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Relic gravitons on Kasner-like branes

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Abstract

We discuss the cosmological amplification of tensor perturbations in a simple example of brane-world scenario, in which massless gravitons are localized on a higher-dimensional Kasner-like brane embedded in a bulk AdS background. Particular attention is paid to the canonical normalization of the quadratic action describing the massless and massive vacuum quantum fluctuations, and to the exact mass-dependence of the amplitude of massive fluctuations on the brane. The perturbation equations can be separated. In contrast to de Sitter models of brane inflation, we find no mass gap in the spectrum and no enhancement for massless modes at high curvature. The massive modes can be amplified, with mass-dependent amplitudes, even during inflation and in the absence of any mode-mixing effect.

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The amplification of the metric quantum fluctuations induced by inflation is expected to produce a cosmic background of relic gravitons, carrying unique information on the primordial state of our universe [1]. This effect is also expected to occur in “brane-world cosmology” models (see, e.g., Ref. [2]). Brane-world cosmology was originally inspired by heterotic M-theory [3], and is phenomenologically motivated by the Randall–Sundrum mechanism [4], which allows the localization of long-range gravitational interactions on the brane through the warping of the higher-dimensional bulk.

Recent studies on inflationary graviton production in the brane-world scenario gave interesting results [5–7] on the possible mixing of massless and massive modes of the tensor perturbation spectrum, and on the possible enhancement of the massless spectral amplitudes occurring above a threshold value of the brane curvature. In these models, the geometry of the brane can be parametrized as a four-dimensional de Sitter slicing of a five-dimensional anti-de Sitter (AdS) manifold. The aim of this Letter is to point out that some of the results are specific to the

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models of inflation considered and cannot be extended to cosmological graviton production in a generic brane-world scenario.

In this Letter we consider a model based on a higher-dimensional Kasner-like brane embedded in an AdS bulk manifold. There are no matter sources on the brane; inflation (i.e., accelerated expansion) is sustained by the accelerated contraction (“spontaneous” dimensional reduction) of the brane internal dimensions. Although the curvature of the brane is time-dependent and the geometry is not de Sitter, we find that the tensor perturbation equation separates in the brane and bulk coordinates. In contrast to the results of [5,6], the amplification of the massless modes is always controlled by the brane curvature in the usual way, without anomalous enhancement at higher curvature. Moreover, there is no mass gap in the spectrum. However, massive modes with mass smaller than the brane curvature scale are also directly amplified during the phase of brane inflation, and may lead to a significant enhancement of the final amplitude of the tensor perturbation background.

Let us consider a $(p+2)$ -dimensional configuration in which a D_p -brane is embedded in a $(p+2)$ -dimensional bulk manifold with one extra spatial dimension and a negative cosmological constant. We denote by $D = p+2$ the total number of dimensions and assume the p -dimensional brane space to be the product of two maximally symmetric manifolds with d and n dimensions ($p = d+n$). The dynamics of this system is described by the action

$$S = S_{\text{bulk}} + S_{\text{brane}} = -\frac{1}{2}M^{(D-2)} \int d^D x \sqrt{|g|} (R + 2\Lambda) - \frac{T_p}{2} \int d^{p+1} \xi \sqrt{|\gamma|} [\gamma^{\alpha\beta} \partial_\alpha X^A \partial_\beta X^B g_{AB}(X) - (p-1)], \quad (1)$$

where Λ is the negative bulk cosmological constant, T_p is the tension of the brane, and M is a mass parameter that characterizes the strength of bulk gravitational interactions (when $p=1$ one recovers the string-gravity self-sustained configuration discussed in [8]). Here $\partial_\alpha X^A$ is a compact notation for $\partial X^A(\xi)/\partial \xi^\alpha$, where ξ^α are the coordinates spanning the $(p+1)$ -dimensional world-volume of the brane, and $X^A = X^A(\xi)$ are functions describing the parametric embedding of the brane into the bulk manifold. $\gamma_{\alpha\beta}(\xi)$ is an auxiliary field representing the metric tensor on the brane. Capital Latin indices run from 0 to $D-1$, Greek indices from 0 to p . For the bulk coordinates we use the notation $x^A \equiv (t, x^i, y^a, z)$, where i, j and a, b run from 1 to d and from $d+1$ to $d+n$, respectively.

The variation of Eq. (1) w.r.t. g , X and γ leads to the Einstein equations

$$G_A{}^B = \Lambda \delta_A^B + \frac{T_p}{M^{(D-2)}} \frac{1}{\sqrt{|g|}} g_{AC} \int d^{p+1} \xi \sqrt{|\gamma|} \gamma^{\alpha\beta} \partial_\alpha X^C \partial_\beta X^B \delta^D(x - X(\xi)), \quad (2)$$

the equation for the brane evolution

$$\partial_\alpha (\sqrt{|\gamma|} \gamma^{\alpha\beta} \partial_\beta X^B g_{AB}(X)) = \frac{1}{2} \sqrt{|\gamma|} \gamma^{\alpha\beta} \partial_\alpha X^B \partial_\beta X^C \partial_A g_{BC}(X), \quad (3)$$

and the equation for the induced metric on the brane

$$\gamma_{\alpha\beta} = \partial_\alpha X^A \partial_\beta X^B g_{AB}(X), \quad (4)$$

respectively. We look for solutions describing a Z_2 -symmetric bulk geometry and a locally anisotropic brane with spatial geometry given by the product of two conformally flat manifolds with d and n dimensions, respectively. Assuming that the brane is rigidly located at $z=0$, we set

$$g_{AB} = f^2(z) \text{diag}(1, -a^2(t)\delta_{ij}, -b^2(t)\delta_{ab}, -1), \quad X^A = \delta_\alpha^A \xi^\alpha. \quad (5)$$

The equation for the brane is trivially satisfied. The induced metric is

$$\gamma_{\alpha\beta}(\xi) = g_{\alpha\beta}(\xi)|_{z=0}. \quad (6)$$

For this background, the (00), (ii), (aa), and (p + 1, p + 1) components of the Einstein equations are

$$-pF' - \frac{1}{2}p(p-1)F^2 + \frac{1}{2}d(d-1)H^2 + \frac{1}{2}n(n-1)G^2 + dnHG = \Lambda f^2 + \frac{T_p}{M^{(D-2)}}f\delta(z), \quad (7)$$

$$\begin{aligned} pF' + \frac{1}{2}p(p-1)F^2 - (d-1)\dot{H} - \frac{1}{2}d(d-1)H^2 - n\dot{G} - \frac{1}{2}n(n+1)G^2 - (d-1)nHG \\ = -\Lambda f^2 - \frac{T_p}{M^{(D-2)}}f\delta(z), \end{aligned} \quad (8)$$

$$\begin{aligned} pF' + \frac{1}{2}p(p-1)F^2 - d\dot{H} - \frac{1}{2}d(d+1)H^2 - (n-1)\dot{G} - \frac{1}{2}n(n-1)G^2 - d(n-1)HG \\ = -\Lambda f^2 - \frac{T_p}{M^{(D-2)}}f\delta(z), \end{aligned} \quad (9)$$

$$\frac{1}{2}p(p+1)F^2 - d\left(\dot{H} + \frac{d+1}{2}H^2\right) - n\left(\dot{G} + \frac{n+1}{2}G^2\right) - dnHG = -\Lambda f^2. \quad (10)$$

Dots (primes) denote differentiation w.r.t. t (z), and $H = \dot{a}/a$, $G = \dot{b}/b$, $F = f'/f$.

The time- and z -dependent parts of the above equations can be separated. The flat Minkowski brane ($H = G = 0$) is a trivial solution. Looking for non-trivial, power-law solutions we set

$$a(t) = (t/t_0)^\alpha, \quad b(t) = (t/t_0)^\beta. \quad (11)$$

The Ricci-flat Kasner solution is

$$\alpha = \frac{1 \pm \sqrt{n(d+n-1)/d}}{d+n}, \quad \beta = \frac{1 \mp \sqrt{d(d+n-1)/n}}{d+n}, \quad (12)$$

where $d\alpha^2 + n\beta^2 = d\alpha + n\beta = 1$. A particular configuration with $d = 3$ inflationary expanding dimensions can be obtained for $\alpha < 0$ and a negative range of the cosmic time parameter t , corresponding to the lower-sign branch of Eq. (12) with $n > 1$ contracting dimensions. Finally, from the z -dependent part of the above equations, we obtain the (Z_2 -symmetric) warp factor for the AdS bulk geometry,

$$f(z) = (1 + |z|/z_0)^{-1}, \quad z_0 = \sqrt{-p(p+1)/2\Lambda}, \quad (13)$$

where Λ , M and T_p are related by

$$2T_p/M^{(D-2)} = \sqrt{-32p\Lambda/(p+1)}. \quad (14)$$

(See also [9] for warped Kasner-like solutions on a brane.)

In order to discuss the propagation of metric perturbations on the above background we now introduce the expansion

$$g_{AB} \rightarrow g_{AB} + \delta g_{AB}, \quad \delta g_{AB} \equiv h_{AB}, \quad \delta g^{AB} = -h^{AB}, \quad (15)$$

and perturb equations (2)–(4) to first order in h_{AB} . The position of the brane is kept fixed by requiring $\delta X^A = 0$ (see, for instance, [10]). In the linear approximation, different components of the tensor fluctuations h_{AB} are decoupled. We are interested, in particular, in the transverse and traceless perturbations h_{ij} of the expanding d -dimensional part of the brane metric (5) which, for $d = 3$, describes the early-time geometry of our present large-scale spacetime. Keeping d generic, and choosing the synchronous gauge, we thus set

$$h_{0A} = 0, \quad h_{aA} = 0, \quad h_{zA} = 0, \quad g^{ij}h_{ij} = 0 = \nabla_j h_i^j \equiv \partial_j h_i^j. \quad (16)$$

The indices of h are raised and lowered with the unperturbed metric, and the covariant derivatives acting on h are constructed with the unperturbed connection.

Assuming that the translations along the n internal dimensions of the brane are isometries not only of g_{AB} , but also of the full perturbed metric $g_{AB} = g_{AB} + \delta g_{AB}$, we can set

$$h_{ij} = h_{ij}(t, x^i, z). \quad (17)$$

The perturbation of Eq. (4) for the induced metric gives

$$\delta\gamma_{\mu\nu} = (\delta_\mu^i \delta_\nu^j h_{ij})_{z=0}. \quad (18)$$

The perturbation of Eq. (3) is identically satisfied. The perturbations of the Einstein equations reduce to $\delta R_A^B = 0$. The propagation equation for tensor metric fluctuations on the expanding part of the brane is

$$\ddot{h}_i^j + (dH + nG)\dot{h}_i^j - \frac{\nabla^2}{a^2} h_i^j - h_i''{}^j - pFh_i'^j = 0, \quad (19)$$

where $\nabla^2 = \delta^{ij} \partial_i \partial_j$ is the flat-space Laplacian. As expected, each physical polarization mode h_i^j satisfies the full covariant d'Alembert equation $\nabla_A \nabla^A h_i^j = 0$, describing the free propagation of a scalar mode on the brane.

The correct normalization of h_i^j to an initial spectrum of vacuum quantum fluctuations is obtained from the quadratic action corresponding to the equations of motion (19). This action is computed by perturbing Eq. (1) up to terms quadratic in the first-order fluctuations δg_{AB} and $\delta\gamma_{\mu\nu}$, at fixed brane position $\delta X^A = 0$. We expand the contravariant components of the metric and the volume density to order h^2 :

$$\delta^{(1)} g^{AB} = -h^{AB}, \quad \delta^{(2)} g^{AB} = h^{AC} h_C^B, \quad \delta^{(1)} \sqrt{|g|} = 0, \quad \delta^{(2)} \sqrt{|g|} = -\frac{1}{4} \sqrt{|g|} h_{AB} h^{AB}, \quad (20)$$

and similarly the connection and the Ricci tensor. The second-order action takes contributions from $\sqrt{|g|}$, R and γ :

$$\begin{aligned} \delta^{(2)} S = & -\frac{M^p}{2} \int d^D x \left[\delta^{(2)} \sqrt{|g|} (R + 2\Lambda) + \sqrt{|g|} (\delta^{(2)} g^{AB} R_{AB} + \delta^{(1)} g^{AB} \delta^{(1)} R_{AB} + g^{AB} \delta^{(2)} R_{AB}) \right] \\ & - \frac{T_p}{2} \int d^{p+1} \xi \left[\delta^{(2)} \sqrt{|\gamma|} (\gamma^{\alpha\beta} \partial_\alpha X^A \partial_\beta X^B g_{AB} - (p-1)) \right. \\ & \left. + \sqrt{|\gamma|} (\delta^{(2)} \gamma^{\alpha\beta} g_{AB} + \delta^{(1)} \gamma^{\alpha\beta} \delta^{(1)} g_{AB}) \partial_\alpha X^A \partial_\beta X^B \right]. \end{aligned} \quad (21)$$

The action for the perturbations of the auxiliary field $\gamma_{\mu\nu}$ leads to the constraint (18). Using the latter, and the unperturbed equations of motion, we obtain

$$\delta^{(2)} S = \frac{M^p}{8} \int d^D x a^d b^n f^p \left[\dot{h}_i^j \dot{h}_j^i - h_i'^j h_j'^i + h_i^j \frac{\nabla^2}{a^2} h_j^i \right], \quad (22)$$

where we have integrated by parts. We can easily check that Eq. (19) is given by the variation of Eq. (22) w.r.t. h . We set $h_{ij} = h_{(a)} \epsilon_{ij}^{(a)}$, where $\epsilon_{ij}^{(a)}$ is the spin-two polarization tensor, and the sum is over all the $(d+1)(d-2)/2$ independent polarization states. Using $\text{Tr}[\epsilon^{(a)} \epsilon^{(b)}] = 2\delta^{ab}$, Eq. (22) can be rewritten as the sum over single-mode actions

$$\delta^{(2)} S = \sum_{(a)} \delta^{(2)} S_{(a)}, \quad \delta^{(2)} S_{(a)} = \frac{M^p}{4} \int d^D x a^d b^n f^p \left[\dot{h}^2 - h'^2 + h \frac{\nabla^2}{a^2} h \right], \quad (23)$$

where the polarization index (a) on the scalar mode h has been omitted in the last equation, for simplicity.

The propagation equation (19) can be separated in the bulk and brane coordinates by setting

$$h(t, x^i, z) = \sum_m h_m(t, x^i, z) = \sum_m v_m(t, x^i) \psi_m(z), \quad (24)$$

where the new variables v, ψ are labelled by the mass eigenvalue m . (The sum is replaced by integration over m for the continuous part of the eigenvalue spectrum.) We then obtain the equations

$$\ddot{v}_m + (dH + nG)\dot{v}_m - \frac{\nabla^2}{a^2}v_m = -m^2v_m, \quad (25)$$

$$\psi_m'' + pF\psi_m' \equiv f^{-p}(f^p\psi')' = -m^2\psi_m. \quad (26)$$

In particular, the rescaled variable $\hat{\psi}_m = \sqrt{M}f^{p/2}\psi_m$ satisfies the Schrodinger-like equation

$$\hat{\psi}_m'' + [m^2 - V(z)]\hat{\psi}_m = 0, \quad V(z) = -\frac{p}{z_0}\delta(z) + \frac{1}{4}p(p+2)(z_0 + |z|)^{-2}, \quad (27)$$

whose effective potential $V(z)$ has a “volcano-like” shape for $z_0 > 0$. As in the standard Randall–Sundrum scenario [4], this potential exactly localizes the massless mode $m = 0$ on the brane. The solutions of Eq. (26) are normalized by requiring the variables $\hat{\psi}_m$ to be orthonormal w.r.t. inner products with measure dz , as in conventional one-dimensional quantum mechanics [4,5,7,10,11]. This is equivalent to require the ψ_m to be orthonormal with measure $M dz f^p$ [12]:

$$\int dz M f^p \psi_m \psi_{m'} = \delta(m, m'), \quad (28)$$

where $\delta(m, m')$ denotes a Kronecker symbol for the discrete part of the spectrum and a Dirac distribution for the continuous one.

Inserting the expansion (24) in Eq. (23), and using the orthonormality condition (28) to integrate over z , the effective action can be written, modulo a total derivative, as a sum over the contributions of all fluctuation modes evaluated on the brane, $\bar{h}_m \equiv h_m(t, x^i, 0) = v_m(t, x^i)\psi_m(0)$:

$$\delta^{(2)}S = \sum_m \delta^{(2)}S_m = \sum_m \frac{M^{d-1}}{4|\psi_m(0)|^2} \int d^{d+1}x a^d b^n \left(\dot{\bar{h}}_m^2 + \bar{h}_m \frac{\nabla^2}{a^2} \bar{h}_m - m^2 \bar{h}_m^2 \right). \quad (29)$$

We have used Eq. (26) to eliminate $\psi_m'^2$, and have omitted the constant dimensionless volume factor $M^n \int d^n y$. (The internal brane sections are assumed to be compactified on a comoving length scale of order M^{-1} .) The above action can be written in canonical form by introducing the conformal-time coordinate $\eta = \int dt/a$ and the canonical variable u_m ,

$$u_m(\eta, x^i) = \xi_m(\eta)\bar{h}_m(\eta, x_i), \quad \xi_m(\eta) = \left(\frac{M}{2}\right)^{(d-1)/2} \frac{a^{(d-1)/2} b^{n/2}}{\psi_m(0)}. \quad (30)$$

Integrating by parts, we find:

$$\delta^{(2)}S = \sum_m \frac{1}{2} \int d\eta d^d x \left(u_m'^2 + u_m \nabla^2 u_m - m^2 a^2 u_m^2 + \frac{\xi_m''}{\xi_m} u_m^2 \right), \quad (31)$$

where, from now on, a prime will denote differentiation w.r.t. η . This is the typical action for a linear fluctuation \bar{h}_m coupled to an external “pump field” $\xi_m(\eta)$ (see, for instance, [14]). According to this action the Fourier modes u_{km} , defined by $\nabla^2 u_{km} = -k^2 u_{km}$, satisfy the canonical evolution equation

$$u_{km}'' + \left(k^2 + m^2 a^2 - \frac{\xi_m''}{\xi_m} \right) u_{km} = 0. \quad (32)$$

Since massless and massive modes do not mix, they can be discussed separately. Let us first consider the massless tensor fluctuations corresponding to the bound state of the potential (27), which describes long-range gravitational interactions confined on the brane. Assuming that the perturbed background is Z_2 -symmetric [13],

the square-integrable, Z_2 -even solution of Eq. (27) with $m = 0$ is

$$\hat{\psi}_0 = c_0 f^{p/2}, \quad \psi_0 = c_0 / \sqrt{M} = \text{const}, \quad (33)$$

where c_0 is an integration constant. By imposing that $\hat{\psi}_0 \in L^2(R)$, according to the normalization (28), we obtain

$$c_0^2 = \left(\int dz f^p \right)^{-1} = \frac{p-1}{2} z_0, \quad \psi_0 = \left(\frac{p-1}{2Mz_0} \right)^{1/2}. \quad (34)$$

The effective coupling parameter for the massless mode \bar{h}_0 is determined by the action (29), and has to be identified with the Planck mass ($M_P \simeq 10^{19}$ GeV) controlling long range gravitational interactions on the brane,

$$M_P^{d-1} = \frac{M^{d-1}}{|\psi_0|^2} = \frac{2z_0}{(p-1)} M^d. \quad (35)$$

For $n = 0$, $d = p = 3$, one recovers the standard Randall–Sundrum relation [4] $M_P^2 = z_0 M^3$ between the Planck mass M_P and the bulk mass M .

In order to discuss the formation of a relic background of gravitational radiation, amplified from the vacuum by the cosmological evolution of the brane, we consider a simple dynamical model where the brane geometry evolves from an initial (inflationary) Kasner-like regime to a final regime characterized by the flat Minkowski metric. (A more realistic picture would require, of course, the presence of matter fields on the brane.) By assuming a transition epoch localized around the conformal time scale $\eta = -\eta_1$, the computation of the pump field (30) gives, for $m = 0$,

$$\begin{aligned} \xi_0(\eta) &= \left(\frac{M_P^{d-1}}{2} \right)^{1/2} \left(-\frac{\eta}{\eta_1} \right)^{1/2}, \quad \eta \ll -\eta_1, \\ \xi_0(\eta) &= \left(\frac{M_P^{d-1}}{2} \right)^{1/2} = \text{const}, \quad \eta \gg -\eta_1. \end{aligned} \quad (36)$$

The effective potential of Eq. (32) evolves from the initial value $\xi''/\xi = -1/(4\eta^2)$ at $\eta \rightarrow -\infty$ to the final value $\xi''/\xi = 0$ at $\eta \rightarrow +\infty$. Note that the initial pump field, $\xi_0 \sim (-\eta)^{1/2}$, is represented by a function of time which is independent from d and n : this “universality” is typical of the Kasner solution and characterizes also the “minimal” pre-big bang configurations of string cosmology [15].

The evolution of the pump field and the effective potential imply, according to Eq. (32), that the Fourier amplitude of the tensor perturbation $\bar{h}_{k0} = u_{k0}/\xi_0$ grows logarithmically in time outside the horizon [16,17], and that the final power of the spectrum is fixed by the number of spatial dimensions in which the perturbations propagate (modulo logarithmic corrections). In fact, the massless solution (34) for ψ_0 is dimensionless, so that u_0 has the correct canonical dimensions ($[u_0] = M_P^{(d-1)/2}$) and can be normalized, asymptotically, to an initial spectrum of quantum vacuum fluctuations by imposing $u_{k0} = \exp(-ik\eta)/\sqrt{2k}$ at $\eta \rightarrow -\infty$. Matching the exact Hankel solution $u_{k0} = (\pi|\eta|/4)^{1/2} H_0^{(2)}(k\eta)$ for $\eta \leq -\eta_1$ to the plane-wave solution for $\eta \geq -\eta_1$, we obtain $\bar{h}_{k0} = u_{k0}/\xi_0$ after the transition. The final spectral distribution is:

$$|\delta_0(k)|^2 = k^d |h_{k0}|^2 \simeq \left(\frac{H_1}{M_P} \right)^{d-1} \left(\frac{k}{k_1} \right)^d (\ln k\eta_1)^2, \quad k < k_1 = \eta_1^{-1}. \quad (37)$$

(See, e.g., [16] for a higher-dimensional computation, and [18] for a recent computation with the same pump field in a different background.)

In the above equation, $H_1 = k_1/a = (a\eta_1)^{-1}$ is the curvature scale of the brane at the transition epoch η_1 , and $k_1 = \eta_1^{-1}$ is the ultraviolet cut-off scale of the spectrum: high-frequency modes with $k > k_1$ are not significantly amplified by the background transition. The final amplitude of the relic background of massless tensor perturbations is determined by the inflation scale in Planck units. There is no difference between this model of brane-world

inflation and more conventional (Kaluza–Klein) models of inflation with higher-dimensional factorized geometry; in particular, there is no extra enhancement of the amplitude when the brane curvature H_1 exceeds the critical value M^d/M_{p}^{d-1} , unlike in de Sitter models of brane inflation [5,6].

Let us now consider massive tensor fluctuations that are not localized on the brane ($m \neq 0$). After imposing the appropriate boundary conditions at $z = 0$ [4,7,10,12] (see also Ref. [19]), the even Z_2 -symmetric solution of Eq. (27) with $m \neq 0$ can be written in terms of first and second kind Bessel functions as

$$\hat{\psi}_m = c_m f^{-1/2} \left[Y_{(p-1)/2}(mz_0) J_{(p+1)/2} \left(\frac{mz_0}{f} \right) - J_{(p-1)/2}(mz_0) Y_{(p+1)/2} \left(\frac{mz_0}{f} \right) \right]. \quad (38)$$

(This is a generalization to p dimensions of the 3-brane solution presented in Ref. [10].) The orthonormality condition (28) gives [10,12,19]

$$c_m = \left(\frac{mz_0}{2} \right)^{1/2} [Y_{(p-1)/2}^2(mz_0) + J_{(p-1)/2}^2(mz_0)]^{-1/2}. \quad (39)$$

It follows

$$\psi_m(0) = \frac{\hat{\psi}_m(0)}{\sqrt{M}} = \frac{F(mz_0)}{\sqrt{M}}, \quad (40)$$

where

$$F(x) = \left(\frac{x}{2} \right)^{1/2} \frac{Y_{(p-1)/2}(x) J_{(p+1)/2}(x) - J_{(p-1)/2}(x) Y_{(p+1)/2}(x)}{[Y_{(p-1)/2}^2(x) + J_{(p-1)/2}^2(x)]^{1/2}}, \quad x = mz_0. \quad (41)$$

According to Eq. (30), the pump field for massive modes

$$\xi_m(\eta) = \left(\frac{M}{2} \right)^{(d-1)/2} \frac{\sqrt{M}}{F(mz_0)} \left(-\frac{\eta}{\eta_1} \right)^{1/2}, \quad (42)$$

has different normalization, but identical time-dependence of the massless pump field. The different coupling parameter, M_m , for any infinitesimal mass interval dm , is defined by the action (29) as

$$M_m^d = \frac{M^{d-1}}{|\psi_m(0)|^2} = \frac{M^d}{F^2(mz_0)}. \quad (43)$$

It may be noted, incidentally, that in the light mass regime ($mz_0 \ll 1$) the use of the small argument limit of the Bessel functions leads to $F^2 \simeq \Gamma^{-2}[(p-1)/2](mz_0/2)^{p-2}$, and to the effective (differential) coupling strength

$$8\pi G_m dm \equiv \frac{dm}{M_m^d} \simeq \frac{dm}{M^d} \Gamma^{-2} \left(\frac{p-1}{2} \right) \left(\frac{mz_0}{2} \right)^{p-2}. \quad (44)$$

For $n = 0$, $d = p = 3$, one recovers the integral measure controlling the well-known Yukawa contribution of light massive modes to short-range gravitational interactions [4,10,12],

$$8\pi G_m dm \simeq \frac{mz_0^2}{2z_0 M^3} dm = 8\pi G_N \frac{mz_0^2}{2} dm, \quad (45)$$

where $8\pi G_N = M_{\text{p}}^{-2} = (z_0 M^3)^{-1}$ is the Newton constant on the brane.

The cosmological amplification of the massive components can be discussed by considering again the transition between the Kasner and Minkowski regimes at the scale η_1 . The canonical equation (32) is still valid, and the evolution in time of the pump field is the same as before. If the mass term of Eq. (32) is negligible w.r.t. the transition curvature scale ($m/H_1 \sim ma_1 \eta_1 \ll 1$), the frequency distribution of the relativistic massive spectrum

(i.e., the modes with $m \ll k/a < H_1$) will be the same as in the massless case. However, the final amplitude will generally be different, because of the different coupling strength of massive modes.

Let us compute the relativistic, massive mode contribution to the Fourier component \bar{h}_k of the metric fluctuation on the brane:

$$\bar{h}_k \equiv \int dm \bar{h}_{km} = \int dm \frac{u_{km}}{\xi_m}. \quad (46)$$

(We are considering the range of modes $m \ll k/a < H_1$.) Taking into account the definitions of ψ_m and ξ_m , the initial normalization to a quantum spectrum of vacuum fluctuations now imposes the condition $u_{km} = \exp(-ik\eta)/\sqrt{kM}$ at $\eta \rightarrow -\infty$. Performing the matching procedure with the massive pump field (42), we can easily obtain the following spectral amplitude of tensor perturbations, in the differential mass interval dm :

$$|\bar{h}_{km}| \simeq \frac{F(mz_0)}{M} \left(\frac{k\eta_1}{kM^{d-1}} \right)^{1/2} \ln(k\eta_1). \quad (47)$$

Integration over m leads to the final spectrum

$$|\delta_m(k)|^2 = k^d |\bar{h}_k|^2 = k^d \left(\int dm |\bar{h}_{km}| \right)^2 \simeq \left(\frac{H_1}{M_*} \right)^{d-1} \left(\frac{k}{k_1} \right)^d (\ln k\eta_1)^2, \quad (48)$$

where

$$M_*^{d-1} = \frac{M^{d-1}}{\left[\int^{H_1} \frac{dm}{M} F(mz_0) \right]^2} = M_P^{d-1} \frac{(p-1)Mz_0}{2 \left[\int^{H_1 z_0} dx F(x) \right]^2}. \quad (49)$$

(We have used Eq. (35) for a direct comparison of M_* to the Planck mass M_P .) Thus we obtain the distribution (37), where H_1/M_P has been replaced by H_1/M_* in the renormalized amplitude.

To discuss the importance of the relativistic massive contributions, let us first consider the case $H_1 z_0 \lesssim 1$, in which the integral $I = \int dx F(x)$ is extended to a mass range which only includes light modes with $mz_0 \leq H_1 z_0 \lesssim 1$. In this case the integration gives a small numerical factor, $I \lesssim 1$. Recalling that $Mz_0 \sim (M_P/M)^{p-1}$, and assuming that our model is characterized by a strong bulk gravity with $M \ll M_P$ (as required, in more general brane-world scenarios, for a possible solution of the hierarchy problem [4,12]), we obtain $M_* \gg M_P$. This means that the massive contribution to the metric fluctuation spectrum is highly suppressed w.r.t. the massless spectrum, in agreement with the well-known decoupling of light massive modes at low energies [12].

By contrast, if the cosmological transition occurs at high curvature scale, $H_1 z_0 \gg 1$, the integral in Eq. (49) also includes the contribution of massive modes with $1 \ll mz_0 < H_1 z_0$. These modes are “light” w.r.t. brane curvature scale H_1 , but “heavy” w.r.t. bulk curvature scale z_0^{-1} . In this case, using the properties of the Bessel functions (or performing numerical integrations) we find that the integral in Eq. (49) is approximated by $\int^{H_1 z_0} dx F(x) \simeq H_1 z_0 / \sqrt{\pi}$. It follows

$$M_*^{d-1} \sim M_P^{d-1} \frac{Mz_0}{H_1^2 z_0^2}. \quad (50)$$

In particular, $M_* < M_P$ when

$$\frac{Mz_0}{H_1^2 z_0^2} \sim \left(\frac{M}{M_P} \right)^{d-1} \left(\frac{M}{H_1} \right)^2 < 1. \quad (51)$$

If $M < M_P$, this condition can be satisfied for high enough transition scales (e.g., for $H_1 \sim M_P$). Therefore, brane-world models with sufficiently strong bulk gravity (i.e., sufficiently small parameter M) may be characterized by strongly enhanced amplifications of metric perturbations due to the contribution of massive gravitons.

A detailed study of the massive mode contribution to the perturbation spectrum, including the non-relativistic sector and the large- m regime ($m > H_1$), will be reported in a forthcoming paper [20]. Here we conclude by noticing that such an enhanced amplification of tensor modes could make the brane-world scenario considered above unstable. The amplified perturbations could indeed destroy the homogeneity of the background, and drive the brane towards a curvature singularity, unless the model is complemented by some appropriate mechanism stopping inflation at a small enough curvature scale $H_1 \lesssim M_*$ (in principle even much smaller than the usual quantum gravity scale M_P). This kind of instability could be relevant to brane models of pre-big bang inflation [21], where the brane has a Kasner-like geometrical structure similar to that discussed in this Letter.

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References

- [1] L.P. Grishchuk, M. Solokin, Phys. Rev. D 43 (1991) 2566.
- [2] D. Langlois, Prog. Theor. Phys. Suppl. 148 (2003) 181.
- [3] P. Hořava, E. Witten, Nucl. Phys. B 460 (1996) 506;
P. Hořava, E. Witten, Nucl. Phys. B 475 (1996) 96.
- [4] L. Randall, R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690.
- [5] D. Langlois, R. Maartens, D. Wands, Phys. Lett. B 489 (2000) 259.
- [6] T. Kobayashi, H. Kudoh, T. Tanaka, Phys. Rev. D 68 (2003) 044025;
A. Frolov, L. Kofman, hep-th/0209133.
- [7] R. Easther, D. Langlois, R. Maartens, D. Wands, JCAP 10 (2003) 014.
- [8] M. Gasperini, N. Sanchez, G. Veneziano, Nucl. Phys. B 364 (1991) 365;
M. Gasperini, N. Sanchez, G. Veneziano, Int. J. Mod. Phys. A 6 (1991) 3853.
- [9] A.V. Frolov, Phys. Lett. B 414 (2001) 213;
A. Fabbri, D. Langlois, D.A. Steer, R. Zegers, hep-th/0407262.
- [10] V. Bozza, M. Gasperini, G. Veneziano, Nucl. Phys. B 619 (2001) 191.
- [11] C. Csaki, J. Erlich, T.J. Hollowood, T. Shirman, Nucl. Phys. B 581 (2000) 309.
- [12] V.A. Rubakov, Phys. Usp. 44 (2001) 871.
- [13] C. van de Bruck, M. Dorca, R.H. Brandenberger, A. Lukas, Phys. Rev. D 62 (2000) 123515.
- [14] V.F. Mukhanov, H.A. Feldman, R.H. Brandenberger, Phys. Rep. 215 (1992) 203.
- [15] M. Gasperini, G. Veneziano, Astropart. Phys. 1 (1993) 317;
M. Gasperini, G. Veneziano, Phys. Rep. 373 (2003) 1.
- [16] R. Brustein, M. Gasperini, M. Giovannini, V. Mukhanov, G. Veneziano, Phys. Rev. D 51 (1995) 6744.
- [17] R. Brustein, M. Gasperini, M. Giovannini, G. Veneziano, Phys. Lett. B 361 (1995) 45.
- [18] M. Gasperini, M. Giovannini, G. Veneziano, Nucl. Phys. B 694 (2004) 206.
- [19] R.A. Battye, C. van der Bruck, A. Mennim, Phys. Rev. D 69 (2004) 064040.
- [20] M. Cavaglià, G. De Risi, M. Gasperini, in preparation.
- [21] S. Foffa, Phys. Rev. D 66 (2002) 063512;
S. Foffa, Phys. Rev. D 68 (2003) 043511.