05 Apr 1995, 1:30 pm - 3:30 pm

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John P. Wolf

Swiss Federal Institute of Technology, Switzerland

Chongmin Song

Swiss Federal Institute of Technology, Switzerland

Proceedings: Third International Conference on Recent Advances in Geotechnical Earthquake Engineering and Soil Dynamics,
April 2–7, 1995, Volume I, St. Louis, Missouri

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Paper No. 5.49

John P. Wolf and Chongmin Song
Institute of Hydraulics and Energy, Department of Civil Engineering, Swiss Federal Institute of Technology Lausanne, CH-1015 Lausanne, Switzerland

SYNOPSIS To calculate the unit-impulse response matrix of an unbounded medium for use in a time-domain analysis of medium-structure interaction, the *consistent* infinitesimal finite-element cell method is developed. Its derivation is based on the finite-element formulation and on similarity. The limit of the cell width is performed analytically yielding a rigorous representation in the radial direction. The discretization is only performed on the structure-medium interface. Explicit expressions of the coefficient matrices for the in-plane motion of anisotropic material are specified. In contrast to the boundary-element formulation, no fundamental solution is necessary and equilibrium and compatibility on the layer interfaces extending from the structure-medium interface to infinity, if present, are incorporated automatically. Excellent accuracy is achieved for an inhomogeneous semi-infinite wedge and a rectangular foundation embedded in an inhomogeneous half-plane.

INTRODUCTION

To analyze dynamic unbounded medium-structure interaction by the substructure method in the time domain, the unit-impulse response matrix of the unbounded medium on the structure-medium interface has to be determined before a transient can be processed. As an alternative to the boundary-element method, which applies an analytical solution incorporating the radiation condition at infinity, the infinitesimal finite-element cell method based solely on the finite-element formulation has been developed in Wolf and Song (1994b, 1994c), which is summarized in the next paragraph.

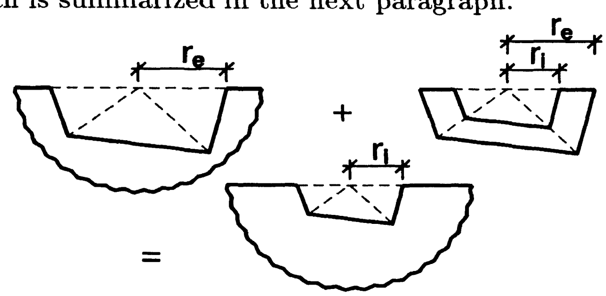

Fig. 1. Fundamental Concept of Infinitesimal Finite-Element Cell Method

The fundamental idea of the infinitesimal finite-element cell method is illustrated in Fig. 1 for the unbounded (semi-infinite) medium taking the irregular structure-medium interface into account. Adding the bounded cell of finite elements to the unbounded medium with the characteristic length r_e (exterior boundary) results in a similar unbounded medium with length r_i (interior boundary). This concept can be applied to their dynamic-stiffness matrices in the frequency domain (Dasgupta (1982)). Assembling the known static-stiffness and mass matrices of the cell and the unknown dynamic-stiffness matrix of the unbounded medium characterized by the length r_e results in the unknown dynamic-stiffness matrix of the unbounded medium with length r_i. As a relationship for the dynamic-stiffness matrices of similar unbounded media characterized by different lengths exists, the infinitesimal finite-element cell method leads to an expression for the dynamic-stiffness matrix of the unbounded medium as a function of the dynamic-stiffness matrix of the cell. Analogously, this applies also to the unit-impulse response matrix after performing the inverse Fourier transformation. This method is a *stand-alone finite-element formulation capable of capturing the radiation condition at infinity* without using analytical solutions. For problems with a boundary extending from the structure-medium interface to infinity such as a half-space with a free surface, this novel method automatically incorporates this boundary condition in contrast to the boundary-element method. Material inhomogeneities which satisfy similarity can be processed without any additional effort. The infinitesimal finite-element cell method can also calculate problems for which the fundamental solution (which is necessary for the boundary-element method) does not exist in closed form. This is, for example, the case for certain anisotropic materials. Only the conventional static-stiffness and mass matrices of the bounded finite-element cell need to be calculated, which are then used in standard matrix operations to obtain the unit-impulse response matrix of the unbounded medium.

In the infinitesimal finite-element cell method in Wolf and Song (1994b, 1994c) the non-dimensionalized cell width measured in the radial direction is selected as a very small number for which the computer will provide reliable results. It is the goal of this paper to perform the limit of the cell width analytically for the in-plane motion. The formulation of this *consistent* infinitesimal finite-element cell method will then depend only on the geometry of the structure-medium interface and on the material properties of the unbounded medium. The resulting formulation is rigorous in the radial direction and ***converges to the exact solution in the finite-element sense*** in the circumferential direction.

STATIC-STIFFNESS AND MASS MATRICES OF INFINITESIMAL FINITE-ELEMENT CELL

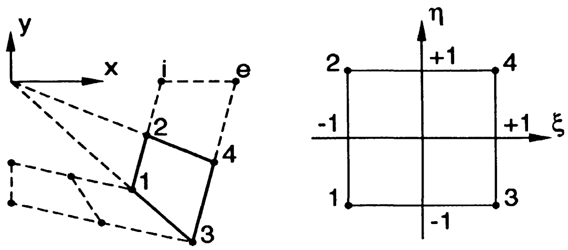

Fig. 2. Quadrilateral Finite Element of Cell with Similarity and Parent Element

For simplicity it is assumed that the cell consists of quadrilateral finite elements. One such element (Fig. 2) is addressed. Due to similarity, the coordinates of the nodes 3 and 4 on the exterior boundary can be expressed by those of the nodes 1 and 2 on the interior boundary as

$$x_3 = (1+w)x_1; \qquad y_3 = (1+w)y_1$$
$$x_4 = (1+w)x_2; \qquad y_4 = (1+w)y_2 \tag{1}$$

with the dimensionless cell width w. The following abbreviations are introduced.

$$a = x_1 y_2 - x_2 y_1 \tag{2}$$

$$\Delta_x = x_2 - x_1; \qquad \Delta_y = y_2 - y_1 \tag{3}$$

$$\overline{x} = \frac{x_1 + x_2}{2}; \qquad \overline{y} = \frac{y_1 + y_2}{2} \tag{4}$$

$$[C_1] = \begin{bmatrix} \Delta_y & 0 \\ 0 & -\Delta_x \\ -\Delta_x & \Delta_y \end{bmatrix}; \qquad [C_2] = \begin{bmatrix} \overline{y} & 0 \\ 0 & -\overline{x} \\ -\overline{x} & \overline{y} \end{bmatrix} \tag{5}$$

Using standard finite-element procedures, the submatrices of the in-plane static-stiffness matrix $[K]$ can be formulated as

$$[K]_{jk} \approx \frac{1}{w}[K_0]_{jk} + [K_1]_{jk} + w[K_2]_{jk} \qquad (j,k = 1,\ldots,4) \tag{6}$$

where

$$[K_0]_{jk} = \xi_j \xi_k \left(1 + \frac{1}{3}\eta_j \eta_k\right)[Q_0] \tag{7}$$

$$[K_1]_{jk} = [K_0]_{jk} - \frac{1}{6}(\xi_j + \xi_k)\,\eta_j \eta_k [Q_0] + \xi_k \eta_j [Q_1] + \xi_j \eta_k [Q_1]^T \tag{8}$$

$$[K_2]_{jk} = \left(1 + \frac{1}{3}\xi_j \xi_k\right)\eta_j \eta_k \left(\frac{1}{12}[Q_0] + [Q_2]\right) \tag{9}$$

with

$$[Q_0] = \frac{1}{4|a|}[C_1]^T[D][C_1]; \quad [Q_1] = \frac{-1}{4|a|}[C_2]^T[D][C_1]$$
$$[Q_2] = \frac{1}{4|a|}[C_2]^T[D][C_2] \tag{10}$$

where $[D]$ is the stress-strain relationship of the, in general, anisotropic material.

The submatrices of the mass matrix are formulated as

$$[M]_{jk} \approx w[M_0]_{jk} \qquad (j,k = 1,\ldots,4) \tag{11}$$

where (unit matrix $[I]$, mass density ρ)

$$[M_0]_{jk} = \frac{\rho|a|}{16}\left(1 + \frac{1}{3}\xi_j \xi_k\right)\left(1 + \frac{1}{3}\eta_j \eta_k\right)[I] \tag{12}$$

In the consistent infinitesimal finite-element cell method it is only necessary to calculate the following coefficient matrices:

$$[F] = [K_0]_{ii} = \frac{2}{3}\begin{bmatrix} 2[Q_0] & [Q_0] \\ [Q_0] & 2[Q_0] \end{bmatrix} \tag{13}$$

$$[B_1] = [K_1]_{ie} + [K_1]_{ee} = \frac{1}{3}\begin{bmatrix} -[Q_0] & [Q_0] \\ [Q_0] & -[Q_0] \end{bmatrix} + 2\begin{bmatrix} -[Q_1] & -[Q_1] \\ [Q_1] & [Q_1] \end{bmatrix} \tag{14}$$

$$[K_2]_{sum} = [K_2]_{ii} + [K_2]_{ie} + [K_2]_{ei} + [K_2]_{ee} = \frac{1}{3}\begin{bmatrix} [Q_0] & -[Q_0] \\ -[Q_0] & [Q_0] \end{bmatrix} + 4\begin{bmatrix} [Q_2] & -[Q_2] \\ -[Q_2] & [Q_2] \end{bmatrix} \tag{15}$$

$$[B_4] = [M_0]_{ii} + [M_0]_{ie} + [M_0]_{ei} + [M_0]_{ee} = \frac{\rho|a|}{6}\begin{bmatrix} 2[I] & [I] \\ [I] & 2[I] \end{bmatrix} \tag{16}$$

In the above derivation, the matrix $[D]$ is in its most general form, which allows a fully anisotropic material to be analyzed.

As for the static-stiffness and mass matrices, the coefficient matrices of the cell can be determined by assembling those of the finite elements. To simplify the nomenclature, the same symbols are used for the assembled coefficient matrices in the following.

CONSISTENT INFINITESIMAL FINITE-ELEMENT CELL EQUATION

Proceeding as described in Section 2.3 of Wolf and Song (1994c) and after taking the limit with respect to the cell width results in the consistent infinitesimal finite-element cell equation

$$\begin{aligned} &\int_0^t [m^\infty(t-\tau)][m^\infty(\tau)]d\tau + [b_1]\int_0^t \int_0^\tau [m^\infty(\tau')]d\tau' d\tau \\ &+ \int_0^t \int_0^\tau [m^\infty(\tau')]d\tau' d\tau [b_2] + t\int_0^t [m^\infty(\tau)]d\tau \\ &- \frac{t^3}{6}[b_3]H(t) - t[b_4]H(t) = 0 \end{aligned} \tag{17}$$

where the coefficient matrices are equal to

$$[b_1] = [F]^{-1}[B_1]; \quad [b_2] = [F]^{-1}[B_1]^T - (s+1)[I]$$
$$[b_3] = -[F]^{-1}[B_1][F]^{-1}[B_1]^T + [F]^{-1}[K_2]_{sum} \tag{18}$$
$$[b_4] = [F]^{-1}[B_4]$$

with the spatial dimension s=2 or 3. This equation corresponds to Eq. (37) of Wolf and Song (1994c), where the time discretization is also addressed. For each time step a system of linear equations is solved. After determining $[m^\infty(t)]$ from (17), the acceleration unit-impulse response matrix follows as

$$[M^\infty(t)] = [F][m^\infty(t)] \tag{19}$$

EXAMPLES

The same geometry as for the examples taking the numerical limit in Wolf and Song (1994c) are used in the following demonstration where the analytical limit of the cell width is performed. Only isotropic examples with inhomogeneity are addressed.

Semi-Infinite Wedge

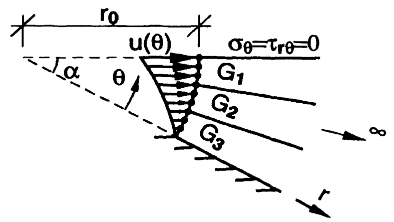

Fig. 3. Inhomogeneous Semi-Infinite Wedge with Prescribed Linear Displacement

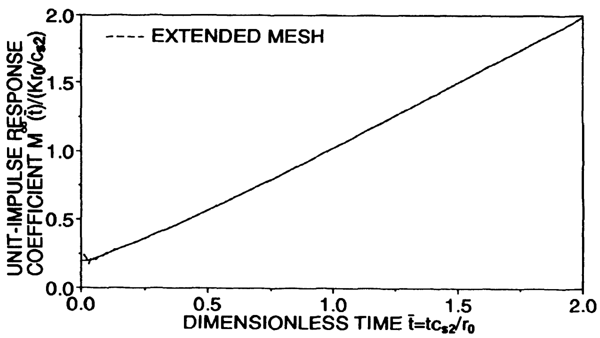

Fig. 4. Nondimensionalized Acceleration Unit-Impulse Response Coefficient of Inhomogeneous Semi-Infinite Wedge

The in-plane motion of a wedge with an opening angle $\alpha = 30^\circ$ and Poisson's ratio $\nu = 0.25$ and with a free and a fixed boundary extending in the radial direction to infinity is addressed (Fig. 3). Three regions of different shear moduli with a soft inner part compatible with similarity is examined ($G_1/G_2 = G_3/G_2 = 10$). On the structure-medium interface 9 elements of equal length are chosen. The unit-impulse response matrix $[M^\infty(t)]$ with respect to the 9 nodes is calculated. To ease the comparison, a linear function of the horizontal displacement in the circumferential direction θ on the arc and zero vertical displacement are prescribed, and the corresponding equivalent unit-impulse response coefficient $M^\infty(t)$ is determined by integrating the nodal forces of the finite elements with the displacement as a weighting function over the structure-medium interface. The time step is selected as $\Delta t = 0.01 r_0/c_s$. $M^\infty(\bar{t})$ calculated with the consistent infinitesimal finite-element cell method agrees well with the result determined with an extended mesh of finite elements with the element length in the radial direction $=0.025 r_0$ and the same time step Δt (Fig. 4 where K is the static-stiffness coefficient, and $\bar{t} = tc_{s2}/r_0$ is the dimensionless time with $c_{s2} = \sqrt{G_2/\rho}$). 230 rows of finite elements are necessary for $\bar{t}$ up to 2. The dynamic-stiffness coefficient $S^\infty(a_0)$ is also calculated from $M^\infty(t)$ by applying the Fourier transformation as described in Wolf and Song (1994c) for comparison. The non-dimensional spring and damping coefficients $k(a_0)$ and $c(a_0)$ shown in Fig. 5 compare very well with the accurate values using the so-called dynamic condensation method (Wolf and Song (1994a)).

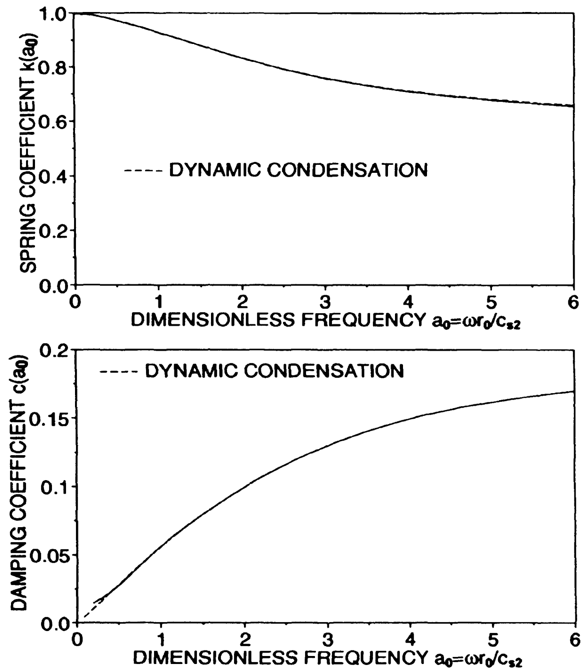

Fig. 5. Dynamic-Stiffness Coefficient in Frequency Domain of Inhomogeneous Semi-Infinite Wedge

Rectangular Foundation Embedded in Half-Plane

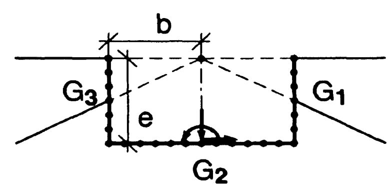

Fig. 6. Rectangular Foundation Embedded in Inhomogeneous Half-Plane

A rectangular foundation embedded in an inhomogeneous half-plane with $G_2/G_1 = G_2/G_3 = 4$ and Poisson's ratio $\nu = 0.25$ and with a ratio of the embedded depth to the half-width $e/b = 1$ is evaluated (Fig. 6). On the structure-medium interface 24 elements are present. The time step is selected as $\Delta t = 0.03b/c_{s1}$ ($c_{s1} = \sqrt{G_1/\rho}$). After determining the unit-impulse response matrix $[M^\infty(t)]$ of order 50 × 50, the rigid-body constraint is introduced to calculate the equivalent coefficients corresponding to the horizontal, vertical and rocking motions. As for the wedge, the equivalent unit-impulse response coefficients are transformed into the frequency domain. The resulting dynamic-stiffness coefficients in the frequency domain non-dimensionalized for the translational motions with G and for the rocking motion with Gb^2 agree well, as shown in Fig. 7, with the results of dynamic condensation.

CONCLUSIONS

Performing the limit of the cell width analytically yields the consistent infinitesimal finite-element cell method which is rigorous in the radial direction. The discretization is performed only to the structure-medium interface, resulting in a reduction of the spatial dimension by 1. The coefficient matrices depend on the geometry of the structure-medium interface and on the material properties of the unbounded medium. Excellent accuracy results even for the inhomogeneous case compatible with similarity, whereby equilibrium and compatibility on the layer interfaces extending from the structure-medium interface to infinity are incorporated automatically.

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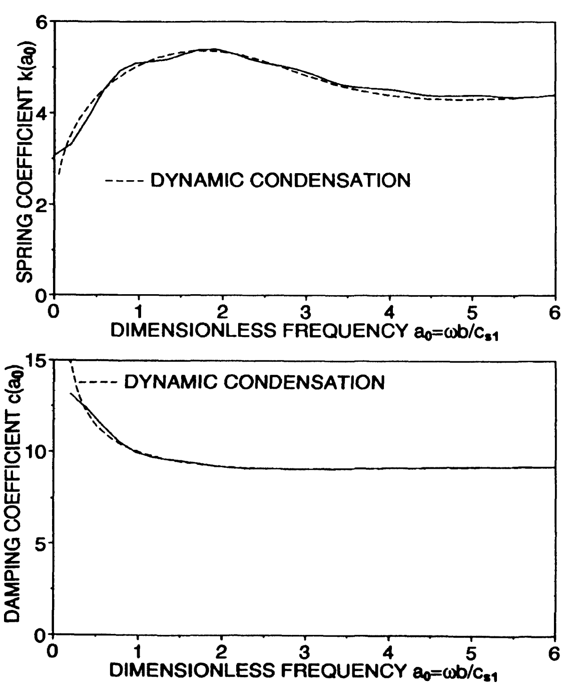
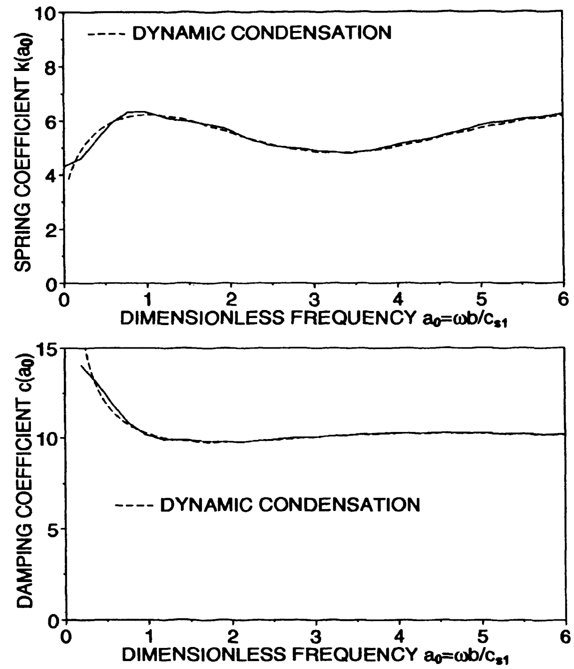
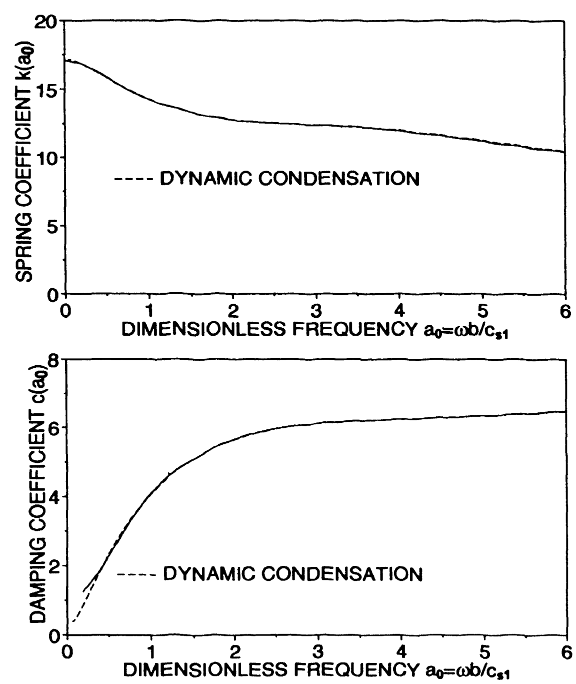


a) Horizontal



b) Vertical



c) Rocking

Fig. 7. Dynamic-Stiffness Coefficient in Frequency Domain of Rigid Rectangular Foundation Embedded in Inhomogeneous Half-Plane