.317	.370	.425	.485	.518	.556	.602	.661	.755	.868
34	-.033	.042	.111	.233	.319	.370	.424	.481	.513	.550	.594	.651	.742	.851
36	-.022	.052	.119	.237	.320	.370	.422	.478	.509	.544	.587	.642	.729	.835
38	-.011	.060	.125	.241	.321	.370	.420	.475	.505	.539	.581	.634	.718	.820
40	-.001	.068	.132	.244	.323	.370	.419	.472	.501	.534	.575	.627	.707	.807
42	.008	.075	.138	.247	.324	.370	.418	.469	.498	.530	.569	.620	.698	.794
44	.016	.082	.143	.250	.325	.370	.416	.467	.494	.526	.564	.613	.689	.783
46	.024	.088	.148	.253	.326	.370	.415	.464	.491	.523	.560	.607	.680	.773
48	.031	.094	.153	.255	.327	.370	.414	.462	.489	.519	.555	.602	.673	.763
50	.038	.100	.157	.257	.328	.370	.413	.460	.486	.516	.551	.596	.665	.754
52	.045	.105	.162	.260	.328	.370	.412	.458	.484	.513	.548	.592	.659	.745
54	.051	.110	.166	.262	.329	.370	.411	.457	.481	.510	.544	.587	.652	.737
56	.056	.115	.169	.264	.330	.370	.410	.455	.479	.507	.541	.583	.646	.729
58	.062	.119	.173	.265	.331	.370	.409	.453	.477	.505	.537	.579	.641	.722
60	.067	.123	.176	.267	.331	.370	.409	.452	.475	.502	.534	.575	.635	.715
64	.077	.131	.183	.270	.332	.370	.407	.449	.472	.498	.529	.568	.625	.703
68	.085	.138	.188	.273	.333	.369	.406	.447	.468	.494	.524	.561	.616	.691
72	.093	.145	.194	.276	.335	.369	.405	.444	.465	.490	.519	.555	.608	.681
76	.101	.151	.198	.279	.335	.369	.403	.442	.462	.487	.515	.550	.601	.672
80	.107	.156	.203	.281	.336	.369	.402	.440	.460	.484	.511	.545	.594	.663

Table A2c: Percentage Points for $\hat{c}[\ln(\hat{b}/b) - \ln(K)]$ with K=.80, $\beta=e^{-K}=.449$

N \ γ	.02	.05	.10	.25	.40	.50	.60	.70	.75	.80	.85	.90	.95	.98
7	-.856	-.582	-.381	-.101	.099	.214	.338	.477	.554	.653	.769	.918	1.178	1.540
8	-.731	-.518	-.334	-.078	.105	.216	.322	.457	.532	.620	.727	.872	1.108	1.408
10	-.580	-.417	-.265	-.039	.120	.218	.311	.427	.493	.567	.657	.785	.982	1.217
12	-.478	-.347	-.216	-.011	.131	.219	.305	.406	.465	.529	.609	.721	.891	1.092
14	-.423	-.296	-.180	.010	.140	.220	.300	.391	.443	.502	.573	.673	.825	1.003
16	-.375	-.256	-.151	.026	.146	.221	.295	.379	.427	.480	.545	.637	.773	.936
18	-.335	-.225	-.127	.039	.152	.221	.292	.369	.413	.463	.523	.607	.733	.884
20	-.304	-.199	-.108	.050	.156	.222	.289	.361	.403	.449	.505	.583	.700	.841
22	-.277	-.177	-.091	.059	.160	.222	.286	.355	.393	.437	.490	.563	.672	.806
24	-.253	-.159	-.077	.066	.163	.222	.283	.349	.385	.427	.477	.546	.649	.775
26	-.233	-.143	-.064	.073	.165	.222	.281	.344	.379	.419	.466	.531	.629	.750
28	-.215	-.128	-.053	.079	.168	.223	.279	.339	.373	.411	.457	.518	.611	.727
30	-.199	-.116	-.043	.084	.170	.223	.277	.335	.367	.404	.448	.507	.595	.707
32	-.185	-.105	-.034	.089	.172	.223	.275	.331	.362	.398	.440	.497	.582	.689
34	-.142	-.094	-.026	.093	.174	.223	.274	.328	.358	.393	.433	.487	.569	.673
36	-.160	-.085	-.018	.097	.175	.223	.272	.325	.354	.388	.427	.479	.558	.659
38	-.149	-.077	-.011	.100	.177	.223	.271	.322	.351	.383	.421	.471	.547	.646
40	-.138	-.069	-.005	.103	.178	.223	.270	.320	.347	.379	.416	.464	.538	.634
42	-.129	-.062	.001	.106	.179	.223	.269	.318	.344	.375	.411	.458	.529	.623
44	-.120	-.055	.006	.109	.180	.223	.267	.315	.341	.371	.406	.452	.521	.612
46	-.112	-.049	.011	.111	.181	.223	.266	.313	.339	.368	.402	.446	.514	.603
48	-.104	-.043	.016	.114	.182	.223	.265	.312	.336	.365	.398	.441	.507	.594
50	-.097	-.038	.020	.116	.183	.223	.265	.310	.334	.362	.394	.437	.500	.586
52	-.091	-.033	.024	.118	.184	.223	.264	.308	.332	.359	.391	.432	.494	.578
54	-.084	-.028	.028	.120	.185	.223	.263	.307	.330	.357	.388	.428	.489	.571
56	-.078	-.024	.032	.122	.185	.223	.262	.305	.328	.354	.385	.424	.483	.564
58	-.072	-.019	.036	.124	.186	.223	.261	.304	.326	.352	.382	.420	.478	.558
60	-.067	-.015	.039	.125	.187	.223	.261	.302	.324	.350	.379	.417	.474	.552
64	-.057	-.008	.045	.128	.188	.223	.259	.300	.321	.346	.374	.410	.465	.540
68	-.048	-.001	.051	.131	.189	.223	.258	.298	.318	.342	.370	.404	.457	.530
72	-.039	-.005	.056	.134	.190	.224	.257	.296	.316	.339	.365	.399	.450	.521
76	-.032	-.011	.061	.136	.191	.224	.256	.294	.313	.336	.362	.394	.443	.513
80	-.025	-.016	.065	.138	.192	.224	.255	.292	.311	.333	.358	.389	.437	.505

Table A2d: Percentage Points for $\hat{c}[\ln(\hat{b}/b) - \ln(K)]$ with K=.90, $\beta=e^{-K}=.407$

N γ	.02	.05	.10	.25	.40	.50	.60	.70	.75	.80	.85	.90	.95	.98
7	-1.070	-.736	-.516	-.226	-.031	.078	.195	.327	.404	.500	.607	.752	.996	1.317
8	-.964	-.652	-.457	-.195	-.018	.081	.189	.309	.379	.465	.560	.695	.911	1.183
10	-.809	-.541	-.379	-.153	-.001	.086	.181	.285	.344	.416	.497	.615	.794	1.011
12	-.701	-.468	-.328	-.125	.010	.090	.175	.269	.321	.384	.456	.560	.717	.902
14	-.621	-.415	-.291	-.105	.018	.092	.171	.256	.303	.360	.426	.520	.660	.824
16	-.559	-.375	-.262	-.089	.025	.094	.168	.247	.290	.342	.403	.489	.616	.766
18	-.509	-.343	-.239	-.077	.030	.096	.165	.239	.279	.328	.384	.464	.581	.720
20	-.467	-.317	-.220	-.066	.035	.097	.162	.232	.270	.316	.369	.443	.552	.683
22	-.433	-.295	-.204	-.057	.039	.098	.160	.226	.262	.305	.356	.426	.527	.651
24	-.403	-.276	-.190	-.050	.042	.099	.158	.221	.256	.297	.345	.410	.506	.624
26	-.377	-.259	-.178	-.043	.045	.100	.156	.217	.250	.289	.335	.397	.488	.601
28	-.355	-.245	-.167	-.038	.048	.100	.155	.213	.245	.282	.326	.386	.472	.581
30	-.335	-.232	-.157	-.032	.050	.101	.153	.209	.240	.276	.319	.375	.458	.562
32	-.317	-.220	-.148	-.028	.052	.101	.152	.206	.236	.271	.312	.366	.445	.546
34	-.301	-.210	-.140	-.024	.054	.102	.151	.203	.232	.266	.306	.357	.433	.532
36	-.287	-.200	-.133	-.020	.056	.102	.150	.201	.229	.261	.300	.350	.423	.518
38	-.274	-.191	-.126	-.016	.057	.102	.149	.198	.225	.257	.295	.353	.413	.506
40	-.262	-.183	-.120	-.013	.059	.103	.148	.196	.222	.253	.290	.336	.405	.495
42	-.251	-.176	-.114	-.010	.060	.103	.147	.194	.220	.250	.286	.330	.396	.485
44	-.240	-.169	-.109	-.007	.062	.103	.146	.192	.217	.247	.282	.325	.389	.475
46	-.231	-.163	-.104	-.005	.063	.103	.145	.190	.215	.244	.278	.319	.382	.467
48	-.222	-.157	-.099	-.002	.064	.104	.144	.188	.213	.241	.275	.315	.375	.458
50	-.214	-.171	-.095	.000	.065	.104	.144	.187	.210	.238	.271	.310	.369	.451
52	-.206	-.146	-.091	.002	.066	.104	.143	.185	.208	.236	.268	.306	.363	.443
54	-.199	-.141	-.087	.004	.067	.104	.142	.184	.207	.233	.265	.302	.358	.437
56	-.192	-.136	-.083	.006	.068	.104	.142	.182	.205	.231	.262	.298	.353	.430
58	-.186	-.132	-.080	.008	.068	.104	.141	.181	.203	.229	.259	.295	.348	.424
60	-.180	-.127	-.076	.009	.069	.105	.141	.180	.202	.227	.257	.291	.343	.418
64	-.169	-.120	-.070	.012	.071	.105	.140	.177	.199	.223	.254	.285	.335	.408
68	-.159	-.113	-.065	.015	.072	.105	.139	.175	.196	.220	.252	.279	.327	.398
72	-.150	-.106	-.060	.018	.073	.105	.138	.173	.193	.217	.248	.274	.320	.389
76	-.142	-.100	-.055	.020	.074	.105	.137	.171	.191	.214	.244	.269	.314	.381
80	-.135	-.095	-.050	.022	.075	.105	.136	.170	.189	.211	.240	.265	.308	.374

Table A2e

Percentage Points, ℓ_γ, such that $P[\hat{c}\ln(\hat{b}/b) < \ell_\gamma] = \gamma$

N \ γ	.02	.05	.10	.25	.40	.50	.60
5	-1.631	-1.247	-.888	-.444	-.241	-.056	.085
6	-1.396	-1.007	-.740	-.385	-.194	-.045	.079
7	-1.196	-0.874	-.652	-.344	-.168	-.038	.074
8	-1.056	-0.784	-.591	-.313	-.150	-.032	.070
9	-0.954	-0.717	-.544	-.289	-.137	-.029	.067
10	-0.876	-0.665	-.507	-.269	-.126	-.026	.065
11	-0.813	-0.622	-.477	-.253	-.118	-.023	.062
12	-0.762	-0.587	-.451	-.239	-.111	-.021	.061
13	-0.719	-0.557	-.429	-.228	-.106	-.019	.059
14	-0.683	-0.532	-.410	-.217	-.100	-.018	.057
15	-0.651	-0.509	-.393	-.208	-.096	-.016	.056
16	-0.624	-0.489	-.379	-.200	-.092	-.015	.054
17	-0.599	-0.471	-.365	-.193	-.089	-.014	.053
18	-0.578	-0.455	-.353	-.187	-.085	-.013	.052
19	-0.558	-0.441	-.342	-.181	-.083	-.013	.051
20	-0.540	-0.428	-.332	-.175	-.080	-.012	.050
22	-0.509	-0.404	-.314	-.166	-.075	-.011	.048
24	-0.483	-0.384	-.299	-.158	-.071	-.009	.047
26	-0.460	-0.367	-.286	-.150	-.068	-.009	.046
28	-0.441	-0.352	-.274	-.144	-.065	-.008	.044
30	-0.423	-0.338	-.264	-.139	-.062	-.007	.043
32	-0.408	-0.326	-.254	-.134	-.059	-.006	.042
34	-0.394	-0.315	-.246	-.129	-.057	-.006	.041
36	-0.382	-0.305	-.238	-.125	-.055	-.005	.040
38	-0.370	-0.296	-.231	-.121	-.053	-.005	.040
40	-0.360	-0.288	-.224	-.118	-.052	-.004	.039
42	-0.350	-0.280	-.218	-.115	-.050	-.004	.038
44	-0.341	-0.273	-.213	-.112	-.048	-.004	.037
46	-0.333	-0.266	-.208	-.109	-.047	-.003	.037
48	-0.325	-0.260	-.203	-.106	-.046	-.003	.036
50	-0.318	-0.254	-.198	-.104	-.045	-.003	.036
52	-0.312	-0.249	-.194	-.102	-.043	-.003	.035
54	-0.305	-0.244	-.190	-.100	-.042	-.002	.035
56	-0.299	-0.239	-.186	-.098	-.041	-.002	.034
58	-0.294	-0.234	-.183	-.096	-.040	-.002	.034
60	-0.289	-0.230	-.179	-.094	-.039	-.002	.033
62	-0.284	-0.226	-.176	-.092	-.039	-.002	.033
64	-0.279	-0.222	-.173	-.091	-.038	-.001	.032
66	-0.274	-0.218	-.170	-.089	-.037	-.001	.032
68	-0.270	-0.215	-.167	-.088	-.036	-.001	.032
70	-0.266	-0.211	-.165	-.086	-.035	-.001	.031
72	-0.262	-0.208	-.162	-.085	-.035	-.001	.031
74	-0.259	-0.205	-.160	-.084	-.034	-.001	.031
76	-0.255	-0.202	-.158	-.083	-.033	-.001	.030
80	-0.248	-0.197	-.153	-.080	-.032	-.000	.030
85	-0.241	-0.190	-.148	-.078	-.031	-.000	.029
90	-0.234	-0.184	-.144	-.075	-.030	.000	.028
100	-0.221	-0.174	-.136	-.071	-.027	.000	.027
120	-0.202	-0.158	-.123	-.064	-.024	.001	.025

Table A2e (cont.)

Percentage Points, ℓ_γ, such that $P[\hat{c}\ln(\hat{b}/b) < \ell_\gamma]=\gamma$

N \ γ	.70	.75	.80	.85	.90	.95	.98
5	.254	.349	.452	.587	.772	1.107	1.582
6	.221	.302	.404	.516	.666	0.939	1.291
7	.200	.272	.362	.465	.598	0.829	1.120
8	.185	.251	.331	.427	.547	0.751	1.003
9	.174	.235	.307	.397	.507	0.691	0.917
10	.165	.222	.288	.372	.475	0.644	0.851
11	.157	.211	.273	.351	.448	0.605	0.797
12	.150	.202	.260	.334	.425	0.572	0.752
13	.145	.194	.249	.319	.406	0.544	0.714
14	.140	.187	.239	.306	.389	0.520	0.681
15	.135	.180	.230	.294	.374	0.499	0.653
16	.131	.175	.223	.284	.360	0.480	0.627
17	.128	.170	.216	.274	.348	0.463	0.605
18	.124	.165	.209	.266	.338	0.447	0.584
19	.121	.161	.204	.258	.328	0.433	0.566
20	.118	.157	.199	.251	.318	0.421	0.549
22	.113	.150	.189	.239	.302	0.398	0.519
24	.109	.144	.181	.228	.288	0.379	0.494
26	.105	.138	.174	.219	.276	0.362	0.472
28	.102	.134	.168	.210	.265	0.347	0.453
30	.098	.129	.163	.203	.256	0.334	0.435
32	.095	.125	.158	.197	.247	0.323	0.420
34	.093	.122	.153	.191	.239	0.312	0.406
36	.090	.118	.149	.185	.232	0.302	0.393
38	.088	.115	.145	.180	.226	0.293	0.382
40	.086	.113	.142	.175	.220	0.285	0.371
42	.084	.110	.139	.171	.214	0.278	0.361
44	.082	.108	.136	.167	.209	0.271	0.352
46	.080	.105	.133	.164	.204	0.264	0.344
48	.079	.103	.130	.160	.199	0.258	0.336
50	.077	.101	.128	.157	.195	0.253	0.328
52	.076	.099	.126	.154	.191	0.247	0.321
54	.074	.097	.123	.151	.187	0.243	0.315
56	.073	.096	.121	.148	.184	0.238	0.309
58	.072	.094	.119	.146	.181	0.233	0.303
60	.071	.092	.117	.143	.177	0.229	0.297
62	.070	.091	.116	.141	.174	0.225	0.292
64	.068	.089	.114	.139	.171	0.221	0.287
66	.067	.088	.112	.137	.169	0.218	0.282
68	.066	.087	.111	.135	.166	0.214	0.278
70	.065	.085	.109	.133	.164	0.211	0.274
72	.464	.084	.108	.131	.161	0.208	0.269
74	.064	.083	.107	.129	.159	0.205	0.266
76	.063	.082	.105	.128	.157	0.202	0.262
80	.061	.080	.103	.125	.153	0.197	0.255
85	.059	.077	.100	.121	.148	0.190	0.246
90	.057	.075	.097	.118	.143	0.185	0.239
100	.054	.071	.093	.112	.136	0.175	0.226
120	.049	.064	.085	.103	.123	0.159	0.205

Table A2f: Percentage Points for $\hat{c}[\ln(\hat{b}/b) - \ln(K)]$ with $K=1.05$, $\beta=e^{-K}=.350$

N \ γ	.02	.05	.10	.25	.40	.50	.60	.70	.75	.80	.85	.90	.95	.98
7	-1.280	-.940	-.709	-.400	-.204	-.090	.019	.141	.208	.292	.388	.517	.725	1.025
8	-1.123	-.854	-.655	-.371	-.191	-.086	.016	.131	.195	.271	.364	.482	.677	.926
10	-.930	-.729	-.568	-.325	-.170	-.079	.012	.113	.170	.235	.316	.416	.583	.782
12	-.814	-.646	-.508	-.293	-.156	-.074	.008	.099	.150	.207	.279	.367	.513	.685
14	-.734	-.587	-.464	-.269	-.145	-.070	.005	.088	.135	.186	.251	.330	.461	.615
16	-.675	-.542	-.430	-.251	-.136	-.067	.003	.079	.122	.170	.229	.302	.420	.562
18	-.929	-.506	-.403	-.237	-.130	-.065	.001	.072	.112	.157	.212	.279	.388	.529
20	-.592	-.478	-.381	-.225	-.124	-.063	-.001	.066	.104	.146	.197	.260	.361	.486
22	-.561	-.454	-.363	-.215	-.119	-.061	-.002	.061	.097	.137	.185	.244	.339	.457
24	-.535	-.433	-.347	-.207	-.116	-.060	-.004	.057	.091	.129	.174	.231	.320	.432
26	-.512	-.416	-.334	-.200	-.112	-.059	-.005	.053	.085	.122	.165	.219	.303	.411
28	-.492	-.400	-.322	-.194	-.109	-.058	-.006	.049	.080	.115	.157	.209	.289	.392
30	-.475	-.387	-.311	-.188	-.107	-.057	-.007	.046	.076	.110	.149	.199	.276	.376
32	-.459	-.375	-.302	-.183	-.104	-.056	-.008	.043	.072	.105	.143	.191	.264	.361
34	-.445	-.364	-.293	-.179	-.102	-.055	-.009	.040	.069	.100	.137	.184	.254	.348
35	-.432	-.354	-.286	-.175	-.100	-.055	-.010	.038	.066	.096	.132	.177	.244	.336
38	-.420	-.345	-.279	-.171	-.098	-.054	-.011	.036	.063	.093	.127	.170	.236	.325
40	-.409	-.337	-.272	-.167	-.097	-.054	-.011	.034	.060	.089	.122	.165	.228	.315
42	-.400	-.329	-.266	-.164	-.095	-.053	-.012	.032	.057	.086	.118	.159	.221	.306
44	-.390	-.322	-.261	-.161	-.094	-.053	-.013	.030	.055	.083	.114	.155	.214	.297
46	-.382	-.315	-.256	-.159	-.093	-.053	-.013	.029	.053	.080	.111	.150	.208	.289
48	-.374	-.309	-.251	-.156	-.092	-.052	-.014	.027	.051	.077	.107	.146	.202	.282
50	-.366	-.304	-.247	-.154	-.091	-.052	-.015	.026	.049	.075	.104	.142	.197	.275
52	-.359	-.298	-.243	-.152	-.090	-.052	-.015	.024	.047	.073	.101	.138	.191	.268
54	-.353	-.293	-.239	-.150	-.089	-.052	-.016	.023	.045	.071	.098	.135	.187	.262
56	-.347	-.289	-.235	-.148	-.088	-.051	-.016	.022	.044	.069	.096	.131	.182	.256
58	-.341	-.284	-.232	-.146	-.087	-.051	-.016	.021	.042	.067	.093	.128	.178	.251
60	-.335	-.280	-.228	-.144	-.086	-.051	-.017	.020	.041	.065	.091	.125	.174	.245
64	-.325	-.273	-.222	-.141	-.085	-.051	-.018	.018	.038	.061	.087	.120	.166	.236
68	-.316	-.265	-.217	-.138	-.084	-.050	-.018	.016	.036	.058	.083	.115	.160	.227
72	-.307	-.259	-.212	-.136	-.082	-.050	-.019	.014	.034	.055	.079	.110	.153	.219
76	-.300	-.253	-.207	-.133	-.081	-.050	-.020	.012	.031	.053	.076	.106	.148	.212
80	-.292	-.248	-.203	-.131	-.080	-.050	-.020	.011	.020	.050	.073	.102	.143	.205

Table A2g: Percentage Points for $\hat{c}[\ln(\hat{b}/b) - \ln(K)]$ with K=1.10, $\beta=e^{-K}=.333$

N γ	.02	.05	.10	.25	.40	.50	.60	.70	.75	.80	.85	.90	.95	.98
7	-1.362	-1.013	-.771	-.454	-.258	-.192	-.034	.086	.152	.232	.327	.451	.645	.936
8	-1.195	-.916	-.708	-.421	-.243	-.181	-.036	.076	.138	.212	.303	.415	.598	.838
10	-.994	-.785	-.619	-.374	-.220	-.164	-.039	.069	.115	.179	.259	.355	.514	.704
12	-.873	-.700	-.559	-.342	-.205	-.153	-.042	.046	.097	.153	.225	.310	.450	.614
14	-.790	-.640	-.515	-.318	-.194	-.145	-.044	.036	.083	.134	.199	.276	.402	.549
16	-.729	-.594	-.481	-.300	-.185	-.139	-.046	.028	.072	.119	.178	.249	.364	.499
18	-.681	-.557	-.454	-.285	-.178	-.134	-.048	.022	.062	.106	.161	.227	.333	.459
20	-.643	-.528	-.431	-.274	-.173	-.130	-.050	.016	.054	.096	.147	.209	.307	.426
22	-.611	-.503	-.412	-.264	-.168	-.127	-.051	.011	.047	.086	.135	.193	.285	.398
24	-.584	-.482	-.396	-.255	-.164	-.124	-.052	.007	.041	.079	.124	.180	.267	.375
26	-.561	-.464	-.382	-.248	-.160	-.121	-.054	.003	.036	.072	.115	.168	.250	.355
28	-.541	-.448	-.370	-.241	-.157	-.119	-.055	.000	.031	.066	.107	.157	.236	.337
30	-.523	-.435	-.359	-.235	-.154	-.117	-.056	-.003	.027	.060	.100	.148	.223	.321
32	-.507	-.422	-.350	-.230	-.152	-.116	-.057	-.006	.023	.055	.093	.140	.212	.307
34	-.493	-.411	-.341	-.226	-.150	-.114	-.058	-.008	.020	.051	.087	.132	.201	.294
36	-.480	-.401	-.333	-.221	-.148	-.113	-.059	-.011	.017	.047	.082	.125	.192	.282
38	-.469	-.392	-.326	-.218	-.146	-.112	-.059	-.013	.014	.043	.077	.119	.183	.272
40	-.458	-.383	-.319	-.214	-.144	-.111	-.060	-.015	.011	.039	.072	.113	.175	.262
42	-.448	-.376	-.313	-.211	-.142	-.110	-.060	-.016	.008	.036	.068	.108	.168	.253
44	-.439	-.369	-.308	-.208	-.141	-.109	-.061	-.018	.006	.033	.064	.103	.161	.245
46	-.431	-.362	-.303	-.205	-.140	-.108	-.061	-.020	.004	.030	.060	.099	.155	.237
48	-.423	-.356	-.298	-.203	-.139	-.107	-.062	-.021	.002	.028	.057	.093	.149	.230
50	-.415	-.350	-.293	-.200	-.137	-.106	-.062	-.022	-.000	.025	.053	.089	.143	.223
52	-.409	-.345	-.289	-.198	-.136	-.106	-.063	-.024	-.002	.023	.050	.085	.138	.217
54	-.402	-.340	-.285	-.196	-.135	-.105	-.063	-.025	-.004	.021	.048	.082	.133	.211
56	-.396	-.335	-.281	-.194	-.134	-.105	-.063	-.026	-.005	.019	.045	.078	.128	.205
58	-.391	-.331	-.278	-.192	-.133	-.104	-.063	-.027	-.007	.017	.042	.075	.124	.200
60	-.385	-.326	-.274	-.190	-.133	-.103	-.064	-.028	-.008	.015	.040	.072	.120	.195
64	-.375	-.319	-.268	-.187	-.131	-.103	-.065	-.030	-.011	.011	.035	.066	.112	.186
68	-.366	-.312	-.263	-.184	-.130	-.102	-.065	-.032	-.014	.008	.031	.061	.105	.177
72	-.358	-.305	-.258	-.181	-.128	-.101	-.066	-.034	-.016	.005	.028	.056	.099	.170
76	-.350	-.299	-.253	-.179	-.127	-.100	-.066	-.035	-.018	.003	.024	.052	.093	.163
80	-.344	-.294	-.249	-.177	-.126	-.100	-.067	-.037	-.020	.000	.021	.048	.088	.156

Table A2h: Percentage Points for $\hat{c}[\ln(\hat{b}/b) - \ln(K)]$ with K=1.15, $\beta=e^{-K}=.317$

N \ γ	.02	.05	.10	.25	.40	.50	.60	.70	.75	.80	.85	.90	.95	.98
7	-1.435	-1.081	-.828	-.504	-.305	-.193	-.083	.034	.100	.176	.268	.387	.586	.857
8	-1.264	-.979	-.767	-.472	-.293	-.186	-.084	.025	.086	.160	.246	.358	.540	.766
10	-1.053	-.840	-.673	-.423	-.269	-.177	-.087	.009	.064	.128	.203	.300	.454	.637
12	-.927	-.751	-.608	-.389	-.252	-.171	-.090	.003	.047	.104	.170	.256	.391	.550
14	-.841	-.688	-.562	-.364	-.240	-.166	-.092	.012	.033	.084	.144	.222	.343	.487
16	-.779	-.641	-.527	-.345	-.231	-.163	-.094	.020	.022	.069	.124	.196	.307	.439
18	-.730	-.604	-.499	-.330	-.223	-.160	-.095	.026	.013	.057	.108	.175	.277	.401
20	-.691	-.574	-.476	-.318	-.217	-.158	-.097	.031	.006	.047	.095	.157	.252	.369
22	-.659	-.549	-.457	-.308	-.212	-.156	-.098	.036	.001	.038	.083	.142	.232	.343
24	-.632	-.528	-.441	-.300	-.208	-.154	-.099	.040	.006	.030	.073	.129	.214	.320
26	-.608	-.509	-.427	-.292	-.205	-.153	-.100	.044	.011	.024	.065	.118	.199	.300
28	-.588	-.493	-.415	-.286	-.201	-.152	-.101	.047	.016	.018	.057	.108	.185	.283
30	-.570	-.479	-.404	-.280	-.199	-.151	-.102	.050	.020	.013	.050	.099	.173	.268
32	-.554	-.467	-.394	-.275	-.196	-.150	-.103	.052	.024	.008	.044	.091	.163	.254
34	-.540	-.456	-.386	-.270	-.194	-.149	-.103	.055	.027	.004	.039	.084	.153	.241
36	-.527	-.445	-.378	-.266	-.192	-.148	-.104	.057	.030	.000	.034	.078	.144	.230
38	-.515	-.436	-.371	-.262	-.190	-.147	-.105	.059	.033	.004	.029	.072	.136	.220
40	-.504	-.428	-.364	-.259	-.189	-.147	-.105	.061	.035	.007	.025	.066	.129	.210
42	-.494	-.420	-.358	-.256	-.187	-.146	-.106	.063	.038	.010	.021	.061	.122	.202
44	-.485	-.413	-.353	-.253	-.186	-.146	-.106	.064	.040	.013	.017	.057	.116	.193
46	-.476	-.406	-.347	-.250	-.184	-.145	-.107	.066	.042	.016	.014	.052	.110	.186
48	-.468	-.400	-.343	-.247	-.183	-.145	-.107	.067	.044	.018	.011	.048	.104	.179
50	-.461	-.394	-.338	-.245	-.182	-.144	-.108	.068	.046	.020	.008	.044	.099	.172
52	-.453	-.389	-.334	-.243	-.181	-.144	-.108	.070	.048	.023	.005	.041	.094	.166
54	-.447	-.384	-.330	-.241	-.180	-.144	-.108	.071	.049	.025	.003	.038	.090	.160
56	-.441	-.379	-.326	-.239	-.179	-.143	-.109	.072	.051	.027	.000	.034	.086	.155
58	-.435	-.374	-.323	-.237	-.178	-.143	-.109	.073	.052	.028	.002	.031	.082	.150
60	-.429	-.370	-.320	-.235	-.178	-.143	-.109	.074	.054	.030	.004	.029	.078	.145
64	-.419	-.362	-.313	-.232	-.176	-.142	-.110	.076	.056	.033	.009	.023	.071	.136
68	-.410	-.355	-.308	-.229	-.175	-.142	-.111	.078	.059	.036	.012	.018	.064	.127
72	-.401	-.348	-.303	-.226	-.174	-.141	-.111	.079	.061	.039	.016	.014	.059	.120
76	-.393	-.342	-.298	-.224	-.173	-.141	-.112	.081	.063	.041	.019	.010	.053	.113
80	-.386	-.337	-.294	-.221	-.172	-.140	-.112	.082	.065	.044	.022	.006	.048	.106

Table A2i: Percentage Points for $\hat{c}[\ln(\hat{b}/b) - \ln(K)]$ with K=2.0, $\beta=e^{-K}=.135$

N	γ .02	.05	.10	.25	.40	.50	.60	.70	.75	.80	.85	.90	.95	.98
7	-2.509	-1.979	-1.626	-1.177	-.936	-.813	-.704	-.588	-.531	-.470	-.400	-.309	-.163	-.000
8	-2.235	-1.806	-1.510	-1.129	-.910	-.796	-.697	-.589	-.535	-.480	-.412	-.324	-.192	-.037
10	-1.903	-1.590	-1.361	-1.055	-.873	-.774	-.687	-.593	-.544	-.495	-.434	-.356	-.242	-.106
12	-1.709	-1.459	-1.268	-1.005	-.848	-.760	-.681	-.596	-.552	-.507	-.452	-.381	-.279	-.158
14	-1.580	-1.370	-1.204	-0.970	-.229	-.750	-.678	-.600	-.559	-.517	-.466	-.402	-.308	-.198
16	-1.487	-1.304	-1.156	-.943	-.815	-.743	-.675	-.603	-.565	-.525	-.478	-.418	-.331	-.230
18	-1.417	-1.253	-1.119	-.923	-.804	-.737	-.673	-.606	-.571	-.532	-.488	-.432	-.350	-.256
20	-1.361	-1.212	-1.089	-.906	-.795	-.733	-.672	-.609	-.575	-.538	-.496	-.444	-.366	-.277
22	-1.316	-1.179	-1.064	-.893	-.788	-.729	-.671	-.611	-.579	-.544	-.504	-.454	-.380	-.296
24	-1.278	-1.151	-1.043	-.881	-.782	-.726	-.671	-.614	-.583	-.549	-.510	-.463	-.392	-.312
26	-1.246	-1.126	-1.025	-.871	-.776	-.724	-.670	-.616	-.586	-.553	-.516	-.470	-.402	-.326
28	-1.219	-1.105	-1.009	-.863	-.772	-.721	-.670	-.617	-.589	-.557	-.521	-.477	-.412	-.338
30	-1.195	-1.087	-.996	-.855	-.768	-.719	-.670	-.619	-.592	-.561	-.526	-.484	-.420	-.349
32	-1.174	-1.071	-.983	-.848	-.764	-.717	-.670	-.621	-.594	-.564	-.530	-.489	-.428	-.359
24	-1.155	-1.056	-.973	-.842	-.761	-.716	-.670	-.622	-.597	-.567	-.534	-.495	-.435	-.368
26	-1.138	-1.043	-.963	-.837	-.758	-.714	-.670	-.623	-.599	-.570	-.538	-.499	-.441	-.377
28	-1.122	-1.031	-.954	-.832	-.756	-.713	-.670	-.625	-.601	-.573	-.541	-.504	-.447	-.384
40	-1.108	-1.020	-.946	-.828	-.753	-.712	-.670	-.626	-.603	-.575	-.544	-.508	-.453	-.392
42	-1.096	-1.010	-.938	-.824	-.751	-.711	-.670	-.627	-.604	-.577	-.547	-.512	-.458	-.398
44	-1.084	-1.001	-.931	-.820	-.749	-.710	-.670	-.628	-.606	-.580	-.550	-.515	-.462	-.404
46	-1.073	- .993	-.925	-.816	-.747	-.709	-.670	-.629	-.607	-.582	-.552	-.518	-.467	-.410
48	-1.063	- .985	-.919	-.813	-.746	-.708	-.671	-.630	-.609	-.584	-.555	-.521	-.471	-.415
50	-1.054	- .977	-.913	-.810	-.744	-.707	-.671	-.631	-.610	-.586	-.557	-.524	-.475	-.420
52	-1.045	- .971	-.908	-.807	-.743	-.707	-.671	-.632	-.612	-.587	-.559	-.527	-.479	-.425
54	-1.037	- .964	-.903	-.805	-.741	-.706	-.671	-.633	-.613	-.589	-.561	-.530	-.482	-.430
56	-1.029	- .958	-.899	-.802	-.740	-.705	-.671	-.633	-.614	-.591	-.563	-.532	-.485	-.434
58	-1.022	- .952	-.894	-.800	-.739	-.705	-.671	-.634	-.615	-.592	-.565	-.535	-.488	-.438
60	-1.015	- .947	-.890	-.798	-.738	-.704	-.672	-.635	-.616	-.594	-.567	-.537	-.491	-.442
64	-1.002	- .937	-.883	-.794	-.736	-.703	-.672	-.636	-.618	-.596	-.570	-.541	-.497	-.449
68	-0.991	- .928	-.876	-.790	-.734	-.702	-.672	-.637	-.620	-.599	-.573	-.545	-.502	-.455
72	-0.981	- .920	-.870	-.787	-.732	-.702	-.672	-.639	-.622	-.601	-.576	-.548	-.507	-.461
76	-0.972	- .913	-.864	-.784	-.731	-.701	-.673	-.640	-.623	-.603	-.578	-.552	-.511	-.466
80	-0.963	- .906	-.859	-.781	-.729	-.700	-.673	-.641	-.625	-.605	-.581	-.555	-.515	-.472

Table A3a

Percentage Points, ℓ, such that $G_K(\ell)=.80$ as a function of β where $K = -\ln(\beta)$

N	β 0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	0.98
10	0.744	0.931	1.133	1.356	1.606	1.897	2.245	2.685	3.289	4.263	5.348
11	0.721	0.905	1.103	1.322	1.568	1.852	2.193	2.623	3.214	4.164	5.222
12	0.702	0.883	1.078	1.293	1.535	1.814	2.149	2.571	3.150	4.082	5.123
13	0.685	0.864	1.057	1.269	1.507	1.782	2.112	2.527	3.097	4.014	5.040
14	0.671	0.848	1.039	1.248	1.483	1.755	2.080	2.489	3.051	3.955	4.972
15	0.658	0.834	1.022	1.230	1.463	1.731	2.052	2.457	3.011	3.904	4.910
16	0.647	0.821	1.008	1.214	1.444	1.710	2.027	2.428	2.976	3.860	4.861
18	0.628	0.800	0.984	1.187	1.413	1.675	1.987	2.380	2.918	3.786	4.767
20	0.613	0.782	0.965	1.164	1.388	1.646	1.954	2.341	2.871	3.726	4.698
22	0.600	0.768	0.948	1.146	1.367	1.622	1.926	2.308	2.831	3.677	4.645
24	0.589	0.755	0.934	1.130	1.349	1.601	1.902	2.281	2.798	3.636	4.598
26	0.579	0.744	0.922	1.116	1.334	1.584	1.882	2.257	2.770	3.601	4.558
28	0.570	0.735	0.911	1.104	1.320	1.569	1.865	2.237	2.745	3.570	4.522
30	0.563	0.726	0.902	1.094	1.309	1.555	1.849	2.219	2.724	3.543	4.490
32	0.556	0.719	0.894	1.085	1.298	1.543	1.835	2.202	2.704	3.519	4.465
34	0.550	0.712	0.886	1.076	1.289	1.533	1.824	2.189	2.688	3.498	4.439
36	0.544	0.706	0.879	1.069	1.280	1.523	1.812	2.175	2.672	3.479	4.421
38	0.539	0.700	0.873	1.061	1.272	1.514	1.802	2.163	2.657	3.462	4.404
40	0.535	0.695	0.867	1.055	1.265	1.506	1.793	2.153	2.645	3.446	4.385
42	0.530	0.690	0.862	1.049	1.259	1.499	1.784	2.143	2.633	3.431	4.368
44	0.526	0.686	0.857	1.044	1.253	1.492	1.776	2.134	2.622	3.418	4.355
46	0.523	0.682	0.853	1.039	1.247	1.485	1.769	2.125	2.612	3.406	4.343
48	0.519	0.678	0.849	1.034	1.242	1.480	1.762	2.117	2.603	3.395	4.330
50	0.516	0.675	0.845	1.030	1.237	1.474	1.756	2.110	2.594	3.384	4.322
52	0.513	0.671	0.841	1.026	1.232	1.469	1.750	2.103	2.586	3.374	4.309
54	0.510	0.668	0.838	1.022	1.228	1.464	1.745	2.097	2.579	3.365	4.297
56	0.508	0.665	0.834	1.019	1.224	1.460	1.740	2.091	2.572	3.357	4.291
58	0.505	0.663	0.831	1.015	1.220	1.456	1.735	2.086	2.565	3.349	4.281
60	0.503	0.660	0.828	1.012	1.217	1.451	1.730	2.080	2.559	3.341	4.275
62	0.500	0.658	0.826	1.009	1.213	1.448	1.726	2.075	2.553	3.334	4.266
64	0.498	0.655	0.823	1.006	1.210	1.444	1.722	2.070	2.547	3.327	4.261
66	0.496	0.653	0.821	1.003	1.207	1.441	1.718	2.066	2.542	3.321	4.253
68	0.494	0.651	0.818	1.001	1.204	1.438	1.715	2.062	2.537	3.314	4.244
70	0.492	0.649	0.816	0.998	1.201	1.434	1.711	2.058	2.532	3.309	4.241
72	0.491	0.647	0.814	0.996	1.199	1.431	1.708	2.054	2.528	3.303	4.234
74	0.489	0.645	0.812	0.994	1.196	1.429	1.704	2.050	2.523	3.298	4.231
76	0.487	0.643	0.810	0.992	1.194	1.426	1.701	2.047	2.519	3.293	4.224
78	0.486	0.642	0.808	0.990	1.192	1.423	1.698	2.043	2.515	3.288	4.221
80	0.484	0.640	0.806	0.988	1.190	1.421	1.696	2.040	2.511	3.283	4.215

Table A3b

Percentage Points, ℓ, such that $G_K(\ell)=.90$ as a function of β where $K = -\ln(\beta)$

	β										
N	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	0.98
10	0.969	1.174	1.396	1.643	1.922	2.248	2.640	3.140	3.830	4.949	6.198
11	0.931	1.131	1.348	1.588	1.859	2.175	2.556	3.040	3.709	4.794	6.009
12	0.899	1.095	1.307	1.542	1.807	2.115	2.487	2.959	3.610	4.666	5.850
13	0.872	1.064	1.272	1.503	1.763	2.065	2.429	2.891	3.527	4.559	5.718
14	0.848	1.037	1.243	1.469	1.725	2.022	2.379	2.832	3.456	4.469	5.609
15	0.827	1.014	1.217	1.441	1.693	1.985	2.337	2.782	3.396	4.390	5.508
16	0.809	0.994	1.194	1.415	1.664	1.952	2.299	2.738	3.343	4.322	5.425
18	0.778	0.959	1.156	1.372	1.616	1.898	2.236	2.665	3.254	4.208	5.285
20	0.752	0.931	1.124	1.337	1.577	1.853	2.186	2.606	3.183	4.116	5.171
22	0.730	0.907	1.098	1.308	1.544	1.817	2.144	2.557	3.124	4.041	5.079
24	0.712	0.887	1.076	1.283	1.516	1.786	2.108	2.515	3.074	3.978	5.002
26	0.696	0.870	1.057	1.262	1.493	1.759	2.078	2.480	3.032	3.924	4.935
28	0.682	0.854	1.040	1.243	1.472	1.735	2.051	2.449	2.995	3.878	4.883
30	0.670	0.841	1.025	1.227	1.453	1.715	2.028	2.422	2.963	3.837	4.832
32	0.659	0.829	1.011	1.212	1.437	1.696	2.006	2.397	2.933	3.802	4.795
34	0.649	0.818	1.000	1.199	1.422	1.680	1.988	2.376	2.908	3.770	4.754
36	0.640	0.808	0.989	1.187	1.409	1.665	1.971	2.357	2.885	3.741	4.719
38	0.632	0.799	0.979	1.176	1.397	1.651	1.956	2.339	2.864	3.715	4.693
40	0.674	0.791	0.970	1.166	1.386	1.639	1.942	2.323	2.845	3.692	4.665
42	0.617	0.783	0.962	1.157	1.376	1.628	1.929	2.308	2.827	3.670	4.640
44	0.611	0.776	0.954	1.148	1.366	1.617	1.917	2.294	2.811	3.650	4.617
46	0.605	0.770	0.947	1.141	1.358	1.608	1.906	2.281	2.796	3.632	4.597
48	0.600	0.764	0.940	1.133	1.350	1.599	1.896	2.270	2.782	3.615	4.580
50	0.594	0.758	0.934	1.127	1.342	1.590	1.886	2.259	2.770	3.599	4.560
52	0.589	0.753	0.928	1.120	1.335	1.582	1.877	2.249	2.757	3.585	4.545
54	0.585	0.748	0.923	1.114	1.328	1.575	1.869	2.239	2.746	3.571	4.533
56	0.581	0.743	0.918	1.109	1.322	1.568	1.861	2.230	2.735	3.558	4.516
58	0.576	0.739	0.913	1.103	1.316	1.562	1.854	2.222	2.726	3.546	4.503
60	0.573	0.734	0.908	1.098	1.311	1.556	1.847	2.214	2.717	3.535	4.488
62	0.569	0.730	0.904	1.093	1.306	1.549	1.840	2.206	2.707	3.524	4.481
64	0.566	0.727	0.900	1.089	1.301	1.544	1.834	2.199	2.699	3.514	4.468
66	0.562	0.723	0.896	1.085	1.296	1.539	1.828	2.192	2.691	3.504	4.455
68	0.559	0.720	0.892	1.081	1.291	1.533	1.822	2.185	2.683	3.496	4.452
70	0.556	0.717	0.889	1.077	1.287	1.529	1.817	2.179	2.676	3.487	4.441
72	0.553	0.713	0.885	1.073	1.283	1.525	1.812	2.174	2.670	3.478	4.430
74	0.551	0.710	0.882	1.069	1.279	1.520	1.807	2.168	2.663	3.471	4.424
76	0.548	0.708	0.879	1.066	1.275	1.516	1.802	2.162	2.656	3.463	4.417
78	0.545	0.705	0.876	1.063	1.272	1.512	1.798	2.158	2.651	3.456	4.408
80	0.543	0.702	0.873	1.060	1.268	1.508	1.793	2.152	2.645	3.449	4.402

Table A3c

Percentage Points, ℓ, such that $G_K(\ell)=.\underline{95}$ as a function of β where $K = -\ln(\beta)$

	β										
N	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	0.98
10	1.441	1.688	1.959	2.262	2.608	3.016	3.515	4.160	5.073	6.618	8.483
11	1.366	1.605	1.867	2.159	2.493	2.885	3.363	3.978	4.838	6.257	7.879
12	1.303	1.536	1.790	2.074	2.398	2.777	3.237	3.827	4.647	5.979	7.454
13	1.251	1.478	1.726	2.002	2.316	2.683	3.129	3.698	4.486	5.758	7.147
14	1.207	1.428	1.670	1.940	2.246	2.604	3.037	3.589	4.352	5.575	6.901
15	1.168	1.385	1.622	1.886	2.185	2.535	2.957	3.495	4.236	5.423	6.703
16	1.134	1.347	1.580	1.838	2.132	2.474	2.887	3.413	4.136	5.292	6.538
18	1.077	1.283	1.509	1.758	2.041	2.371	2.768	3.273	3.968	5.081	6.288
20	1.030	1.232	1.451	1.694	1.969	2.290	2.675	3.165	3.837	4.914	6.081
22	0.992	1.189	1.403	1.640	1.908	2.220	2.595	3.071	3.727	4.780	5.939
24	0.959	1.153	1.363	1.595	1.858	2.163	2.530	2.996	3.636	4.669	5.811
26	0.931	1.121	1.328	1.556	1.814	2.114	2.474	2.930	3.559	4.574	5.705
28	0.906	1.094	1.298	1.523	1.777	2.071	2.426	2.874	3.492	4.492	5.612
30	0.884	1.070	1.271	1.493	1.744	2.034	2.383	2.826	3.435	4.421	5.533
32	0.865	1.049	1.248	1.467	1.715	2.001	2.346	2.782	3.383	4.359	5.466
34	0.847	1.030	1.227	1.444	1.688	1.972	2.312	2.743	3.337	4.303	5.405
36	0.832	1.012	1.208	1.423	1.665	1.945	2.282	2.708	3.296	4.254	5.352
38	0.817	0.997	1.190	1.404	1.644	1.921	2.255	2.677	3.259	4.209	5.303
40	0.804	0.982	1.175	1.386	1.625	1.900	2.231	2.649	3.226	4.167	5.250
42	0.792	0.969	1.160	1.370	1.607	1.880	2.208	2.623	3.195	4.129	5.210
44	0.781	0.957	1.147	1.356	1.591	1.862	2.188	2.600	3.167	4.095	5.171
46	0.770	0.946	1.134	1.342	1.575	1.845	2.169	2.577	3.141	4.063	5.136
48	0.761	0.935	1.123	1.329	1.562	1.830	2.151	2.557	3.117	4.034	5.102
50	0.752	0.925	1.112	1.318	1.549	1.816	2.135	2.539	3.096	4.006	5.066
52	0.743	0.916	1.102	1.307	1.537	1.802	2.120	2.521	3.075	3.981	5.039
54	0.736	0.908	1.093	1.297	1.525	1.789	2.105	2.505	3.055	3.957	5.012
56	0.728	0.900	1.084	1.287	1.515	1.778	2.092	2.490	3.038	3.934	4.983
58	0.721	0.892	1.076	1.278	1.505	1.767	2.080	2.476	3.021	3.913	4.958
60	0.714	0.885	1.068	1.270	1.496	1.756	2.068	2.462	3.005	3.893	4.933
62	0.708	0.878	1.061	1.262	1.487	1.747	2.058	2.450	2.990	3.874	4.908
64	0.702	0.872	1.054	1.254	1.478	1.737	2.047	2.438	2.976	3.857	4.892
66	0.697	0.866	1.047	1.247	1.471	1.728	2.037	2.426	2.962	3.840	4.870
68	0.691	0.860	1.041	1.240	1.463	1.720	2.028	2.416	2.950	3.824	4.849
70	0.686	0.854	1.035	1.234	1.456	1.712	2.019	2.406	2.938	3.808	4.829
72	0.681	0.849	1.029	1.227	1.449	1.705	2.011	2.396	2.927	3.794	4.810
74	0.676	0.844	1.024	1.221	1.443	1.698	2.003	2.387	2.916	3.780	4.793
76	0.672	0.839	1.019	1.216	1.437	1.691	1.995	2.378	2.905	3.767	4.775
78	0.668	0.834	1.014	1.210	1.431	1.684	1.988	2.370	2.995	3.754	4.759
80	0.664	0.830	1.009	1.205	1.425	1.678	1.981	2.362	2.886	3.742	4.744

Table A3d

Percentage Points, ℓ, such that $G_K(\ell)=\underline{.98}$ as a function of β where $K = -\ln(\beta)$

N	β 0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	0.98
10	1.182	1.405	1.649	1.921	2.230	2.592	3.031	3.592	4.373	5.648	7.083
11	1.128	1.345	1.581	1.844	2.142	2.491	2.914	3.455	4.207	5.440	6.839
12	1.083	1.294	1.525	1.780	2.071	2.410	2.820	3.345	4.073	5.265	6.619
13	1.045	1.252	1.477	1.727	2.011	2.342	2.742	3.252	3.961	5.118	6.429
14	1.012	1.215	1.436	1.681	1.959	2.283	2.674	3.173	3.864	4.992	6.268
15	0.983	1.183	1.401	1.642	1.915	2.233	2.617	3.106	3.782	4.882	6.121
16	0.957	1.155	1.369	1.607	1.876	2.189	2.566	3.046	3.709	4.787	5.999
18	0.914	1.107	1.316	1.548	1.810	2.114	2.481	2.947	3.589	4.629	5.791
20	0.879	1.068	1.273	1.500	1.757	2.054	2.412	2.866	3.491	4.501	5.629
22	0.849	1.035	1.238	1.461	1.712	2.004	2.355	2.799	3.411	4.396	5.493
24	0.824	1.008	1.207	1.427	1.674	1.962	2.306	2.743	3.343	4.308	5.381
26	0.802	0.984	1.181	1.397	1.642	1.925	2.265	2.694	3.284	4.234	5.290
28	0.783	0.963	1.157	1.372	1.613	1.893	2.228	2.652	3.234	4.169	5.208
30	0.766	0.944	1.137	1.349	1.588	1.865	2.196	2.615	3.189	4.112	5.140
32	0.751	0.928	1.119	1.329	1.566	1.840	2.168	2.582	3.150	4.062	5.076
34	0.737	0.913	1.102	1.311	1.546	1.817	2.142	2.552	3.114	4.018	5.026
36	0.725	0.899	1.088	1.295	1.528	1.797	2.119	2.525	3.082	3.977	4.975
38	0.713	0.887	1.074	1.280	1.511	1.778	2.097	2.501	3.053	3.941	4.934
40	0.703	0.876	1.062	1.266	1.496	1.761	2.079	2.479	3.027	3.908	4.892
42	0.694	0.865	1.050	1.254	1.482	1.745	2.061	2.458	3.002	3.878	4.858
44	0.685	0.856	1.040	1.242	1.469	1.731	2.045	2.440	2.980	3.850	4.825
46	0.677	0.847	1.030	1.231	1.457	1.718	2.029	2.422	2.959	3.825	4.796
48	0.669	0.838	1.021	1.221	1.446	1.705	2.015	2.405	2.940	3.801	4.771
50	0.662	0.831	1.012	1.212	1.436	1.693	2.002	2.390	2.922	3.779	4.746
52	0.655	0.824	1.005	1.203	1.426	1.683	1.990	2.376	2.905	3.759	4.723
54	0.649	0.817	0.997	1.195	1.417	1.673	1.978	2.363	2.890	3.739	4.699
56	0.643	0.810	0.990	1.187	1.408	1.663	1.967	2.350	2.875	3.722	4.682
58	0.638	0.804	0.983	1.180	1.400	1.654	1.957	2.339	2.861	3.705	4.661
60	0.632	0.798	0.977	1.173	1.393	1.646	1.948	2.328	2.949	3.688	4.641
62	0.628	0.793	0.971	1.166	1.385	1.637	1.938	2.317	2.836	3.674	4.630
64	0.623	0.788	0.966	1.160	1.378	1.630	1.930	2.307	2.824	3.560	4.613
66	0.618	0.783	0.960	1.154	1.372	1.623	1.922	2.298	2.813	3.646	4.596
68	0.614	0.778	0.955	1.149	1.366	1.615	1.914	2.289	2.802	3.634	4.586
70	0.610	0.774	0.950	1.143	1.360	1.609	1.906	2.280	2.793	3.622	4.572
72	0.606	0.770	0.946	1.138	1.354	1.603	1.899	2.272	2.783	3.610	4.559
74	0.602	0.766	0.941	1.134	1.349	1.597	1.892	2.264	2.774	3.599	4.548
76	0.599	0.762	0.937	1.129	1.344	1.591	1.886	2.257	2.765	3.589	4.537
78	0.595	0.758	0.933	1.124	1.339	1.585	1.880	2.250	2.757	3.579	4.528
80	0.592	0.755	0.929	1.120	1.334	1.580	1.874	2.243	2.749	3.569	4.515

Table A4a

Percentage Points, ℓ, such that $G_K(\ell) = \underline{.02}$ as a function of β where $K = -\ln(\beta)$

$\bar{N}$	β 0.02	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
10	-2.949	-2.470	-2.024	-1.712	-1.458	-1.236	-1.034	-0.845	-0.664	-0.488	-0.314
11	-2.802	-2.346	-1.921	-1.622	-1.379	-1.167	-0.973	-0.791	-0.616	-0.446	-0.278
12	-2.683	-2.246	-1.837	-1.550	-1.315	-1.110	-0.922	-0.746	-0.577	-0.412	-0.248
13	-2.585	-2.163	-1.768	-1.490	-1.262	-1.063	-0.880	-0.709	-0.544	-0.382	-0.222
14	-2.502	-2.093	-1.709	-1.439	-1.217	-1.023	-0.845	-0.677	-0.515	-0.357	-0.199
15	-2.432	-2.034	-1.659	-1.395	-1.179	-0.988	-0.813	-0.649	-0.490	-0.334	-0.179
16	-2.368	-1.980	-1.614	-1.356	-1.144	-0.958	-0.786	-0.625	-0.469	-0.315	-0.163
18	-2.264	-1.891	-1.540	-1.291	-1.087	-0.906	-0.741	-0.584	-0.432	-0.283	-0.134
20	-2.185	-1.823	-1.482	-1.240	-1.041	-0.865	-0.703	-0.550	-0.402	-0.256	-0.109
22	-2.123	-1.769	-1.435	-1.199	-1.003	-0.831	-0.672	-0.522	-0.376	-0.232	-0.088
24	-2.072	-1.725	-1.397	-1.164	-0.972	-0.802	-0.646	-0.497	-0.353	-0.211	-0.068
26	-2.029	-1.687	-1.364	-1.134	-0.945	-0.777	-0.623	-0.476	-0.334	-0.193	-0.052
28	-1.995	-1.657	-1.337	-1.110	-0.922	-0.756	-0.603	-0.457	-0.316	-0.176	-0.036
30	-1.966	-1.631	-1.314	-1.088	-0.902	-0.737	-0.585	-0.441	-0.300	-0.161	-0.022
32	-1.940	-1.608	-1.293	-1.069	-0.885	-0.721	-0.569	-0.426	-0.286	-0.148	-0.009
34	-1.917	-1.588	-1.275	-1.053	-0.869	-0.706	-0.555	-0.412	-0.273	-0.136	0.003
36	-1.897	-1.569	-1.259	-1.037	-0.854	-0.692	-0.543	-0.400	-0.262	-0.125	0.013
38	-1.879	-1.553	-1.244	-1.024	-0.842	-0.680	-0.531	-0.389	-0.251	-0.114	0.023
40	-1.862	-1.538	-1.230	-1.011	-0.830	-0.669	-0.520	-0.379	-0.241	-0.105	0.032
42	-1.847	-1.525	-1.218	-1.000	-0.819	-0.659	-0.511	-0.369	-0.232	-0.096	0.041
44	-1.834	-1.512	-1.207	-0.989	-0.809	-0.649	-0.501	-0.361	-0.224	-0.088	0.049
46	-1.821	-1.501	-1.196	-0.980	-0.800	-0.641	-0.493	-0.353	-0.216	-0.081	0.056
48	-1.809	-1.490	-1.187	-0.971	-0.791	-0.632	-0.485	-0.345	-0.209	-0.074	0.063
50	-1.799	-1.481	-1.178	-0.962	-0.783	-0.625	-0.478	-0.338	-0.202	-0.067	0.069
52	-1.789	-1.472	-1.170	-0.954	-0.776	-0.618	-0.471	-0.332	-0.196	-0.061	0.075
54	-1.780	-1.463	-1.162	-0.947	-0.769	-0.611	-0.465	-0.325	-0.190	-0.055	0.081
56	-1.771	-1.455	-1.155	-0.940	-0.763	-0.605	-0.459	-0.319	-0.184	-0.049	0.086
58	-1.763	-1.448	-1.148	-0.934	-0.756	-0.599	-0.453	-0.314	-0.178	-0.044	0.092
60	-1.756	-1.441	-1.142	-0.928	-0.751	-0.593	-0.448	-0.309	-0.173	-0.039	0.097
62	-1.748	-1.435	-1.136	-0.922	-0.745	-0.588	-0.443	-0.304	-0.169	-0.034	0.101
64	-1.742	-1.428	-1.130	-0.917	-0.740	-0.583	-0.438	-0.299	-0.164	-0.030	0.106
66	-1.736	-1.423	-1.125	-0.912	-0.735	-0.578	-0.433	-0.295	-0.160	-0.025	0.110
68	-1.730	-1.417	-1.119	-0.907	-0.730	-0.574	-0.429	-0.290	-0.155	-0.021	0.114
70	-1.724	-1.412	-1.115	-0.902	-0.726	-0.570	-0.424	-0.286	-0.151	-0.017	0.118
72	-1.719	-1.407	-1.110	-0.898	-0.722	-0.565	-0.420	-0.282	-0.147	-0.014	0.122
74	-1.713	-1.402	-1.106	-0.894	-0.718	-0.561	-0.417	-0.279	-0.144	-0.010	0.125
76	-1.709	-1.398	-1.101	-0.890	-0.714	-0.558	-0.413	-0.275	-0.140	-0.007	0.129
78	-1.699	-1.391	-1.096	-0.885	-0.710	-0.555	-0.410	-0.272	-0.139	-0.004	0.131
80	-1.695	-1.386	-1.092	-0.881	-0.707	-0.551	-0.407	-0.269	-0.135	-0.001	0.134

Table A4b

Percentage Points, ℓ, such that $G_K(\ell)=.\underline{05}$ as a function of β where $K = -\ln(\beta)$

	β										
N	0.02	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
10	-2.542	-2.146	-1.772	-1.507	-1.289	-1.097	-0.920	-0.753	-0.591	-0.432	-0.273
11	-2.440	-2.056	-1.694	-1.438	-1.226	-1.040	-0.868	-0.706	-0.548	-0.393	-0.237
12	-2.358	-1.985	-1.632	-1.382	-1.175	-0.993	-0.825	-0.666	-0.512	-0.360	-0.207
13	-2.293	-1.927	-1.581	-1.336	-1.133	-0.954	-0.789	-0.633	-0.481	-0.331	-0.181
14	-2.238	-1.878	-1.538	-1.296	-1.097	-0.921	-0.758	-0.604	-0.454	-0.306	-0.158
15	-2.191	-1.837	-1.501	-1.263	-1.066	-0.892	-0.731	-0.579	-0.431	-0.284	-0.137
16	-2.151	-1.802	-1.470	-1.234	-1.039	-0.867	-0.708	-0.557	-0.410	-0.265	-0.119
18	-2.086	-1.743	-1.417	-1.186	-0.994	-0.825	-0.668	-0.519	-0.375	-0.232	-0.088
20	-2.034	-1.696	-1.375	-1.147	-0.958	-0.791	-0.636	-0.489	-0.346	-0.204	-0.062
22	-1.991	-1.658	-1.341	-1.115	-0.928	-0.762	-0.609	-0.463	-0.322	-0.181	-0.040
24	-1.956	-1.625	-1.312	-1.088	-0.903	-0.738	-0.586	-0.442	-0.301	-0.161	-0.021
26	-1.925	-1.598	-1.287	-1.065	-0.881	-0.718	-0.567	-0.423	-0.283	-0.144	-0.004
28	-1.900	-1.575	-1.266	-1.045	-0.862	-0.700	-0.550	-0.407	-0.267	-0.129	0.010
30	-1.877	-1.554	-1.247	-1.027	-0.846	-0.684	-0.535	-0.392	-0.253	-0.115	0.023
32	-1.857	-1.536	-1.230	-1.012	-0.831	-0.670	-0.521	-0.379	-0.241	-0.103	0.035
34	-1.836	-1.518	-1.214	-0.998	-0.818	-0.658	-0.510	-0.368	-0.230	-0.093	0.045
36	-1.817	-1.501	-1.200	-0.985	-0.806	-0.647	-0.499	-0.358	-0.221	-0.084	0.054
38	-1.801	-1.487	-1.188	-0.973	-0.795	-0.637	-0.489	-0.349	-0.212	-0.075	0.063
40	-1.786	-1.474	-1.176	-0.963	-0.785	-0.627	-0.481	-0.340	-0.203	-0.067	0.071
42	-1.771	-1.462	-1.165	-0.953	-0.776	-0.619	-0.473	-0.333	-0.196	-0.060	0.078
44	-1.759	-1.451	-1.156	-0.944	-0.768	-0.611	-0.465	-0.325	-0.189	-0.053	0.085
46	-1.747	-1.441	-1.147	-0.936	-0.760	-0.604	-0.458	-0.319	-0.182	-0.047	0.091
48	-1.736	-1.431	-1.138	-0.928	-0.753	-0.597	-0.452	-0.312	-0.176	-0.041	0.097
50	-1.727	-1.423	-1.131	-0.921	-0.746	-0.590	-0.445	-0.307	-0.171	-0.035	0.102
52	-1.717	-1.414	-1.123	-0.914	-0.740	-0.585	-0.440	-0.301	-0.165	-0.030	0.107
54	-1.709	-1.407	-1.117	-0.908	-0.734	-0.579	-0.434	-0.296	-0.160	-0.025	0.112
56	-1.699	-1.399	-1.110	-0.902	-0.729	-0.574	-0.430	-0.291	-0.156	-0.021	0.117
58	-1.692	-1.392	-1.104	-0.897	-0.724	-0.569	-0.425	-0.287	-0.151	-0.016	0.121
60	-1.684	-1.386	-1.099	-0.892	-0.719	-0.564	-0.421	-0.283	-0.147	-0.012	0.125
62	-1.678	-1.380	-1.093	-0.887	-0.714	-0.560	-0.416	-0.278	-0.143	-0.008	0.129
64	-1.671	-1.374	-1.088	-0.882	-0.710	-0.556	-0.412	-0.275	-0.139	-0.004	0.133
66	-1.665	-1.369	-1.084	-0.878	-0.706	-0.552	-0.408	-0.271	-0.136	-0.001	0.136
68	-1.660	-1.364	-1.079	-0.873	-0.702	-0.548	-0.405	-0.267	-0.132	0.003	0.140
70	-1.654	-1.359	-1.075	-0.869	-0.698	-0.544	-0.401	-0.264	-0.129	0.006	0.143
72	-1.648	-1.354	-1.070	-0.866	-0.694	-0.541	-0.398	-0.261	-0.126	0.009	0.146
74	-1.643	-1.350	-1.066	-0.862	-0.691	-0.538	-0.395	-0.258	-0.123	0.012	0.149
76	-1.638	-1.345	-1.063	-0.859	-0.688	-0.535	-0.392	-0.255	-0.120	0.015	0.152
78	-1.634	-1.341	-1.059	-0.855	-0.685	-0.532	-0.389	-0.252	-0.117	0.018	0.155
80	-1.630	-1.338	-1.056	-0.852	-0.682	-0.529	-0.386	-0.249	-0.114	0.020	0.157

Table A4c

Percentage Points, ℓ, such that $G_K(\ell)=.\underline{10}$ as a Function of β where $K = -\ln(\beta)$

N	β 0.02	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
10	-2.241	-1.876	-1.530	-1.284	-1.082	-0.902	-0.736	-0.579	-0.426	-0.275	-0.123
11	-2.172	-1.816	-1.477	-1.237	-1.038	-0.862	-0.699	-0.545	-0.395	-0.246	-0.097
12	-2.117	-1.767	-1.434	-1.198	-1.002	-0.829	-0.669	-0.516	-0.368	-0.221	-0.074
13	-2.071	-1.726	-1.398	-1.165	-0.972	-0.801	-0.642	-0.492	-0.345	-0.200	-0.054
14	-2.033	-1.692	-1.368	-1.137	-0.946	-0.777	-0.620	-0.471	-0.326	-0.182	-0.037
15	-2.000	-1.663	-1.342	-1.113	-0.924	-0.756	-0.600	-0.452	-0.308	-0.165	-0.021
16	-1.971	-1.637	-1.319	-1.092	-0.904	-0.737	-0.583	-0.436	-0.293	-0.151	-0.008
18	-1.924	-1.595	-1.281	-1.057	-0.871	-0.706	-0.554	-0.408	-0.266	-0.126	0.016
20	-1.883	-1.559	-1.249	-1.028	-0.845	-0.682	-0.530	-0.386	-0.246	-0.106	0.035
22	-1.849	-1.529	-1.223	-1.004	-0.823	-0.661	-0.511	-0.368	-0.228	-0.089	0.051
24	-1.820	-1.504	-1.201	-0.984	-0.804	-0.644	-0.494	-0.352	-0.213	-0.075	0.065
26	-1.796	-1.482	-1.182	-0.967	-0.788	-0.628	-0.480	-0.338	-0.200	-0.062	0.078
28	-1.774	-1.463	-1.166	-0.952	-0.774	-0.615	-0.468	-0.326	-0.188	-0.051	0.089
30	-1.755	-1.447	-1.151	-0.939	-0.761	-0.603	-0.456	-0.316	-0.178	-0.041	0.098
32	-1.738	-1.432	-1.138	-0.927	-0.750	-0.593	-0.447	-0.306	-0.169	-0.032	0.107
34	-1.723	-1.419	-1.126	-0.916	-0.741	-0.584	-0.438	-0.298	-0.161	-0.024	0.115
36	-1.710	-1.407	-1.116	-0.906	-0.732	-0.575	-0.429	-0.290	-0.153	-0.016	0.122
38	-1.698	-1.396	-1.107	-0.898	-0.723	-0.567	-0.422	-0.283	-0.146	-0.010	0.129
40	-1.687	-1.387	-1.098	-0.890	-0.716	-0.560	-0.415	-0.276	-0.140	-0.003	0.135
42	-1.677	-1.378	-1.090	-0.883	-0.709	-0.554	-0.409	-0.270	-0.134	0.003	0.141
44	-1.668	-1.370	-1.083	-0.876	-0.703	-0.548	-0.403	-0.264	-0.128	0.008	0.146
46	-1.659	-1.362	-1.076	-0.870	-0.697	-0.542	-0.398	-0.259	-0.123	0.013	0.151
48	-1.651	-1.355	-1.070	-0.864	-0.691	-0.537	-0.393	-0.254	-0.118	0.018	0.156
50	-1.644	-1.349	-1.064	-0.858	-0.686	-0.532	-0.388	-0.250	-0.114	0.022	0.160
52	-1.637	-1.343	-1.059	-0.853	-0.681	-0.527	-0.384	-0.245	-0.110	0.026	0.164
54	-1.631	-1.337	-1.054	-0.849	-0.677	-0.523	-0.379	-0.241	-0.106	0.030	0.168
56	-1.624	-1.331	-1.049	-0.844	-0.673	-0.519	-0.376	-0.238	-0.102	0.034	0.172
58	-1.619	-1.327	-1.044	-0.840	-0.669	-0.515	-0.372	-0.234	-0.099	0.037	0.175
60	-1.614	-1.322	-1.040	-0.836	-0.665	-0.512	-0.368	-0.231	-0.095	0.041	0.179
62	-1.609	-1.317	-1.036	-0.832	-0.661	-0.508	-0.365	-0.227	-0.092	0.044	0.182
64	-1.603	-1.313	-1.032	-0.829	-0.658	-0.505	-0.362	-0.224	-0.089	0.046	0.185
66	-1.599	-1.309	-1.028	-0.825	-0.655	-0.502	-0.359	-0.221	-0.086	0.049	0.187
68	-1.595	-1.305	-1.025	-0.822	-0.652	-0.499	-0.356	-0.219	-0.083	0.052	0.190
70	-1.590	-1.301	-1.022	-0.819	-0.649	-0.496	-0.354	-0.216	-0.081	0.055	0.193
72	-1.587	-1.298	-1.019	-0.816	-0.646	-0.494	-0.351	-0.213	-0.078	0.057	0.195
74	-1.583	-1.295	-1.016	-0.813	-0.644	-0.491	-0.348	-0.211	-0.076	0.060	0.198
76	-1.579	-1.291	-1.013	-0.811	-0.641	-0.489	-0.346	-0.209	-0.074	0.062	0.200
78	-1.576	-1.288	-1.010	-0.808	-0.639	-0.486	-0.344	-0.207	-0.071	0.064	0.202
80	-1.572	-1.285	-1.007	-0.806	-0.636	-0.484	-0.342	-0.204	-0.069	0.066	0.204

Table A4d

Percentage Points, ℓ, such that $G_K(\ell)=\underline{.25}$ as a Function of β where $K = -\ln(\beta)$

	β										
N	0.02	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
10	-1.866	-1.537	-1.221	-0.993	-0.804	-0.635	-0.478	-0.327	-0.179	-0.032	0.117
11	-1.826	-1.502	-1.191	-0.968	-0.781	-0.614	-0.459	-0.310	-0.164	-0.018	0.130
12	-1.792	-1.474	-1.167	-0.946	-0.761	-0.596	-0.443	-0.295	-0.150	-0.006	0.141
13	-1.765	-1.450	-1.147	-0.928	-0.745	-0.581	-0.429	-0.282	-0.139	0.005	0.151
14	-1.741	-1.430	-1.129	-0.912	-0.731	-0.568	-0.417	-0.271	-0.129	0.014	0.159
15	-1.720	-1.412	-1.114	-0.899	-0.719	-0.557	-0.407	-0.262	-0.120	0.022	0.167
16	-1.703	-1.396	-1.101	-0.887	-0.708	-0.547	-0.397	-0.254	-0.112	0.029	0.173
18	-1.673	-1.370	-1.078	-0.867	-0.689	-0.531	-0.382	-0.239	-0.099	0.042	0.185
20	-1.648	-1.349	-1.060	-0.850	-0.675	-0.517	-0.370	-0.228	-0.088	0.052	0.195
22	-1.628	-1.332	-1.045	-0.837	-0.663	-0.506	-0.359	-0.218	-0.079	0.061	0.203
24	-1.612	-1.318	-1.033	-0.826	-0.652	-0.496	-0.350	-0.209	-0.071	0.068	0.210
26	-1.598	-1.306	-1.022	-0.816	-0.643	-0.488	-0.342	-0.202	-0.064	0.075	0.216
28	-1.586	-1.295	-1.013	-0.808	-0.635	-0.481	-0.335	-0.195	-0.058	0.081	0.222
30	-1.574	-1.285	-1.004	-0.800	-0.629	-0.474	-0.329	-0.190	-0.052	0.086	0.227
32	-1.566	-1.277	-0.997	-0.794	-0.622	-0.468	-0.324	-0.185	-0.047	0.091	0.231
34	-1.557	-1.270	-0.991	-0.788	-0.617	-0.463	-0.319	-0.180	-0.043	0.095	0.235
36	-1.550	-1.264	-0.985	-0.783	-0.612	-0.459	-0.315	-0.176	-0.039	0.099	0.239
38	-1.542	-1.257	-0.980	-0.778	-0.608	-0.455	-0.311	-0.172	-0.035	0.102	0.242
40	-1.537	-1.252	-0.975	-0.773	-0.604	-0.451	-0.307	-0.169	-0.032	0.105	0.246
42	-1.531	-1.247	-0.971	-0.769	-0.600	-0.447	-0.304	-0.166	-0.029	0.108	0.249
44	-1.525	-1.242	-0.967	-0.766	-0.596	-0.444	-0.301	-0.163	-0.026	0.111	0.251
46	-1.521	-1.238	-0.963	-0.762	-0.593	-0.441	-0.298	-0.160	-0.023	0.114	0.254
48	-1.516	-1.234	-0.959	-0.759	-0.590	-0.438	-0.295	-0.157	-0.021	0.116	0.256
50	-1.512	-1.231	-0.956	-0.756	-0.588	-0.436	-0.293	-0.155	-0.019	0.118	0.258
52	-1.509	-1.227	-0.953	-0.753	-0.585	-0.433	-0.290	-0.153	-0.016	0.121	0.261
54	-1.504	-1.224	-0.950	-0.751	-0.583	-0.431	-0.288	-0.151	-0.015	0.122	0.262
56	-1.501	-1.221	-0.948	-0.748	-0.580	-0.429	-0.286	-0.149	-0.013	0.124	0.264
58	-1.498	-1.219	-0.945	-0.746	-0.578	-0.427	-0.284	-0.147	-0.011	0.126	0.266
60	-1.495	-1.216	-0.943	-0.744	-0.576	-0.425	-0.282	-0.145	-0.009	0.128	0.268
62	-1.492	-1.213	-0.941	-0.742	-0.574	-0.423	-0.281	-0.143	-0.007	0.129	0.269
64	-1.490	-1.211	-0.939	-0.740	-0.572	-0.421	-0.279	-0.142	-0.006	0.131	0.271
66	-1.487	-1.209	-0.937	-0.738	-0.571	-0.420	-0.277	-0.140	-0.004	0.132	0.272
68	-1.485	-1.207	-0.935	-0.736	-0.569	-0.418	-0.276	-0.139	-0.003	0.134	0.274
70	-1.483	-1.205	-0.933	-0.735	-0.568	-0.416	-0.274	-0.137	-0.001	0.135	0.275
72	-1.480	-1.203	-0.931	-0.733	-0.565	-0.415	-0.273	-0.136	-0.000	0.136	0.276
74	-1.478	-1.201	-0.930	-0.732	-0.565	-0.414	-0.272	-0.135	0.001	0.138	0.278
76	-1.477	-1.199	-0.928	-0.730	-0.563	-0.412	-0.271	-0.133	0.002	0.139	0.279
78	-1.474	-1.198	-0.927	-0.729	-0.562	-0.411	-0.269	-0.132	0.003	0.140	0.280
80	-1.473	-1.196	-0.925	-0.728	-0.561	-0.410	-0.268	-0.131	0.005	0.141	0.281

Table A5

75% Lower Confidence Limits for R(t)

$\hat{R}(t)$	n = 8	10	12	15	18	20	25	30	40	50	75	100
.50	.399	.411	.419	.428	.434	.438	.445	.449	.456	.461	.467	.472
.52	.417	.429	.437	.446	.453	.457	.465	.468	.475	.481	.487	.491
.54	.435	.446	.455	.464	.472	.476	.484	.487	.495	.500	.507	.511
.56	.452	.465	.474	.483	.491	.495	.503	.506	.515	.520	.526	.531
.58	.471	.483	.492	.501	.510	.514	.522	.526	.534	.540	.546	.551
.60	.489	.501	.510	.520	.529	.533	.542	.546	.554	.560	.566	.571
.62	.507	.520	.529	.539	.549	.553	.562	.565	.574	.579	.586	.591
.64	.526	.539	.548	.559	.568	.572	.581	.585	.594	.600	.606	.611
.66	.544	.558	.568	.578	.588	.592	.601	.605	.614	.620	.627	.631
.68	.563	.577	.587	.598	.608	.612	.621	.625	.635	.640	.647	.651
.70	.583	.596	.607	.618	.628	.632	.641	.646	.655	.660	.667	.672
.72	.602	.616	.627	.638	.648	.653	.662	.666	.676	.681	.688	.692
.74	.622	.636	.648	.659	.668	.673	.682	.687	.697	.701	.708	.713
.76	.643	.657	.668	.680	.690	.694	.703	.708	.717	.722	.729	.734
.78	.663	.678	.690	.701	.711	.715	.725	.729	.738	.743	.750	.754
.80	.685	.699	.712	.723	.732	.737	.746	.750	.760	.764	.771	.775
.82	.707	.721	.734	.745	.754	.759	.768	.772	.781	.785	.792	.796
.84	.730	.744	.757	.767	.777	.781	.790	.794	.803	.807	.814	.818
.86	.754	.768	.780	.791	.800	.804	.813	.817	.825	.829	.836	.839
.88	.779	.792	.805	.815	.823	.828	.836	.839	.848	.851	.857	.861
.90	.804	.818	.830	.840	.848	.852	.859	.863	.870	.874	.880	.883
.92	.833	.846	.856	.866	.873	.877	.884	.887	.894	.897	.902	.905
.94	.863	.875	.885	.894	.900	.903	.909	.912	.918	.921	.925	.927
.96	.897	.908	.916	.923	.928	.931	.936	.938	.943	.945	.948	.950
.98	.937	.945	.950	.956	.960	.962	.965	.967	.970	.971	.973	.974

Table A5 (cont.)

80% Lower Confidence Limits for R(t)

$\hat{R}(t)$	n = 8	10	12	15	18	20	25	30	40	50	75	100
.50	.377	.391	.399	.410	.418	.423	.432	.437	.446	.451	.459	.465
.52	.393	.408	.417	.429	.437	.442	.451	.456	.465	.471	.479	.484
.54	.411	.425	.435	.447	.456	.461	.470	.475	.484	.491	.499	.504
.56	.428	.443	.453	.466	.475	.480	.490	.494	.504	.510	.518	.524
.58	.446	.461	.471	.485	.494	.499	.509	.514	.524	.530	.538	.544
.60	.464	.479	.490	.503	.513	.518	.529	.534	.543	.550	.558	.564
.62	.481	.497	.508	.523	.532	.537	.549	.553	.563	.570	.578	.584
.64	.499	.516	.527	.542	.552	.557	.568	.573	.583	.590	.599	.604
.66	.517	.534	.546	.562	.572	.577	.588	.593	.604	.610	.619	.625
.68	.537	.554	.566	.582	.592	.597	.608	.613	.624	.630	.639	.645
.70	.556	.573	.586	.601	.612	.617	.629	.634	.644	.651	.660	.665
.72	.576	.593	.606	.622	.633	.637	.649	.654	.665	.671	.681	.686
.74	.596	.613	.626	.642	.653	.658	.670	.675	.686	.692	.701	.707
.76	.617	.634	.648	.663	.674	.679	.691	.696	.707	.713	.722	.728
.78	.638	.656	.669	.684	.696	.701	.712	.717	.729	.735	.743	.749
.80	.659	.678	.691	.706	.717	.723	.734	.739	.750	.756	.764	.770
.82	.682	.700	.714	.729	.740	.745	.756	.761	.772	.778	.786	.792
.84	.705	.723	.737	.752	.763	.768	.778	.784	.794	.800	.807	.813
.86	.729	.748	.762	.776	.787	.791	.801	.807	.817	.822	.829	.835
.88	.755	.774	.787	.801	.811	.816	.825	.830	.840	.844	.851	.856
.90	.783	.800	.814	.826	.837	.841	.850	.855	.863	.868	.874	.879
.92	.813	.829	.842	.854	.863	.867	.875	.880	.887	.892	.897	.901
.94	.845	.860	.872	.883	.891	.894	.902	.906	.912	.916	.921	.924
.96	.881	.894	.905	.914	.921	.924	.930	.933	.939	.942	.945	.948
.98	.924	.936	.943	.949	.955	.957	.961	.963	.967	.969	.971	.973

Table A5 (cont.)

85% Lower Confidence Limits for R(t)

$\hat{R}(t)$	n = 8	10	12	15	18	20	25	30	40	50	75	100
.50	.350	.365	.376	.390	.399	.406	.416	.422	.433	.440	.450	.457
.52	.367	.382	.394	.408	.418	.425	.435	.441	.453	.460	.470	.477
.54	.383	.399	.411	.426	.437	.443	.454	.460	.472	.479	.490	.496
.56	.400	.417	.430	.445	.456	.462	.473	.480	.492	.499	.509	.516
.58	.417	.435	.448	.464	.475	.481	.492	.499	.511	.519	.529	.536
.60	.434	.453	.466	.483	.494	.500	.512	.519	.531	.539	.549	.556
.62	.451	.471	.484	.502	.513	.519	.532	.538	.551	.559	.569	.576
.64	.469	.490	.503	.521	.532	.539	.552	.558	.571	.579	.590	.597
.66	.487	.509	.522	.540	.552	.559	.572	.578	.592	.599	.610	.617
.68	.505	.528	.541	.560	.572	.579	.592	.599	.612	.619	.630	.637
.70	.524	.547	.561	.580	.592	.599	.612	.619	.633	.640	.651	.658
.72	.544	.567	.581	.601	.613	.619	.633	.640	.653	.661	.671	.679
.74	.563	.587	.601	.621	.634	.640	.654	.661	.674	.682	.692	.700
.76	.584	.608	.623	.642	.655	.662	.675	.683	.696	.703	.713	.721
.78	.605	.629	.644	.664	.677	.683	.697	.704	.717	.724	.735	.742
.80	.627	.651	.667	.687	.699	.705	.719	.726	.739	.746	.756	.763
.82	.650	.674	.690	.710	.722	.728	.741	.749	.761	.768	.778	.785
.84	.674	.698	.714	.733	.745	.751	.764	.772	.784	.791	.800	.807
.86	.699	.723	.739	.758	.769	.775	.788	.795	.807	.813	.822	.829
.88	.725	.749	.765	.783	.794	.800	.812	.819	.831	.837	.845	.852
.90	.753	.778	.793	.810	.821	.826	.838	.844	.855	.860	.868	.874
.92	.785	.809	.822	.838	.848	.853	.864	.870	.880	.885	.892	.897
.94	.820	.842	.854	.869	.878	.883	.892	.898	.906	.910	.916	.921
.96	.860	.879	.890	.902	.911	.914	.922	.927	.934	.937	.942	.945
.98	.909	.924	.932	.942	.947	.950	.956	.959	.964	.966	.969	.971

Table A5 (cont.)

90% Lower Confidence Limits for R(t)

$\hat{R}(t)$	n = 8	10	12	15	18	20	25	30	40	50	75	100
.50	.316	.336	.348	.365	.378	.385	.396	.404	.418	.426	.438	.447
.52	.332	.352	.365	.382	.396	.403	.415	.423	.437	.445	.457	.467
.54	.348	.369	.382	.400	.414	.421	.433	.442	.456	.464	.477	.486
.56	.364	.385	.399	.418	.432	.439	.452	.461	.476	.484	.497	.506
.58	.380	.401	.417	.436	.450	.457	.471	.481	.495	.504	.517	.526
.60	.397	.419	.435	.455	.469	.477	.490	.500	.515	.524	.537	.546
.62	.414	.437	.453	.473	.488	.496	.510	.520	.535	.544	.557	.567
.64	.432	.455	.472	.492	.507	.516	.529	.540	.555	.564	.577	.587
.66	.450	.474	.491	.512	.526	.535	.549	.560	.575	.584	.598	.607
.68	.468	.493	.511	.532	.546	.555	.569	.580	.596	.605	.618	.628
.70	.486	.512	.530	.552	.566	.575	.589	.601	.616	.626	.639	.649
.72	.504	.532	.550	.573	.586	.596	.610	.622	.637	.646	.660	.670
.74	.524	.552	.571	.593	.607	.617	.631	.643	.658	.668	.681	.691
.76	.544	.573	.592	.615	.628	.638	.653	.665	.680	.690	.702	.712
.78	.566	.595	.613	.637	.651	.660	.675	.687	.702	.711	.724	.734
.80	.588	.618	.635	.660	.674	.683	.698	.709	.724	.733	.746	.755
.82	.611	.641	.659	.683	.697	.706	.721	.732	.746	.756	.768	.777
.84	.636	.666	.683	.707	.722	.730	.745	.755	.769	.778	.790	.799
.86	.662	.692	.709	.732	.747	.755	.769	.780	.793	.802	.813	.821
.88	.689	.719	.736	.759	.773	.781	.795	.805	.818	.825	.837	.844
.90	.719	.748	.765	.787	.800	.808	.821	.831	.843	.851	.861	.868
.92	.751	.780	.796	.817	.829	.837	.849	.859	.869	.876	.885	.892
.94	.787	.815	.831	.849	.861	.867	.879	.887	.897	.903	.911	.916
.96	.829	.855	.870	.887	.896	.901	.911	.918	.926	.931	.937	.942
.98	.885	.906	.917	.930	.937	.941	.948	.953	.959	.962	.966	.969

Table A5 (cont.)

95% Lower Confidence Limits for R(t)

$\hat{R}(t)$	n = 8	10	12	15	18	20	25	30	40	50	75	100
.50			.308	.329	.343	.353	.366	.379	.394	.404	.420	.432
.52		.308	.325	.346	.361	.371	.384	.398	.413	.423	.439	.452
.54	.300	.323	.341	.363	.378	.389	.402	.416	.432	.442	.459	.471
.56	.316	.339	.358	.381	.396	.407	.421	.435	.451	.461	.478	.491
.58	.331	.355	.376	.398	.414	.425	.440	.454	.471	.481	.498	.510
.60	.347	.372	.393	.416	.432	.443	.459	.473	.490	.500	.517	.530
.62	.363	.389	.411	.434	.450	.462	.478	.493	.510	.519	.537	.551
.64	.380	.406	.428	.452	.469	.480	.497	.512	.530	.539	.558	.571
.66	.396	.424	.445	.471	.488	.499	.517	.532	.550	.559	.579	.592
.68	.414	.443	.464	.490	.507	.519	.536	.552	.570	.580	.599	.612
.70	.432	.461	.483	.510	.527	.538	.557	.573	.591	.601	.620	.633
.72	.450	.481	.502	.530	.547	.559	.577	.594	.612	.622	.642	.654
.74	.469	.500	.523	.550	.568	.580	.598	.616	.633	.644	.663	.675
.76	.489	.520	.544	.572	.590	.602	.620	.638	.654	.666	.684	.697
.78	.509	.542	.567	.594	.612	.625	.643	.661	.676	.688	.707	.719
.80	.529	.564	.590	.617	.636	.648	.666	.683	.700	.711	.729	.741
.82	.552	.587	.614	.641	.660	.672	.689	.706	.724	.734	.752	.763
.84	.576	.611	.638	.667	.685	.697	.714	.730	.748	.758	.775	.786
.86	.602	.638	.664	.693	.710	.723	.740	.755	.772	.783	.799	.809
.88	.629	.666	.692	.721	.737	.750	.767	.781	.798	.808	.823	.833
.90	.661	.696	.722	.751	.766	.780	.795	.809	.824	.834	.848	.857
.92	.695	.729	.755	.782	.798	.811	.825	.838	.853	.862	.874	.882
.94	.735	.767	.792	.817	.832	.845	.858	.869	.882	.890	.901	.908
.96	.782	.812	.835	.857	.872	.882	.893	.903	.915	.921	.930	.935
.98	.844	.869	.890	.907	.918	.926	.935	.943	.950	.955	.962	.965

Table A5 (cont.)

98% Lower Confidence Limits for R(t)

$\hat{R}(t)$	n = 8	10	12	15	18	20	25	30	40	50	75	100
.50				.293	.305	.317	.331	.349	.366	.379	.401	.415
.52				.307	.322	.334	.348	.366	.384	.397	.420	.434
.54			.300	.323	.339	.351	.366	.384	.403	.417	.439	.453
.56			.309	.339	.356	.369	.384	.402	.422	.436	.458	.473
.58		.309	.324	.356	.373	.386	.402	.421	.441	.455	.478	.493
.60		.324	.340	.374	.391	.404	.420	.440	.461	.475	.498	.514
.62	.305	.340	.357	.391	.408	.423	.439	.460	.481	.494	.517	.534
.64	.320	.356	.374	.409	.427	.442	.458	.480	.500	.514	.538	.554
.66	.335	.373	.392	.428	.446	.462	.478	.500	.519	.534	.558	.575
.68	.350	.390	.410	.446	.465	.481	.499	.519	.540	.554	.578	.595
.70	.369	.408	.429	.466	.484	.501	.519	.540	.561	.574	.600	.616
.72	.387	.426	.447	.485	.504	.521	.540	.561	.582	.595	.620	.637
.74	.405	.445	.468	.505	.524	.542	.562	.582	.604	.617	.642	.659
.76	.425	.465	.489	.526	.546	.563	.584	.605	.625	.639	.664	.681
.78	.445	.487	.510	.547	.568	.585	.606	.628	.648	.662	.687	.702
.80	.466	.509	.532	.569	.591	.609	.631	.651	.672	.685	.709	.725
.82	.488	.532	.557	.593	.615	.633	.655	.676	.695	.709	.733	.748
.84	.511	.557	.583	.618	.639	.658	.681	.701	.720	.733	.757	.771
.86	.535	.583	.610	.645	.666	.686	.708	.727	.746	.759	.781	.795
.88	.561	.611	.639	.675	.694	.715	.736	.755	.772	.786	.806	.820
.90	.590	.642	.670	.706	.723	.745	.765	.784	.800	.813	.833	.845
.92	.623	.677	.707	.740	.758	.778	.797	.814	.831	.842	.861	.871
.94	.664	.716	.748	.778	.796	.814	.831	.847	.863	.873	.890	.899
.96	.714	.763	.793	.823	.838	.855	.871	.884	.899	.906	.920	.928
.98	.785	.828	.854	.880	.892	.906	.919	.928	.940	.945	.955	.960

Table A6

Percentage Points, ℓ_γ, such that $P[(\hat{c}_1/c_1)/(\hat{c}_2/c_2) < \ell_\gamma] = \gamma$

N \ γ	.60	.70	.75	.80	.85	.90	.95	.98
5	1.158	1.351	1.478	1.636	1.848	2.152	2.725	3.550
6	1.135	1.318	1.418	1.573	1.727	1.987	2.465	3.146
7	1.127	1.283	1.370	1.502	1.638	1.869	2.246	2.755
8	1.119	1.256	1.338	1.450	1.573	1.780	2.093	2.509
9	1.111	1.236	1.311	1.410	1.534	1.711	1.982	2.339
10	1.104	1.220	1.290	1.380	1.486	1.655	1.897	2.213
11	1.098	1.206	1.273	1.355	1.454	1.609	1.829	2.115
12	1.093	1.195	1.258	1.334	1.428	1.571	1.774	2.036
13	1.088	1.186	1.245	1.317	1.406	1.538	1.727	1.972
14	1.048	1.177	1.233	1.301	1.386	1.509	1.688	1.917
15	1.081	1.170	1.224	1.288	1.369	1.485	1.654	1.870
16	1.077	1.164	1.215	1.277	1.355	1.463	1.624	1.829
17	1.075	1.158	1.207	1.266	1.341	1.444	1.598	1.793
18	1.072	1.153	1.200	1.257	1.329	1.426	1.574	1.762
19	1.070	1.148	1.194	1.249	1.318	1.411	1.553	1.733
20	1.068	1.144	1.188	1.241	1.308	1.396	1.534	1.708
22	1.064	1.136	1.178	1.227	1.291	1.372	1.501	1.663
24	1.061	1.129	1.169	1.216	1.276	1.351	1.473	1.625
26	1.058	1.124	1.162	1.206	1.263	1.333	1.449	1.593
28	1.055	1.119	1.155	1.197	1.252	1.318	1.428	1.566
30	1.053	1.114	1.149	1.190	1.242	1.304	1.409	1.541
32	1.051	1.110	1.144	1.183	1.233	1.292	1.393	1.520
34	1.049	1.107	1.139	1.176	1.224	1.281	1.378	1.500
36	1.047	1.103	1.135	1.171	1.217	1.272	1.365	1.483
38	1.046	1.100	1.131	1.166	1.210	1.263	1.353	1.467
40	1.045	1.098	1.127	1.161	1.204	1.255	1.342	1.453
42	1.043	1.095	1.124	1.156	1.198	1.248	1.332	1.439
44	1.042	1.093	1.121	1.152	1.193	1.241	1.323	1.427
46	1.041	1.091	1.118	1.149	1.188	1.235	1.314	1.416
48	1.040	1.088	1.115	1.145	1.184	1.229	1.306	1.405
50	1.039	1.087	1.113	1.142	1.179	1.224	1.299	1.396
52	1.038	1.085	1.111	1.139	1.175	1.219	1.292	1.387
54	1.037	1.083	1.108	1.136	1.172	1.215	1.285	1.378
56	1.036	1.081	1.106	1.133	1.168	1.210	1.279	1.370
58	1.036	1.080	1.104	1.131	1.165	1.206	1.274	1.363
60	1.035	1.078	1.102	1.128	1.162	1.203	1.268	1.355
62	1.034	1.077	1.101	1.126	1.159	1.199	1.263	1.349
64	1.034	1.076	1.099	1.124	1.156	1.196	1.258	1.342
66	1.033	1.075	1.097	1.122	1.153	1.192	1.253	1.336
68	1.032	1.073	1.096	1.120	1.151	1.189	1.249	1.331
70	1.032	1.072	1.094	1.118	1.148	1.186	1.245	1.325
72	1.031	1.071	1.093	1.116	1.146	1.184	1.241	1.320
76	1.030	1.069	1.090	1.112	1.141	1.179	1.233	1.310
80	1.030	1.067	1.088	1.109	1.137	1.174	1.227	1.301
90	1.028	1.063	1.082	1.102	1.128	1.164	1.212	1.282
100	1.026	1.060	1.078	1.097	1.121	1.155	1.199	1.266
120	1.023	1.054	1.071	1.087	1.109	1.142	1.180	1.240

Table A7

Percentage Points, z_γ, such that $H_1(z_\gamma) = \gamma$

N γ	.60	.70	.75	.80	.85	.90	.95	.98
5	.228	.476	.608	.777	.960	1.226	1.670	2.242
6	.190	.397	.522	.642	.821	1.050	1.404	1.840
7	.164	.351	.461	.573	.726	.918	1.315	1.592
8	.148	.320	.415	.521	.658	.825	1.086	1.421
9	.126	.296	.383	.481	.605	.757	.992	1.294
10	.127	.277	.356	.449	.563	.704	.918	1.195
11	.120	.261	.336	.423	.528	.661	.860	1.115
12	.115	.248	.318	.401	.499	.625	.811	1.049
13	.110	.237	.303	.383	.474	.594	.770	.993
14	.106	.227	.290	.366	.453	.567	.734	.945
15	.103	.218	.279	.352	.434	.544	.704	.904
16	.099	.210	.269	.339	.417	.523	.676	.867
17	.096	.203	.260	.328	.403	.505	.654	.834
18	.094	.197	.251	.317	.389	.488	.631	.805
19	.091	.101	.244	.308	.377	.473	.611	.779
20	.089	.186	.237	.299	.366	.349	.593	.755
22	.085	.176	.225	.284	.347	.435	.561	.712
24	.082	.168	.215	.271	.330	.414	.534	.677
26	.079	.161	.206	.259	.316	.396	.510	.646
28	.076	.154	.198	.249	.303	.380	.490	.619
30	.073	.149	.191	.240	.292	.366	.472	.595
32	.071	.144	.185	.232	.282	.354	.455	.574
34	.069	.139	.179	.225	.273	.342	.441	.555
36	.067	.135	.174	.218	.265	.332	.427	.537
38	.065	.131	.169	.212	.258	.323	.415	.522
40	.064	.127	.165	.206	.251	.314	.404	.507
42	.062	.124	.160	.202	.245	.306	.394	.494
44	.061	.121	.157	.196	.239	.298	.384	.482
46	.059	.118	.153	.192	.234	.292	.376	.470
48	.058	.115	.150	.188	.229	.285	.367	.460
50	.057	.113	.147	.184	.224	.279	.360	.450
52	.056	.110	.144	.180	.220	.273	.353	.440
54	.055	.108	.141	.176	.215	.268	.346	.432
56	.054	.106	.138	.173	.212	.263	.340	.423
58	.053	.104	.136	.170	.208	.258	.334	.416
60	.052	.102	.134	.167	.204	.254	.328	.408
62	.051	.100	.131	.164	.201	.250	.323	.402
64	.050	.099	.129	.162	.198	.246	.317	.395
66	.049	.097	.127	.159	.195	.242	.313	.389
68	.049	.095	.125	.157	.192	.238	.308	.383
70	.048	.094	.123	.154	.190	.235	.304	.377
72	.047	.092	.122	.152	.187	.231	.299	.372
76	.046	.090	.118	.148	.182	.225	.291	.361
80	.045	.087	.115	.144	.178	.219	.284	.352
90	.042	.082	.109	.136	.168	.207	.268	.332
100	.040	.077	.103	.128	.160	.196	.255	.315
120	.036	.070	.094	.117	.147	.179	.233	.287

Table A8: Probabilities of Correct Selection

N \ αC_s	1.10	1.15	1.20	1.25	1.30	1.35	1.40	1.45	1.50	1.55	1.60	1.70	1.80	1.90	2.00
8	.570	.601	.631	.660	.687	.712	.732	.755	.774	.793	.809	.837	.860	.879	.911
10	.580	.614	.447	.681	.705	.736	.760	.783	.805	.820	.836	.867	.889	.908	.936
12	.589	.626	.662	.699	.722	.756	.782	.806	.825	.843	.859	.889	.909	.927	.953
14	.596	.637	.676	.714	.738	.774	.801	.824	.844	.862	.878	.906	.925	.942	.964
16	.602	.646	.689	.727	.754	.790	.818	.840	.860	.878	.894	.920	.938	.953	.973
18	.607	.654	.700	.738	.769	.804	.832	.854	.874	.892	.907	.932	.949	.962	.980
20	.612	.661	.710	.748	.783	.817	.844	.866	.886	.909	.918	.942	.958	.970	
22	.617	.668	.719	.757	.796	.828	.855	.871	.897	.914	.928	.950	.965	.977	
24	.622	.675	.727	.766	.807	.838	.865	.887	.906	.921	.937	.956	.971		
26	.627	.682	.743	.774	.817	.847	.874	.896	.914	.931	.944	.961	.976		
28	.632	.688	.741	.782	.826	.855	.883	.904	.921	.938	.950	.966	.980		
30	.636	.694	.748	.790	.834	.863	.891	.911	.928	.944	.955	.971			
32	.640	.700	.755	.798	.841	.870	.898	.918	.935	.949	.959	.976			
34	.644	.706	.761	.806	.848	.877	.904	.924	.941	.953	.963	.980			
36	.648	.712	.767	.814	.854	.883	.910	.930	.946	.957	.967				
38	.652	.717	.773	.821	.860	.887	.915	.935	.950	.966	.970				
40	.656	.722	.779	.827	.865	.894	.920	.939	.953	.964	.973				
42	.660	.727	.784	.832	.870	.899	.925	.943	.956	.967	.977				
44	.664	.732	.789	.837	.875	.904	.929	.947	.959	.970	.980				
46	.668	.736	.794	.842	.880	.908	.933	.950	.962	.973					
48	.672	.740	.799	.846	.885	.912	.937	.953	.965	.976					
50	.676	.744	.804	.850	.889	.916	.941	.956	.968	.978					
52	.679	.748	.809	.854	.893	.920	.944	.959	.971	.980					
54	.682	.752	.813	.858	.897	.924	.947	.961	.973						
56	.685	.756	.817	.862	.901	.927	.950	.963	.975						
58	.688	.760	.821	.866	.905	.930	.953	.965	.977						
60	.691	.764	.825	.870	.909	.933	.955	.967	.979						
62	.694	.768	.829	.874	.912	.936	.957	.969	.980						
64	.697	.771	.833	.878	.915	.939	.959	.971							
66	.699	.774	.837	.882	.918	.842	.961	.973							
68	.702	.777	.840	.885	.921	.945	.963	.975							
72	.707	.783	.846	.891	.927	.950	.967	.978							
76	.711	.789	.852	.897	.931	.954	.969								
80	.715	.795	.858	.902	.935	.958	.971								

APPENDIX B

Subroutine to Compute Estimates of b, c and the Reliability

Name: SUBROUTINE WEIBL

Purpose: To find the maximum likelihood estimates of the scale and shape parameters in the Weibull distribution and the reliability at a given time, T.

Method: The Newton-Raphson procedure is used to find the maximum likelihood estimate of the shape parameter. The program uses Menon's estimate of the shape parameter as the initial estimate.

Calling Sequence:

CALL WEIBL(X, N, T, SHAPE, SCALE, RELI)

where X = array consisting of the sample values from the Weibull distribution

N = size of the sample

T = time

SHAPE = maximum likelihood estimate of the shape parameter

SCALE = maximum likelihood estimate of the scale parameter

RELI = maximum likelihood estimate of the reliability at time T

Program:

```
    SSLNX = 0.0
    SLNX = 0.0
    DO 3 I=1, N
    ALNX(I) = ALOG(X(I))
    SLNX = SLNX + ALNX(I)
    W(I) = ALNX(I)*ALNX(I)
  3 SSLNX = SSLNX + W(I)
    AVLX = SLNX/N
```

```
      BEST = .3183099*SQRT(6.*(SSLNX-SLNX*AVLX)/(N-1.))
      SHAPE = 1./BEST
      SHAPE = SHAPE - .005
  306 SH = SHAPE
      SLXB = 0.0
      SXB = 0.0
      SLX2 = 0.0
      DO 10 K=1, N
      WP = X(K)**SH
      SLXB = SLXB + ALNX(K)*WP
      SXB = SXB + WP
   10 SLX2 = SLX2 + WP*W(K)
      Y = 1./SH + AVLX - SLXB/SXB
      YP = -1./SH**2 - (SXB*SLX2 - SLXB**2)/(SXB**2)
      SHAPE = SH - Y/YP
      IF(ABS(SHAPE - SH) - .00005) 499, 499, 306
  499 SXB = 0.0
      DO 12 K=1, N
      WP = X(K)**SHAPE
   12  SXB = SXB + WP
      SCALE = (SXB/N)**(1./SHAPE)
      RELI = EXP(-(T/SCALE)**SHAPE)
      RETURN
```

VITA

Darrel Ray Thoman was born September 26, 1938 at Hebron, Nebraska. He graduated from Hebron High School in May 1956 and entered Hastings College, Hastings, Nebraska. In May 1960 he graduated Cum Laude receiving a Bachelor of Arts degree with majors in Mathematics and Physics. From September 1960 to August 1962 he attended the University of Kansas and graduated Phi Beta Kappa with a Master of Arts degree in Mathematics. In September 1962 he assumed the position of Instructor of Mathematics at William Jewell College at Liberty, Missouri. During the summers of 1964 and 1965 he was granted N.S.F. fellowships to attend institutes in Computer Science at the University of Missouri at Rolla and in Statistics at Oklahoma State University. Additional graduate work was also taken at the University of Missouri at Kansas City. In January 1966 he took a leave of absence from his position as Assistant Professor of Mathematics at William Jewell College to continue graduate studies at the University of Missouri at Rolla. From September 1966 to September 1967 he was an N.S.F. Science Faculty Fellow. With the exception of this period he has been employed as Instructor of Mathematics at the University of Missouri at Rolla from January 1966 to the present.

On August 30, 1959, he was married to the former Harriet E. Taylor of Hastings, Nebraska.