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Extension of some theorems of complex functional analysis to linear spaces over the quaternions and Cayley numbers

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Abstract

In this work certain aspects of Functional Analysis are considered in the setting of linear spaces over the division rings of the real Quaternions and the real Cayley algebra.

The basic structure of Banach spaces over these division rings and the rings of bounded operators on these spaces is developed. Examples of finite and infinite dimensional spaces over these division rings are given. Questions concerning linear functionals, the Hahn-Banach Theorem and Reflexivity are considered. The Stone-Weierstrass Theorem is proven for functions with values in a real Cayley Dickson algebra of dimension n.

The concepts of inner product spaces and Hilbert spaces over the Quaternions and the Cayley algebra are developed. An extensive study of Hilbert spaces over the Quaternions is carried out.

In the case of Hilbert spaces over the Quaternions, the Riesz-Representation Theorem and the Jordan-von Neumann Theorem are proven. In addition, spectral theorems for both self-adjoint and normal operators are proven for finite dimensional Hilbert spaces.

These results are extended to infinite dimensional spaces for the cases of compact self-adjoint operators and compact normal operators. The spectrum of an arbitrary bounded Hermitian operator on a Hilbert space over the Quaternions is shown to be non-void.

A generalization of the Fourier Transform for functions in $L^1_Q(-\infty,\infty)$ and $L^2_Q(-\infty,\infty)$ is given. The Plancherel Theorem is proven for functions in $L^2_Q(-\infty,\infty)$.

Finally, the Jordan-von Neumann theorem is proven for a Hilbert space over the Cayley algebra.

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I. Introduction

Theoretical physics has been the source of some of the most interesting and complex mathematical problems of this century. Nowhere is this more evident than in the discipline of Quantum Mechanics. It has been the motivation for such fundamental theories as the theory of Distributions and the theory of Non-associative algebras to name only two.

Since its initial development, the mathematical structure of Quantum Mechanics has been imbedded in Complex Hilbert Spaces. But as early as 1936 (Birkhoff, G. and von Neumann, J., 1936) it was conjectured that Complex Hilbert Spaces might not be the most natural setting for the theory. This idea was largely ignored by physicists and mathematicians alike for over 20 years. However, in 1962, J. M. Jauch and his coworkers published a paper in the Journal of Mathematical Physics on the Foundations of Quaternion Quantum Mechanics. This paper was followed by two others and was the impetus for much of the research being done today. Numerous other physicists have followed in their footsteps and among these are G. Emch (Emch, G., 1963) and V. S. Varadarajan (Varadarajan, V. S., 1968).

As a result of these investigations many questions concerning the structure of Hilbert spaces over the Quaternions have arisen. In addition, questions about Hilbert spaces over other non-commutative and non-associative scalars have also been raised.

Motivated by the work of Jauch, we have investigated in this paper some of the aspects of Quaternionic Hilbert spaces. Choosing not to restrict ourselves just to Hilbert spaces over the Quaternions we have considered questions of functional analysis in the setting of linear spaces over the division rings of the real Quaternions and the real Cayley numbers. In this study, those theorems of elementary functional analysis which depend primarily on the algebraic structure of the scalars have been of primary interest. Therefore, most of the generalizations are algebraic rather than topological.

II. Review of the Literature

O. Teichmüller attributed the idea of a Hilbert space over the division ring of real Quaternions to H. Wachs in a paper dealing with theory of such spaces, in 1938 (Teichmüller, O., 1938). According to Teichmüller, Wachs made the conjecture in 1934 that a theory of these spaces could be developed. In his paper in 1938, which was his Ph.D. dissertation at Göttingen, Teichmüller studied the properties of these spaces and operators on them.

Teichmüllers definition of a Hilbert space over Q required the axiom of separability. Among the topics discussed were the following: The basic properties of a Quaternion valued inner product, the proof of the Cauchy-Schwartz inequality, the Riesz representation theorem, the concept of dimension and the projection theorem. In addition he defined the concept of a an imaginary operator and discussed the "normal" form of such operators. He defined T to be imaginary if $T^* = -T$, $T^2 = -E$ where E is the identity operator. Using the results of spectral theory on real Hilbert spaces developed by Riesz, Rellich, and von Neumann, Teichmüller proved the following theorem which he called the fundamental theorem for normal operators.

<u>Theorem</u> Let N be a normal operator in a Wachsschen space R. Then there exists operators A, B, T_0 and a subspace R_1 with the properties,

(a) A and B are self adjoint operators, B is non-negative definite; R_1 is the closed Hull of the domain of B, T_0 is an imaginary operator from R_1 to R_1.

(b) Each spectral projector of A or B commutes with N and N*; R_1 reduces N and N*, and in R_1, T_0 commutes with N and N*.

(c) If C is a bounded operator and C commutes with N and N* then C commutes with A and B; R_1 reduces C, and the restriction of C to R_1 commutes with T_0.

(d) $N = A + T_0 B$

$N^* = A - T_0 B$

In the same period that Teichmuller was doing his work on Wachsschen spaces, E. H. Moore (Moore, E. H., 1935) was studying finite and infinite dimensional spaces over the real Quaternions. He introduced the concept of Quaternion valued inner product and considered the question of conoical forms and generalized inverses of matrices with Quaternion entries.

Beginning in 1962 there was a series of three papers by Jauch, Finklestein et. al., in the Journal of Mathematical Physics (Finklestein, D., Jauch, J., Schiminovich, S., and Speiser, D., 1962, 1963) in which the formulation of the mathematical structure of Quantum Mechanics was developed in the setting of a Hilbert space over the real Quaternions. Their definition of a Hilbert space over Q was essentially that of Teichmuller.

In the first paper the Cauchy Schwartz inequality was proven and the results of the projection theorem were used although it was not stated explicitly. Their basic consideration was to determine the appropriate analog of the Schrodinger equation in the case under consideration. A generalized version of Stone's Theorem and the representation of semi-groups of operators on a Hilbert space was given and the Schrodinger equation was obtained in the usual way from this result. In addition, they also attempted to assign a physical significance to the new symmetries induced on the linear space by the automorphisms of the Quaternions.

In the second paper they considered the representation of compact groups by matrices with real Quaternion entries. In addition to proving certain standard theorems about group representations, they developed a criterion under which such representations reduce over the skew field of real Quaternions. The criterion was given in terms of the Schur-Frobenius classification of groups.

The last paper in the series was primarily a consideration of the physical consequences of the concept of

Q-covariance which is developed in this paper. In addition, they show that in the case of a Hilbert space over the Quaternions, the lattice of closed subspaces has a symmetry group which is connected, whereas it is not for the case of complex Hilbert space.

S. Natarajan and K. Viswanath (Natarajan, S. and Viswanath, K., 1969) also considered the question of Quaternionic group representations. They considered only compact metric groups. A generalized Peter-Weyl Theorem was proven and the problem of finding all irreducible representations of an arbitrary compact metric group was considered. It was shown that in the case of Abelian groups that the representations are all one dimensional.

Emch, has studied the structure of a Quaternionic Relativistic Quantum Mechanics and the representations of the Lorentz Group. (Emch, G. 1963). He utilized the mathematical tools developed by Jauch and his co-workers.

Other Physicists have considered aspects of Quaternionic Quantum Mechanics. Bargmann has given a proof of Wigner's Theorem on Symmetry Operations for the case of Quaternionic Hilbert Space. (Bargmann, V. 1964).

The study of matrices with Quaternion entries has relevance to this area of research since a linear transformation on a finite dimensional linear space over Q can be represented by a matrix with Quaternion entries. Since 1936 there have been several papers on such matrices but none of these have been framed in the context of linear operators on linear spaces over the Quaternions.

One of the first works was by Wolf in 1936 (Wolf, L., 1936). In this paper necessary and sufficient conditions are given under which, for two matrices A and B, there exists a nonsingular matrix S such that $SAS^{-1} = B$. The criterion for non-singularity of Quaternion matrices is that of Moore, (Moore, E. H., 1935).

The next paper in this area was by Lee in 1949, (Lee, H. C., 1949). In this work Lee considered the eigenvalue problem for Quaternion matrices and the related question of canonical forms. Using the symplectic representation, Lee proved that an n x n matrix with real Quaternion entries has 2n complex eigenvalues. The usefulness of this work is somewhat restricted by the fact that Lee requires the eigenvalues to be complex and not general Quaternions.

Using the results and methods of Lee, Wiegmann (Wiegmann, N. A., 1955) obtained additional results about canonical forms for matrices with Quaternion entries. Among the topics considered were the Jordan Canonical form, similarity for Quaternion matrices, polar forms, and finally the concept of unitary equivalence.

In two later papers, Wiegmann (Wiegmann, N. A., 1955, 1956) determined the structure of unitary and orthogonal matrices.

The most relevant result in the area of Quaternionic matrices is due to J. L. Brenner (Brenner, J. L., 1951). Brenner proved that every Quaternion matrix has at least one eigenvalue. In the proof of this result, Brenner had to appeal to a very important theorem due to Eilenberg and Niven which is the Fundametal Theorem of algebra for the Quaternions. (Eilenberg, S. and Niven, I., 1944).

One of the first papers in functional analysis dealing with linear spaces over the Quaternions was by Soukhomlinov (Soukhamlinov, G. A., 1938). In this paper Soukhomlinov proved the analog of the Hahn-Banach theorem for linear spaces over Q as well as the complexes.

A generalization of the Soukhamlinov theorem was given by T. Ono (Ono, T., 1953). Ono proved the extension theorem for semi-linear functionals on linear spaces over the Quaternions. A semi-linear functional f is additive but $f(x \cdot a) = f(x) \cdot a'$ where a' is an automorphism of a.

There have been numerous other generalizations of the Hahn-Banach theorem. One of the more interesting was by Harte (Harte, R. E., 1965). Harte proved an extension theorem of the Hahn-Banach type for linear functionals, on a linear space over a Banach algebra, which take their values in the dual space of the algebra.

Another paper along these lines is by Bonsall and Goldie (Bonsall, F. and Goldie, A., 1953). They proved an extension theorem for linear functionals on linear spaces over algebras which "represent their linear functionals".

Numerous other generalizations in this area have been given and among these are the papers by G. Vincent Smith (Vincent-Smith, G., 1965), A. W. Ingleton (Ingleton, A. W., 1952), and A. F. Monna (Monna, A. F., 1946).

The "Stone-Weierstrass Theorem" has been proven for continuous Quaternion valued functions on a compact

Hausdorff space by Holladay (Holladay, J. C., 1957).

One of the more interesting examples of a linear space over the Quaternions was studied by Fueter in a series of papers beginning in 1936, (Fueter, R., 1936). Fueter defined the concept of an analytic function of Quaternion variable. Although he did not base the concept of analyticity on the derivative of a function, he did obtain many results analogous to those of complex function theory.

The first paper on linear spaces over Cayley numbers was by Goldstine and Horwitz (Goldstine, H. H. and Horwitz, L. P.). They gave a definition of a linear space over Cayley numbers and defined an inner product with values in the Cayley numbers. In addition to the usual postulates for a real valued inner product, they assumed that (i) $(ax,x) = a(x,x)$, (ii) $(ax,y) = a(x,y)$ for a real, (iii) $(x,y) = \overline{(y,x)}$ and (iv) $Re[(ax,y)] = Re[a(x,y)]$. Since this inner product is not homogeneous with respect to general Cayley numbers they defined a new product (,) given by $(x,y) = Re[(x,y)]$. Using the latter inner product they reproved or simply stated many well known results for real Hilbert spaces. Their definition of a subspace was simply a real subspace. Using known results from spectral theory in real Hilbert spaces they obtained a spectral resolution for elements of the Cayley Operator group, a concept defined in the paper.

In a second paper Goldstine and Horwitz (Goldstine, H. H., and Horwitz, L. P., 1966) study the concept of a Hilbert space over a finite dimensional associative algebra. The spectral resolution for a bounded Hermitian operator is developed and theory of preceeding paper is shown to be a special case of the theory in this paper.

Horwitz and Biedenharn (Horwitz, L. P., and Biedenharn, L. C., 1965) study the structure of a Quantum Theory described by a Hilbert space over an arbitrary associative algebra with a unit. They show that the minimal ideals of the algebra play a role analagous to the bases in a complex Hilbert space.

Saworotnow (Saworotnow, P. P., 1968) generalizes the spaces studied by Horwitz and Biedenharn. Saworotnow studies the concept of Hilbert modules, which is a module over a real H* algebra. He shows that the scalars considered by Horwitz and Biedenharn are a special case of an H* algebra and hence his theory is more general.

III. Definitions and Terminology

In this Chapter the basic definitions and terminology that are to be used throughout this paper are given.

A. Rings and algebras

<u>Definition</u> A ring consists of a set $R \equiv \{a,b,c,\ldots,\}$ together with two binary operations $+,\cdot$. In addition the following properties are satisfied.

(a) With respect to $+$ R is an abelian group with the neutral element denoted by 0. The element inverse to $x \in R$ is denoted by $-x$.

(b) For any three elements a, b, and c in R

$$a\cdot(b+c) = a\cdot b + a\cdot c$$
$$(b+c)\cdot a = ba + ca.$$

<u>Definition</u> Let R be a ring. The commutator $[a,b]$ of any two elements of R is defined as $[a,b] = ab-ba$. The associator of any three elements of R is defined as $(a,b,c) = (ab)c - a(bc)$. If $[\,,\,]$ vanishes identically on R, R is commutative. If $(\;,\;,\;)$ vanishes identically on R, R is associative.

<u>Definition</u> An algebra A over a field F is a ring which is also a linear space over F and the following is satisfied.

(a) $\alpha \cdot a = a \cdot \alpha \ \forall \ a \in A, \ \alpha \in F$

(b) $(\alpha \cdot a) \cdot b = a \cdot (\alpha \cdot b) \ \forall \ a,b \in A, \ \alpha \in F$

Definition Let A be an algebra over a field F. A is called an algebra with involution if there exists an operation "-" on A satisfying the properties for all a,b $\in$ A and $\alpha \in F$.

$$\overline{(a+b)} = \overline{a} + \overline{b}$$

$$\overline{(a \cdot b)} = \overline{b} \cdot \overline{a}$$

$$\overline{(\overline{a})} = a$$

$$\overline{\alpha \cdot a} = \alpha \cdot \overline{a}.$$

Throughout this paper, the various algebras discussed will be assumed to have the reals as the underlying field.

B. The Real Quaternions and Real Cayley numbers.

1. algebraic aspects.

The Real Quaternions (Cayley numbers) form an algebra over the real field of dimension 4(8). If the basis elements are denoted by $\{e_0, e_1, e_2, e_3\}$ ($\{e_0, e_1, e_2, \ldots, e_7\}$) the multiplication table for the algebra is given by:

Quaternion algebra.

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	$-e_0$	e_3	$-e_2$
e_2	e_2	$-e_3$	$-e_0$	e_1
e_3	e_3	e_2	$-e_1$	$-e_0$

Cayley algebra.

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

From the multiplication table, it is clear that the Quaternion algebra (Cayley algebra) is associative but non-commutative (non-commutative and non-associative). It is also clear that each algebra has a unit, e_0.

Each Quaternion (Cayley-number) "a" can be expressed as $a = a_0e_0 + \sum_{k=1}^{3} a_ke_k$ ($a = a_0e_0 + \sum_{k=1}^{7} a_ke_k$) where the a_j are real numbers. Each Quaternion (Cayley number) "a" has an involution which is given by

$$\bar{a} = a_0e_0 - \sum_{k=1}^{3} a_ke_k \quad (\bar{a} = a_0e_0 - \sum_{k=1}^{7} a_ke_k).$$

The "trace" of a Quaternion (Cayley number) "a" is defined as $t(a) = a + \bar{a}$. It easily follows that if $a = a_0e_0 + \sum_{k=1}^{3} a_ke_k$ then $2a_j = -t(e_ja)$ $j = 1,2,3$, and $2a_0 = a + \bar{a}$. Also, the following theorem can easily be proven.

<u>Theorem 1</u>. If a is any Quaternion with $a = a_0e_0 + \sum_{k=1}^{3} a_ke_k$. Then

(i) $\sum_{k=0}^{3} e_kae_k = -2\bar{a}$

(ii) $\sum_{k=0}^{3} e_ka\bar{e}_k = 2t(a)$.

The (algebraic) norm of any Quaternion (Cayley number) "a" is defined by $n(a) = a\bar{a} = \bar{a}a$. Clearly, $n(a) = (\sum_{k=0}^{3} a_k^2)e_0$, ($n(a) = (\sum_{k=0}^{7} a_k^2)e_0$).

Every Quaternion (Cayley number) "a" possesses an inverse defined by $a^{-1} = \frac{\bar{a}}{n(a)}$. As a consequence of the existence of inverses the Quaternions (Cayley numbers) form a non-commutative (non-commutative and non-associative) division algebra (Kurosh, 1960).

Although the Cayley algebra is non-associative it does have the important property (Kurosh, 1960) that $(a,a,b) = (b,a,a) = 0$ for any two Cayley numbers a,b. In addition it can be shown that $(a^{-1},a,b) = (a,a^{-1},b) = 0$. Such algebras are called alternative. In fact, it turns out that every alternative division ring is either associative or an eight dimensional Cayley algebra over its center. (The center is the subset of the algebra which commutes and associates with every element in the algebra. The center of the algebras considered in this paper is always the reals.)

<u>Theorem 2</u>. Let a be any quaternion such that $n(a) = 1$. Then there exists a real number θ and a quaternion I such that $I^2 = -e_0$ and $a = \cos\theta\, e_0 + \sin\theta\, I$.

Proof. Since $a = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3$ and

$$a_0^2+a_1^2+a_3^2+a_4^2 = 1, \text{ one can take } \theta = \tan^{-1} \frac{(a_1^2+a_2^2+a_3^2)^{1/2}}{a_0}$$

and I to be $I = \frac{a_1e_1+a_2e_2+a_3e_3}{(a_1^2+a_2^2+a_3^2)^{1/2}}$. Then it follows

easily that $a = \cos\theta\, e_0 + \sin\theta\, I$.

<u>Corollary</u> Any quaternion a can be written as $a = r(\cos\theta\, e_0 + \sin\theta\, I)$, where r, θ are real numbers, I a quaternion such that $I^2 = -e_0$. I, θ, r all depend on a.

<u>Definition</u> Let I be any quaternion such that $I^2 = -e_0$. Let θ be any real number, then $e^{I\theta}$ is defined as $e^{I\theta} = \cos\theta\, e_0 + \sin\theta\, I$. It is easily verified that $e^{I\theta}$ has the following properties.

(a) $e^{I\theta_1} \cdot e^{I\theta_2} = e^{I(\theta_1+\theta_2)}$

(b) $e^{I\cdot\theta} = e_0$

(c) $\overline{e^{I\theta_1}} = e^{-I\theta_1}$

(d) $n(e^{I\theta}) = 1$.

2. Topological properties.

Since both the Quaternions and the Cayley numbers are algebras over the reals they are linear spaces over the reals. Moreover, they are both finite dimensional.

The Quaternions and the Cayley numbers both form normed linear spaces over the reals. If a is any

Quaternion (Cayley number) the norm (topological) of a is defined as $|a| = [n(a)]^{1/2}$. It is easy to verify that "$|\ |$" satisfies the properties of a norm. Using the completeness of the reals the following theorem can easily be proven.

Theorem 1. With respect to "$|\ |$", the Quaternions (Cayley numbers) form a Banach Space.

If a,b are any two Quaternions (Cayley numbers) then (a,b) is defined to be $(a,b) = \frac{1}{2}t(a\bar{b})$. It is elementary to show that (,) has the property of a real inner product. Moreover, since $(a,a) = \frac{1}{2}t(a\bar{a}) = |a|^2$ the following is true.

Theorem 2. With respect to (,) the Quaternions (Cayley numbers) form a real Hilbert Space.

C. The Real Cayley-Dickson algebras.

If A is an algebra over a field F of dimension n, a new algebra B of dimension 2n can be constructed over the same field by a process known as the Cayley-Dickson process (Schafer, 1966). Algebras constructed in this manner are called Cayley-Dickson algebras. Although only the real field will be used in this paper, it should be noted that the process is valid for arbitrary base field (Schafer, 1966) and more generally for algebras over a commutative, associative ring (Penico, 1968).

The process will now be described. Let A be an algebra of dimension n over the reals with unit 1 and involution "-". Define B to be the set of all ordered pairs (a_1,a_2) with $a_1,a_2 \in A$.

If $b = (a_1,a_2)$ is any element of B and α is any real number, define $\alpha \cdot b = (\alpha \cdot a_1, \alpha \cdot a_2)$. If $b' = (a_3,a_4)$ is any other element of B then define $b+b' = (a_1+a_3, a_2+a_4)$ and $b \cdot b' = (a_1a_3+\mu \cdot a_4 \cdot \bar{a}_2, \bar{a}_1a_4+a_3a_2)$ where μ is a real number different from 0. B now becomes an algebra with respect to the operations of scalar multiplication, $+,\cdot$. The identity for B is the element (1,0). Moreover, the sub-algebra $B' = \{(a,0) \mid a \in A\}$ is isomorphic to A. The element $e = (0,1)$ is an element of B with the property $e^2 = \mu(1,0) = \mu \cdot 1$, 1 being the identity of B. If the elements of A are identified with the elements of B' then every $b \in B$ can be expressed as $b = a_1 + ea_2$. Multiplication is then given by $(a_1+ea_2)(a_3+ea_4) = (a_1a_3+\mu a_4\bar{a}_2) + e(\bar{a}_1a_4+a_3a_2)$. An involution can be defined in B as follows: For $b \in B$, $b = a_1 + ea_2$, define $\bar{b} = \bar{a}_1 - ea_2$. It is easily shown that $^-$ satisfies the requirements of an involution.

It can be shown (Schafer, 1966) that the new algebra is alternative if and only if the initial algebra is associative.

The Quaternions and Cayley numbers arise naturally from this process. If the initial algebra A is the complex numbers and μ is taken to be $\mu = -1$, the resulting algebra B is the algebra of real Quaternions. If A is the real Quaternions and μ is -1 then B is the real Cayley algebra. Hereafter, the following notations will be used for the various number systems.

Re - real numbers

K - complex numbers

Q - Quaternions

C - Cayley numbers.

D. Linear Spaces Over the Real Quaternions.

Definition A (right) linear space over Q is an additive abelian group in which there is defined an operation of scalar multiplication by elements of Q. Scalar multiplication is assumed to obey the following laws for all $x,y \in L$, $a,b \in Q$.

(i) $(x+y)\cdot a = x\cdot a + y\cdot a$ (iv) $x\cdot e_0 = x$

(ii) $x(a+b) = x\cdot a + x\cdot b$

(iii) $x\cdot(a\cdot b) = (xa)\cdot b$

A left space is defined similarly. The choice of scalar multiplication is more or less arbitrary.

Example 1. Let $L = \{ (x_1, x_2, \ldots, x_n) \mid x_j \in Q \}$. If x, y are any two elements of L with $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ then $x \cdot \alpha = (x_1 \cdot \alpha, x_2 \cdot \alpha, \ldots, x_n \cdot \alpha)$ $x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$.

Example 2. Let X be any nonvoid set and $F(X)$ the collection of all Quaternion valued functions defined on X. If $f, g \in F(X)$ then $(f \cdot \alpha)(x) = f(x) \cdot \alpha$ and $(f+g)(x) = f(x) + g(x)$.

It should be noted that the fundamental theory of finite dimensional linear spaces over associative division algebras is well known (Jacobson, 1953). The definitions of basis, dimension, subspace, etc. are exactly the same as those in the complex case and will not be given here.

E. Linear Spaces Over the Real Cayley Algebra.

Definition A (right) linear space over the real Cayley algebra is

(i) a right linear space over $(\mathrm{Re})e_0$

(ii) L is closed with respect to a multiplication on the right by elements of the Cayley algebra.

(iii) This scalar multiplication is assumed to obey the following rules (for all $x,y \in L$, $\alpha, \beta \in C$).

(a) $(x+y)\cdot\alpha = x\cdot\alpha + y\cdot\alpha$

$x\cdot(\alpha+\beta) = x\cdot\alpha + x\cdot\beta$

$x\cdot e_0 = x.$

(b) Define $[x,\alpha,\beta] = (x\cdot\alpha)\beta - x\cdot(\alpha\beta)$.

Then $[x,\alpha,\alpha^{-1}] = [x,\alpha^{-1},\alpha] = [x,\alpha,\alpha] = 0.$

(iv) There exists a non-trivial real subspace L_R of L such that $[x,\alpha,\beta] = 0$ for all $x \in L_R$ and $\alpha,\beta \in C$.

(v) Every $x \in L$ can be expressed uniquely as $x = \sum_{j=0}^{7} x_j e_j$ where the $x_j \in L_R$ and the e_j are the Cayley basis elements.

<u>Definition</u> A Cayley-Subspace M of L is a real subspace of L such that for every pair $x,y \in L$ and every $\alpha \in C$, $x + y \in L$ and $x\cdot\alpha \in L$.

Example 1. Let $L = \{ (x_1,x_2,\ldots,x_n) \mid x_j \in C \}$. If x,y are any two elements of L and α is any Cayley number then define $x + y = (x_1+y_1,\ldots,x_n+y_n)$ and $x\cdot\alpha = (x_1\cdot\alpha, x_2\cdot\alpha,\ldots x_n\cdot\alpha)$. With these definitions L is a linear space over the Cayley

Numbers. L_R in this case is the real subspace
$L_R = \{(x_1,\ldots,x_n) \mid x_j \in \text{Re}\}$.

Example 2. Let X be any nonvoid set and F(X) the collection of all C valued functions defined on X. For $f,g \in F(X)$ and $\alpha \in C$, define $(f+g)(x) = f(x) + g(x)$ and $(f\cdot\alpha)(x) = f(x)\cdot\alpha$. With these definitions F(X) is a linear space over C. $F_R(X)$ is the real subspace of F(X) consisting of real valued functions on X.

F. Linear Transformations and Linear Functionals.

In the following, let F be either Q or C or a subalgebra of Q or C.

<u>Definition</u> Let L_1 and L_2 be (right) linear spaces over F. A mapping $T\colon L_1 \to L_2$ is called

(i) additive if $T(x+y) = T(x) + T(y)$ $\forall\, x,y \in L_1$

(ii) D homogeneous if $T(x\cdot\alpha) = T(x)\cdot\alpha$ for every $\alpha \in D$ where D is a subring of F.

(iii) F homogeneous if $T(x\cdot\alpha) = T(x)\cdot\alpha$ $\forall\, \alpha \in F$.

(v) Linear if both (i) and (iii) hold.

<u>Definition</u> Let L be a linear space over F. A mapping $f\colon L \to F$ is called a linear functional if $f(x+y) = f(x) + f(y)$ and $f(x\cdot\alpha) = f(x)\cdot\alpha$ for every $x,y \in L$, $\alpha \in F$.

Examples of linear transformations and linear functionals will be given in a later section.

<u>Definition</u> Let L be a linear space over F and T a transformation of L into L. A subspace M of L is called an invariant subspace if T is $T(M) \subset M$.

IV. Normed Linear Spaces over Q and C

A. Definitions and Examples.

In the following, F will denote either the real Quaternions or Cayley numbers.

<u>Definition</u> A linear space over F is called a normed linear space if there exists a function $\| \|: L \longrightarrow Re$ with the following properties.

(i) $\|x\| \geq 0 \ \forall \ x \in L$ and $\|x\| = 0$ iff $x = 0$.

(ii) $\|x \cdot \alpha\| = \|x\| \cdot |\alpha| \ \forall \ x \in L$ and $\alpha \in F$.
($| \ |$ is the norm in F).

(iii) $\|x+y\| \leq \|x\| + \|y\| \ \forall \ x,y \in L$.

As usual, if p is defined by $p(x,y) = \|x-y\|$, then p is metric and the space is topologized by this metric.

In view of the inequality $\big| \|x\| - \|y\| \big| \leq \|x-y\|$ the norm $\| \|$ is a continuous real valued function of its argument.

Definition A linear space L over F with a topology T is called a topological linear space if addition and scalar multiplication are continuous with respect to the topology on L.

Using the properties of the norm the following theorem is easy to prove.

Theorem 1. Any normed linear space L over F is a topological space if the topology is defined by the metric $p(x,y) = \|x-y\|$.

1. Examples of Normed Linear spaces.

Example 1. Let L be the linear space over F defined by $L = \{(x_1,\ldots,x_n) \mid x_j \in F\}$ with operations defined pointwise. For any $x \in L$, define $\|x\| = (\sum_{k=1}^{n} |x_k|^2)^{1/2}$.

It is easy to verify that this function satisfies the properties of a norm.

Example 2. Let X be any topological space and B(X) consist of the collection of all F-valued bounded functions on X. (i.e. $\sup_{x \in X} |f(x)| < \infty$ for every $f \in B(X)$.) For $f \in B(X)$ define $\|f\|_u = \sup_{x\ X} |f(x)|$. The set B(X) becomes a linear space over F if $(f+g)(x) = f(x)+g(x)$ and $(f\cdot\alpha)(x) = f(x)\cdot\alpha$. With respect to $\|\cdot\|_u$, B(X) is a normed linear space over F.

Example 3. Let L be the collection of all continuous F valued functions defined on $[a,b]$. L is a linear space over F if the operations are defined pointwise. L becomes a normed linear space if for $f \in L$ $\|f\| = \int_a^b |f(x)|\,d$.

Example 4. Consider the set $[0,2\pi]$ with ordinary Lebesque measure. Any Quaternion valued function f on $[0,2\pi]$ can be expressed as $f(x) = \sum_{k=0}^{3} f_k(x)e_k$ where the f_k are real valued functions on $[0,2\pi]$.

Definition A Q-valued function $f = \sum_{k=0}^{3} f_k e_k$ defined on $[0,2\pi]$ is Lebesque-measurable iff each of the f_k are Lebesque measurable.

Definition Let $P > 0$. Then $L_Q^P[0,2\pi]$ is defined to be the class of all measurable Q-valued functions defined on $[0,2\pi]$ such that $|f|^P \in L'_{Re}$ for all $f \in L_Q^P[0,2\pi]$. It is clear that if $f \in L_Q^P$ then $f\cdot\alpha$ is also in L_Q^P for every $\alpha \in Q$. Since $|f+g|^P \leq 2^P(|f|^P + |g|^P)$, L_Q^P is also closed under addition. Therefore the class $L_Q^P[0,2\pi]$ is a right linear space over Q.

<u>Definition</u> For $f \varepsilon L_Q^P$, $\|f\|_P$ is defined as

$$\|f\|_P = \left(\int_0^{2\pi} |f|^P \right)^{1/P} .$$

From the definition of $\| \|$ its easy to see that $\|f\cdot\alpha\|_P = \|f\|_P \cdot |\alpha|$. To show that $\| \|_P$ satisfies the other properties of a norm is non-trivial.

The proofs of the following two theorems carry over from the usual proofs directly and will not be given (Royden, 1969).

<u>Theorem 2.</u> (Holder, $P > 1$).

Let $f \varepsilon L_Q^P$, $g \varepsilon L_Q^{P'}$ where $\frac{1}{P} + \frac{1}{P'} = 1$, and $P > 1$.

Then $f\cdot g \varepsilon L_Q^1$ and

(i) $\left| \int_0^{2\pi} f\cdot g \right| \leq \int_0^{2\pi} |f\cdot g|$ (ii) $\int_0^{2\pi} |f\cdot g| \leq \|f\|_P \cdot \|g\|_{P'}$

(iii) $\left| \int_0^{2\pi} f\cdot g \right| \leq \|f\|_P \cdot \|g\|_{P'}$

<u>Theorem 3</u>. (Minkowski).

Let $f,g \varepsilon L_Q^P$. Then $\|f+g\|_P \leq \|f\|_P + \|g\|_P$.

Now if functions that are equal almost everywhere are identified, the space $L_Q^P[0,2\pi]$ becomes a normed linear space over Q.

B. The Ring of Linear Transformations on a Normed Linear Space over F.

1. Semi-linear Transformations.

The definition of a linear transformation has been given in III. There is another type transformation which proves to be important in later work.

Definition Let L_1, L_2 be linear spaces over Q. A transformation $T: L_1 \to L_2$ is called semi-linear if

(i) $T(x+y) = T(x) + T(y)$.

(ii) $T(x \cdot \alpha) = T(x)\alpha'$ where $'$ is an automorphism of α .

It should be noted that every auto morphism of Q is an inner auto morphism. That is, $\alpha' = \beta\alpha\beta^{-1}$ for some $\beta \in Q$. (MacDuffee, 1940). Also, since $|\beta\alpha\beta^{-1}| = |\alpha|$ for every $\alpha, \beta \in Q$, every auto morphism is continuous with respect to the norm topology for Q.

Example 1. An example of a semi linear transformation on a linear space L over Q is scalar multiplication. For if T_λ is defined by $T_\lambda(x) = x \cdot \lambda$, then T_λ is additive and $T_\lambda(x \cdot \alpha) = (x \cdot \alpha) \cdot \lambda = (x \cdot \lambda) \cdot (\lambda^{-1} \cdot \alpha \cdot \lambda)$ $= T_\lambda(x) \cdot \alpha'$.

One very important property of a semi-linear transformation T is that it is always homogeneous with respect to real scalars. Thus, any semi-linear transformation in always a real-linear transformation.

Let L_1 be a linear space over Q and $S(L_1)$ the class of all semi-linear transformations of L_1 into L_1. Then the following is true.

Theorem 1. The class $S(L_1)$ is a semi-group with respect to the operation of composition and $S(L_1)$ possesses an identity.

2. Continuous Linear Transformations.

In the following F will denote either Q or C.

<u>Definition</u> Let L_1, L_2 be normed linear spaces over F with norms $\| \|_1$, $\| \|_2$ respectively. Let T be a linear transformation of L_1 into L_2. T is called continuous if for every sequence $\{x_n\}$ from L_1 such that $x_n \to x$, $T(x_n) \to T(x)$.

The following theorem gives some equivalent conditions for a transformation to be continuous. The theorem is stated only for additive, real homogeneous transformations but since any linear (or even semi-linear in the case of linear spaces over Q) transformation satisfies these conditions, the theorem is quite general.

Also, since any linear space over F is also a linear space over Re, the usual proof suffices (Simmons, 1963) and therefore will not be given.

<u>Theorem 1</u>. Let L_1, L_2 be normed linear spaces (with respective norms $\| \|_1$ and $\| \|_2$) over F. Let T be a real linear transformation of L_1 into L_2. Then the following are equivalent.

(i) T is continuous.

(ii) T is continuous at the zero vector.

(iii) There exists a real number K_T that $\|T(x)\|_2 \leq K_T \|x\|_1$ for all $x \varepsilon L_1$.

(iv) If $S_1 = \{ x \mid \|x\|_1 \leq 1 \}$ then $T(S_1)$ is bounded in L_2.

3. The Topological Structure of R*(L).

In the usual complex linear space case the collection of all linear transformations on a linear space form a complex algebra. Since the linear spaces under consideration here are one sided spaces over non-commutative division algebras this will no longer hold. However, the collection of linear transformation on such spaces forms an interesting structure as will now be shown.

Let R(L) be the linear transformations of L into L. For $T_1, T_2 \varepsilon R(L)$ define $(T_1+T_2)(x) = T_1(x) + T_2(x)$ and $(T_1 \cdot T_2)(x) = T_1(T_2(x))$. Then R(L) is a ring with respect to the operations of + and $\cdot$. It's clear that R(L) is an associative ring but it may be non-commutative.

<u>Definition</u> Let R*(L) be the subset of R(L) that consists of all bounded linear transformations on L. For each $T \varepsilon R^*(L)$ define $\|T\|_* = \sup_{\|x\|=1} \|T(x)\|$.

It's clear that R^* is a subring of R and it will be shown that R^* is actually a topological ring. That is, the operations of multiplication and addition are continuous.

<u>Theorem 1</u>. The real valued function $\| \|_*$ has the following properties.

(i) $\|T\|_* \geq 0$ and $\|T\|_* = 0$ iff T is the 0-transformation.

(ii) $\|T_1+T_2\|_* \leq \|T_1\|_* + \|T_2\|_*$ for all $T_1, T_2 \in R^*$.

Proof. (i) follows from the fact that $\|Tx\| \geq 0$ for all $x \in L$ and $\|Tx\| = 0$ iff $Tx = 0$. (ii) follows from the inequality $\|T_1(x) + T_2(x)\| \leq \|T_1(x)\| + \|T_2(x)\|$ for all $x \in L$.

A metric can now be defined in R^* if one defines $\rho(T_1,T_2) = \|T_1-T_2\|_*$ for any pair $T_1, T_2 \in R^*$. This induces a metric topology on R^* and moreover with this topology R^* is a topological ring.

<u>Lemma 1</u>. If $T_1, T_2 \in R^*$ then $\|T_1 \cdot T_2\|_* \leq \|T_1\|_* \|T_2\|_*$

Proof. $\|T_1 \cdot T_2\|_* = \sup\limits_{\|x\|=1} \{ T_1(T_2(x)) \}$

$$\|T_1 \cdot T_2\|_* \leq \sup_{\|x\|=1} \{ \|T_1\|_* \|T_2(x)\| \}$$

$$\|T_1 T_2\|_* \leq \|T_1\|_* \sup_{\|x\|=1} \{\|T_2(x)\|\} = \|T_1\|_* \|T_2\|_*.$$

<u>Theorem 2</u>. Multiplication and addition are continuous operations in the topology of R*.

Proof. That addition is continuous follows immediately from the inequality $\|T_1+T_2\|_* \leq \|T_1\|_* + \|T_2\|_*$

Let $\{T_n\}$ and $\{T_n'\}$ be sequences from R* such that $\lim_{n\to\infty} T_n = T$ and $\lim_{n\to\infty} T_n' = T'$. (The limits here are to be interpreted as $\| \|_*$ limits.)

$$\|T_n T_n' - T\cdot T\|_* = \|T_n(T_n' - T') + (T_n - T)T'\|_*.$$

From Lemma 1 and the triangle inequality for $\| \|_*$ it follows that

$$\|T_n T_n' - T\cdot T'\|_* \leq \|T_n\|_* \|T_n'-T'\|_* + \|T_n-T\|_* \|T'\|_*.$$

From this it is clear that $\lim_n T_n\cdot T_n' = T\cdot T'$.

It will now be shown that if L is a complete metric space with respect to the metric $\rho(x,y) = \|x-y\|$ then R* is complete in its metric.

<u>Lemma 2.</u> If $\{T_n\}$ is a Cauchy sequence in R* then $\{T_n(x)\}$ is a Cauchy sequence in L for any $x \in L$.

Proof. Let $\{T_n\}$ be a Cauchy sequence in R*. Then

$$\|T_n(x)-T_m(x)\| = \|(T_n-T_m)(x)\| \leq \|T_n-T_m\|_* \|x\|$$

and clearly $\{T_n(x)\}$ will be a Cauchy sequence in L.

<u>Theorem 3</u>. The metric space R* is complete whenever L is complete.

Proof. Let $\{T_n\}$ be a Cauchy sequence in R*. Then by Lemma 2 $\{T_n(x)\}$ is a Cauchy sequence in L for each $x \in L$. But since L is complete there exists a $y \in L$ such that $y = \lim_{n\to\infty} T_n(x)$. Hence, define $T(x) = y = \lim_{n\to\infty} T_n(x)$. Since each of T_n are linear transformations it is clear that $T \in R$.

It must be shown that $T \in R^*$ and that $\{T_n\}$ converges to T in the metric for R*.

To show that T is bounded, the fact that $\{T_n\}$ is "Cauchy" will be used. There exists an integer N_0 such that $\|T_n-T_m\|_* < 1$ for all $n,m > N_0$. But it follows from this that $\|T_n\|_* \leq 1 + \|T_m\|_*$ and hence for each $x \in L$, $\|T_n(x)\| \leq (1 + \|T_m\|_*)\|x\|$ for all $n,m > N_0$. Using the continuity of the norm it follows that $\|T(x)\| \leq (1 + \|T_m\|_*)\|x\|$ for all $m \varepsilon N_0$. Therefore, T is bounded and a member of R*.

Since $\|T_n-T\|_* = \sup_{\|x\|=1} \{\|T_n(x) - T(x)\|\}$

it's clear from the definition of T as the $\|\ \|$ limit of the sequence $T_n(x)$ that $T_n \rightarrow T$ in $\|\ \|_*$ limit.

C. Linear Functionals, the Hahn Banach Theorem, and the Conjugate Space.

1. The Soukhamlinov Theorem and its Consequences.

Unless stated otherwise, all linear spaces in this section will be right spaces over the real Quaternions.

Definition A mapping $f: L \rightarrow Q$ where L is right linear space is a linear functional if

(a) $f(x+y) = f(x) + f(y)$

(b) $f(x \cdot \mu) = f(x) \cdot \mu$.

The set of all linear functionals on L is denoted by $L^{\#}$.

The set $L^{\#}$ can be made into a left linear space over Q as follows. Define $(f+g)(x) = f(x) + g(x)$ when $f, g \in L^{\#}$ and $(\lambda \cdot f)(x) = \lambda f(x)$. It is easily verified that with these operations $L^{\#}$ is a left linear space over Q.

It is because of the non-commutative properties of Q that $L^{\#}$ must be a left linear space rather than a right space. For if one defines $(f \cdot \lambda)(x) = f(x) \cdot \lambda$, then $(f \cdot \lambda)(x \cdot \mu) = f(x \cdot \mu) \cdot \lambda = f(x) \mu \cdot \lambda \neq f(x) \lambda \cdot \mu = (f \cdot \lambda)(x) \mu$. Thus, under such multiplication $L^{\#}$ is not necessarily closed.

The question now arises as to whether $L^{\#}$ is void or trivial, since $f(x) = 0 \ \forall \ x \in L$ is a linear functional. The answer to this question as well as to questions about the nature of certain subspaces of $L^{\#}$ follows from a theorem due to Sukhomlinov, (Sukhomlinov, 1938). It is known as the Hahn-Banach Theorem for the case of real and complex linear spaces.

Lemma 1. Let f be a linear functional on a linear space L over Q. Then there exists a real valued, real linear functional f_0 on L such that $f(x) = f_0(x)e_0 - \sum_{i=1}^{3} f_0(xe_i)e_i$ for every $x \in L$.

Proof. Since $f\colon L \longrightarrow Q$, f can be expressed as follows. For each $x \in L$, $f(x) = \sum_{k=0}^{3} f_k(x)e_k$ where the f_k are real valued linear functionals on L.

Since f is linear, $f(xe_j) = f(x)e_j$ for $j = 0,1,2,3$. Hence, $f(xe_j) = \sum_{k=0}^{3} f_k(xe_j)e_k = (\sum_{k=0}^{3} f_k(x)e_k)e_j$. After a little algebra it follows that $f_j(x) = -f_0(xe_j)$ for $j = 1,2,3$. Therefore, $f(x) = f_0(x)e_0 - \sum_{j=1}^{3} f_0(xe_j)e_j$.

Lemma 2. Let L be a linear space over Q. Let h be a real valued, real linear functional on L. Then the functional $H(x) = h(xe_0)e_0 - \sum_{k=1}^{3} h(xe_k)e_k$ is linear on L.

Proof. It is clear from the properties of h that H is additive and homogeneous with respect to real scalars. $H(xe_1) = h([xe_1]e_0)e_0 - \sum_{k=1}^{3} h([x\ e_1]e_k)e_k$ and by the associative postulate for scalar multiplication this can be written as $H(x\ e_1)$ $= h(x(e_1e_0)e_0 - \sum_{k=1}^{3} h(x\ (e_1e_k))e_k$. Writing out the summation yields,

$H(xe_1) = h(xe_1)e_0 + h(xe_0)e_1 - h(xe_3)e_2 + h(xe_2)e_3$.

But this last statement can be written as

$H(xe_1)=(h(xe_0)e_0-h(xe_1)e_1-h(xe_2)e_2-h(xe_3)e_3)e_1$, or

$H(xe_1) = H(x)e_1$. Similar computations yield

$H(xe_j) = H(x)e_j$ for $j = 2,3$. Therefore H is homogeneous with respect to scalars from Q and consequently H is linear.

<u>Theorem 1</u>. (Soukhamlinov) Let L be a (right) linear space over Q and M a subspace of L. Suppose f is a linear functional whose domain is M and has the property $|f(x)| \leq p(x)$ for all $x \in M$ where p is a real valued functional defined on all of L with the properties;

(i) $p(x) \geq 0$, (ii) $p(x+y) \leq p(x) + p(y)$, and

(iii) $p(x\cdot\alpha) = p(x)|\alpha|$. Then there exists an extension F of f to all of L with the property that $|F(x)| \leq p(x)$.

Proof. Since f is linear it follows from Lemma 1 that $f(x) = f_0(xe_0)e_0 - \sum_{k=1}^{3} f_0(xe_k)e_k$, where f_0 is a real valued real linear functional on M.

By hypothesis $|f(x)| \leq p(x)$ for all $x \in M$ and consequently $f_0(x) \leq p(x)$ for all $x \in M$. The "real" Hahn-Banach Theorem can now be applied to f_0 to yield the existance of a real-linear functional F_0 which extends F_0 to all of L with the property that $F_0(x) \leq p(x)$ for all $x \in L$.

Define the functional F on L as follows. $F(x) = F_0(xe_0)e_0 - \sum_{k=1}^{3} F_0(xe_k)e_k$ for all $x \in L$. By Lemma 2, F(x) is a linear functional. Moreover, for $x \in M$ $F_0(xe_k) = f_0(xe_k)$ for $k = 0,1,\ldots 3$. Hence, $F(x) = f(x)$ and therefore F is an extension of f.

If $F(x) = 0$ for all $x \in L$ then $F(x) \leq p(x)$ trivially. Suppose that F is not identically zero. Let $x \in L$. Choose $\mu \in Q$ such that $\mu = \dfrac{F(x)}{|F(x)|}$. Then $|\mu| = 1$ and $F(x) = |F(x)|\mu$. It now follows that $F(x)\bar{\mu} = F(x\bar{\mu}) = |F(x)|\,|\mu|^2 = |F(x)|$. Therefore $F(x\bar{\mu})$ is real and consequently $F(x\bar{\mu}) = F_0(x\bar{\mu})e_0$. But $F_0(x\bar{\mu}) \leq p(x\bar{\mu}) = p(x)|\bar{\mu}| = p(x)$. Using the fact that $F(x\bar{\mu}) = |F(x)|$ it follows that $|F(x)| \leq p(x)$. Since this is true for every $x \in L$ the proof is complete.

<u>Definition</u> Let L be a normed linear space over F = Q or C. A linear f on L is called bounded if $|f(x)| \leq k_f \|x\|$ for all $x \in L$, where $k_f > 0$ and depends only on f. L* will denote the class of all bounded linear functionals on L. For $f \in L^*$ the norm is defined to be

$$\|f\| = \sup_{\|x\|=1} |f(x)|.$$

It follows easily from the analogous theorem for bounded linear transformations that a linear functional f on L is continuous iff its bounded.

<u>Corollary 1</u>. Let M be a subspace of a normed linear space N, and f a bounded linear functional on M. Then f can be extended to a linear functional F defined on all of N such that $\|F\| = \|f\|$.

Proof. Define $p(x) = \|f\| \cdot \|x\|$. Then $|f(x)| \leq p(x)$ for all $x \in M$. By Sukhamlinov's Theorem there exists a functional F that extends f and $|F(x)| \leq p(x) = \|f\| \cdot \|x\|$ for all $x \in L$. It follows that $\|F\| = \sup_{\|x\|=1} |F(x)| \leq \|f\|$, But since F is an extension of f, $\|F\| \geq \|f\|$.

<u>Corollary 2</u>. If N is a normed linear space, x_0 a non zero element of N, there exists a linear functional F on N such that $F(x_0) = \|x_0\|$ and $\|F\| = 1$.

Proof. Let $M = \{x_0\alpha \mid \alpha \varepsilon Q\}$ and define $f(x_0 \cdot \alpha) = \alpha \cdot \|x_0\|$. Then $f(x_0) = \|x_0\|$ and $\|f\| = 1$. Now apply corollary 1.

As remarked previously, the initial goal of this section was to answer the question of whether the space $L^{\#}$ is non-trivial. The question can be answered affirmatively in the following sense.

<u>Theorem 2</u>. Let L be a linear space over Q and L* the class of continuous linear functional on L. Then L* is not trivial.

Proof. Let $x \varepsilon L$ and $x \neq 0$. Let $M = \{ x \cdot \alpha \mid \alpha \varepsilon Q \}$. Then M is a subspace of L. Define the functional f on L as follows.

$f(x \cdot \alpha) = \|x\| \alpha$. Clearly f is a bounded linear functional on M. By corollary 1, f can be extended to a functional F defined on all of L without change of norm. Hence $F \varepsilon L^*$.

The next application of the Sukhamlinov Theorem is related to the separation of convex sets in linear spaces.

<u>Definition</u> A subset K of a linear space L over Q is convex for every pair of real numbers α, β, for which $\alpha + \beta = 1$, $x\alpha + y\beta \varepsilon K$ for all $x, y \varepsilon K$.

Definition A subset K of L has an internal point if for each $y \in L$ there exists an $\varepsilon > 0$ such that $x + y\alpha \in K$ for $|\alpha| \leq \varepsilon$.

Definition Let L be a linear space over Q and f a functional on L. The functional f separates the subsets M, N of L if there exists a real constant c such that $t(f(M)) \geq c$ and $t(f(N)) \leq c$. Here $t(f(x)) = f(x) + \overline{f(x)}$.

The following theorem is well known for real and complex linear spaces (Dunford & Schwartz, 1958). Only the case for real spaces will be needed.

Theorem 3. Let M and N be disjoint convex subsets of a linear space L (over Re), and let M have an internal point. Then there exists a non zero linear functional which separates M and N. This Theorem can be extended to the case of linear spaces over Q as follows.

Theorem 4. Let M and N be disjoint convex subsets of a linear space L over Q, and let M have an internal point. Then there exists a non zero linear functional which separates M and N.

Proof. L may be regarded as a linear space over R_e and consequently by the preceeding theorem there exists a functional f on L such that

(a) $f(x+y) = f(x) + f(y)$

(b) $f(x\cdot\alpha) = f(x)\alpha$ for α real. Moreover there exists a real constant c such that $f(M) \geq c$ while $f(N) \leq c$.

As in the proof of the Sukhamlinov Theorem, define the functional $F(x) = f(xe_0)e_0 - \sum_{k=1}^{3} f(xe_k)e_k$. This is a linear functional and F separates M and N.

2. The Canonical Embedding of a linear space L over Q in $(L^*)^*$.

If L is any right linear space over Q then L^* is a left space over Q. The conjugate of L^* denoted by L^{**} will now be considered.

Clearly, L^{**} must again be a right space over Q. The question naturally arises as to how L^{**} and L are related.

As in the case for real and complex spaces every $x \in L$ gives rise to an element F_x in L^{**}. The mapping is given by $x \rightarrow F_x$ where $F_x(f) = f(x)$ for any $f \in L^*$. Clearly F_x is linear.

The norm on L** is defined as follows:

$\|F_x\| = \sup_{\|f\|=1} \{ |F_x(f)| \} = \sup_{\|f\|=1} \{ |f(x)| \}$. From this it follows that $\|F_x\| \leq \sup_{\|f\|=1} \{ \|f\| \cdot \|x\| \} = \|x\|$.

But by corollary 2 of the Sukhamlinov Theorem it follows that $\|F_x\| \geq \|x\|$ and consequently $\|F_x\| = \|x\|$. Therefore, the mapping of L into L** is norm preserving. The Mapping $J: L \longrightarrow L^{**}$ given by $J(x) = F_x$ is a linear mapping. For if $x,y \in L$, $J(x+y) = F_{x+y}$ and if $f \in L^*$, $F_{x+y}(f) = f(x+y) = f(x) + f(y) = F_x(f) + F_y(f) \Rightarrow J(x+y) = J(x) + J(y)$. Also, for $\lambda \in Q$ and $x \in L$, $J(x\lambda)\cdot f = F_{x\lambda}(f) = f(x\lambda) = f(x)\lambda = [J(x)\lambda]\cdot f$. Consequently, J is linear and by the preceeding remarks it is an isometry.

D. Linear Functionals on Linear Spaces over C.

<u>Definition</u> Let L be a linear space over C. A functional $f\colon L \to C$ is called linear if

(i) $f(x+y) = f(x) + f(y)$,

(ii) $f(xa) = f(x)\cdot a$ for all $x,y \in L$ and $a \in C$.

If (i) holds and (ii) hold for all $a \in D$ where D is a subalgebra of C, then f is called D-linear.

It should be noted at this point that $L^{\#}$, the class of all linear functionals on L, does not form a linear space over C. It will become a linear space over Re under the usual pointwise definitions of addition and scalar multiplication.

The reason that $L^{\#}$ is not a linear space over C appears to be a direct result of the fact that C is not an associative algebra. If scalar multiplication is defined in the usual manner, that is, $(a\cdot f)(x) = a\cdot f(x)$, then since the associator $(a,f(x),b)$ is not zero in general, such a mapping will not yield a homogeneous functional.

The fact that $L^{\#}$ is not a linear space over C eliminates the possibility of embedding L in L^{**}. Therefore, some of the structure theorems for normed linear spaces over Q (or K) will not carry over.

In the following, some results about linear functionals on linear spaces over C will be given and finally an extension theorem analogous to the Soukhamlinov Theorem will be proven.

<u>Lemma 1</u>. Let L be a linear space over C and f an additive functional on L to C. If f is real valued and homogeneous on L_R then f is linear on L.

Proof. Recall that $[x,a,b] = 0 \ \forall \ x \in L_R$, $a,b \in C$. For $x \in L_R$ $f(xa) = f(x)a$. From the axioms for a linear space over C, for $x \in L$, $x = \sum_{k=0}^{7} x_k e_k$ where the $x_k \in L_R$. Now consider for example $f(xe_1)$. Since $x = \sum_{k=0}^{7} x_k e_k$, xe_1 can be written as

$$xe_1 = \sum_{k=0}^{7} x_k(e_k e_1). \quad \text{Hence,}$$

$$xe_1 = -x_1 e_0 + x_0 e_1 + x_3 e_2 - x_2 e_3 + x_5 e_4 - x_4 e_5 + x_6 e_7 - x_7 e_6.$$

It follows that

$f(xe_1) = -f(x_1)e_0 + f(x_0)e_1 + f(x_3)e_2 - f(x_2)e_3 + f(x_5)e_4 - f(x_4)e_5 + f(x_6)e_7 - f(x_7)e_6$. But if f is real valued on L_R

$f(xe_1) = [f(x_0)e_0 + f(x_1)e_1 + f(x_2)e_2 + f(x_3)e_3 + f(x_4)e_4 + f(x_5)e_5 + f(x_6)e_6 + f(x_7)e_7]e_1$. Or,

$f(xe_1) = f(\sum_{k=0}^{7} x_k e_k)e_1 = f(x)e_1$. Using the same methods it can be shown that $f(xe_j) = f(x)e_j$ for

$j = 1,2,\ldots 7$. Therefore, f is homogeneous and consequently linear.

Lemma 2. Let L be a linear space over C. Let f be a real valued, real linear functional on L such that $f(xe_j) = 0$ whenever $x \in L_R$ and $j \neq 0$. Then the functional $F(x) = f(x)e_0 - \sum_{j=1}^{7} f(xe_j)e_j$ is linear on L.

Proof. It follows immediately from the properties of f that F is additive and real homogeneous. Let $x \in L_R$. Then $F(xe_1) = f(xe_1)e_0 - \sum_{j=1}^{7} f((xe_1)e_j)e_j$. Since $x \in L_R$, $(xe_1)e_j = x(e_1e_j)$. Thus, $F(xe_1) = f(xe_1) - \sum_{j=1}^{7} f(x(e_1e_j))e_j$. This equation can be written as

$$F(xe_1) = f(xe_1)+f(xe_0)e_1-f(xe_3)e_2+f(xe_2)e_3-f(xe_5)e_4 + f(xe_4)e_5+f(xe_7)e_6-f(xe_6)e_7.$$

But then

$$F(xe_1) = \left[f(xe_0)e_0-f(xe_1)e_1-f(xe_2)e_2-f(xe_3)e_3 - f(xe_4)e_4-f(xe_5)e_5-f(xe_6)e_6-f(xe_7)e_7 \right] e_1.$$

That is, $F(xe_1) = F(x)e_1$. Again, using the same technique, it can be shown that $F(xe_j) = F(x)e_j$ for $l = 2,3,\ldots,7$ and $x \in L_R$. Therefore, F is homogeneous on L_R.

Since $f(xe_j) = 0$ for $j \geq 1$ and $x \in L_R$, F is real valued on L_R. But then it follows from Lemma 1 that F is linear.

Lemma 3. Let L be a linear space over C. If f is a linear functional on L, then for every $x \in L_R$ $(f(x), e_j, e_k) = 0$ for all j, k.

Proof. Let $x \in L_R$. Then $(xe_j)e_k = x(e_je_k)$ for all j, k. Hence, $f((xe_j)e_k) = f(x(e_je_k))$. But since f is homogeneous $f(x(e_je_k)) = f(x)(e_je_k)$ and $f(xe_j)e_k = [f(x)e_j]\, e_k$. Therefore, $f(x)(e_je_k) = [f(x)e_j]\, e_k$.

Lemma 4. Let L be a linear space over C and let f be a linear functional on L. Then there exists a real valued, real linear functional f_0 on L such that $f(x) = f_0(x)e_0 - \sum_{j=1}^{7} f_0(xe_j)e_j$ for each $x \in L$. Also, $f_0(xe_j) = 0$ for each $x \in L_R$.

Proof. Since f takes its value in C and f is linear, f(x) can be written as $f(x) = f_0(x)e_0 + \sum_{k=1}^{7} f_k(x)e_k$ where the f_j are real valued, real linear functionals on L. By hypothesis $f(xa) = f(x)a$ for all $a \in C$. Thus, $f(xe_j) = f(x)e_j$ for $j=0,1,2,\ldots,7$.

Consider $f(xe_1)$. $f(xe_1) = f_0(xe_1)e_0 + \sum_{k=1}^{7} f_k(xe_1)e_k$ $= (\sum_{k=1}^{7} f_k(x)e_k)e_1$. From this last equation, it follows that $f_1(x) = -f_0(xe_1)$. Similar computations yield $f_j(x) = -f_0(xe_j)$. Therefore, $f(x) = f_0(x)e_0 - \sum_{j=1}^{7} f_0(xe_j)e_j$. Also, from lemma 3 and the fact f is linear it follows that $f_0(xe_j) = 0$ when $x \in L_R$.

With these lemmas the main result of this section can be proven. The statement of the theorem is almost the same as the Soukhamlinov Theorem for linear spaces over Q. However, there are some critical differences and these will be pointed out.

<u>Theorem 1</u>. Let L be a linear space over C and p a functional defined on L with the properties,

(i) $p(x+y) \leq p(x) + p(y)$

(ii) $p(x) \geq 0$

(iii) $p(xa) = p(x)\cdot|a|$ for all $x,y \in L$, and $a \in C$.

Let M be a subspace of L and f a linear functional defined on M such that $|f(x)| \leq p(x)$ for all $x \in M$. Then there exists a functional F that extends f to all of L such that

(1) $F(x+y) = F(x) + F(y)$

(2) $F(x \cdot a) = F(x)a$ for $a \in Re$

(3) $F(x \cdot a) = F(x)a$ for $x \in L_R$ and $a \in C$

(4) $|F(x)| \leq p(x)$ for all $x \in L_R$

Proof. From Lemma 4 there exists a real-valued, real linear functional f_0 defined on M such that $f(x) = f_0(x)e_0 - \sum_{j=1}^{7} f_0(xe_j)e_j$ for all $x \in M$ and $f_0(xe_j) = 0$ whenever, $x \in M_R$ and $j = 1,2,\ldots,7$.

Since $|f(x)| \leq p(x)$ for all $x \in M$ it is clear that $f_0(x) \leq p(x)$ for all $x \in M$. The Hahn-Banach Theorem can now be applied to yield the existence of real-linear functional F_0 which extends f_0 to all of L with the property that $F_0(x) \leq p(x)$ for all $x \in L$.

Define the functional F on L as folows.
$F(x) = F_0(x)e_0 - \sum_{j=1}^{7} F_0(xe_j)e_j$ for all $x \in L$. From the proof of Lemma 2 it is clear that F is additive, real homogeneous, and $F(xa) = F(x)a$ for all $x \in L_R$ and $a \in C$. Moreover, for $x \in M$, $F_0(xe_j) = f_0(xe_j)$ for $j = 0,1,2\ldots,7$. Hence $F(x) = f(x)$ whenever $x \in M$.

If $F(x) = 0$ for $x \in L$ then $|F(x)| \leq p(x)$ trivially. Suppose that $F(x) \neq 0$. Let $x \in L_R$ and choose $a \in C$ such that $a = \frac{F(x)}{|F(x)|}$. Then $|a| = 1$ and $F(x) = |F(x)|a$.

It now follows that $F(x\bar{a}) = F(x)\bar{a} = |F(x)||a|^2 = |F(x)|$. Therefore, $F(x\bar{a})$ is real and consequently $F(x\bar{a}) = F_0(x\bar{a})$. But $F_0(x\bar{a}) \leq p(x\bar{a}) = p(x)|\bar{a}| = p(x)$. Hence $|F(x)| \leq p(x)$. Since this is true for any $x \in L_R$, the proof is complete.

The differences between this Theorem and the Soukhamlinov Theorem are clear now. The extended functional is not homogeneous on all of L. Moreover, it is bounded by p only on L_R. These facts weaken the result considerably.

E. Banach Spaces over Q(and C) and Function Algebras

1. Definition and Examples

In the following, unless stated otherwise, F will denote either Q or C.

Definition A normed linear space L over F with norm is called a Banach space if L is a complete metric space relative to the metric $\rho(x,y) = \|x-y\|$

Example 1. Consider the linear space $L_Q^P[0,2\pi]$. It has already been noted that this space is a normed linear space with respect to the norm $\|f\|_p = \left[\int_0^2 |f|^p\right]^{\frac{1}{p}}$.

In view of the following theorem $L_Q^P[0,2\pi]$ is a Banach space.

<u>Theorem 1</u>. The $L_Q^P[0,2\pi]$ space $(p \geq 1)$ is complete.

Proof. Let $\{f_n\}$ be a Cauchy sequence in L_Q^P. For each n, f_n can be written as $f_n = \sum_{j=0}^{3} f_{nj} e_j$ where the $f_{nj} \in L_{Re}^P$ for each j. From Minkowski's inequality $\|f+g\|_p \leq \|f\|_p + \|g\|_p$, it follows that each of the sequences $\{f_{nj}\}$ $j = 0,1,2,3$ are Cauchy sequences in L_{Re}^P. But from the completeness of L_{Re}^P (Royden, 1969) there exists function f_j $j = 0,1,2,3$ in L_{Re}^P such that $\|\;\|_p\text{-}\lim_n f_{nj} = f_j$. Applying the Minkowski inequality again it follows that the function $f = \sum_{j=0}^{3} f_j e_j$ is in L_Q^P and $\|\;\|_p\text{-}\lim_n f_n = f$. Therefore,

L_Q^P is complete.

<u>Example 2</u>. Let X be a topological space and consider the class of function on X with values in F. Each function f has a representation of the form $f = \sum_{j=0}^{n_F} f_j e_j$ where the f_j are real valued functions on X and $n_F = 3$ or 7 depending on whether $F = Q$ or C.

Let $B(X,F)$ be the class of all bounded functions on X with values in F (that is, for each $f \in B(X,F)$ there exists a real number N such that $|f(x)| \leq N$ where $|\;|$ denotes the norm (topological) of F). If $(f+g)(x) = f(x) + g(x)$

and $(f\cdot\alpha)(x) = f(x)\cdot\alpha$, then B(X,F) is a linear space over F.

If for each $f \in B(X,F)$ $\|f\|_u = \sup_{x \in X} |f(x)|$,

then B(X,F) becomes a normed linear space over F. Moreover, B(X,F) is a Banach space with this norm.

<u>Theorem 2</u>. B(X,F) is complete in the norm $\| \|_u$.

Proof. Let $\{f_n\}$ be a Cauchy sequence from B(X,F). Then given any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that $\|f_n - f_m\|_u < \varepsilon$ for all $n,m \geq N(\varepsilon)$. But it then follows that for all $x \in X$ that $|f_n(x) - f_m(x)| < \varepsilon$ and thus $\{f_n(x)\}$ is a Cauchy sequence in F. By the completeness of F there exists $f(x) \in F$ given by $f(x) = \lim_n f_n(x)$. Since each f_n is bounded it follows that f is bounded and therefore in B(X,F). That is, B(X,F) is complete.

<u>Definition</u> Let X be a Topological Space. A function $f: X \rightarrow F$ is continuous at $x_0 \in X$ if given any $\varepsilon > 0$ there exists a neighborhood $N(x_0)$ such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in N(x_0)$.

Example 3. Let $C(X,F)$ be the subset of $B(X,F)$ consisting of bounded continuous functions on X. Then $C(X,F)$ is also a linear space over F. Using the normed defined on $B(X,F)$, $C(X,F)$ is a normed linear space. It will now be shown that $C(X,F)$ is a closed subspace of the complete (metric) space $B(X,F)$ and therefore complete.

Theorem 3. $C(X,F)$ is a closed subspace of $B(X,F)$.

Proof. Let $f \in \overline{C(X,F)}$. Then given $\varepsilon > 0$ there exists $f_0 \in C(X,F)$ such that $\|f - f_0\|_u < \varepsilon/3$. Let $x \in X$. From the continuity of f_0 there exists a neighborhood $N(x)$ such that for all $x' \in N(x)$, $|f(x') - f(x)| < \varepsilon/3$. Consequently, $|f(x')-f(x)| \leq |f(x')-f_0(x')| + |f_0(x')-f_0(x)| + |f(x) - f_0(x)| < \varepsilon$ for all $x' \in N(x)$. Since ε and x were arbitrary f is continuous on X and $C(X,F) = \overline{C(X,F)}$.

Corollary $C(X,F)$ is a Banach Space over F.

2. The Function Algebras $C(X,F)$

Let F be a Cayley-Dickson algebra over Re of dimension n. Each $a \in F$ can be expressed as $a = \sum_{j=0}^{n-1} a_j e_j$ where the $a_j \in$ Re and e_j are the basis elements of F.

A norm can be defined in F if for each $a \in F$, $|a| = \left[\sum_{j=0}^{n-1} a_j^2 \right]^{1/2}$. Using the completeness of the Reals it is not hard to show that F becomes an n-dimensional Banach space over Re.

<u>Definition</u> Let X be a topological space. A function f from X to F is continuous at $x_0 \in X$ if for any $\varepsilon > 0$ there exists a neighborhood $N(x_0; \varepsilon)$ such that for all $x \in N(x_0; \varepsilon)$ $|f(x_0) - f(x)| < \varepsilon$.

Let X be a topological space. C(X,F) will denote the class of all bounded continuous F-valued functions on X. C(X,F) becomes an algebra over Re if the operations are defined pointwise. C(X,F) becomes a normed algebra if for each $f \in C(X,F)$ $\|f\|_u = \sup_{x \in X} |f(x)|$. Using essentially the same proof as in Theorem 3 of the previous section C(X,F) can be shown to be complete with respect to this norm. Since $\|f \cdot g\|_u \leq \|f\|_u \, \|g\|_u$ the operations are continuous and hence C(X,F) is a topological algebra.

Since F is finite dimensional every $f \in C(X,F)$ has a representation of the form $f = \sum_{j=0}^{n-1} f_j e_j$ where the f_j are real valued functions on X and the e_j are the basis elements of F. Moreover, it is clear that $f \in C(X,F)$

iff $f_j \in C(X,Re)$ for each j.

The algebra $C(X,F)$ has an involution "-" defined by $\overline{f}(x) = \overline{f(x)}$ where "-" is the involution in F. Since $|\overline{f}(x)| = |\overline{f(x)}| = |f(x)|$ for each $f \in C(X,F)$ and $x \in X$ it follows that "-" is a continuous operation in $C(X,F)$.

In general, $C(X,F)$ will be neither commutative nor associative. Also, $C(X,F)$ will be infinite dimensional for general X. Therefore, $C(X,F)$ is a non-associative analog of a Banach Algebra. It appears that no one has studied structures of this type to date.

An analog of the Stone-Weierstrass Theorem will now be given for algebras of the type $C(X,F)$, where F is any finite dimensional Cayley-Dickson algebra over Re.

Definition Let A be an algebra of functions defined on a topological space X to a Cayley-Dickson algebra F. Then A separates points (in X) if to each pair of points x,y in X there exists an element $f \in A$ such that $f(x) \neq f(y)$.

The following is well known (Simmons, 1963).

Theorem 4. Let X be a compact Hausdorff space, and let A be a closed subalgebra of $C(X,Re)$ which separates points and contains a non-zero constant function. Then $A = C(X,Re)$.

To generalize this theorem let F be any Cayley-Dickson algebra of dimension n over Re. Let X be any topological space. As noted previously, the class $C(X,F)$ is an algebra. Any function $f \varepsilon C(X,F)$ can be represented as $f(x) = f_0(x)e_0 + \sum_{i=1}^{n-1} f_i(x)e_i$ where the $f_j \varepsilon C(X,Re)$ for each j. Each element of $f \varepsilon C(X,F)$ has a conjugate defined by $\overline{f}(x) = f_0(x)e_0 - \sum_{i=1}^{n-1} f_i(x)e_i$.

Theorem 5. Let X be a compact Hausdorff space, and let A be a closed subalgebra of $C(X,F)$ which separates points and contains the constant functions and $f \varepsilon A \Rightarrow \overline{f} \varepsilon A$. Then $A = C(X,F)$.

Proof. Let A_{Re} denote the class of all real valued functions in $C(X,F)$. Clearly A_{Re} is a closed subalgebra of $C(X,F)$. Moreover if $f \varepsilon C(X,F)$, $f(x) = f_0(x) + \sum_{i=1}^{n-1} f_i(x)e_i$ and $f_i \varepsilon C(X,Re)$ for each i. Consequently, if $A_{Re} = C(X,Re)$ then it follows that $A = C(X,F)$.

To show that $A_{Re} = C(X,Re)$ it must first be shown that A_{Re} separates points. Let x,y be any two distinct points in X. Since A separates points there exists $f \varepsilon C(X,F)$ such that $f(x) \neq f(y)$. But $f(x) = f_0(x)e_0 + \sum_{i=1}^{n-1} f_i(x)e_i$ and consequently $f_j(x) \neq f_j(y)$ for at least one value of j. Since $f_k \varepsilon A_{Re}$ for each k it follows that A_{Re} separates points. Now to show A_{Re} contains at least one non zero constant function, let f be any non zero constant function of A. The function $f \cdot \overline{f} \varepsilon A_{Re}$ and is a (real valued) non zero constant function. Applying the preceding theorem it follows that $A_{Re} = C(X,Re)$. Therefore $A = C(X,F)$.

F. Open Mapping, Closed Graph, and Uniform Boundness Theorems.

Let L be a Banach space over the field F where F is either Q or C. Then L is also a real linear space. If T is a linear transformation of the linear space L_1 into the linear space L_2 then T is clearly a real-linear transformation of L_1 into L_2. Hence, the following theorems (Simmons, 1963) are true, since their proofs depends only on the fact that transformations are real homogeneous and additive.

<u>Theorem 1</u>. (Open Mapping Theorem)

A continuous linear transformation T of a Banach space L_1 over F on to Banach space L_2 over F is an open mapping.

<u>Corollary</u>. If L_1, L_2 are Banach spaces over F and T is a one to one continuous linear transformation from L_1 onto L_2, then T has a continuous inverse.

Proof. Let O be open, L_1 = Range (T^{-1}), then
$(T^{-1})^{-1}(O) = T(O)$ is open in L_2 = Domain (T^{-1}).

<u>Theorem 2</u>. (Closed Graph Theorem)

Let L_1, L_2 be Banach spaces over F and T a linear transformation of L_1 into L_2. If

(a) $x_n \to x$ in L_1

(b) $T(x_n) \to y \in L_2$

then $T(x) = y$, i.e. the graph of T is closed.

<u>Theorem 3</u>. (Uniform Boundness Theorem)

Let L_1 be a Banach space over F and L_2 a normed linear space over F. If $\{T_i\}$ is a non-empty set of continuous linear transformation of L_1 into L_2 with the property that $\{T_i(x)\}$ is a bounded subset of L_2 $\forall$ $x \in L_1$, then $\{\|T_i\|\}$ is a bounded subset of Re.

<u>Theorem 4</u>. A non-empty subset X of linear space L over Q is bounded iff f(X) is a bounded subset of Q $\forall$ $f \in L^*$.

2. The Conjugate of a Linear Transformation in L over Q.

Let L be a normed linear space over Q and suppose T is a linear transformation of L into L.

Define a mapping $T^{\#}: L^{\#} \longrightarrow L^{\#}$ as follows:
Given $f \varepsilon L^{\#}$, let $(T^{\#}f)(x) = f(Tx) \; \forall x \varepsilon L$.
$T^{\#}$ is a linear transformation. For $f_1, f_2 \varepsilon L^{\#}$ and $\alpha_1, \alpha_2 \varepsilon Q$, then

$$\begin{aligned}[T^{\#}(\alpha_1 f_1 + \alpha_2 f_2)](x) &= (\alpha_1 f_1 + \alpha_2 f_2)(Tx) \\ &= \alpha_1 f_1(Tx) + \alpha_2 f_2(Tx) \\ &= [\alpha_1 \cdot T^{\#}(f_1)](x) + [\alpha_2 \cdot T^{\#}(f_2)](x).\end{aligned}$$

To show that $T^{\#}$ is actually a map of $L^{\#}$ into $L^{\#}$ consider the following

$$(T^{\#}f)(x\alpha) = f(T(x\alpha)) = f(T(x)\alpha) = f(T(x))\alpha = [Tf(x)]\alpha.$$

and
$$\begin{aligned}T^{\#}f(x_1+x_2) &= f(T(x_1+x_2)) = f(T(x_1) + T(x_2)) \\ &= f(T(x_1)) + f(T(x_2)) = T^{\#}_f(x_1) + T^{\#}f(x_2).\end{aligned}$$

The question now arises as to when $T^{\#}$ maps L^* into L^*.

<u>Theorem 1</u>. $T^{\#}(L^*) \subset L^*$ iff T is continuous.

Proof. T is continuous $\Longleftrightarrow$ $T(S)$ is bounded
$\Longleftrightarrow$ $f(T(S))$ is bounded $\forall f \varepsilon L^*$
$\Longleftrightarrow$ $T^{\#}f(S)$ is bounded $\forall f \varepsilon L^* \Longleftrightarrow T^{\#}f \varepsilon L^* \forall f \varepsilon L^*$

Now let T be a continuous linear transformation on L.

By the preceding theorem $T(L^*) \subset T(L^*)$. Therefore, the conjugate T^* of T is defined by $T^* = T^{\#}|_{L^*}$.

If $R(L)$, $R(L^*)$ denote the rings of bounded linear transformations on L and L^* respectively then the mapping $T \to T^*$ is a norm preserving map of the ring $R(L)$ into $R(L^*)$.

The conjugate of an operator has the following properties.

(a) $(T_1+T_2)^* = T_1^* + T_2^*$

(b) $(T_1 \cdot T_2)^* = T_2^* \cdot T_1^*$

G. Linear Functionals as Differentials of a Norm.

In the following, only linear spaces L over $F = Q$ will be treated. This section contains a slight extension of the work of R. C. James (James, R. C., 1951).

Definition The differential of the norm in a Banach space B(over Q or subfields of Q) at a point x_0 B is defined by

$$D(x_0;y) = \lim_{t\ 0} \frac{\|x_0+yt\| - \|x_0\|}{t}$$

provided the limt exists.

Lemma 1. If L is a right linear space over Q and $f \in L^{\#}$

then f has the following form,

$$f(x) = f_0(x)e_0 - f_0(xe_1)e_1 - f_0(xe_2) - f_0(xe_3)$$

where

$$f_0(x) = \tfrac{1}{2}t\,[\,f(x)] = \tfrac{1}{2}[f(x) + \overline{f(x)}].$$

Lemma 2. $D \mp (x;y) = \lim_{t \to 0\mp} \Delta(x,t,y)$ exist for each $x,y \in B$, where $\Delta(x,t,y) = \frac{\|x+yt\| - \|x\|}{t}$.

Proof. Let $t_1 > t_2 > 0$ and consider the difference

$$\Delta(x,t_1,y) - \Delta(x,t_2,y) = \frac{t_2\|x+yt_1\| - t_1\|x+yt_2\| + (t_1-t_2)\|x\|}{t_1t_2}$$

Since

$$\|xt_2+yt_1t_2\| = \|xt_1-xt_1+xt_2+yt_2t_1\|$$
$$\geq \|xt_1+yt_1t_2\| - \|x(t_1-t_2)\|$$

and

$$\|xt_2+yt_1t_2\| - \|xt_1+yt_1t_2\| \geq -(t_1-t_2)\|x\|$$

it follows that $\Delta(x,t_1,y) - \Delta(x,t_2,y) \geq 0$ whenever $t_1 > t_2 > 0$. Hence, for decreasing t, the function is non-increasing. Also,

$$\Delta(x,t,y) = \frac{\|x+yt\| - \|x\|}{t} \geq \frac{\|x\| - t\|y\| - \|x\|}{t} \geq -\|y\|$$

and consequently Δ is bounded below. If

$D_+(x;y) \equiv \lim_{t \to 0^+} \Delta(x,t,y)$ and $D_-(x;y) \equiv \lim_{t \to 0^-} \Delta(x,t,y)$

it is clear from the preceding remarks that both of these limits exist.

<u>Lemma 3</u>. Suppose B is a real Banach space and $f \in B^*$. Furthermore, suppose there exists $x_0 \in B$ for which $f(x_0) = \|f\| \, \|x_0\|$. Then for every $y \in B$

$$\|f\| \, D_-(x_0;y) \leq f(y) \leq \|f\| \, D_+(x_0;y).$$

Proof. For any $t > 0$ $f(x_0 \pm yt) \leq \|f\| \cdot \|x_0 \pm yt\|$ or

$f(x_0) \pm f(y)t \leq \|f\| \, \|x_0 \pm yt\|$. But

$f(x_0) = \|f\| \cdot \|x_0\|$ so

$\|f\| \cdot \|x_0\| \pm f(y)t \leq \|f\| \cdot \|x_0 \pm yt\|$ and finally

this can be expressed as

$\pm f(y)t \leq \|f\| [\, \|x_0 \pm yt\| - \|x_0\|]$.

Recalling the definition of $\Delta(x,t,y)$ it is clear that $\|f\| \, \Delta(x_0,-t,y) \leq f(y) \leq \|f\| \, \Delta(x_0,t,y)$ and on taking the limit as $t \to 0^+$ this expression yields $\|f\| \, D_-(x_0;y) \leq f(y) \leq \|f\| \, D_+(x_0;y)$.

<u>Definition</u> The norm " $\| \ \|$ " in a Banach space B (over Q) is differentiable at $x_0 \varepsilon$ B if

$$\lim_{t \ \ 0} \frac{\|x_0+yt\| - \|x_0\|}{t}$$

exists for every $y \varepsilon$ B.

From Lemma 3 and the preceding definitions the following Theorem is clear.

<u>Theorem 1</u>. Let B be a real Banach space and $f \varepsilon B^*$. If there exists $x_0 \varepsilon$ B such that $f(x_0) = \|f\| \cdot \|x_0\|$ and $D(x_0;y)$ exists, then for $y \varepsilon$ B $f(y) = \|f\| \cdot D(x_0;y)$.

<u>Theorem 2</u>. If $D(x_0;y)$ exists for $x_0 \varepsilon$ B, then $D(x_0; \ )$ is in B^* with $\|D(x_0; \ )\|=1$ and $D(x_0;x_0) = \|x_0\|$

Proof. $$|\Delta(x_0,t,y)| = \left|\frac{\|x_0+yt\| - \|x_0\|}{t}\right| \leq \left|\frac{\|x_0\| + \|yt\| - \|x_0\|}{t}\right|$$

which implies

$$\frac{|\Delta(x_0,t,y)|}{\|y\|} \leq 1$$

for all $y \neq 0$ and consequently

$$\lim_{t \ \ 0} \frac{|\Delta(x_0,t,y)|}{\|y\|} \leq 1 \Rightarrow \|D(x_0; \ )\| \leq 1.$$

But $D(x_0;x_0) = \lim_{t \quad 0} (\frac{1+t-1}{t}) \| x_0 \| = \| x_0 \|$

and consequently $\| D(x_0; \ ) \| = 1$. It has been shown then that $D(x_0;)$ is a bounded functional of its second argument and it will be sufficient to show that $D(x_0; \ )$ is additive to prove that it is a member of B*.

To do this consider the following:

$$D(x_0;y+z) - D(x_0;y) - D(x_0;z)$$
$$= \lim_{t \to 0} \frac{\|x_0+(y+z)t\| + \|x_0-yt\| + \|x_0-zt\| - 3\|x_0\|}{t}$$

But

$$\|x_0+(y+z)t\| + \|x_0-yt\| + \|x_0-zt\| - 3\|x_0\| \geqslant 0.$$

Since the expression is decreasing as $t \to 0$ the limit must be zero and therefore $D(x_0; \ )$ is additive.

As a consequence of these Theorems the following is true.

(a) if the norm of the Banach space B(over R) is differentiable at each $x \in B$

(b) and given any $f \in B^*$ theorem exists $x_0 \in B$ for which $f(x_0) = \|f\| \|x_0\|$.

Then for every $y \in B$ $f(y) = \|f\| D(x_0;y)$. Therefore for such Banach spaces (over Re) there is a representation for the bounded linear functionals on B.

There is a class of Banach spaces for which (b) always holds. These are known as uniformly convex spaces.

<u>Definition</u> A Banach space (over Q or subfield of Q) is uniformly convex if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x+y\| < 2 - \delta$ whenever $\|x\| = \|y\| = 1$ and $\|x-y\| > \varepsilon$.

<u>Theorem 3</u>. If the norm in a uniformly convex Banach Space B (over Re) is differentiable at each non-zero element of B, then for every $f \in B^*$ there exists a unique $x_f \in B$ such that $\|x_f\| = 1$ and $f(y) = \|f\| D(x_f;y)$ for all $y \in B$.

Remark. If it can be shown that for every $f \in B^*$ there exists a unique $x_f \in B$ with $\|x_f\| = 1$ and $f(x_f) = \|f\|$ then the conclusion will follow from Theorems 1, 2.

Proof. Since the $\|f\| = \sup_{\|x\|=1} |f(x)|$ there exists a a sequence $\{x_n\}$ with $\|x_n\| = 1$ for every n and $\lim_n f(x_n) = \|f\|$. By hypothesis B is uniformly convex so for any $\varepsilon > 0$ a $\delta > 0$ can be chosen such that for all $\|x\| = \|y\| = 1$

$\|x-y\| > \varepsilon \Rightarrow \|x+y\| < 2 - \delta$. Now choose an N such that $f(x_n) > \|f\| (1 - \frac{\delta}{2})$ when $n > N$. If $n,m > N$ $f(x_n+x_m) = f(x_n) + f(x_m) > 2\|f\|(1 - \frac{\delta}{2})$

$$= \|f\|(2 - \delta).$$

But by definition of $\|f\|$, $f(x_n+x_m) \leq \|f\| \|x_n+x_m\|$. Hence, $\|x_n+x_m\| > 2 - \delta$ and by uniform convexity $\|x_n-x_m\| < \varepsilon$. Therefore, $\{x_n\}$ is a Cauchy Sequence and since B is complete, there exists $x_0 \varepsilon B$ such that $\lim_n x_n = x_0$. But f is continuous, and consequently $\lim_n f(x_n) = \|f\| = f(x_0)$ and it is clear that $\|x_0\| = 1$.

From this Theorem the following important Corollary can now be derived.

<u>Corollary</u>. Let B be a (right) Banach Space over Q which is uniformly convex. Suppose the norm is differentiable for each $x \varepsilon B$. Then for each $F \varepsilon B^*$ there exists a unique x_F in B for which

$$F(y) = \|F_0\| [D(x_F;y) - D(x_F;ye_1)e_1 - D(x_F;ye_2)e_2 - D(x_F;ye_3)e_3]$$

where $F_0(x) = \frac{1}{2}t(F(x))$ and y is any element of B.

Proof. F_0 is a bounded real linear functional on the Banach B which is also a Banach space over R. By Lemma 1

$F(y) = F_0(y) - \sum_{i=1}^{3} F_0(ye_i)e_i$. But by Theorem 3 there exists some $x_F \in B$ such that $F_0(y) = \|F_0\| \cdot D(x_F;y)$ for all $y \in B$.

The question now arises as to whether there exist any uniformly convex Banach spaces over Q. The answer is yes, and examples of such spaces will be given in a later section.

V. Hilbert Spaces over Q

A. Definitions and a Fundamental Theorem.

<u>Definition</u> Let L be a (right) linear space over Q. L will be called an inner product space if there exists a function (,); $L \times L \to Q$ with the properties

(a) $(x,y+z) = (x,y) + (x,z)$

(b) $(x,y\alpha) = (x,y)\alpha$

(c) $(x,y) = \overline{(y,x)}$

(d) $(x,x) > 0 \quad x \neq 0$

It follows immediately from these results that

(e) $(x+y,z) = (x,z) + (y,z)$ and

(f) $(x\alpha,y) = \bar{\alpha}(x,y)$

The next lemma is known (Teichmüller, 1938). The following proof is due to Penico (Penico, 1968).

<u>Lemma 1</u>. (CBS inequality).

If x,y are any two elements of L then

$$|(x,y)| \leq (x,x)^{\frac{1}{2}} (y,y)^{\frac{1}{2}} .$$

Proof. If x or y is the 0-vector the result is clear. Suppose now that neither x nor y is 0. Then

$$0 \leq (x+y\alpha\ ,\ x+y\alpha)(y,y) = (x,x)(y,y) - (x,y)(y,x)$$
$$+ [(x,y) + (y,y)\bar{\alpha}][\ (y,x) + y,y)\alpha]$$

for any $\alpha \in Q$. Choose α such that the product of square brackets vanish. It then follows that $(x,y)(y,x) \leq (x,x)(y,y)$ and the result of the theorem is clear.

Lemma 2. If α is any quaternion then $\alpha + \bar{\alpha} \leq [\alpha\bar{\alpha}]^{\frac{1}{2}} = |\alpha|$.

Theorem 1. Any inner product space L over Q is a normed space.

Proof. Define $\|x\| = (x,x)^{\frac{1}{2}}$ for $x \in L$. By direct calculation it follows that $\|x\| = 0$ iff $x = 0$ and $\|x\alpha\| = \|x\|\cdot|\alpha|$. To show that the triangle inequality holds, Lemmas 1, 2 must be used.

$$\|x+y\|^2 = (x+y,x+y) = (x,x) + (x,y) + (y,x) + (y,y)$$
$$= \|x\|^2 + (x,y) + \overline{(x,y)} + \|y\|^2$$
$$\|x+y\|^2 \leq \|x\|^2 + 2\,|(x,y)| + \|y\|^2 \text{ (Lemma 2)}.$$
$$\text{Or } \|x+y\|^2 \leq \|x\|^2 + 2\,\|x\|\cdot\|y\| + \|y\|^2 \text{ (Lemma 3)}.$$
$$\|x+y\| \leq \|x\| + \|y\| .$$

These results show that any inner product space is a normed space and consequently a metric topological space.

If $(L,(\ ,\ ))$ is complete with respect to the metric defined by $\rho(x,y) = \|x-y\|$ then L will be called a Hilbert Space, (or Wach's Space, see Teichmuller).

Example It has been noted previously that $L_Q^2[0,2\pi]$ is a Banach space over Q when the norm is defined by $\|f\|_2 = (\int_0^{2\pi} |f|^2)^{\frac{1}{2}}$. An inner product can be introduced into $L_Q^2[0,2\pi]$ as follows; for $f,g \in L_Q^2[0,2\pi]$ define $(f,g) = \int_0^{2\pi} \overline{f}\cdot g$. It is readily verified that $(\ ,\ )$ satisfies the properties of an inner product. Moreover, since $(f,f) = \int_0^{2\pi} \overline{f}\cdot f = \int_0^{2\pi} |f|^2 = \|f\|_2^2$, $L_Q^2[0,2\pi]$ is actually a Hilbert Space.

It should be remarked at this point that the important inequality given in Lemma 1 is valid in much more general circumstances. In particular the inequality is true for a class of modules over the general (real) Cayley-Dickson Algebras (Penico, 1968).

As in the case of linear spaces over the Real or Complex field the structure of inner product and in particular Hilbert spaces yields a much richer theory.

Theorem 2. (Parallelogram Law).
Let L be an inner product space over Q. If $x, y \in L$ then $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

Proof. Direct Computation.

The question that now arises is; given a Banach space over Q, when is it a Hilbert space? The answer to the question was given by Jordan & Von Neumann for the case of complex Banach spaces in 1937. The result is given in the following Theorem.

Theorem 3. (Jordan-Von Neumann).
If B is a complex Banach space whose norm satisfies the parallelogram law then B is a Hilbert Space.

It happens that in the case of Banach spaces over Q that the above theorem remains valid as will now be shown.

Theorem 4. If B is a Banach space over Q whose norm satisfies the parallelogram law then there exists an inner product $(\, , \,)$ in B such that $(x,x) = \|x\|^2$ (i.e. B is a Hilbert Space).

Proof. From theorem 3 it is known that the function $(x,y)_R = \frac{1}{4}[\|x+y\|^2 - \|x-y\|^2]$ defines a real inner product on B.

Define: $(x,y) = (x,y)_R - \sum_{i=1}^{3} (x,ye_i)_R e_i$ for $x,y \in B$.

Now it must be verified that (,) satisfies the postulates for an inner product.

(i) $(x,y+z) = (x,y+z)_R - \sum_{k=1}^{3} (x,(y+z)e_k)_R e_k$

$$(x,y+z) = (x,y)_R + (x,z)_R - \sum_{k=1}^{3} (x,ye_k)_R e_k - \sum_{k=1}^{3} (x,ze_k)_R e_k$$

$(x,y+z) = (x,y) + (x,z)$. In this argument the additivity of (,) has been used.

(iii) $(x,y) = \frac{1}{4} \sum_{k=0}^{3} \|x+ye_k\|^2 - \|x-ye_k\|^2$

but for $k \neq 0$, $(x,ye_k)_R = \frac{1}{4}\left[\|x+ye_k\|^2 - \|x-ye_k\|^2 \right]$

and $(x,y\cdot e_k)_R = \frac{1}{4}\left[\|(y-xe_k)e_k\|^2 - \|(y+xe_k)(-e_k)\|^2 \right]$

$$(x,ye_k)_R = -\frac{1}{4}\left[\|(y+xe_k)\|^2 - \|(y-xe_k)\|^2 \right]$$

or $(x,ye_k)_R = \underline{-\ (y,xe_k)_R.}$

Now, $\overline{(x,y)} = (x,y)_R + \sum_{k=1}^{3} (x,ye_k)_R e_k$

$$\overline{(x,y)} = \overline{(x,y)}_R - \sum_{k=1}^{3} \bar{e}_k \overline{(x,ye_k)}_R$$

But $(\ , \ )_R$ is real and symmetric

$\overline{(x,y)} = (y,x)_R - \sum_{k=1}^{3} - (x,ye_k)_R e_k$. Using the that $(x,ye_k)_R = -(y,xe_k)_R$ it follows that

$$\overline{(x,y)} = (y,x)_R - \sum_{k=1}^{3} (y,xe_k)_R e_k = (y,x).$$

(ii) To show that $(x,y\alpha) = (x,y)\alpha$ consider the following

$$(x,ye_1) = (x,ye_1)_R - (x,ye_1 \cdot e_1)_R e_1 - (x,ye_1 \cdot e_2)_R e_2 - (x,ye_1 \cdot e_3)_R e_3$$

$$(x,ye_1) = (x,ye_1)_R + (x,y)_R e_1 - (x,ye_3)_R e_2 + (x,ye_2)_R e_3$$

$$(x,ye_1) = [(x,y)_R - (x,ye_1)_R e_1 - (x,ye_2)_R e_2 - (x,ye_3)_R e_3] e_1$$

or $(x,ye_1) = (x,y)e_1$.

By similar computations, it can be shown that

$(x,ye_j) = (x,y)e_j$ for $j = 2,3$.

Since any quaternion α can be written as $\alpha = \sum_{i=0}^{3} \alpha_i e_i$ where the α_i are real and since $(\ , \ )_R$ is additive and real homogeneous it follows that for any α, $(x,y\alpha) = (x,y)\alpha$.

(iv) $(x,x) = (x,x)_R - \sum_{k=1}^{3} (x,xe_k)_R e_k$

But $(x,xe_k)_R = \frac{1}{4}[\|x+xe_k\|^2 - \|x-xe_k\|^2]$

$$" \quad = \frac{1}{4}[|1+e_k|^2 - |1-e_k|^2]\cdot\|x\|^2$$

$$" \quad = \frac{1}{4}[(1+e_k)\overline{(1+e_k)} - (1-e_k)\overline{(1-e_k)}]\|x\|^2$$

$$" \quad = \frac{1}{4}[(1+e_k)(1-e_k) - (1-e_k)(1+e_k)]\cdot\|x\|^2$$

$(x,xe_k)_R = 0$ for $k = 1,2,3.$

Hence, $(x,x) = (x,x)_R = \|x\|^2$ and therefore

$(x,x) > 0 \quad x \neq 0.$

This completes the proof of the theorem.

Theorem 5. If L is an inner product space over Q then

$$4(x,y) = \sum_{i=0}^{3} (\|x+ye_k\|^2 - \|x-ye_k\|^2)\bar{e}_k \quad \text{where}$$

$\|x\|^2 = (x,X).$

Proof.

For $k = 0$, $\|x+ye_k\|^2 - \|x-ye_k\|^2 = (x+y,x+y) - (x-y,x-y)$

$$" \quad " \quad = (x,x) + (y,x) + (x,y) + (y,y)$$

$$- (x,x) + (y,x) + (x,y) - (y,y)$$

$$" \quad " \quad = 2(y,x) + 2(x,y)$$

for $k \neq 0$,

$$
\begin{aligned}
(\|x+ye_k\|^2 - \|x-ye_k\|^2)e_k &= [(x+ye_k, x+ye_k) - (x-ye_k, x-ye_k)]\, e_k \\
\text{"} \qquad\qquad \text{"} \qquad &= ((x,x)+\bar{e}_k(y,x)+(x,y)e_k+(y,y))e_k \\
&\quad (-(x,x)+\bar{e}_k(y,x)+(x,y)e_k-(y,y))e_k \\
&= 2\bar{e}_k(y,x)e_k - 2(x,y) \\
&= -2(x,y) - 2e_k(y,x)e_k
\end{aligned}
$$

Hence,
$$
\begin{aligned}
\sum_{k=0}^{3} (\|x+ye_k\|^2 - \|x-ye_k\|^2)e_k &= 2(y,x) + 2(x,y) \\
&\quad - 2(x,y) - 2e_1(y,x) \\
&\quad - 2(x,y) - 2e_2(y,x)e_2 \\
&\quad - 2(x,y) - 2e_3(y,x)e_3
\end{aligned}
$$

or
$$
\begin{aligned}
\sum_{k=0}^{3} (\|x+ye_k\|^2 - \|x-ye_k\|^2)e_k &= 2(y,x) - 4(x,y) \\
&\quad - 2e_1(y,x)e_1 - 2e_2(y,x)e_2 \\
&\quad - 2e_3(y,x)e_3
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{3} (\|x+ye_k\|^2 - \|x-ye_k\|^2)e_k &= 4(y,x) - 4(x,y) - 2\sum_{k=0}^{3} e_k(y,x)e_k \\
&= 4(y,x) - 4(x,y) - 2\left[-2(\overline{y,x})\right] \\
&= 4(y,x)
\end{aligned}
$$

Hence, $4(x,y) = \sum_{k=0}^{3} (\|x+ye_k\|^2 - \|x-ye_k\|^2)\bar{e}_k$. This completes the proof of the theorem.

B. Orthogonality in Hilbert Spaces Over Q.

1. Orthoganality and the Projection Theorem.

<u>Definition</u> Let H be a Hilbert space over Q. Two elements x,y of H are called orthogonal if $(x,y) = 0$.

<u>Definition</u> Let S be any subset of H. Then $S^{\perp}$ is defined to be $S^{\perp} = \{ y \mid (x,y) = 0 \text{ for every } x \in S \}$. $(S^{\perp})^{\perp}$ will be written $S^{\perp\perp}$.

The following are easy consequences of these definitions.

(a) $\{0\}^{\perp} = H$ and $H^{\perp} = \{0\}$; $S^{\perp}$ is a subspace of H.

(b) $S \cap S^{\perp} \subseteq \{0\}$

(c) $S_1 \subset S_2 \Rightarrow S_2^{\perp} \subset S_1^{\perp}$

(d) $S^{\perp}$ is a closed subspace of H.

It is a well known result (Simmons, 1963) that for Hilbert spaces over the complex field that if M is any closed subspace of H then $H = M \oplus M^{\perp}$. This result is true for Hilbert spaces over Q as will now be shown. That this result is true seems to have been recognized first by Jauch and his collaborators.

<u>Definition</u> A subset K of a linear space L over Q is convex if given any $x,y \in k$ then $x(1-\lambda) + x\lambda \in K$ for all real numbers $0 \leq \lambda \leq 1$.

The next two lemmas are known results for real and complex Hilbert spaces (Simmons, 1963). Their results can be carried over directly to the case of Hilbert spaces over Q since their proof depends only on the fact that the space is a Real Hilbert Space.

Lemma 1. A closed convex subset K of a Hilbert space over Q contains a unique element of smallest norm.

Lemma 2. Let M be a closed subspace of Hilbert space H over Q. Let $x \in H - M$ and d the distance from x to M. Then there exists a unique element $y' \in M$ such that $d = \|x-y'\|$.

The following theorem is a very important result which will be needed not only for establishing the primary result of this section but will also be needed to establish the Riesz representation theorem.

The theorem is a well known result (Simmons) for complex Hilbert spaces and the proof for Hilbert spaces over Q is almost the same. However, it will be given for completeness since it is scalar dependent.

Theorem 1. If M is a proper closed subspace of a Hilbert space H over Q, then there exists a non-zero element $z \in H$ such that $z \perp M$.

Proof. Let $x \in M$ and $d = \inf_{y \in M} \| x-y \|$ (the distance from x to M). By Lemma 2 there exists $y' \in M$ such that $\|x-y'\| = d$. Define $z' = x-y'$. Clearly $z' \neq 0$ since $d > 0$. To show that $z' \perp M$ let $\alpha \in Q$ and $y \in M$. Then $\|z'-y\alpha\| = \|x-(y'+y\alpha)\| \geqslant d = \|z'\|$. From this it follows that $\|z'-y\|^2 - \|z'\|^2 \geqslant 0$. Using the definition of $\| \; \|$ this can be written as;

$$(z'-y\alpha, z'-y\alpha) - (z', z') \geqslant 0,$$

or

$$- \bar{\alpha}(y, z') - (z', y)\alpha + |\alpha|^2 \|y\|^2 \geqslant 0.$$

Now let $\alpha = \beta(z', y)$ where β is any real number. Then,

$$-2\beta |(z', y)|^2 + \beta^2 |(z', y)|^2 \|y\|^2 \geqslant 0.$$

Now let $a = |(z', y)|^2$, $b = \|y\|^2$. This last equation becomes

$$-2\beta a + \beta^2 ab = a(\beta b - 2) \geqslant 0$$

for all $\beta \in Re$. If $a > 0$ then this last inequality will be false if β is taken sufficiently small and positive. Hence, a must be zero and consequently $z' \perp y$.

The fundamental results of this section can be summarized in the following two theorems. The second is known as the projection theorem, in case of complex Hilbert spaces. Now that the previous theorem has been given, it's conclusions may be used to prove the next two theorems in exactly the same manner as in the case of complex Hilbert spaces. For this reason, their proofs will not be given.

Theorem 2. If M and N are closed linear subspaces of a Hilbert space over Q such that $M \perp N$, then the subspace $M + N$ is closed.

Theorem 3. If M is closed linear subspace of a Hilbert space H over Q, then $M \oplus M^{\perp} = H$.

The significance of these results lie first in the fact that they insure the existence of projections in any Hilbert space over Q.

The second important result that these Theorems yield is related to the lattice structure of the collection $\mathcal{M}$ of closed subspaces of H. The collection $\mathcal{M}$ is ordered by the inclusion relation and the lattice operations $\vee$, $\wedge$ are defined by

(a) $M \wedge N = M \cap N$

(b) $M \vee N = [M \cup N]^{-}$.

The fact that M forms a lattice is well known (Birkhoff and Von-Neumann 1936, Jauch 1963, Varadarajan 1969). The last Theorem implies the lattice M is complemented (Birkhoff 1936). This is a very important result for the structure of Quantum Mechanics (Jauch 1963, Birkhoff-Von-Neumann 1937).

2. Orthonormal Sets in H.

<u>Definition</u> Let Λ be an index set and $\{x_i\}_{i\in\Lambda}$ a subset of a Hilbert space H. $\{x_i\}_{i\in\Lambda}$ will be called an orthonormal set if $(x_i,x_j) = \delta_{ij}$.

Some of the results on orthonormal sets in Hilbert spaces over Q to be presented in this section are due to Teichmuller (Teichmuller 1938). The first theorem is of fundamental importance for the rest of the development.

<u>Theorem 1</u>. Let $x_1,x_2,\ldots,x_n$ be a finite orthonormal set in a Hilbert space H (over Q). If $x \in H$, then

(a) $\sum_{i=1}^{n} |(x,x_i)|^2 \leq \|x\|^2$ (Bessel's inequality)

(b) $x - \sum_{i=1}^{n} x_i(x_i,x) \perp x_j$ for each j.

Proof. If $x \in H$, then $0 \leq \|x - \sum_{i=1}^{n} x_i(x_i,x)\|^2$ or

$$0 \leq \left(x - \sum_{i=1}^{n} x_i(x_i,x),\ x - \sum_{j=1}^{n} x_j(x_j,x)\right)$$

$$0 \leq \|x\|^2 - \sum_{i=1}^{n} |(x_i,x)|^2 - \sum_{i=1}^{n} |(x_i,x)|^2 + \sum_{i=1}^{n} \overline{(x_i,x)}(x_i,x_j)(x_j,x)$$

Using the orthonormality of the x_i it follows that

$$0 \leq \|x\|^2 - 2\sum_{i=1}^{n} |(x_i,x)|^2 + \sum_{i=1}^{n} |(x_i,x)|^2,$$

or $0 \leq \|x\|^2 - \sum_{i=1}^{n} |(x_i,x)|^2$ and (a) follows.

To show (b) consider $(x - \sum_{i=1}^{n} x_i(x_i,x),x_j)$.

$$(x - \sum_{i=1}^{n} x_i(x_i,X),\ x_j) = (x,x_j) - \sum_{i=1}^{n} \overline{(x_i,x)}(x_i,x_j) = 0.$$

<u>Theorem 2</u>. If $\{x_i\}_{i\varepsilon\Lambda}$ is an orthonormal set in a Hilbert Space H, and if $x \varepsilon H$, then the set $S_i = \{x_i |\ (x,x_i)^2 \neq 0\}$ is either empty or countable.

Proof. For each positive integer n, let

$S_n = \{x_i |\ |(x,x_i)|^2 > \frac{\|x\|^2}{n}$. Then, S_n can contain at most n-1 elements for otherwise Bessel's inequality would be contradicted. The conclusion follows from the fact that $S = \bigcup_{n=1}^{\infty} S_n$.

Using this result, Bessel's inequality and the other result of Theorem 1 can be generalized to arbitrary orthonormal sets in H. The proofs given for complex spaces may be carried over directly and will not be given.

<u>Definition</u> An orthonormal set $\{x_i\}_{i\epsilon\Lambda}$ in H is complete if it is maximal in the partially ordered set of all orthonormal sets for H. This class is ordered by inclusion.

The following Theorem was proven for Hilbert Spaces over Q by Teichmuller.

<u>Theorem 3</u>. Every non-zero Hilbert space over Q contains a complete orthonormal set.

Complete orthonormal sets in Hilbert Spaces over Q possess the same properties of such sets in complex Hilbert spaces. In particular the following theorem is valid.

<u>Theorem 4</u>. Let H be a Hilbert space over Q, and let $\{x_i\}_{i\epsilon\Lambda}$ be an orthonormal set in H. The following conditions are equivalent.

(a) $\{x_i\}_{i\varepsilon\Lambda}$ is complete.

(b) $x \perp \{x_i\}_{i\varepsilon\Lambda} \Rightarrow x = 0$

(c) if $x \varepsilon H$, then $x = \sum x_i(x_i,x)$

(d) if $x \varepsilon H$, then $\|x\|^2 = \sum |(x_i,x)|^2$ (Parseval's Relation)

3. An Orthonormal set for $L_Q^2[0,2\pi]$.

Let $L_Q^2[0,2\pi]$ denote the Hilbert space of Q-valued functions on $[0,2\pi]$ for which $\int_0^{2\pi}|f|^2<\infty$. Recall that the inner product for this space is given by

$(f,g) = \int_0^{2\pi} \overline{f}g\,dx$ and the norm is $\|f\| = \left[\int_0^{2\pi}|f|^2\right]^{\frac{1}{2}}$.

To give an example of an orthonormal set for $L_Q^2[0,2\pi]$ let I be any quaternion for which $I^2 = -1$, and define $\exp(Ix) = \cos x + I \sin x$ $(x \varepsilon \mathrm{Re})$. Then the set $\{\exp(Inx)\}$ $n = 0,\pm1,\pm2,\ldots.$ has the following properties:

$$\int_0^{2\pi} \exp(-Imx)\exp(Inx) = \begin{cases} 0 & n \neq m \\ 2\pi & n = m \end{cases}$$

Therefore, the set of functions $\left\{\frac{\exp(Inx)}{\sqrt{2\pi}}\right\}$ $n = 0, \pm 1, \pm 2, \ldots$ is an orthonormal set for $L_Q^2[0, 2\pi]$.

C. H* and the Riesz-Representation Theorem.

If H is a Hilbert Space over Q then H* denotes the class of continuous linear functional on H. If $y \in H$ then $f_y(x) = (y,x)$ is clearly a linear functional on H. Moreover, it follows from the Cauchy-Bunyakowski-Schwartz inequality that f_y is a bounded linear functional and consequently $f_y \in H^*$ for each $y \in H$.

Since $|f_y(x)| = |(y,x)| \leq \|y\| \|x\|$ it folows that $\|f_y\| \leq \|y\|$. If $y = 0$ then $\|f_y\| = 0 = \|y\|$. If $y \neq 0$, $\|f_y\| = \sup \{ |f_y(x)| \mid \|x\| = 1$ and therefore $\|f_y\| \geq | f_y(y \cdot \|y\|^{-1}) = |(y, y \cdot \|y\|^{-1})| = \|y\|$.
It follows then that the mapping $J(y) = f_y$ is a norm preserving map of H into H*. In the next Theorem it will be shown that mapping J is actually a mapping of H onto H*. This very important result is known as the Riesz-Representation Theorem in the case of complex Hilbert Spaces.

Theorem (Riesz). Let H be a Hilbert space over Q and $f \in H^*$. There then exists a unique $y \in H$ such that $f(x) = (y,x)$ for all $x \in H$.

Proof. It is clear from the properties of the inner product that if y exists it is unique. If f is identically zero on H then y may be chosen to be 0 (the null vector).

Now assume $f \neq 0$. Then the null space of f, $N(f) = \{ x \mid f(x) = 0 \}$ is a closed subspace of H. Theorem 1.1 can be employed to yield the existence of a vector $y_0 \in H$ such that $y_0 \perp N(f)$. It will now be shown that $y_0\alpha$ will meet the requirements of the Theorem if α is properly chosen. Clearly, if $y = y_0\alpha$ then for any $x \in N(f)$, $f(x) = 0$ and $(y,x) = 0$. Hence, y satisfies the condition for $x \in N(f)$. To determine the proper scalar α, consider the case $x = y$. Then $f(y_0) = (y_0\alpha, y_0) = \bar{\alpha}\|y_0\|^2$ so $\alpha = \overline{f(y_0)}\|y_0\|^{-2}$. From the projection Theorem it is known that each $x \in H$ can be expressed as $x = x' + y_0\beta$ for $x' \in N(f)$ and some $\beta \in Q$. Now, $f(x) = f(x'+y_0\beta)$ $= f(x') + f(y_0)\beta$. But since $x' \in N(f)$; $f(x') = (y,x')$ and by the choice of α, $f(y_0) = (y,y_0)$ it follows that $f(x) = (y,x') + (y,y_0)\beta = (y, x'+y_0\beta)$. This is true for any $x \in H$ and the theorem is proven.

This Theorem will have a great deal of importance in the treatment of operators on H. In particular, it is one of the most important of the structure theorems for Hilbert Spaces over Q.

<u>Corollary</u> Every Hilbert Space is a reflexive Banach Space.

VI. Operators On Hilbert Spaces Over Q

A. Existence of Adjoints

Let T be a bounded linear transformation on H. Let $y \varepsilon H$ and define the functional $f(x) = (y, Tx)$. $f(x)$ is clearly a bounded linear functional on H. Using the Riesz-Representation Theorem, it follows that there exists a unique $z \varepsilon H$ such that $f(x) = (z,x)$ for every $x \varepsilon H$. The vector z clearly depends on the y chosen initially. To emphasize this, z will be written as $z = T^*y$. Where T^* is a mapping defined on H. The mapping T^* is unique by virtue of the properties of the inner product (,).

It will now be shown that the mapping T^* is actually a bounded linear transformation on H. Let $x,y,z \varepsilon H$ then $(T^*(y+z),x) = (y+z,Tx) = (y,Tx) + (z,Tx)$.
Hence, $(T^*(y+z),x) = (T^*y,x) + (T^*z,x) = (T^*y+T^*z,x)$ and therefore $T^*(y+z) = T^*(x) + T^*(z)$. If $\alpha \varepsilon Q$, $y \varepsilon H$ then $(T^*(y\alpha),x) = (y\alpha, Tx) = \bar{\alpha}(T^*y,x) = ((T^*y)\alpha, x)$.

Since this last statement is true for every $x,y \varepsilon H$ and every $\alpha \varepsilon Q$ it follows that $T^*(y\alpha) = T^*(y)\alpha$. Therefore T^* is a linear transformation. Now to show T^* is bounded. Let $y \varepsilon H$, then

$$\|T^*y\|^2 = (T^*y,T^*y) = (TT^*y,y) \leq \|T(T^*y)\| \cdot \|y\| .$$

Since T is bounded; $\|T^*y\|^2 \leq \|T\| \|T^*y\| \|y\|$ and consequently $\|T^*y\| \leq \|T\| \|y\|$ for all $y \in H$. Therefore, T^* is bounded.

Just as in the case of complex Hilbert Spaces (Simmons 1963) the following theorem is true and the proof carries over directly from the complex case.

<u>Theorem 1</u>. The adjoint operation $T \to T^*$ has the following properties:

(a) $(T_1+T_2)^* = T_1^* + T_2^*$

(b) $(T_1 \cdot T_2)^* = T_2^* \cdot T_1^*$

(c) $T^{**} = T$

(d) $\|T^*\| = \|T\|$

(e) $\|T^*T\| = \|T\|^2$

It should be noted that one important conclusion usually added to the above theorem for complex spaces is not stated. It is the statement that for any Scalar , and bounded linear operator T, $(\alpha \cdot T)^* = \overline{\alpha} \cdot T^*$.

The reason for this is that the class of bounded linear operators on a Hilbert space over Q is not (in general) a linear space over Q. That is, αT is not a linear operator under any suitable definition. This is due to the fact that Q is a non-commutative ring. The lack of this powerful result will cause many difficulties in the spectral theory of linear operators in Hilbert spaces over Q.

It should be remarked however that in finite dimensional (right) spaces over Q the class of linear transformation is a left space over Q. The reason for this is that every linear transformation on such a space is representable by a matrix with quaternion entries. Therefore if T is a linear operator, $\alpha \in Q$, $\alpha \cdot T$ may be identified with $(\alpha \cdot T_{ij})$.

It should be noted at this point that certain classes of operators on $L_Q^2[0,2\pi]$ can be considered as linear spaces over Q. For example, consider the transformation $K \cdot u = \int_0^{2\pi} k(x,t)u(t)dt$ where $u \in L_Q^2[0,2\pi]$ and $k(x,t)$ is a Lebesque integrable function over the rectangle $[0,2\pi] \times [0,2\pi]$ and $k(x,t)$ has values in Q. K will be called an integral operator on $L_Q^2[0,2\pi]$. If K_1, K_2 are two such operators then $(K_1+K_2)u$ will be defined as $K_1u + K_2u$. Scalar multiplication will be defined by; $(\alpha \cdot K)u = \int_0^{2\pi} \alpha \cdot k(x,t)u(t)dt$ for any $\alpha \in Q$. Under these operations the class of integral operators on L_Q^2 form a left space over Q.

Example 1. Formal calculation of the adjoint for an integral operator on $L_Q^2[0,2\pi]$. Let $Ku = \int_0^{2\pi} k(x,t)u(t)dt$.

$$(ku,v) = \int_0^{2\pi} \overline{\left[\int_0^{2\pi} k(x,t)u(t)dt \right]} \; v(x)dx$$

$$= \int_0^{2\pi} \int_0^{2\pi} \overline{u(t)}\; \overline{k(x,t)}dt \; u(x)dx$$

$$= \int_0^{2\pi} \overline{u(x)} \left[\int_0^{2\pi} \overline{k(t,x)}\; v(t)dt \right] dx$$

$$= (u,k^*v) \; . \quad \text{Thus,} \quad K^*v = \int_0^{2\pi} \overline{K(t,x)}\; v(t)dt.$$

<u>Example 2</u>. Formal calculation of the adjoint for an unbounded operator. Let LA $[a,b]$ be the subspace of $L_Q^2[a,b]$ consisting of all functions that are absolutely continuous and have the addition property that $f(a) = f(b) = 0$ for every $f \in$ LA. Let D_t be the operator defined by

$$D_t \; f(t) = \frac{d}{dt} \; f(t).$$

Now let $\alpha \in Q$ and consider $((\alpha \cdot D_t)f,g)$.

$$((\alpha \cdot D_t)f,g) = \int_a^b \overline{\alpha \frac{df}{dt}} \; g \; dt = \int_a^b \overline{\frac{df}{dt}} \; \bar{\alpha} \; g \; dt. \quad \text{But}$$

$$\frac{d}{dt}[\bar{f}\cdot\bar{\alpha}\ g] = \frac{d}{dt}(\bar{f}\bar{\alpha})\cdot g + (\bar{f}\cdot\bar{\alpha})\frac{dg}{dt} \quad \text{so}$$

$$(\frac{d}{dt}\bar{f})\alpha\cdot g = \frac{d}{dt}[\bar{f}\bar{\alpha}]g - \bar{f}\bar{\alpha}\frac{dg}{dt}. \quad \text{Hence,}$$

$$((\alpha\cdot D_t)f,g) = \int_a^b \frac{d}{dt}(\bar{f}\bar{\alpha}\cdot g) - \int_a^b \bar{f}\bar{\alpha}\frac{dg}{dt}. \quad \text{The}$$

first integral vanishes so $((\alpha\cdot D_t)f,g) = \int_a^b \bar{f}(-\bar{\alpha}\frac{d}{dt}g)dt.$

Hence, $((\alpha\cdot D_t)f,g)=(f,(-\alpha\cdot D_t)g)$. That is $(\alpha\cdot D_t)^* = -\bar{\alpha}\cdot D_t$. Now if $\alpha + \bar{\alpha} = t(\alpha) = 0$, it follows that $(\alpha D_t)^* = (\alpha D_t)$. It should be noted that the operator $\alpha\cdot D_t$ is unbounded on $LA[a,b]$.

<u>Theorem 2</u>. Let H be an inner product over Q and let T be a linear transformation on H.

Then $\sum_{k=0}^{3} e_k[(T(x+ye_k),x+ye_k)-(T(x-ye_k),x-ye_k)]$

$= 8(Ty,x) - 4(y,Tx)$

Proof. For $k = 0,1,2,3$

$(T(x+ye_k),x+ye_k)=(Tx,x)+(Tx,y)e_k+\bar{e}_k(Ty,x)+\bar{e}_k(Ty,y)e_k$

and $(T(x-ye_k),x-ye_k)=(Tx,x)-(Tx,y)e_k-\bar{e}_k(Ty,x)+\bar{e}_k(Ty,y)e_k$.

Hence,

$$\sum_{k=0}^{3} e_k [\ (T(x+ye_k),x+ye_k) - (T(x-ye_k),x-ye_k)\ ]$$

$$= 2\sum_{k=0}^{3} e_k [\ (Tx,y)e_k + \bar{e}_k(Ty,x)\ ]$$

$$= 2\sum_{k=0}^{3} e_k(Tx,y)e_k + 8(Ty,x)$$

$$= 2(-2\ (\overline{Tx,y})) + 8(Ty,x)$$

$$= 8(Ty,x) - 4(\overline{Tx,y})$$

$$= 8(Ty,x) - 4\ (y,Tx)$$

Corollary. If H is an inner product space over Q and T is an operator such that $(Ty,x) = (y,Tx)$ for every $x,y \in H$ then $(Ty,x) = \frac{1}{4}\sum_{k=0}^{3} e_k \left[\ (T(x+ye_k),x+ye_k) - (T(x-ye_k),x-ye_k)\right]$.

Corollary. If H is an inner product space over Q and T is an operator on H such that $(Ty,y) = 0$ for all $y \in H$ then $T = 0$.

Proof. From Theorem 2 it follows that $8(Ty,x) - 4(y,Tx) = 0$ and $8(Tx,y) - 4(x,Ty) = 0$ for every $x,y \in H$. But then $2(Ty,x) = (y,Tx)$ and $(y,Tx) = \frac{1}{2}(Ty,x)$ which implies $2(Tx,y) = \frac{1}{2}(Tx,y)$. Thus $T = 0$.

It is worth noting that this corollary is not true in the case of real inner product spaces. The technique used in the proof of this corollary is due to Penico.

B. Bilinear and Hermitian Bilinear Forms On Linear Spaces Over Q.

In this section certain results about Bilinear (or sesquilinear) forms on linear spaces over Q will be developed. These results will be used for certain aspects of spectral theory.

<u>Definition</u> Let L be a linear space over Q. A functional ϕ defined on L X L to Q will be called bilinear (or sesquilinear) if

(a) $\phi(x+y,z) = \phi(x,z) + \phi(y,z)$

$\phi(x,y+z) = \phi(x,y) + \phi(x,z)$

(b) $\phi(x\alpha,y) = \bar{\alpha}\,\phi(x,y)$

$\phi(x,y\alpha) = \phi(x,y)\alpha$

<u>Definition</u> If ϕ is a bilinear functional $\hat{\phi}(x)$ will be defined as $\hat{\phi}(x) = \phi(x,x)$. A bilinear functional will be called Hermitian if $\phi(x,y) = \phi(y,x)$. It follows that if $\phi(x,y)$ is Hermitian, then $\phi(x,x) = \phi\overline{(x,x)}$ and therefore $\hat{\phi}$ is real.

<u>Theorem 1</u>. If ϕ is a Hermitian bilinear functional on L then $\phi(x,y) = \frac{1}{4}\sum_{k=0}^{3} \bar{e}_k [\hat{\phi}(x+ye_k) - \hat{\phi}(x-ye_k)]$.

Proof. $\hat{\phi}(x+ye_k) - \hat{\phi}(x-ye_k) = \phi(x+ye_k,x+ye_k) - \phi(x-ye_k,x-ye_k)$

but $\phi(x+ye_k,x+ye_k)=\phi(x,x)+\bar{e}_k\,\phi(y,x)+\phi(x,y)e_k+\bar{e}_k\,\phi(y,y)e_k$

and $\phi(x-ye_k,x-ye_k)=\phi(x,x)-\bar{e}_k\,\phi(y,x)-\phi(x,y)e_k+\bar{e}_k\,\phi(y,y)e_k$

Hence, $\hat{\phi}(x+ye_k)-\hat{\phi}(x-ye_k)=2\bar{e}_k\,\phi(y,x)+2\phi(x,y)e_k$.

For $k=0$, $\hat{\phi}(x+ye_k) - \hat{\phi}(x-ye_k) = 2\phi(y,x) + 2\phi(x,y)$

and for $k \neq 0$, $e_k[\hat{\phi}(x+ye_k)-\hat{\phi}(x-ye_k)] = 2\phi(y,x)+2e_k\phi(x,y)$.

It follows that,
$$\begin{aligned}\sum_{k=0}^{3} e_k[\hat{\phi}(x+ye_k)-\hat{\phi}(x-ye_k)] &= 2\phi(y,x)+2\phi(x,y)\\ &+2\phi(y,x)+2e_1\phi(x,y)e_1\\ &+2\phi(y,x)+2e_2\phi(x,y)e_2\\ &+2\phi(y,x)+2e_3\phi(x,y)e_3.\end{aligned}$$

Thus, $\sum_{k=0}^{3} e_k[\hat{\phi}(x+ye_k)-\hat{\phi}(x-ye_k)] = 8\phi(y,x)+2[-2\,\overline{\phi(x,y)}\,]$

or $\sum_{k=0}^{3} \bar{e}_k[\hat{\phi}(x+ye_k)-\hat{\phi}(x-ye_k)] = 4\phi(y,x)$. Taking the conjugate of both sides yields,

$$\phi(x,y) = \frac{1}{4}\sum_{k=0}^{3} \bar{e}_k[\hat{\phi}(x+ye_k) - \hat{\phi}(x-ye_k)].$$

<u>Corollary</u>. If two Hermitian Bilinear functionals ϕ and Ψ have the property that $\hat{\phi} = \hat{\Psi}$ then $\phi = \Psi$.

<u>Definition</u> A bilinear functional is bounded iff there exists a real number $\alpha > 0$ such that $|\phi(x,y)| \leq \alpha \|x\| \|y\|$. If ϕ is a bounded linear functional, the norm of ϕ is defined by $\|\phi\| = \sup\limits_{\substack{x \neq 0 \\ y \neq 0}} \frac{|\phi(x,y)|}{\|x\| \|y\|}$.

The induced quadratic form is said to be bounded if there exists a real number $\alpha > 0$ such that $\hat{\phi}(x) \leq \alpha \|x\|^2$ for all x. If $\hat{\phi}$ is bounded, $\|\hat{\phi}\| = \sup\limits_{x \neq 0} \frac{|\hat{\phi}(x)|}{\|x\|}$

If ϕ is a bilinear functional then $|\phi(x_1,y_1) - \phi(x_2,y_2)| = |\phi(x_1-x_2,y_1-y_2)|$. If ϕ is bounded, $|\phi(x_1,y_1) - \phi(x_2,y_2)| \leq \alpha \|x_1-x_2\| \cdot \|y_1-y_2\|$ and clearly is continuous. Using essentially the same technique as for the case of linear functionals the converse of this last statement can also be shown to be true. Hence, the following theorem can be given.

<u>Theorem 2</u>. A bilinear functional ϕ is continuous iff it is bounded. Also, the associated quadratic form $\hat{\phi}$ is continuous iff it is bounded.

The boundedness of ϕ and $\hat{\phi}$ are related through the following theorem.

<u>Theorem 3</u>. The quadratic form $\hat{\phi}$ associated with the bilinear functional ϕ is bounded iff ϕ is bounded. Moreover, if $\phi, \hat{\phi}$ are bounded the following relation holds. $\|\hat{\phi}\| \leq \|\phi\| \leq 4\|\hat{\phi}\|$.

Proof. If ϕ is bounded there exists a real number $\alpha > 0$ such that $|\phi(x,y)| \leq \alpha \|x\| \|y\|$ and consequently $|\hat{\phi}(x)| \leq \alpha \cdot \|x\|^2$ for all x. Thus $\hat{\phi}$ is bounded.

If $\hat{\phi}$ is bounded, then

$$|\phi(x,y)| \leq \tfrac{1}{4}\|\hat{\phi}\| \sum_{k=0}^{3} (\|x+ye_k\|^2 + \|x-ye_k\|^2)$$

but, $\|x+ye_k\|^2 + \|x-ye_k\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all x,y. Therefore, $|\phi(x,y)| \leq \tfrac{1}{4}\|\hat{\phi}\| \; 8(\|x\|^2 + \|y\|^2)$
$= 2\|\hat{\phi}\| \; (\|x\|^2 + \|y\|^2) \Rightarrow \|\phi\| \leq 4\|\hat{\phi}\|$.

If ϕ is a Hermitian bilinear functional the relation between $\|\phi\|$ and $\|\hat{\phi}\|$ is even stronger. In particular, the following is true.

<u>Theorem 4</u>. If ϕ is a bounded Hermitian bilinear functional, then $\|\phi\| = \|\hat{\phi}\|$.

Proof. In the preceding theorem, it has been shown that $\|\hat{\phi}\| \leq \|\phi\|$. For any x,y, $t[\phi(x,y)] = \hat{\phi}(\frac{x+y}{2}) - \hat{\phi}(\frac{x-y}{2})$. Hence, $|t[\phi(x,y)]| \leq \frac{\|\hat{\phi}\|}{4}(\|x+y\|^2 + \|x-y\|^2)$, or $|t[\phi(x,y)]| \leq \tfrac{1}{2}\|\hat{\phi}\| (\|x\|^2 + \|y\|^2)$ frowm which it follows that $t[\phi(x,y)] \leq o$. Now let

x,y be any two vectors and α a quaternion with $|\alpha| = 1$ and $|\phi(x,y)| = \phi(x,y)\alpha$. Then it follows that $\phi(x,y\alpha) = |\phi(x,y)| = |t[\phi(x,y\alpha)]| \leq \|\hat{\phi}\|$. This inequality establishes the theorem.

The preceding theorms on bilinear functionals will now be applied to yield an important theorem pertaining to bounded linear operators on a Hilbert space over Q.

<u>Theorem 5</u>. Let H be a Hilbert space over Q and A a bounded linear operator on H. If ϕ is defined by $\phi(x,y) = (Ax,y)$ then ϕ is a bounded bilinear functional and $\|\phi\| = \|A\|$. If, conversely, ϕ is a bounded bilinear functional, then there exists a unique bounded linear operator A such that $\phi(x,y) = (A\cdot x,y)$.

Proof. If A is a bounded linear operator on H, then $\phi(x,y) = (Ax,y)$ is a bilinear functional on H. $|\phi(x,y)| = |(Ax,y)| \leq \|A\|\,\|x\|\,\|y\|$ for all $x,y \in H$ and consequently $\|\phi\| \leq \|A\|$. But $\|Ax\|^2 = (Ax,Ax)$ and $(Ax,Ax) = \phi(x,Ax) \leq \|\phi\|\cdot\|x\|\cdot\|Ax\|$, from which, $\|Ax\| \leq \|\phi\|\cdot\|x\|$ and hence $\|A\| \leq \|\phi\|$. These two inequalities establish that $\|A\| = \|\phi\|$.

Now suppose that ϕ is a bounded bilinear functional on H. The functional $f_x(y) = \phi(x,y)$ is a bounded linear functional of H. By the Riesz-Representation theorem there exists a unique $z \in H$ for which $f_x(y) = (z,y)$ for all $y \in H$. Since z clearly depends on x, z will be written as $z = Ax$.

It will now be shown that the transformation defined by $z = Ax$ is actually a bounded linear transformation on H.

A is additive:
$$\begin{aligned}(A(x_1+x_2),y) &= \phi(x_1+x_2,y) = \phi(x_1,y) + \phi(x_2,y)\\ &= (Ax_1,y) + (Ax_2,y)\\ &= (Ax_1+Ax_2,y)\end{aligned}$$
and consequently, $A(x_1+x_2) = Ax_1+Ax_2$

A is homogeneous:
$$\begin{aligned}(A(x\alpha),y) &= \phi(x\alpha,y) = \bar{\alpha}\phi(x,y)\\ &= \bar{\alpha}(Ax,y) = ((Ax)\alpha,y)\\ &\Longrightarrow A(x\alpha) = (Ax)\alpha .\end{aligned}$$

A is bounded: $\|Ax\|^2 = (Ax,Ax) = \phi(x,Ax) \leq \|\phi\|\cdot\|x\|\cdot\|Ax\|$ from which it follows that $\|Ax\| \leq \|\phi\|\cdot\|x\|$ for all x. Therefore, A is bounded.

This last result establishes the theorem.

VII. Spectral Representations

A. Self-adjoint And Normal Operators.

In this section elementary properties of self-adjoint and normal operators will be investigated.

<u>Definition</u> Let A be a bounded linear transformation (operator) on H. A will be called

(a) Self adjoint if $A = A^*$

(b) Normal if $A\,A^* = A^*A$.

<u>Definition</u> If A is any linear operator on H and $x \in H$ $(x \neq 0)$ for which $Ax = x\lambda$ for some $\lambda \in Q$ then x is an eigenvector of A and λ the eigenvalue of A corresponding to x.

The case of self-adjoint operators will first be considered.

<u>Theorem 1</u>. The eigenvalues of a self-adjoint operator are real.

Proof. Suppose $Ax = x\mu$. Then $(Ax,x) = (x\mu,x)$ and $(Ax,x) = (x,Ax) = (x,x\mu)$. Consequently, $\bar{\mu}(x,x) = (x,x)\mu$ and since (x,x) is real, $(\bar{\mu}-\mu)(x,x) = 0$ from which it follows that $\mu = \bar{\mu}$.

Theorem 2. The eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof. Suppose $Ax = x\lambda$ and $Ay = y\mu$ with $\lambda \neq \mu$. Hence, $\lambda(x,y) = (x,y)\mu$ since λ, μ are real. But then $(\lambda - \mu)(x,y) = 0$ and since $\lambda \neq \mu$, $(x,y) = 0$.

The previous two theorems are well know results for the case of complex linear spaces and these results are also known for linear spaces over Q. (Teichmuller, 1938; Jauch, 1963).

The results analogous to the above theorems will now be considered for the case of normal operators. It will be shown that there is a major difference in the context of the theorems for normal operators and this is due to the lack of commutativity of the quaternions.

First it should be noted that if $Ax = x\mu$ where A is a normal operator, then $A(x\alpha) = (x\mu)\alpha = (x\alpha)(\alpha^{-1}\lambda\alpha)$. That is, the scalar multiple of an eigenvector with eigenvalue μ of a normal operator is an eigenvector of A with eigenvalue $\alpha^{-1}\mu\alpha$. Here, normality is not required.

In view of this result, it is clear that given an "eigen pair" of a normal operator A, uncountably many "eigenpairs" of A can be constructed. This is not a good situation but fortunately the eigenpairs can be separated into equivalence classes.

Let $\Lambda(A)$ be the totality of eigenvalues of A. Given any two elements $\lambda_1, \lambda_2 \varepsilon \Lambda(A)$, then $\lambda_1 \sim \lambda_2$ iff $\lambda_1 = \mu^{-1} \lambda_2 \mu$ for some $\mu \varepsilon Q$. It can be shown easily that $\sim$ is an equivalence on $\Lambda(A)$. If $\mathcal{E}(A)$ denotes the class of all eigenvectors of A then any two elements $x_1, x_2 \varepsilon \mathcal{E}(A)$ are said to be equivalent if $x_1 = x_2 \cdot \alpha$ for some $\alpha \varepsilon Q$. Clearly, equivalent eigenvectors correspond to equivalent eigenvalues.

In order to use these concepts the following lemma will be needed.

<u>Lemma 1</u>. If $Ax = x\mu$ then $A^*x = x\bar{\mu}$ when A is normal.

Proof.

$$\|Ax-x\mu\|^2 - \|A^*x-x\bar{\mu}\|^2 = (Ax-x\mu, Ax-x\mu) - (A^*x-x\bar{\mu}, A^*x-x\bar{\mu})$$

$$\|Ax-x\mu\|^2 - \|A^*x-x\bar{\mu}\|^2 = (Ax,Ax)-(Ax,x\mu)-(x\mu,Ax)+|\mu|^2(x,x)$$

$$-(A^*x,A^*x)+(A^*x,x\bar{\mu})+(x\bar{\mu},A^*x)$$

$$-|\mu|^2 (x,x).$$

Using the normality of A, this last expression can be written as:

$$\|Ax-x\mu\|^2 - \|A^*x-x\bar{\mu}\|^2 = \|Ax\|^2 - (Ax,x)\mu - \bar{\mu}(x,Ax) - \|Ax\|^2 + (x,Ax)\bar{\mu} + \mu(Ax,x)$$

or $\|Ax-x\mu\|^2 - \|A^*x-x\bar{\mu}\|^2 = [(x,Ax),\bar{\mu}] + [\mu,(Ax,x)]$, where $[\, ,]$ denotes the commutator bracket. Clearly now, if $Ax = x\mu$ the commutators on the right side of this last equation vanish identically. The theorem is then established.

Recall that for the case of complex Hilbert spaces this lemma is a trivial corollary of the theorem on the properties of adjoints. In particular, it is a result of the fact that $(\mu\cdot Ax,y) = (x,\bar{\mu}A^*y)$ for any operator A. It has already been noted in a previous section why this result is not true for Hilbert Spaces over Q.

The main result of this section can now be established.

Theorem 3. If x_1 and x_2 are eigenvectors of a normal operator A corresponding to inequivalent eigenvalues then $(x_1,x_2) = 0$.

Proof. $(Ax_1,x_2) = \bar{\lambda}_1(x_1,x_2) = (x_1,A^*x_2) = (x_1,\bar{\lambda}_2 x_2)$. Hence, $\bar{\lambda}_1(x_1,x_2) = (x_1,x_2)\bar{\lambda}_2$ or equivalently $(x_2,x_1)\lambda_1 = \lambda_2(x_2,x_1)$. Now, if $(x_2,x_1) \neq 0$ then $\lambda_1 = (x_2,x_1)^{-1}\lambda_2(x_2,x_1)$ which contradicts the hypothesis that λ_1 and λ_2 are inequivalent. Hence, $(x_1,x_2) = 0$. That is, the inequivalent eigenvectos of a normal operator are orthogonal.

B. The Spectral Theorems in Finite Dimensional Spaces.

It is well known that a linear transformation (Jacobson, 1953) acting on a finite dimensional vector space over a division ring Δ can be represented by a matrix with entries from Δ. Thus, the eigenvalue problem $Tx = x\mu$ is equivalent to the matrix eigenvalue problem $\sum_{j=1}^{n} T_{ij} x_j = x_i\mu \quad i = 1,2,\ldots n$. J. L. Brenner (Brenner, 1951) has shown, using a result of Eilenberg and Niven (Eilenberg, S., Niven, I., 1944) that any nonsingular matrix (A_{ij}) (with quaternion entries) has at least one eigenvalue. This is exactly what is needed to generalize the Theorems on spectral resolution of certain complex matrices to the case under consideration.

The first case to be considered is that of a self-adjoint operator on a finite dimensional Hilbert space over Q.

<u>Theorem 1</u>. Let H be a Hilbert space over Q of dimension N. Let A be a self-adjoint operator on H. Then there exists a basis for H consisting of eigenvectors of A.

Proof. From the result of Brenner it follows that there exists at least one $x_1 \in H$ and $\lambda_1 \in Q$ for which $Ax_1 = x_1\lambda_1$. If $\dim(H) = 1$, the basis may be taken to be $\{ x_1 \|x_1\|^{-1} \}$. For the case $\dim(H) > 1$ induction will be used.

The conclusion of the Theorem is assumed to be true for all spaces L of $\dim(L) < \dim(H)$. Let $M_1 = [x_1]$ and then $H = M_1 \oplus M_1^{\perp}$ (by the projection Theorem). Clearly $\dim(M_1) < \dim(H)$. Since M_1 is an invariant subspace of A, M_1 will be an invariant subspace of $A^* = A$. Consider now the restriction $A|M_1^{\perp}$ of A to $M_1^{\perp}$. With $A^* = A$ and $A^*|_{M_1^{\perp}} = A|_{M_1^{\perp}}$ it follows from the hypothesis that M_1 has a basis $\{x_k\}_{k=1}^{n}$ consisting of eigenvectors of $A|_{M_1^{\perp}}$. But $A|_{M_1^{\perp}}$ is merely the restriction of A to $M_1^{\perp}$ and consequently each eigenvector of $A|_{M_1^{\perp}}$ is also an eigenvector of A. Therefore $\{x_k\}_{k=1}^{n}$ is a basis for H.

This Theorem can be used to obtain a "spectral" representation for A.

If $y \in H$, then y can be written as $y = \sum_{k=1}^{n} x_k(x_k,y)$ where the x_k are eigenvectors of A.

$$Ay = \left(\sum_{k=1}^{n} x_k(x_k,y)\right) = \sum_{k=1}^{n} (Ax_k)(x_k,y) = \sum_{k=1}^{n} x_k \lambda_k(x_k,y).$$

Since the λ_k are real, $Ay = \sum_{k=1}^{n} x_k(x_k,y)\lambda_k$. It is important to note that the λ_k need not be distinct.

Define $P_k y = \sum_{\lambda_i=\lambda_k} x_i(x_i,y)$. Then the following lemmas are valid.

<u>Lemma 1</u>. $P_k^2 = P_k$

Proof. $P_k(P_k y) = \sum_{\lambda_j=\lambda_k} x_j(x_j, \sum_{\lambda_i=\lambda_k} x_i(x_i,y))$.

$$P_k(P_k y) = \sum_{\lambda_j=\lambda_k}\sum_{\lambda_i=\lambda_k} x_j(x_j,x_i)(x_i,y)$$

$$= \sum_{\lambda_j=\lambda_k}\sum_{\lambda_i=\lambda_k} x_j \quad \delta_{ji}(x_i,y)$$

Hence, $P_k^2 y = \sum_{\lambda_j=\lambda_k} x_j(x_j,y) = P_k y$.

<u>Lemma 2</u>. $\sum_{k=1}^{n} P_k = I$

Proof. $\sum_{k=1}^{n} P_k y = \sum_{k=1}^{n} P_k y = \sum_{k=1}^{n} \sum_{\lambda_i=\lambda_k} x_i(x_i,y)$

$$= \sum_{k=1}^{n} x_k(x_k,y) = I\cdot y.$$

Lemma 3. $P_k P_j = 0 \quad k \neq j$

Proof. $P_j y = \sum_{\lambda_i=\lambda_j} x_i(x_i,y)$

$$P_k(P_j y) = \sum_{\lambda_i=\lambda_k} x_i\Big(x_i, \sum_{\lambda_i=\lambda_j} x_i(x_i,y)\Big)$$

$$= \sum_{\lambda_i=\lambda_k} \sum_{\lambda_i=\lambda_j} (x_i,x_i)(x_i,y)$$

$$P_k(P_j y) = \sum_{\lambda_i=\lambda_k} \sum_{\lambda_i=\lambda_j} \delta_{ii}(x_i,y) = 0 \text{ unless } k = j.$$

Lemma 4. P_k is Self-adjoint for each k.

Proof. Let x,y be any two vectors in H. Then

$$P_k x = \sum_{\lambda_j=\lambda_k} x_j(x_j,x), \quad y = \sum_{k=1}^{n} x_k(x_i,y).$$

$$(P_k x, y) = \sum_{\lambda_j = \lambda_k} \sum_{i=1} \overline{(x_j, x)}(x_j, x_i)(x_i, y).$$

On the other hand, $P_k y = \sum_{\lambda_j = \lambda_k} x_j(x_j, y)$, $x = \sum_{k=1}^{n} x_k(x_k, x)$.

$$(x, P_k y) = \sum_{\lambda_j = \lambda_k} \sum_{i=1} \overline{(x_i, x)}(x_i, x_j)(x_j, y) = (P_k x, y).$$

The operators P_k satisfy the usual requirement of projection operators on a complex linear space (Berberian, S. K., 1961) and will be refered to as orthogonal projections.

As noted before, for each $y \in H$, Ay can be written as $Ay = \sum_{k=1}^{n} x_k(x_k, y)\ \lambda_k$. If one assumes that A has m distinct eigenvalues Ay can be written as

$$Ay = \sum_{j=1}^{m} \Big(\sum_{\lambda_i = \lambda_j} x_i, y) \Big)\ \lambda_j = \sum_{j=1}^{m} (P_j x)\ \lambda_j\ .$$ If

$\lambda_j \cdot P_j(x)$ is defined as $P_j(x) \cdot \lambda_j$ then Ay can be expressed as $Ay = \sum_{j=1}^{m} (\lambda_j \cdot P_j)(y)$. Since this is true for every $y \in H$ then it makes sense to write $A = \sum_{j=1}^{m} \lambda_j \cdot P_j$. This is called the spectral representation of A.

The main results of this section can now be summed up in the following Theorem.

<u>Theorem 2</u>. Let A be a self-adjoint operator on a finite dimensional Hilbert space H. Then there exists distinct eigenvalues $\lambda_1, \lambda_2, \dots \lambda_m$ and orthogonal projections $P_1, P_2, \dots, P_m$ such that $A = \sum_{j=1}^{m} \lambda_j \cdot P_j$.

From the spectral representation of an operator it is easy to evaluate operations such as $A^2 y$. It is easy to verify that $A^2 = \sum_{j=1}^{m} \lambda_j^2 \cdot P_j$. In general the following is valid for Hermitian operators A.

<u>Corollary</u>. Let p(t) be a polynomial in the variable t of degree n with real coefficients. If p(A) denotes the associated polynomial operator then $p(A) = \sum_{j=1}^{m} p(\lambda_j) P_j$.

The case of normal operators will now be considered. Some of the results of the previous section will carry over, but not all. It is the case of normal operators which brings out the difficulties arising from the fact that the quaternions are not commutative.

Theorem 3. Let A be a normal operator on a Hilbert Space H over Q with dimension n. Then, there exists a basis for H consisting of eigenvectors of A.

Proof. Brenner's result again yields the existence of at least one eigenpair $\{\lambda_1, x_1\}$ for A. If $\dim(H) = 1$, $\{ x_1 \cdot \|x_1\|^{-1} \}$ will suffice as a basis for H. If $\dim(H) > 1$ an induction argument can be given.

It will be assumed that the conclusion is true for all spaces L with $\dim(L) < d(H)$. Let $\{\lambda_1, x_1\}$ be an eigenpair of A. Define $M_1 = [x_1]$. Then by the projection theorem, $H = M_1 \oplus M_1^{\perp}$. M_1 is an invariant subspace of A and consequently $M_1^{\perp}$ is an invariant subspace of A*. It has been shown that the eigenvectors of A are also eigenvectors of A* and it follows that $M_1^{\perp}$ is an invariant subspace of $(A^*)^* = A$. The proof can now be completed by the corresponding argument for self-adjoint operators.

It should be clear that the basis consists of inequivalent eigenvectors of A.

In view of this result, any $y \in H$ can be written as $y = \sum_{j=1}^{n} x_j(x_j,y)$. Hence, $Ay = A(\sum_{j=1}^{n} x_j(x_j,y))$ or

$Ay = \sum_{j=1}^{n} (Ax_j)(x_j,y) = \sum_{j=1}^{n} x_j \cdot \lambda_j(x_j\ y)$. Contrary to the self-adjoint case, the λ_j are not necessarily all real and consequently $x_j\ \lambda_j(x_j,y)$ can't be written as $x_j(x_j,y)\cdot\lambda_j$. This fact is critical. It means that the nature of the operator can't (in general) be explained by means of certain sums of projection operators. Therefore, the most that can be said (apparently) in this case is given in the following Theorem.

<u>Theorem 4</u>. Let H be a finite dimensional Hilbert space of dimension over Q. Let A be a normal operator on H. Then there exists a collection of inequivalent orthogonal eigenvectors $x_1,x_2,\ldots,x_n$ and eigenvalues $\lambda_1, \lambda_2,\ldots, \lambda_n$ of A such that $Ay = \sum_{k=1}^{n} x_k\, \lambda_k(x_k,y)$ for every $y \in H$.

<u>Corollary</u>. If J is a positive integer then $A^J_y = \sum_{k=1}^{n} x_k\, \lambda_k^J(x_k,y)$. In general if p(t) is a polynomial of degree J with real coefficients then

$$p(A)y = \sum_{k=1}^{n} x_k p(\lambda_k)(x_k, y).$$

C. Compact Self-adjoint Operators and Their Spectral Representation.

In this section a spectral representation for a certain type of self-adjoint operator will be given. In particular compact operators will be studied. The first thing that must be considered is the existence of eigenvalues.

In the section on Self-adjoint Bilinear forms the following theorem was proven.

Theorem 1. If A is a bounded self-adjoint operator on H and $\phi(x,y) = (Ax,y)$, then ϕ is a bounded bilinear functional on H and $\|\phi\| = \|A\|$. It was also proven that;

Theorem 2. If ϕ is a bounded self-adjoint, bilinear functional on H then $\|\phi\| = \|\hat{\phi}\|$ where $\hat{\phi}(x) = \phi(x,x)$.

With the use of these two theorems the following important theorem can be proven.

<u>Theorem 3</u>. If A is a self-adjoint operator on H then $\|A\| = \sup_{\|x\|=1} |(Ax,x)|$.

Proof. Let $\phi(x,y) = (Ax,y)$ for every $x,y \in H$. Then by the first theorem $\|\phi\| = \|A\|$. By the second theorem $\|\phi\| = \|\hat{\phi}\|$ and $\|\hat{\phi}\| = \sup_{\|x\|=1} |(Ax,x)|$. Hence, $\|A\| = \sup_{\|x\|=1} |(Ax,x)|$.

Using this theorem, it will now be possible to show that eigenvalues exist for certain self-adjoint operators.

<u>Definition</u> Let X,Y be normed linear spaces over F (where F = Re, K, Q). Suppose T is a linear operator with dom$(T) = X$ and Range$(T) = Y$. Then T is compact if for each bounded sequence $\{x_n\}$ from X, $\{Tx_n\}$ contains a convergent subsequence in Y. This is equivalent to saying T takes bounded sets into compact sets.

The following two lemmas are well known results for compact operators on Hilbert spaces over Re and K. Their proofs can be easily carried over to the case of Q-Hilbert spaces and will not be given. (Berberian, S., 1961).

Lemma 1. If S and T are compact then S + T is compact.

Lemma 2. Any operator with finite dimensional range is compact.

Definition Let $Ku = \int_a^b K(x,t)\, u(t)dt$ be a bounded integral operator on $L_Q^2[a,b]$. If K(x,t) $\sum_{k=1}^{n} p_k(x)\, q_k(t)$, with the p_k and q_k being Q-valued functions such that

$$\int_a^b |q_k(t)|^2 dt < \infty \quad \text{and} \quad \int_a^b |p_k(x)|^2 dx < \infty \quad ,$$

then K(x,t) is called a separable kernel. In this case K will also be called a separable integral operator.

Example. Any separable kernel generates a compact operator on $L^2_Q[a,b]$.

Proof.

$$Ku = \int_a^b \left(\sum_{k=1}^{n} p_k(x)\, q_k(t)\right)u(t)dt = \sum_{k=1}^{n} p_k(x) \int_a^b q_k(t)u(t)dt$$

$Ku = \sum_{k=1}^{n} p_k(x)a_k$ where $a_k = \int_a^b q_k(t)u(t)dt$. Hence, the range of K is finite dimensional and therefore K is compact.

Theorem 3. Let A be compact, self-adjoint and $A \neq 0$. Then either $\|A\|$ or $-\|A\|$ is an eigenvalue of A and there exists an x for which $|(Ax,x)| = \|x\|$.

Proof. Since $\|A\| = \sup_{\|x\|=1} |(Ax,x)|$ there exists a sequence of vectors $\{x_n\}$ with $\|x_n\| = 1$ such that $\lim_{n\to\infty} (Ax_n,x_n) = \lambda$ with $|\lambda| = \|A\|$. But,

$$\|Ax_n - x_n\lambda\|^2 = (Ax_n - x_n\lambda, Ax_n - x_n\lambda) = \|Ax_n\|^2 - 2(Ax_n,x_n)\lambda + \lambda^2.$$

From this it follows that;

$$\|Ax_n - x_n\lambda\|^2 \leq \|A\|^2 - 2(Ax_n,x_n)\lambda + \lambda^2 \text{ and}$$

consequently, $\lim_{n\to\infty} \| Ax_n - x_n\lambda \|^2 \leq \lim_{n\to\infty} [\ \|A\|^2 - 2(Ax_n,x_n)\lambda + \lambda^2] = \|A\|^2 - \lambda^2$, whence $\lim_{n\to\infty} \| Ax_n - x_n\lambda \|^2 = 0$, from which $\lim_{n\to\infty} (Ax_n - x_n\lambda) = 0$.

Since A is compact and $\{x_n\}$ is a bounded sequence, $\{Ax_n\}$ must contain a convergent subsequence $\{Ax_{n_k}\}$, where $\{x_{n_k}\}$ is the appropriate subsequence of $\{x_n\}$. Moreover, with the fact that $\lambda \neq 0$, it is clear that $\{x_{n_k}\}$ is convergent. Let $x = \lim_{k\to\infty} x_{n_k}$ then $\|x\| = 1$ and $\lim_{k\to\infty} Ax_{n_k} = \lim_{k\to\infty} x_{n_k}\cdot\lambda = (\lim_{k\to\infty} x_{n_k})\lambda = x\lambda$.

But A is bounded and thus continuous so $\lim_{k\to\infty} Ax_{n_k} = A(\lim_{k\to\infty} x_{n_k}) = Ax \Rightarrow Ax = x\lambda$. Therefore, A has at least one eigenvalue and $\lambda\|x\|^2 = \lambda = |(Ax,x)| = \|x\|$.

Now that it is known that every compact self-adjoint operator on a Hilbert space H over Q has at least one eigenpair, the procedure used in the finite dimensional case will be used to generate a sequence of such eigenpairs.

Let $\{\lambda_1, x_1\}$ denote the eigenpair whose existence was demonstrated in the preceding theorems. Define $H_1 = H$ and $H_2 = \{\, y \mid (x_1,y) = 0 \,\}$. H_2 is an invariant subspace of A and the restriction $A|_{H_2}$ is also a compact symmetric operator. If $A|_{H_2} \neq 0$, the preceding theorem can be employed to $A|_{H_2}$ to assert the existence of $x_2 \varepsilon H_2$ and $\lambda_2 \varepsilon Q$ such that $A|_{H_2}\ x_2 = x_2 \cdot \lambda_2$, and again, $|\lambda_2| = \|A|_{H_2}\|$ so $|\lambda_2| \leq |\lambda_1|$. Continuing this process, a sequence of non-zero eigenvalues $\{\lambda_n\}$ and eigenvectors $\{x_n\}$ with $\|x_n\| = 1$ for every n is obtained. In addition a sequence of subspaces $\{H_k\}$ of H is obtained where H_{k+1} consists of the set of elements of H_k which are orthogonal to $x_1, x_2, \ldots, x_k$. For each k $|\lambda_k| \geqslant |\lambda_{k+1}|$ since $|\lambda_k| = \|A|_{H_k}\|$.

If $A|_{H_{m+1}}$ is the zero operator the process stops. In this case the range of A, is given by range $(A) = [\{x_k\}_1^n]$. For if $x \varepsilon H$, let $y_n = x - \sum_{k=1}^{n} x_k(x_k,x)$. Then

$(x_j,y_n) = (x_j,x) - \sum_{k=1}^{n} (x_j,x_k)(x_k,x)$ or $(x_j,y_n) = (x_j,x) - \sum_{k=1}^{n} \delta_{jk}(x_k,x) = (x_j,x) - (x_j,x) = 0$. But this last result implies $y_n \varepsilon H_{n+1}$ and since $A|_{H_{n+1}} = 0$,

$Ay_n = 0 = A(x - \sum_{k=1}^{n} x_k(x_k,x))$, or $Ax = \sum_{k=1}^{n} x_k\ \lambda_k(x_k,x)$.

Thus, for each x, $Ax = \sum_{k=1}^{n} x_k \beta_k$ so range (A) is spanned by $\{x_k\}_1^n$.

It will now be assumed that the process does not terminate. The process then yields an infinite sequence of eigenvalues $\{\lambda_k\}$, for which $|\lambda_k| \geq |\lambda_{k+1}|$ for $k = 1,2,\ldots$. One of the following cases must occur.

(a) $\lim_{k\to\infty} \lambda_k = 0$

(b) $|\lambda_k| \geq \varepsilon$ for some $\varepsilon > 0$ and for each K.

Suppose (b) is true. Consider the sequence $\{x_n \cdot \lambda_n^{-1}\}$. The sequence is bounded, for $\|x_n \cdot \lambda_n^{-1}\| = \|x_n\| \, |\lambda_n^{-1}|$. Since A is compact, $\{A(x_n \cdot \lambda_n^{-1})\} = \{x_n(\lambda_n \cdot \lambda_n^{-1})\} = \{x_n\}$ must contain a convergence subsequence. But this can't be since $\|x_m - x_n\|^2 = \|x_n\|^2 + \|x_m\|^2 = 2$. Therefore, if A is compact, $\lim_{k\to\infty} \lambda_k = 0$.

If the process does not terminate at some n let $y_n = x - \sum_{k=1}^{n} x_k(x_k,x)$. Then $\|y_n\|^2 = \|x\|^2 - \sum_{k=1}^{n} |(x_k,x)|^2$ so $\|y_n\|^2 \leq \|x\|^2$ for each n. Since $y_n \in H_{n+1}$,

$|\lambda_{n+1}| = \|A|_{H_{n+1}}\|$ it follows that $\|Ay_n\|$

$\leq |\lambda_{n+1}|\cdot\|y_n\| \leq |\lambda_{n+1}|\cdot\|x\|$. Thus, $\lim Ay_n = 0$.

But $A_{yn} = Ax = \sum_{k=1}^{n} Ax_k(x_k,x)$ which implies

$$Ax = \sum_{k=1}^{n} A_{xk}(x_k,x) = \sum_{k=1}^{\infty} x_k\,\lambda_k(x_k,x) = \sum_{k=1}^{\infty} x_k(x_k,x)\,\lambda_k.$$

Suppose that λ is an eigenvalue of A corresponding to eigenvector x which does not occur in the sequence $\{\lambda_k\}$. Then x is orthogonal to every one of the x_k since A is self-adjoint. But if $Ax = \sum_{k=1}^{\infty} x_k(x_k,x)\cdot\lambda_k$, $Ax \equiv 0$ which precludes it's being an eigenvalue.

An eigenvalue can not occur infinitely often in the sequence $\{\lambda_k\}$ since $\lim_{k\to\infty}\lambda_k = 0$.

It is clear that if an eigenvalue λ is repeated n times the corresponding eigenmanifold has dimension n.

These results can be summed up in the following theorem.

<u>Theorem 4</u>. (Spectral Theorem for compact self-adjoint operators). Let A be a compact (non-zero) self adjoint operator. Then there exists a sequence $\{\lambda_k\}$ of real

eigenvalues of A which may or may not be finite. If the sequence is infinite, $\lim_{k\to\infty} |\lambda_k| = 0$. The expansion $Ax = \sum_{k=1}^{\infty} x_k(x_k,x)\lambda_k$ is valid for each $x \in H$. Each non-zero eigenvalue occurs in the sequence $\{\lambda_k\}$. The eigenmanifold corresponding to a particular λ_i is finite dimensional and its dimension is exactly the number of times this particular eigenvalue is repeated in the sequence.

D. Existence of Eigenvalues for Compact Normal Operators.

To show that a compact normal operator on a Hilber Space over Q has eigenvalues, a generalization of method due to Bernau and Smithies (Bernau, S. J., and Smithies, F., 1963) will be used. It depends on the exponential representation of unit quaternions.

Any quaternion q for which $|q| = 1$ can be written as $q = \cos\theta + I \sin\theta$. In particular if $q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3$ and $\sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = 1$ then $\cos\theta = q_0$, $\sin\theta = \sqrt{q_1^2 + q_2^2 + q_3^2}$, and

$I = \dfrac{q_1e_1+q_2e_2+q_3e_3}{\sqrt{q_1^2 + q_2^2 + q_3^2}}$. Note that $|q_0| \le |q| = 1$ and

$\sqrt{q_1^2 + q_2^2 + q_3^2} \leq |q| = 1$. Recalling the definition of $e^{I\theta}$, then $q = e^{I\theta}$.

<u>Definition</u> Let T be a linear operator on a Hilbert space over Q. $\rho(T)$ is defined as

$$\rho(T) = \sup_{\|x\|=1} |(Tx,x)|.$$

<u>Lemma 1</u>. Let λ, θ be real numbers, $\lambda \neq 0$ and T a linear operator on H. If G_1, G_2 are defined as

$$G_1 = (T^2x\lambda e^{2I\theta} + Tx\lambda^{-1}e^{I\theta},\ Tx\lambda e^{I\theta} + x\lambda^{-1})$$

$$G_2 = (T^2x\lambda e^{2I\theta} - Tx\lambda^{-1}e^{I\theta},\ Tx\cdot\lambda\cdot e^{I\theta} - x\cdot\lambda^{-1})$$

then $\|Tx\|^2 + e^{-2I\theta}(T^2x,x) = \frac{1}{2}(G_1 - G_2)$.

Proof.

$$G_1 = (T^2x\lambda e^{2I\theta}, Tx\lambda e^{I\theta}) + (T^2x\lambda e^{2I\theta}, x\lambda^{-1})$$
$$+ (Tx\lambda^{-1}e^{I\theta}, Tx\lambda e^{I\theta}) + (Tx\lambda^{-1}e^{I\theta}, x\lambda^{-1})$$

using properties of the inner product G can be written as

$$G_1 = \lambda e^{-2I\theta}(T^2x, Tx)\lambda e^{I\theta} + \lambda e^{-2I\theta}(T^2x,x)\lambda^{-1}$$
$$+ \lambda^{-1}e^{-I\theta}(Tx,Tx)\lambda e^{I\theta} + \lambda^{-1}e^{-I\theta}(Tx,x)\lambda^{-1}.$$

Hence,

$$G_1 = \lambda e^{-2I\theta}(T^2x,Tx)\lambda e^{I\theta} + e^{-2I\theta}(T^2x,x) + \|Tx\|^2 + \lambda^{-2}e^{-I\theta}(Tx,x)$$

An Analogous computation yields

$$G_2 = \lambda e^{-2I\theta}(T^2x,Tx)\lambda e^{I\theta} - e^{-2I\theta}(T^2x,x) - \|Tx\|^2 + \lambda^{-2}e^{-I\theta}(Tx,x).$$

Therefore $\frac{1}{2}(G_1-G_2) = \frac{1}{2}(2e^{-2I\theta}(T^2x,x) + 2\|Tx\|^2)$,

or $\frac{1}{2}(G_1-G_2) = e^{-2I\theta}(T^2x,x) + \|Tx\|^2$.

<u>Lemma 2</u>. If T is a linear operator on a Hilbert space over Q then $\|Tx\|^2 + |(T^2x,x)| \leq 2\rho(T)\|Tx\|\|x\|$ for all $x \in H$.

Proof. From the definition of $\rho(T)$ it is clear that for every $y \in H$ $|(Ty,y)| \leq \rho(T)\|y\|^2$. If y_1, y_2 are defined as $y_1 = Tx\cdot\lambda\cdot e^{I\theta} + x\lambda^{-1}$, $y_2 = Tx\lambda e^{I\theta} - x\lambda^{-1}$ then $G_1 = (Ty_1,y_2)$ and $G_2 = (Ty_2,y_2)$. Since $\|Tx\|^2 + e^{-2I\theta}(T^2x,x) = \frac{1}{2}(G_1-G_2)$ it follows that $|\|Tx\|^2 + e^{-2I\theta}(T^2x,x)| \leq \frac{1}{2}(|(Ty_1,y_1)| + |(Ty_2,y_2)|)$. This last inequality can be rewritten as

$$|\|Tx\|^2 + e^{-2I\theta}(T^2x,x)| \leq \tfrac{1}{2}\rho(T)(\|y_1\|^2 + \|y_2\|^2).$$

Recalling the definitions of y_1, y_2 this last statement becomes

$$\left|\,\|Tx\|^2 + e^{-2I\theta}(T^2x,x)\right| \le \tfrac{1}{2}\rho(T)\left[\,\|Tx\,\lambda\, e^{I\theta} + x\,\lambda^{-1}\|^2 + \|Tx\,\lambda\, e^{I\theta} - x\,\lambda^{-1}\|^2\right]$$

But for every pair of elements u,v of H
$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$. Using this relation,

$$\left|\,\|Tx\|^2 + e^{-2I\theta}(T^2x,x)\right| \le \tfrac{1}{2}\rho(T)\left[\,2\|Tx\,\lambda\, e^{I\theta}\|^2 + 2\|x\,\lambda^{-1}\|^2\right]$$

or $\left|\,\|Tx\|^2 + e^{-2I\theta}(T^2x,x)\right| \le \rho(T)\left[\,\lambda^2\|Tx\|^2 + \lambda^{-2}\|x\|^2\right]$.

Now choose θ such that $e^{-2I\theta}(T^2x,x) = |(T^2x,x)|$
and λ such that $\lambda^2\|Tx\| = \|x\|$. Having done this the preceding inequality yields

$$\|Tx\|^2 + |(T^2x,x)| \le 2\rho(T)\,\|Tx\|\,\|x\|.$$

<u>Lemma 3</u>. Let T be a linear operator on H. Then

(a) $\|T\| \le 2\rho(T)$
(b) $\rho(T^2) \le [\rho(T)]^2$

Proof. (a) follows immediately from the second lemma.

To show that (b) is true let $\|x\| = 1$ in the second lemma. Then $\|Tx\|^2 + |(T^2x,x)| \leq 2\rho(T)\|Tx\|$.

From this it follows that

$$\|Tx\|^2 - 2\rho(T)\|Tx\| + [\rho(T)]^2 + |(T^2x,x)| \leq [\rho(T)]^2,$$

or

$[\|Tx\| - \rho(T)]^2 + |(T^2x,x)| \leq [\rho(T)]^2$. Hence,

$|(T^2x,x)| \leq [\rho(T)]^2$. But $\rho(T^2) = \sup\limits_{\|x\|=1}\{|(T^2x,x)|\}$

so clearly $\rho(T^2) \leq [\rho(T)]^2$.

In the three preceding lemmas the normality of T was not required. Requiring now that T be normal, yields the following theorem.

<u>Theorem 1</u>. Let T be a bounded normal operator on H, then $\|T\| = \rho(T)$.

Proof. For each $x \in H$, $|(Tx,x)| \leq \|T\|\cdot\|x\|^2$ so $\rho(T) \leq \|T\|$. Since T is normal $\|T^2x\| = \|T^*Tx\|$. Therefore, $\|T^2\| = \|T^*T\| = \|T\|^2$. By induction, if $p = 2^k$, it follows that $\|T^p\| = \|T\|^p$.

From lemma 4 part (b) it follows that $\rho(T^p) \leq [\rho(T)]^p$.

Applying lemma 4 part (a) $\|T\| = \|T^p\|^{1/p}$

$$\leq [2\rho(T^p)]^{1/p} \leq 2^{\frac{1}{p}}\rho(T);$$

letting $k \to \infty$ yields $\|T\| \leq \rho(T)$. This inequality together with $\rho(T) \leq \|T\|$ gives the desired inequality.

The primary result of this section can now be obtained if T is assumed to be compact as well as normal.

Theorem 2. Let T be a non-zero compact normal operator; then T has at least one non-zero eigenvalue.

Proof. Since $\rho(T) = \|T\|$ there exists a sequence $\{x_n\}$ from H such that $\|x_n\| = 1$ and $\lim_{n\to\infty} |(Tx_n, x_n)| = \|T\|$. It can be assume that $\lim_{n\to\infty}(Tx_n, x_n) = \lambda$ where $|\lambda| = \|T\| \neq 0$. (It may be necessary to take a subsequence here.) The sequence $\{x_n\}$ is bounded and since T is compact it may be assumed that $\{Tx_n\}$ converges to some $y \in H$. That is $y = \lim_{n\to\infty} Tx_n$. But $\|Tx_n\| \leq \|T\|$ and consequently $\|y\| \leq \|T\| = |\lambda|$. From this observation it follows that $\|Tx_n - x_n\lambda\|^2 = (Tx_n - x_n\lambda, Tx_n - x_n\lambda)$.

$$\|Tx_n - x_n\lambda\|^2 = (Tx_n, Tx_n) - \bar{\lambda}(x_n, Tx_n) - (Tx_n, x_n)\lambda + \bar{\lambda}\lambda(x_n, x_n).$$

Whence,
$$\lim_{n\to\infty} \|Tx_n - x_n\lambda\|^2 = \|y\|^2 - \bar{\lambda}\lambda - \bar{\lambda}\lambda + \bar{\lambda}\lambda$$
$$= \|y\|^2 - |\lambda|^2 \leq 0.$$

Since $\|Tx_n - x_n\lambda\|^2 \geq 0$ for every n it follows that $\lim_{n\to\infty} \|Tx_n - x_n\lambda\|^2 = 0$ and consequently $\lim_{n\to\infty} Tx_n - x_n\lambda = 0$

and $\|y\| = |\lambda| > 0$. Moreover, since $Tx_n \to y$ and $\lambda \neq 0$, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n \lambda^{-1} = y\lambda^{-1}$. Whence,

$Ty = (\lim_{n\to\infty} Tx_n)\lambda = y\lambda$. That is λ is an eigenvalue of T which is the desired result.

Now that it has been shown that every compact normal operator on a Hilbert space over Q has at least one eigenvalue, the spectral theorem for compact normal operators can be proven using essentially the same technique as for compact self-adjoint operators. Since the technique is the same the proof will not be carried out. However, there is a slight difference in the conclusion. The theorem will now be stated and the difference pointed out .

Theorem 3. Let A be a compact (non-zero) normal operator on a Hilbert Space H. Then there exists a sequence $\{\lambda_k\}$ of inequivalent eigenvalues of A which may or may not be infinite. If the sequence is infinite, $\lim_{k\to\infty} \lambda_k = 0$. The expansion $Ax = \sum_{k=1}^{\infty} x_k \lambda_k(x_k, x)$ is valid for each $x \in H$. Each non-zero eigenvalue occurs in the sequence $\{\lambda_k\}$. The eigenspace corresponding to a particular λ_i is finite dimensional and its dimension is exactly the number of times this particular eigenvalue is repeated in the sequence.

The only difference between the conclusion of this theorem and theorem 4 of the preceding section is the form of the representation for Ax. Since the λ_k are not necessarily real the product $\lambda_k(x_k,x)$ can't be written as $(x_k,x)\lambda_k$.

E. Bounded Hermitian Operators on Hilbert Spaces Over Q.

In the usual treatments of the spectral theorem for bounded Hermitian operators (Halmos, P., 1957, Lorch, 1962, Bachman, G. and Narici, L., 1966). various techniques are used to show that the spectrum of such an operator is non-void. Simmons (Simmons, G., 1963) shows via Banach algebra theory and complex function theory that any operator on a complex Hilbert space has a non void spectrum. The use of complex function theory is not possible for the case of a Hilbert space over the Quaternions.

In this section the spectrum of an operator on a Hilbert space over Q will be defined and it will be shown that, at least in the case of Hermitian operators, the spectrum is non-void. The technique is a variation of that used by Halmos (Halmos, P., 1957) for complex Hilbert Space.

<u>Definition</u> A bounded, additive, real homogeneous transformation on a Hilbert space H (over Q) has an inverse if there exists a bounded, additive, real homogeneous transformation B such that AB = BA = I, where I is the identity transformation on H.

The following two theorems are very important and they are well known for linear operators on a complex Hilbert space (Halmos, P., 1957). Their proofs carry over immediately to the case of a Hilbert space over Q and will not be given.

<u>Theorem 1</u>. If A is a bounded, additive, real homogeneous transformation on a Hilbert space H over Q and b is a positive real number such that $\|Ax\| \geq b\|x\|$ for every $x \in H$ then the range of A is closed.

<u>Theorem 2</u>. A bounded, additive, real homogeneous transformation A on a Hilbert space H over Q is invertible iff the range of A is dense in H and there exists a positive real number b such that $\|Ax\| \geq b\|x\|$ for every $x \in H$.

<u>Definition</u> Let A be a bounded linear transformation on a Hilbert space H over Q. The spectrum S(A) is defined to be $S(A) = \{ \lambda \in Q \mid A - \lambda \cdot I$ is not invertible $\}$. Here $(\lambda \cdot I)(x) = x\lambda$. Note that $A - \lambda \cdot I$ is a bounded, additive, real homogeneous transformation on H for every $\lambda \in Q$.

<u>Definition</u> Let A be a bounded linear transformation on a Hilbert space H over Q. The approximate point spectrum of A, AS(A) is defined as $AS(A) = \{ \lambda \in Q \mid \forall \varepsilon > 0, \exists$ x such that $\|Ax - x\lambda\| < \varepsilon \|x\| \}$.

<u>Theorem 3</u>. If A is a bounded linear operator on a Hilbert space H over Q then $AS(A) \subset S(A)$.

Proof. If $\lambda \notin S(A)$ then $(A - \lambda \cdot I)$ is invertible. Hence, $\|x\| = \|(A - \lambda \cdot I)^{-1}(A - \lambda \cdot I)x\| \leq \|(A - \lambda \cdot I)^{-1}\| \, \|(A - \lambda \cdot I)x\|$ for every $x \in H$. This implies $\|(A - \lambda \cdot I)x\| \geq \varepsilon \|x\|$ with $\varepsilon = \|(A - \lambda \cdot I)^{-1}\|^{-1}$. Since this is true for every $x \in H$, $\lambda \notin AS(A)$. The proof is complete.

The following theorem is a weaker result than is usually obtained in the case of complex Hilbert space (Halmos, P., 1957). However, it will be sufficient for the purposes of this section.

Theorem 4. If A is a normal operator on a Hilbert space H over Q and $\lambda \in AS(A)$ with $\lambda \in Re$ then $\lambda \in S(A)$.

Proof. Suppose $\lambda \in Re$ and $\lambda \notin AS(A)$. Then the following facts are evident.

(1) $A - \lambda \cdot I$ is linear

(2) $A - \lambda \cdot I$ is normal and $(A - \lambda \cdot I)^* = A^* - \lambda \cdot I$

(3) $\| A - \lambda \cdot I \| = \| A^* - \lambda \cdot I \|$

Since $\lambda \notin AS(A)$ there exists a positive real number ε such that $\| Ay - y\lambda \| \geq \varepsilon \| y \|$. But then $\| A^*y - y\lambda \| \geq \varepsilon \| y \|$. To show that $A - \lambda \cdot I$ is invertible it is sufficient (by Theorem 2) to show that the range of $A - \lambda \cdot I$ is dense. This can be done by showing that the orthogonal complement of the range of $A - \lambda \cdot I$ is $\{0\}$. If $((A - \lambda \cdot I)x, y) = 0$ for all x, then $(x, (A^* - \lambda \cdot I)y) = 0$ for all x. Hence $A^*y - y\lambda = 0$. But since $\| (A^* - \lambda \cdot I)y \| \geq \varepsilon \| y \|$, y must be 0 and the proof is complete.

Theorem 5. If A is a bounded linear transformation on H the $S(A^2) = [S(A)]^2 = \{ \lambda^2 \mid \lambda \in S(A) \}$.

Proof. If $\lambda \in S(A)$ then $A - \lambda \cdot I$ is not invertible. But $(A^2 - \lambda^2 \cdot I)x = (A - \lambda \cdot I)(A + \lambda \cdot I)x = (A + \lambda \cdot I)(A - \lambda \cdot I)x$

and consequently $A^2 - \lambda^2 \cdot I = (A - \lambda \cdot I)(A + \lambda \cdot I)$. From this representation it follows that $A^2 - \lambda^2 \cdot I$ can not be invertible. For if there exists an operator B such that $(A^2 - \lambda^2 \cdot I)B = B(A^2 - \lambda^2 \cdot I) = I$ then
$I = (A - \lambda \cdot I)(A + \lambda \cdot I)B = B(A + \lambda \cdot I)(A - \lambda \cdot I)$ and this implies $A - \lambda \cdot I$ is invertible which is a contradiction. Therefore, $A^2 - \lambda^2 \cdot I$ is not invertible and $\lambda^2 \in S(A^2)$. From this it follows that $[S(A)]^2 \subset S(A^2)$.

If $\lambda \in S(A^2)$ then $A^2 - \lambda \cdot I$ is not invertible. Let μ be any quaternion such that $\mu^2 = \lambda$. Then
$(A^2 - \lambda \cdot I) = (A + \mu \cdot I)(A - \mu \cdot I) = (A - \mu \cdot I)(A + \mu \cdot I)$ and either μ or $-\mu$ are in $S(A)$. In either case
$\lambda = (\pm \mu)^2 \in [S(A)]^2$. Whence $S(A^2) \subset [S(A)]^2$ and the proof is complete.

Lemma 1. If A is a Hermitian operator on a Hilbert space over Q and λ is a real number, then
$\|A^2x - x\lambda^2\|^2 = \|A^2x\|^2 - 2\lambda^2\|Ax\|^2 + \lambda^4\|x\|^2$.

Theorem 6. If A is a Hermitian operator on a Hilbert space over Q, then $\pm \|A\| \in AS(A)$.

Proof. Since $\|A\| = \sup_{\|x\|=1} \|Ax\|$ there exists a sequence of unit vectors $\{x_n\}$ such that $\lim_{n\to\infty} \|Ax_n\| = \|A\|$. Setting $\lambda = \|A\|$ and applying Lemma 1 it follows that $\|A^2x_n - x_n\lambda^2\|^2 \leq (\|A\|\,\|Ax_n\|)^2 - 2\lambda^2\|Ax_n\|^2 + \lambda^4$. But then $\|A^2x_n - x_n\lambda^2\|^2 \leq \lambda^4 - \lambda^2\|Ax_n\|^2$. Hence $\lim_{n\to\infty} \|A^2x_n - x_n\lambda^2\| = 0$ and consequently $\|A\|^2 \in AS(A^2)$. By Theorem 5, $\pm\|A\| \in AS(A)$.

Corollary. If A is a Hermitian operator on a Hilbert space H over Q then the spectrum S(A) is non-void.

Proof. By Theorem 3, $AS(A) \subset S(A)$ and Theorem 6 implies $\pm\|A\| \in AS(A) \subset S(A)$.

VIII. The Fourier Transform

A. The Fourier Transform in $L_Q'(-\infty, \infty)$.

In this section the definition of a Fourier Transform for functions in $L_Q'(-\infty,\infty)$ will be given. In addition a theorem will be proven that gives conditions under which the inverse of the transform can be calculated. The treatment given here is a generalization of that given by Goldberg (Goldberg, R. R., 1961).

<u>Definition</u> Let $f \varepsilon L_Q'$. Let I be a fixed quaternion such that $I^2 = -e_0$. The Fourier Transform $\hat{f}$ of an $f \varepsilon L_Q'$ is defined as $\hat{f}(x) = \int_{-\infty}^{\infty} e^{Ixt} f(t)dt$.

<u>Proposition 1.</u> $\hat{f}$ is bounded for each $f \varepsilon L_Q'$.

Proof. $$|\hat{f}(x)| = \left|\int_{-\infty}^{\infty} e^{Ixt} f(t)dt\right| \leq \int_{-\infty}^{\infty}|f(t)|\,dt = \|f\|_1$$

<u>Proposition 2.</u> $\hat{f}$ is continuous for each $f \varepsilon L_Q'$.

Proof. $$\hat{f}(x+h) - \hat{f}(x) = \int_{-\infty}^{\infty} (e^{I(x+h)t} - e^{Ixt})f(t)dt$$

$$\hat{f}(x+h) - \hat{f}(x) = \int_{-\infty}^{\infty} e^{Ixt}(e^{Iht} - 1)f(t)dt$$

$$|\hat{f}(x+h) - \hat{f}(x)| \leq \int_{-\infty}^{\infty} |e^{Iht} - 1|\,|f(t)|dt$$

Since $|e^{Iht} - 1|\cdot|f(t)| \leq 2|f(t)|$ and

$\lim\limits_{h \to 0} |e^{Iht} - 1| \cdot |f(t)| = 0$, it follows from Lebesques Dominated Convergence Theorem that $\lim\limits_{h \to 0} \hat{f}(x+h) = \hat{f}(x)$.

<u>Theorem 1</u>. If $\{f_n\}$ is a sequence from L_Q' and

$\|f_n - f\|_1 \to 0$ as $n \to \infty$, then $\lim\limits_{n \to \infty} f_n(x) = f(x)$

uniformly on Re.

Proof. For any $x \in$ Re, $|f_n(x) - f(x)| \leq \|f_n - f\|_1$.

<u>Theorem 2</u>. Let a,b be fixed real numbers. If $f \in L_Q'$ then

$$\text{(i)} \quad \widehat{f(t+a)} = e^{-Iax} \hat{f}(x)$$

$$\text{(ii)} \quad \hat{f}(x+b) = \widehat{e^{Ibt} f(t)}$$

<u>Theorem 3</u>. (Riemann-Lebesque).

If $f \in L_Q'$ then $\lim\limits_{x \to \pm} \hat{f}(x) = \lim\limits_{x \, \pm} \left[\int_{-\infty}^{\infty} e^{Ixt} f(t)dt \right] = 0.$

Proof. (1) $\hat{f}(x) = \int_{-\infty}^{\infty} e^{Ixt} f(t)dt$. Since $e^{I(x+\pi)} = -e^{Ix}$

$$(2) \quad -\hat{f}(x) = \int_{-\infty}^{\infty} e^{Ix(t+\frac{\pi}{x})} f(t)dt = \int_{-\infty}^{\infty} e^{Ixt} f(t - \tfrac{\pi}{x})dt$$

It follows that

$$(3) \quad 2f(x) = \int_{-\infty}^{\infty} e^{Ixt}(f(t) - f(t - \tfrac{\pi}{x}))dt. \quad \text{Hence}$$

(4) $2|\hat{f}(x)| \quad \int_{-\infty}^{\infty} |f(t) - f(t - \frac{\pi}{x})| dt.$ Since $f \in L_Q^1$

(5) $\lim_{x \to \pm\infty} \left[\int_{-\infty}^{\infty} |f(t) - f(t - \frac{\pi}{x})| dt \right] = 0.$ The result

follows from (3), (4), and (5).

Corollary. If $f \in L_Q^1$ then $\lim_{x \to \pm\infty} \int_{-\infty}^{\infty} f(t) \sin xt \, dt$

$$= \lim_{x \to \pm\infty} \int_{-\infty}^{\infty} f(t) \cos xt \, dt = 0.$$

Definition A quaternion valued function f is of bounded variation on $[a,b]$ if $\sup \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| < \infty$ where the supremum is taken over all partitions of $[a,b]$.

It's an easy consequence of the definition that a Quaternion valued function $f(x) = \sum_{k=0}^{3} f_k(x) e_k$ is of bounded variation on $[a,b]$ iff each of the real valued functions $f_k(x)$ are of bounded variation on $[a,b]$.

The following is an easy consequence of the corresponding theorem for real valued function (Goldberg, R. R., 1961).

Lemma 1. Let g be a quaternion valued function of bounded variation on $[0,d]$ for some $d > 0$. Then

$$\lim_{R\to\infty} \frac{1}{\pi} \int_0^d g(t) \frac{\sin Rt}{t} dt = \tfrac{1}{2}g(0+).$$

The following theorem gives as a corollary a condition for inversion the Fourier Transform.

<u>Theorem 4</u>. If $f \epsilon L_Q^{'}$ and f is of bounded variation in some neighborhood of a point u, then

$$\lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} e^{-Iux} \hat{f}(x)dx = \tfrac{1}{2}(f(u^+) + f(u^-)).$$

Proof. For any $R > 0$ let

$$S_R(u) = \frac{1}{2\pi} \int_{-R}^{R} e^{-Iux} \hat{f}(x)dx = \frac{1}{2\pi} \int_{-R}^{R} e^{-Iux} dx \int_{-\infty}^{\infty} e^{Ixt} f(t)dt.$$

Since $f \epsilon L_Q^{'}$ and f is continuous the iterated integral is convergent and hence by the Tonelli-Hobson theorem (Royden, H. L., 1969)

$$S_R(u) = \frac{1}{2\pi} \int_{-R}^{R} e^{-Ix(u-t)} dx \int_{-\infty}^{\infty} f(t)dt.$$

Now, $\frac{1}{2\pi} \int_{-R}^{R} e^{-Ix(u-t)}dx = \frac{1}{2\pi} \int_{-R}^{R} \cos x(u-t)dx - \frac{I}{2\pi} \int_{-R}^{R} \sin x(u-t).$

Hence, $\frac{1}{2\pi} \int_{-R}^{R} e^{-Ix(u-t)}dx = \frac{\sin R(u-t)}{(u-t)}$. Making a change of variable $S_R(u)$ becomes

$$S_R(u) = \frac{1}{\pi} \int_0^{\infty} t^{-1}\left[f(u+t) + f(u-t)\right] \sin Rt\, dt = I_1 + I_2, \text{ where}$$

$$I_1 = \frac{1}{\pi} \int_0^{\delta} t^{-1}\left[f(u+t) + f(u-t)\right] \sin Rt\, dt$$

$$I_2 = \frac{1}{\pi} \int_{\delta}^{\infty} t^{-1}\left[f(u+t) + f(u-t)\right] \sin Rt\, dt.$$

$\delta > 0$ and chosen such that f is of bounded variation in $[u-\delta, u+\delta]$. By Lemma 1,

$\lim_{R\to\infty} I_1 = \frac{1}{2}\left[f(u+) + f(u-)\right]$. Since $t^{-1}[f(u+t)+f(u-t)] \varepsilon L_Q^1[\delta, \infty)$

$\lim_{R\to\infty} I_2 = 0$ by the Riemann-Lebesque Lemma. Hence,

$\lim_{R\to\infty} S_R(u) = \frac{1}{2}\left[f(u+) + f(u-)\right]$.

<u>Corollary</u>. If in addition to the hypothesis of this theorem f is continuous at u,

$$f(t) = \int_{-\infty}^{\infty} e^{-Ixt}\, \hat{f}(x)dx.$$

B. The Fourier Transform on $L_Q^2(-\infty, \infty)$.

In this section the Fourier Transform will be extended to $L_Q^2(-\infty,\infty)$.

Definition For $f \in L_Q^1 \cap L_Q^2$; $\hat{f}(x) \equiv \int_{-\infty}^{\infty} e^{Ixt} f(t)dt$.

The following is a very important technical lemma.

Lemma 1. For any real numbers $\varepsilon > 0$ and a

$$\int_{-\infty}^{\infty} e^{Iat} e^{-\varepsilon t^2} dt = \left(\frac{\pi}{\varepsilon}\right)^{\frac{1}{2}} e^{-a^2/4\varepsilon}$$

Proof. $e^{Iat} = \cos at + I \sin at$

$$\int_{-\infty}^{\infty} e^{Iat} e^{-\varepsilon t^2} dt = \int_{-\infty}^{\infty} \cos at\, e^{-\varepsilon t^2} dt + I \int_{-\infty}^{\infty} \sin at\, e^{-t^2} dt$$

But $\int_{-\infty}^{\infty} \cos at\, e^{-\varepsilon t^2} dt = 2 \int_{0}^{\infty} \cos at\, e^{-\varepsilon t^2} dt$. The integral on the right hand side is a well known integral. Hence $\int_{-\infty}^{\infty} \cos at\, e^{-\varepsilon t^2} dt = \left(\frac{\pi}{\varepsilon}\right)^{\frac{1}{2}} e^{-a^2/4\varepsilon}$. Since the function $f(t) = \sin at\, e^{-\varepsilon t^2}$ is an odd function the integral $\int_{-\infty}^{\infty} \sin at\, e^{-\varepsilon t^2} dt = 0$.

The following lemmas will provide enough information to show the relation between the norms of $\hat{f}$ and f.

Lemma 2. If $f \in L_Q^2$ and $F(u) = \int_{-\infty}^{\infty} \overline{f(t)}\, f(t+u)dt$ then $F(0) = \|f\|_2^2$.

Lemma 3. If $f \in L_Q^2$ and $F(u) = \int_{-\infty}^{\infty} \overline{f(t)}\, f(t+u)dt$, then F is continuous at $u = 0$.

Proof. $|F(u) - F(0)| = \left| \int_{-\infty}^{\infty} \overline{f(t)} [f(t+u) - f(t)]\, dt \right|$.

$|F(u) - F(0)| \leq \int_{-\infty}^{\infty} |f(t)|\, |f(t+u) - f(t)| dt$. But, by the "CBS inequality".

$$|F(u) - F(0)|^2 \leq \int_{-\infty}^{\infty} |f(t)|^2\, dt \int_{-\infty}^{\infty} | f(t+u) - f(t)|^2\, dt.$$

By the continuity of the norm, it is clear that the right side of the last equation approaches zero as $u \to 0$. Hence $\lim_{u \to 0} F(u) = F(0)$.

Lemma 4. If $f \in L_Q^2$ and $F(u) = \int_{-\infty}^{\infty} \overline{f(t)}\, f(t+u)dt$, then

$|F(u)|^2 \leq (\|f\|_2)^4$.

Proof. Apply the CBS inequality.

Lemma 5. If $f \varepsilon L_Q^1 \cap L_Q^2$ and $\hat{f}(x) = \int_{-\infty}^{\infty} e^{Ixt} f(t)dt$ then

$$\int_{-\infty}^{\infty} e^{-x^2/n} |\hat{f}(x)|^2 dx = (\pi n)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-nu^2/4} F(u)du \quad \text{where}$$

$$F(u) = \int_{-\infty}^{\infty} \overline{f(t)}\, f(t+u)dt.$$

Proof.

$$|\hat{f}(x)|^2 = \overline{\hat{f}(x)}\, \hat{f}(x) = \overline{\int_{-\infty}^{\infty} e^{Ixt} f(t)dt} \int_{-\infty}^{\infty} e^{Ixu} f(u)du$$

$$|\hat{f}(x)|^2 = \int_{-\infty}^{\infty} \overline{f(t)}\, e^{-Ixt}\, dt \int_{-\infty}^{\infty} e^{Ixu} f(u)du.$$

Then,

$$\int_{-\infty}^{\infty} e^{-x^2/n}|\hat{f}(x)|^2 dx = \int_{-\infty}^{\infty} dx\, e^{-x^2/n} \int_{-\infty}^{\infty} \overline{f(t)}\, e^{-Ixt} dt \int_{-\infty}^{\infty} e^{Ixu} f(u)du$$

Since $f \varepsilon L_Q^1$, the iterated integral converges absolutely and the order of integration may be changed to yield;

$$\int_{-\infty}^{\infty} e^{-x^2/n} |\hat{f}(x)|^2 dx = \int_{-\infty}^{\infty} \overline{f(t)} dt \int_{-\infty}^{\infty} e^{-x^2/n}\, e^{Ix(u-t)} dx \int_{-\infty}^{\infty} f(u)du.$$

Using the result of Lemma 1, $\int_{-\infty}^{\infty} e^{-x^2/n} e^{Ix(u-t)} dx = (\pi n)^{\frac{1}{2}} e^{\frac{-n(t-u)^2}{4}}$.

Hence,

$$\int_{-\infty}^{\infty} e^{-x^2/n} |\hat{f}(x)|^2 dx = (\pi n)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{\frac{-n(u-t)^2}{4}} \int_{-\infty}^{\infty} \overline{f(t)}\, f(u)\,dt\, du.$$

Making a change of variable, this last statement can be written as

$$\int_{-\infty}^{\infty} e^{-x^2/n} |\hat{f}(x)|^2 dx = (\pi n)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{nu^2}{4}} \int_{-\infty}^{\infty} \overline{f(t)}\, f(t+u)\,dt\, du,$$

and then if $F(u) \equiv \int_{-\infty}^{\infty} \overline{f(t)}\, f(t+u)\,dt$, the proof is complete.

<u>Corollary</u>. If $f \in L_Q^1 \cap L_Q^2$ and $\hat{f}(x) = \int_{-\infty}^{\infty} e^{Ixt} f(t)dt$

then $\int_{-\infty}^{\infty} e^{-x^2/n} |\hat{f}(x)|^2 dt = 2\pi^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-u^2} F(2n^{-\frac{1}{2}} u)du$

where $F(u) = \int_{-\infty}^{\infty} \overline{f(t)}\, f(t+u)dt$.

Proof. Make the change of variable $u \to 2n^{-\frac{1}{2}}u$ in the result of the preceding lemma.

<u>Theorem 1</u>. Let $f \varepsilon L_Q^1 \cap L_Q^2$. Then $\hat{f} \varepsilon L_Q^2$ and

$$||f||_2 = (2\pi)^{\frac{1}{2}} ||f||_2.$$

Proof. By the corollary to Lemma 5,

$$\int_{-\infty}^{\infty} e^{-x^2/n} |\hat{f}(x)|^2 \, dx = 2\pi^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-u^2} F(2n^{-\frac{1}{2}} u) du \quad \text{where}$$

$F(u) = \int_{-\infty}^{\infty} \overline{f(t)} f(t+u) dt$. Consider the sequence

$g_n(u) = e^{-u^2} F(2n^{-\frac{1}{2}} u)$. $g_n(u)$ is dominated by $e^{-u^2} ||f||_2^2$.

Applying Lebesques Dominated Convergence theorem to the sequence $g_n(u)$ yields

$$2\pi^{\frac{1}{2}} \lim_n \int_{-\infty}^{\infty} g_n(u) du = 2\pi^{\frac{1}{2}} \int_{-\infty}^{\infty} \lim_n g_n(u) du = 2\pi^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-u^2} ||f||_2^2 \, du.$$

The last integral is obtained using the results of Lemmas 2 and 3. Applying Fatous theorem,

$$\int_{-\infty}^{\infty} |\hat{f}(x)|^2 \, dx = \int_{-\infty}^{\infty} \lim_{n \to \infty} e^{-x^2/n} |\hat{f}(x)|^2 dx \leq \lim_{n \to \infty} \int_{-\infty}^{\infty} e^{-x^2/n} |\hat{f}(x)|^2 \, dx.$$

Hence,

$$\int_{-\infty}^{\infty} |\hat{f}(x)|^2 dx \leq \lim_{n \to \infty} \int_{-\infty}^{\infty} e^{-x^2/n} |\hat{f}(x)|^2 dx = 2\pi ||f||_2^2.$$

whence $\|\hat{f}\|_2^2 = \int_{-\infty}^{\infty} |\hat{f}(x)|^2 dx \leq \|f\|_2^2$, which implies $\hat{f} \in L_Q^2$. Moreover, since $\lim_{n\to\infty} \int_{-\infty}^{\infty} e^{-x^2/n} |\hat{f}(x)|^2 dx = 2\pi \|f\|_2^2$, Lebesques Dominated Convergence theorem may be applied to yield $\int_{-\infty}^{\infty} |\hat{f}(x)|^2 = \|\hat{f}\|_2^2 = 2\pi \|f\|_2^2$. This completes the proof of the theorem.

<u>Theorem 2</u>. Let $f \in L_Q^2$. For $N = 1,2,\ldots.$, define f_N by

$$f_N(t) = \begin{cases} f(t) & |t| \leq N \\ 0 & |t| > N \end{cases}$$

Then $f_N \in L_Q^1 \cap L_Q^2$ and $\hat{f}_N \in L_Q^2$. Furthermore, the sequence $\{\hat{f}_N\}$ is a Cauchy sequence in L_Q^2.

Proof. For any fixed N, $\int_{-\infty}^{\infty} |f_N(t)|\, dt = \int_{-N}^{N} |f(t)|\, dt.$

By the CBS inequality,

$$\int_{-\infty}^{\infty} |f_N(t)|\, dt \leq \left[\int_{-N}^{N} |f(t)|^2 dt \int_{-N}^{N} dt\right]^{\frac{1}{2}} \leq \|f\|_2 (2N)^{\frac{1}{2}} < \infty \quad .$$

It follows that $f_N \in L_Q^1$. Since $|f_N(t)| \leq |f(t)|$ and

$f \in L_Q^2$, it's clear that $f_N \in L_Q^2$. Thus, $f_N \in L_Q^1 \cap L_Q^2$ and by Theorem 1, $\hat{f}_N \quad L_Q^2$. The proof of the first part of the theorem is complete.

To show that $\{f_N\}$ is an L_Q^2 Cauchy sequence, Theorem 1 can be applied to yield $\|\hat{f}_M - \hat{f}_N\|_2^2 = 2\pi \|f_M - f_N\|_2^2$. Using the definition of f_N,

$$\|f_M - f_N\|_2^2 = 2\pi \left| \int_{-N}^{-M} |f(t)|^2 dt + \int_M^N |f(t)|^2 dt \right| . \quad \text{Clearly,}$$

$$\lim_{\substack{N\to\infty \\ M\to\infty}} \|f_n - f_N\|_2^2 = 0.$$

Since L_Q^2 is complete the limit of a Cauchy sequence in L_Q^2 will always converge to an element of L_Q^2. Therefore, the following definition makes sense.

<u>Definition</u> For $f \in L_Q^2$, $\hat{f}$ is defined as

$$\hat{f}(x)' = \| \|_2 - \lim_{N\to\infty} \int_{-N}^{N} e^{Ixt} f(t)dt.$$

<u>Theorem 3</u>. (Parsevals Relation). If $f \in L_Q^2$, then

$$\|\hat{f}\|_2 = (2\pi)^{\frac{1}{2}} \|f\|_2 .$$

Proof. For $f \in L_Q^2$ f_N is defined as in Theorem 1. Then

$\lim_{N\to\infty} \|\hat{f}_N - \hat{f}\|_2 = 0$. But for each N $\big| \|f_N\|_2 - \|f\|_2 \big| \le \|f_N - f\|_2$. Hence, $\lim_{N\to\infty} \|\hat{f}_N\|_2 = \|\hat{f}\|_2$. But from the definition of f_N, it is clear that $\lim_{N\to\infty} \|f_N\|_2 = \|f\|_2$. Since $f_N \in L_Q^1 \cap L_Q^2$, $\|\hat{f}_N\|_2 = (2\pi)^{\frac{1}{2}} \|f_N\|_2$. Therefore,

$$\|\hat{f}\|_2 = \lim_{N\to\infty} \|\hat{f}_N\|_2 = (2\pi)^{\frac{1}{2}} \lim_{N\to\infty} \|f_N\|_2 = (2\pi)^{\frac{1}{2}} \|f\|_2.$$

The proof is complete.

It should be remarked that Theorem establishes the fact that the Fourier Transform is nearly isometry of L_Q^2 an to itself.

C. The Inversion Theorem for Functions in $L_Q^2(-\infty,\infty)$

In this section sufficient conditions for inversion of the Fourier Transform for functions in L_Q^2 will be given. An exact form for the inverse transform will also be given.

Lemma 1. If $f \in L_Q^2$ and $\tilde{f}(x) \equiv \overline{f}(-x)$ then $\tilde{f} \in L_Q^2$ and $\|\tilde{f}\|_2^2 = \|f\|_2^2$.

Proof. $|\tilde{f}(x)|^2 = |\overline{f}(-x)|^2 = |f(-x)|^2$. Hence,

$$\int_{-\infty}^{\infty} |\tilde{f}(x)|^2 dx = \int_{-\infty}^{\infty} |f(-x)|^2 dx = \int_{-\infty}^{\infty} |f(t)|^2 dt$$ and the results follow.

Lemma 2. If f and $g \varepsilon L_Q^2$ and f_M, g_N are defined by

$$f_M(x) = \begin{cases} f(x) & |x| \leq M \\ 0 & |x| > M \end{cases} \qquad g_N(x) = \begin{cases} g(x) & |x| \leq N \\ 0 & |x| > N \end{cases}$$

then $$\int_{-\infty}^{\infty} \overline{\hat{\tilde{f}}}_M(x)\; g_N(x)dx = \int_{-\infty}^{\infty} f_M(x)\; \hat{g}_N(x)dx.$$

Proof. For each M,N f_M and g_N are in L_Q^2. Thus,

$$\hat{f}_M(x) = \int_{-\infty}^{\infty} e^{Ixt}\, f_M(t)dt \quad \text{and} \quad g_N(x) = \int_{-\infty}^{\infty} e^{Ixt}\, g_N(t)dt$$

$$\int_{-\infty}^{\infty} \overline{\hat{\tilde{f}}}_M(x)\; g_N(x)dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{e^{Ixt}\, \tilde{f}_M(t)}dt\; g_N(x)dx$$

$$\text{"} \qquad \text{"} \qquad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\tilde{f}}_M(t)dt\; e^{-Ixt}\; g_N(x)dx$$

$$\text{"} \qquad \text{"} \qquad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_M(-t)dt\; e^{-Ixt}\; g_N(x)dx$$

Hence,

$$\int_{-\infty}^{\infty} \overline{\hat{\tilde{f}}}_M(x) g_N(x)dx = \int_{-\infty}^{\infty} f_M(t) \int_{-\infty}^{\infty} e^{Ixt}\; g_N(x)dx\; dt.$$

Finally, $\int_{-\infty}^{\infty} \overline{\hat{\tilde{f}}}_M(x)\; g_N(x)dx = \int_{-\infty}^{\infty} f_M(t)\; \hat{g}_N(t)dt$. A change of variable yield the desired result.

<u>Corollary 1</u>. If f and g ε L_Q^2, $\int_{-\infty}^{\infty} \overline{\hat{\tilde{f}}}(x)\; g(x)dx = \int_{-\infty}^{\infty} f(t)\hat{g}(t)dt$.

<u>Corollary 2</u>. If f, g ε L_Q^2, $\int_{-\infty}^{\infty} \overline{f}(t)\hat{g}(t)dt = \int_{-\infty}^{\infty} \tilde{\hat{f}}(x)g(x)dx$.

Proof. From Corollary 1,

$$\int_{-\infty}^{\infty} \overline{f}(t)\hat{g}(t)dt = \int_{-\infty}^{\infty} \overline{\hat{\tilde{\overline{f}}}}(x)g(x)dx.$$

However, $\overline{\hat{\tilde{\overline{f}}}}(x) = \overline{\hat{\overline{\overline{f}}}}(-x) = \overline{\hat{f}}(-x) = \tilde{\hat{f}}(x)$.

Now the main theorem can be proven. It will yield the inversion theorem as a corollary.

<u>Theorem 1</u>. Let f ε L_Q^2 and let $g = \overline{\tilde{\hat{f}}}$. Then $f = \frac{1}{2\pi}\;\hat{g}$.

Proof. $\left\| f - \frac{1}{2\pi}\hat{g} \right\|_2^2 = \int_{-\infty}^{\infty} (\overline{f} - \frac{1}{2\pi}\overline{\hat{g}})(f - \frac{1}{2\pi}\hat{g})dt$.

$$\left\| f - \frac{1}{2\pi}\hat{g} \right\|_2^2 = \int_{-\infty}^{\infty} |f|^2\, dt + \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} |\hat{g}|^2\, dt$$

$$- \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} \overline{f}\; g\; dt + \int_{-\infty}^{\infty} \overline{\hat{g}}\; f\; dt\right).$$

Hence,

$$\|f-\frac{1}{2\pi}\hat{g}\|_2^2 = \|f\|_2^2 + \frac{1}{4\pi^2}\|\hat{g}\|_2^2 - \frac{1}{2\pi}\left(\int_{-\infty}^{\infty}\bar{f}\hat{g}\,dt + \overline{\int_{-\infty}^{\infty}\bar{f}\,\hat{g}\,dt}\right).$$

Corollary 2 can now be applied to yield,

$$\|f-\frac{1}{2\pi}\hat{g}\|_2^2 = \|f\|_2^2 + \frac{1}{4\pi^2}\|\hat{g}\|_2^2 - \frac{1}{2\pi}\left(\int_{-\infty}^{\infty}\tilde{\hat{f}}\,g\,dt + \int_{-\infty}^{\infty}\tilde{\hat{f}}\,g\,dt\right).$$

But $g = \overline{\tilde{\hat{f}}}$, so

$\|f-\frac{1}{2\pi}\hat{g}\|_2^2 = \|f\|_2^2 + \frac{1}{4\pi^2}\|\hat{g}\|_2^2 - \frac{1}{\pi}\int_{-\infty}^{\infty}|\tilde{\hat{f}}|^2\,dt.$ This can be written as

$\|f-\frac{1}{2\pi}\hat{g}\|_2^2 = \|f\|_2^2 + \frac{1}{4\pi^2}\|\hat{g}\|_2^2 - \frac{1}{\pi}\|\tilde{\hat{f}}\|_2^2.$ From Lemma 1, and from Parsevals Theorem, $\|\tilde{\hat{f}}\|_2^2 = \|\hat{f}\|_2^2 = 2\pi\|f\|_2^2.$ In addition $\|\hat{g}\|_2^2 = 2\pi\|g\|_2^2.$ But $g = \overline{\tilde{\hat{f}}}$, so $\|\hat{g}\|_2^2 = (4\pi)^2\|f\|_2^2.$ Therefore,

$$\|f-\frac{1}{2\pi}\hat{g}\|_2^2 = \|f\|_2^2 + \|f\|_2^2 - 2\|f\|_2^2 = 0.$$

The proof is complete.

<u>Corollary 1.</u> (Inversion Theorem)

If $f \in L_Q^2$ then $f(t) = \|\;\|_2\text{-}\lim_{N\to\infty} \frac{1}{2\pi}\int_{-N}^{N} e^{-Ixt}\hat{f}(x)dx.$

Proof. Let $g = \overline{\tilde{\hat{f}}}$. Then by the preceding theorem $f = \frac{1}{2\pi}\hat{g}$. Hence,

$$f(t) = \|\,\|_2\text{-}\lim_{N\to\infty} \frac{1}{2\pi}\int_{-N}^{N} e^{Ixt} g(x)dx = \|\,\|_2 \lim_{N\to\infty} \frac{1}{2\pi}\int_{-N}^{N} e^{Ixt}\, \overline{\tilde{\hat{f}}}(x)dx.$$

$$f(t) = \|\,\|_2\text{-}\lim_{N\to\infty} \frac{1}{2\pi}\int_{-N}^{N} e^{Ixt}\, \overline{\overline{\hat{f}}}(-x)dx = \|\,\|_2\text{-}\lim_{N\to\infty} \frac{1}{2\pi}\int_{-N}^{N} e^{Ixt}\hat{f}(-x)dx.$$

Making a change of variable,

$$f(t) = \|\,\|_2\text{-}\lim_{N\to\infty} \frac{1}{2\pi}\int_{-N}^{N} e^{-Ixt} f(x)dx.$$

<u>Corollary 2</u>. Every $f \in L_Q^2$ is the Fourier Transform of a unique element in L_Q^2.

The results of this section are summarized in the following theorem.

<u>Theorem 2</u>. If $f \in L_Q^2$ then there exists a function $\hat{f} \in L_Q^2$ such that

$$\hat{f}(x) = \|\,\|_2\text{-}\lim_{N\to\infty} \int_{-N}^{N} e^{Ixt} f(t)dt$$

$$f(t) = \| \; \|_2 \text{-} \lim_{N\to\infty} \frac{1}{2\pi}\int_{-N}^{N} e^{-Ixt} f(x)dx \quad \text{and}$$

$\|\hat{f}\|_2 = (2\pi)^{\frac{1}{2}} \|f\|_2$. Every $f \in L_Q^2$ can be expressed as $f = \hat{g}$ for a unique $g \in L_Q^2$.

IX. Hilbert Spaces over C

In this section a few elementary results about inner product spaces over C will be given. The definition of an inner product space over C is from Goldstine and Horwitz (Goldstine, H. H., Horwitz, L. P., 1964).

<u>Definition</u> A linear space L over C is an inner product space if there exists a function (,): $L \times L \longrightarrow C$ such that

(i) $(x,x) \geq 0$ and $(x,x) = 0$ iff $x = 0$

(ii) $(x,y+z) = (x,y) + (x,z)$

(iii) $(x,x\alpha) = (x,x)\alpha$; $(x,ya) = (x,y)\cdot a$ for a real.

(iv) $(x,y) = \overline{(y,x)}$

(v) $t\,[(x,y\alpha)] = t\,[(x,y)\alpha]$ $(t(a) = a + \bar{a})$

It follows easily from the definition that

(a) $(x+y,z) = (x,z) + (y,z)$

(b) $(x\alpha,x) = \bar{\alpha}(x,x)$.

It should be noted that one of the usual properties for an inner product is missing from the preceding definition. It is not assumed that $(x,ya) = (x,y)a$ for $a \in C$. The reason for this is that none of the canonical examples have this property and also it is not known to this author whether it is possible to construct a function on a linear space over C with properties (i) through (v) and the additional property that $(x,ya) = (x,y)a$ for all $x,y \in L$ and $a \in C$.

Example 1. Let $L = \{ (x_1,x_2,\ldots,x_n) \mid x_j \in C \}$ with addition and scalar multiplication defined pointwise. An example of an inner product on this space is given by

$$(x,y) = \sum_{j=1}^{n} \overline{x}_j \, y_j \text{ where } x = (x_1,x_2,\ldots,x_n) \text{ and}$$

$y = (y_1,y_2,\ldots y_n)$.

Example 2. Let L be the class of C valued continuous functions defined on $[a,b]$, with the operations of addition and scalar multiplicated defined pointwise. For f and g in L define $(f,g) = \int_a^b \overline{f(x)}g(x)dx$.

The very fact that the inner product is not homogeneous with respect to scalars seriously affects the geometrical structure of Hilbert spaces over C. Moreover, It's clear that no linear functional can be represented by the inner product and hence the Riesz Representation Theorem is impossible. It's clear that if any satisfactory theory is desirable then certain other assumptions are necessary.

It will now be shown that the inner product does give rise to a metric on the linear space. To do this, the following lemmas will be needed.

Lemma 1. $(x,ya) + (y,xa) = [(x,y) + (y,x)]\cdot a$

Proof. From (iii) of the definition $(x+y), [x+y]\, a)$ $= (x+y, x+y)a$. Hence, $(x,xa) + (y,xa) + (x,ya) + (y,xa)$ $+ (y,ya) = (x,x)a + (y,y)a + [(x,y) + (y,x)]\, a.$

Lemma 2. $(x\bar{a},ya) = a(x,y)a$

Proof. By Lemma 1, $(x,ya) + (y,xa) = [(x,y) + (y,x)]\cdot a.$ Replace x by $x\bar{a}$ in this result. Then it follows that $(x\bar{a},ya) + (y,(x\bar{a})a) = [(x\bar{a},y) + (y,x\bar{a})]\, a.$ Hence, $(x\bar{a},ya) + (y,x)(\bar{a}a) = t\, [\, (y,x\bar{a})]\, a = t\, [(y,x)\bar{a}$, by (v). This can be written as

$(x\bar{a},ya) + (y,x)(\bar{a}a) = [(y,x)\bar{a} + a(\overline{y,x})]\, a$ and the result follows.

<u>Corollary 1</u>. $(xa,xa) = \bar{a}(x,x)a = \bar{a}a(x,x) = |a|^2 (x,x)$.

<u>Lemma 3</u>. (CBS inequality).
$|(x,y)| \le (x,x)^{\frac{1}{2}} (y,y)^{\frac{1}{2}}$ for all $x,y \in L$.

Proof. If either x or y are 0 then the inequality is true trivially. Let $(y,y) = 1$. Then for every choice of $x \in L$ and $a \in C$ $\quad 0 \le (x-ya,x-ya) = (x,x) - [(ya,x)+(x,ya)] + (ya,ya)$.

Or, $\quad 0 \le (x,x) - t[(x,ya)] + |a|^2 (y,y) = (x,x) - t[(x,y)a] + |a|^2$.

Now choose $a = (\overline{x,y})$. It follows that

$$0 \quad (x,x) - [(x,y)(\overline{x,y}) + (x,y)(\overline{x,y})] + (x,y)(\overline{x,y}).$$

Hence, $(x,y)(y,x) \le (x,x)$ or equivalently $|(x,y)| \le (x,x)^{\frac{1}{2}}$.
For any $z \in L$ with $x \ne 0$ it follows that
$|(x,z\,(z,z)^{-\frac{1}{2}})| \le (x,x)^{\frac{1}{2}}$ and hence $|(x,z)| \le (x,x)^{\frac{1}{2}}(z,z)^{\frac{1}{2}}$.

<u>Theorem 1</u>. If L is an inner product space over C with inner product (,), then if $\|x\| \equiv (x,x)^{\frac{1}{2}}$, L becomes a normed linear space over C.

Proof. If $\|x\|$ is defined by $\|x\| = (x,x)^{\frac{1}{2}}$ then $\|x\| \geq 0$ for all x and $\|x\| = 0$ iff $x = 0$ from the properties of the inner product. From Corollary 1 $\|xa\|^2 = (xa,xa) = |a|^2(x,x) = |a|^2\|x\|^2$, so $\|xa\| = \|x\|\,|a|$. $\|x+y\|^2 = (x+y,x+y) = (x,x) + t[(x,y)] + (y,y) = \|x\|^2 + t[(x,y)] + \|y\|^2$. By Lemma 3 it follows that $t[(x,y)] \leq 2|(x,y)| \leq 2\|x\|\cdot\|y\|$. Hence, $\|x+y\|^2 \leq (\|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2) = (\|x\| + \|y\|)^2$ and consequently $\|x+y\| \leq \|x\| + \|y\|$.

<u>Lemma 4</u>. If $f\colon L \times L \longrightarrow C$, then $f(x,y) = \sum_{j=0}^{7} f_j(x,y)y$ and $f_o(x,y) = \frac{1}{2}t[f(x,y)]$, $f_j(x,y) = -\frac{1}{2}t[e_j f(x,y)]$.

The following is a generalization of the polarization identity which is well known for the case of complex linear spaces.

Theorem (Polarization Identity). If L is an inner product space over C and $\|x\| \equiv (x,x)^{\frac{1}{2}}$, then

$$\sum_{j=0}^{7} (e_j\|x+ye_j\|^2 - e_j\|x - ye_j\|^2) = 4(\overline{x,y})$$

Proof. Let a be any real number. Then

$$(x+ye_ja,x+ye_ja) = (x,x) + a\,t[(x,ye_j)] + (ye_j,ye_j)a^2.$$

For $a = \pm 1$, $(x-ye_j, x-ye_j) = \|x\|^2 \pm t\,[(x,y)e_j] + \|y\|^2$.
Then for $j = 0,1,2,\ldots,7$,

$e_j\|x+ye_j\|^2 - e_j\|x-ye_j\|^2 = 2e_j t\,[(x,y)e_j]$. Therefore,

$$\sum_{j=0}^{7} (e_j\|x+ye_j\|^2 - e_j\|x-ye_j\|^2) = 2\sum_{j=0}^{7} e_j t\,[(x,y)e_j].$$

Applying lemma 4 it follows that

$$\sum_{j=0}^{7} (e_j\|x+ye_j\|^2 - e_j\|x-ye_j\|^2) = 2\cdot 2(\overline{x,y}) = 4(\overline{x,y}).$$

Definition A Hilbert space over C is a complete inner product space.

Theorem 3. (Parallelogram law)
Let L be an inner product space over C. If $x,y \in L$ then
$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

The following theorem is a generalization of the Jordan-Von Neumann theorem for complex linear spaces to case of inner product spaces over C.

Theorem 4. If L is a Banach space over C with norm $\| \ \|$ and $\| \ \|$ satisfies the parallelogram law, there exists an inner product $(\ , \ )$ on L such that $(x,x) = \|x\|^2$.

Proof.

Define $[x,y] = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$.

$$(x,y) = [x,y]e_0 - \sum_{j=1}^{7} [x,ye_j]\, e_j$$

(i) clearly $(\ ,\ )$ is additive (since $[\ ,\ ]$ is additive).

(ii) $[x,xe_j] = \|x+xe_j\|^2 - \|x-xe_j\|^2$

$$= \|x\|^2\left[\,|e_0+e_j|^2 - |e_0-e_j|^2\,\right] = 0 \text{ for } j\neq 0.$$

Hence, $(x,x) = [x,x]\,e_0 = \|x\|^2$.

(iii) $(x,xe_n) = [x,xe_n]\,e_0 - \sum_{j=1}^{7} [x,(xe_n)e_j]\,e_j$

for $n\neq 0$, $[x,xe_n] = 0$. Also, $[x,(xe_n)e_j]$

$$= \tfrac{1}{4}(\|x-(xe_n)e_j\|^2 - \|x-(xe_n)e_j\|^2)$$

$$[x,(xe_n)e_j] = \tfrac{1}{4}(\|(x\bar{e}_j+xe_n)e_j\|^2 - \|(x\bar{e}_j-xe_n)e_j\|^2)$$

$$[x,(xe_n)e_j] = \tfrac{1}{4}(\|x\|^2)\left[\,(\bar{e}_j+e_n)(e_j+\bar{e}_n) - (\bar{e}_j-e_n)(e_j-\bar{e}_n)\,\right]$$

$$[x,(xe_n)e_j] = \frac{\|x\|^2}{4}\left[\,\bar{e}_je_j+e_ne_j+e_n\bar{e}_n+\bar{e}_j\bar{e}_n-\bar{e}_je_j + e_ne_j+\bar{e}_j\bar{e}_n-e_n\bar{e}_n\,\right]$$

$$[x,(xe_n)e_j] = \frac{\|x\|^2}{2} t(e_n e_j) = -\|x\|^2 \delta_{nj}$$

Hence

$$(x,xe_n) = \|x\|^2 e_n = (x,x)e_n.$$

(iv) $[x,ye_k] = \frac{1}{4}(\|x+ye_k\|^2 - \|x-ye_k\|^2)$

$$[x,ye_k] = \frac{1}{4}(\|(x\bar{e}_k+Y)e_k\|^2 - \|(x\bar{e}_k-y)e_k\|^2)$$

$$[x,ye_k] = \frac{1}{4}(\|y-xe_k\|^2 - \|y+xe_k\|^2) = -[y,xe_k]$$

Now $\overline{(x,y)} = [\overline{x,y}]\, e_0 - \sum_{j=1}^{7} \overline{[x,ye_j]\, e_j}$

$$\overline{(x,y)} = [y,x] + \sum_{j=1}^{7} [x,ye_j]\, e_j$$

$$\overline{(x,y)} = [y,x] - \sum_{j=1}^{7} [y,xe_j]\, e_j = (y,x).$$

(v) $t[(x,y)e_n] = (x,y)e_n + \bar{e}_n(y,x)$ by definition

$$(x,y)e_n = [x,y]\, e_n - \sum_{j=1}^{7} [x,ye_j]\,(e_j e_n)$$

$$\bar{e}_n(y,x) = [x,y]\, \bar{e}_n - \sum_{j=1}^{7} [y,xe_j]\,(\bar{e}_n e_j).$$

But from (iv), $[y,xe_j] = -[x,ye_j]$, so

$$\bar{e}_n(y,x) = [x,y]\,\bar{e}_n + \sum_{j=1}^{7} [x,ye_j]\,(\bar{e}_n e_j).$$

For $n \neq 0$, $\bar{e}_n = -e_n$ and hence,

$$\bar{e}_n(y,x) = -[x,y]\,e_n - \sum_{j=1}^{7} [x,ye_j]\,(e_n e_j).$$

Whence,

$$t[(x,y)e_n] = -\sum_{j=1}^{7} [x,ye_j]\cdot(e_je_n+e_ne_j), \text{ or}$$

$$t[(x,y)e_n] = +2[x,ye_j]\,\delta_{nj} = 2[x,ye_n].$$ That

is, $t[(x,y)e_n] = 2[x,ye_n]$.

Now, $t[(x,ye_n)] = (x,ye_n) + (ye_n,x)$. But

$$(x,ye_n) = [x,ye_n]\,e_0 - \sum_{j=1}^{n} [x,(ye_n)e_j]\,e_j$$

$$(ye_n,x) = [ye_n,x] - \sum_{j=1}^{n} [ye_n,xe_j]\cdot e_j.$$

$$[x,(ye_n)e_j] = \tfrac{1}{4}(\|x+(ye_n)e_j\|^2 - \|x-(ye_n)e_j\|^2).$$

This can be rewritten as

$$[x,(ye_n)e_j] = \tfrac{1}{4}(\|-xe_j+ye_n\|^2 - \|xe_j+ye_n\|^2).$$

Moreover,

$$[ye_n, xe_j] = \tfrac{1}{4}(\|ye_n+xe_j\|^2 - \|ye_n-xe_j\|^2).$$

Hence $[x,(ye_n)e_j] + [ye_n,xe_j] = 0$. Therefore,

$$t[(x,ye_n)] = [x,ye_n] + [ye_n,x] = 2[x,ye_n]$$
$$= t[(x,y)e_n]$$

Since this is true for $n = 0,1,2,\ldots 7$, it follows that $t[(x,ya)] = t[(x,y)a]$ and consequently $(\ ,\ )$ has the properties of inner product space over C.

<u>Definition</u> A normed linear space L over C with norm $\|\ \|$ is called uniformly convex if to each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\|x+y\| \leq 2(1-\delta(\varepsilon))$ when $\|x\| = 1$, $\|y\| = 1$ and $\|x-y\| \geq \varepsilon$.

<u>Theorem</u> An inner product space L over C is a uniformly convex normed linear space over C.

Proof. $\|x+y\|^2 = 2\|x\|^2 + 2\|y\|^2 - \|x-y\|^2$ from Theorem 3. If $\|x\| = \|y\| = 1$, then $\|x+y\|^2 = 4 - \|x-y\|^2$. Hence, $\|x+y\| = 2 - [2 - (4-\|x-y\|^2)^{\frac{1}{2}}]$. If $\|x\| = \|y\| = 1$ and $\|x-y\| > \varepsilon$ then $\|x+y\| < 2(1-\delta(\varepsilon))$ if $\delta(\varepsilon) = \dfrac{2-(4-\|x-y\|^2)^{\frac{1}{2}}}{2}$.

X. Conclusions and Suggestions for Further Study

It is clear now that many of the results of functional analysis that are usually given for complex linear spaces are also true for linear spaces over the Quaternions. However, it appears that much less can be said for linear spaces over the Cayley numbers. This is due basically to the fact that scalar multiplication is not an associative operation. It seems to this author that more has to be done with specific examples of these spaces before a general theory can be developed.

As a result of this study some interesting but unresolved questions have arisen.

In Chapter VII, Spectral Theorems for compact normal and compact self-adjoint operators on Hilbert Spaces over Q have been given. It is natural to ask if analogous theorems are true for arbitrary bounded self adjoint and normal operators. One of the first difficulties encountered in this problem is proving that the spectrum is non-void. It was shown in Chapter VII that the spectrum of a bounded self adjoint operator is non-empty, but the case for a normal operator is open. Even if this result is true for normal operators many difficulties remain and the approach one should take is not clear.

One of the more interesting structures encountered in this work is the algebra $C(X,F)$, where F is a Cayley-Dickson Algebra of dimension n. $C(X,F)$ possesses all of the usual properties of a real Banach algebra except it is not associative. The Stone-Weierstrass theorem was proven for this algebra but no other results were obtained. It appears that no one to date has made a study of algebras of this type and this might be an interesting though formidable task.

In Chapter VIII a Fourier Transform for functions in $L_Q^1(-\infty,\infty)$ and $L_Q^2(-\infty,\infty)$ was studied. Many questions concerning this transformation were not considered. It would be very interesting if the results of Chapter VIII could extend to $L_Q^1(G)$ and $L_Q^2(G)$ where G is a locally compact abelian group. It also appears that this transformation might have some application to Quaternion Quantum Mechanics.

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