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Instability of the $R^3 \times S^1$ vacuum in low-energy effective string theory

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We present and discuss a Euclidean solution of the low-energy effective string action that can be interpreted as a semiclassical decay process of the ground state of the theory.

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In Ref. [1] the authors found an instanton solution of a four-dimensional, modulus field-dependent, low-energy effective string theory. That solution describes either a wormhole connecting two asymptotically flat regions or the nucleation of a baby universe starting from an original flat region. Our aim here is to show how this instanton can also describe a different physical process taking place in the theory. Indeed, using a different analytical continuation to the hyperbolic space, the solution of Ref. [1] can be interpreted as a semiclassical decay process of the ground state (vacuum) of the theory. The existence of a process of semiclassical decay is important since it may lead to the instability of the vacuum of the theory. Furthermore, a careful analysis of the geometric and topological features of the instanton will enable us to identify the wormhole solution of Ref. [1] as a Hawking-type wormhole [2] connecting two asymptotic regions of $R^3 \times S^1$ topology.

In this paper we will follow an approach similar to the one used by Witten in Ref. [3] to prove the semiclassical instability of the Kaluza-Klein vacuum in five dimensions. Even though the theory considered here has little to do with the Kaluza-Klein theory in five dimensions, both instantons have common geometrical and topological features and consequently most of the mathematical techniques used in [3] can also be implemented in our case.

Our starting point is the Euclidean action $[(16\pi G)^{-1} \equiv M_{\rm Pl}^2/16\pi = 1]$

$$S_E = \int_\Omega d^4x\sqrt{|g|}e^{-2\phi}\left[-R + \frac{8k}{1-k}(\nabla\phi)^2 + \varepsilon\frac{3+k}{1-k}F^2\right] -2\int_{2\Omega} d^3x\sqrt{h}e^{-2\phi}(\mathbf{K}-\mathbf{K_0}) \;, \tag{1}$$

where R is the curvature scalar, ϕ is the dilaton field, $F_{\mu\nu}$ is the usual electromagnetic (EM) field tensor, and k is a coupling constant, $-1 \leq k \leq 1$. The boundary term is required by unitarity (see, e.g., [4]). $\varepsilon = \pm 1$ is a parameter whose meaning will be clear in a moment.

Action (1) follows from the modulus-dependent low-energy effective string theory considered in [5] once one eliminates the modulus from the action by choosing an appropriate ansatz consistent with the field equations [6,1]. The action describes a Jordan-Brans-Dicke theory coupled to the electromagnetic field and reduces to well-known theories according to the value of k [6–9].

The meaning of the parameter ε needs some further explanation. As shown in [1], in order to write the contribution of the EM field to the Lagrangian in a space with a signature $(+,+,+,+)$, we have to choose the sign of the term F^2 according to the electric or magnetic configuration of the field. Indeed, the EM field in Euclidean space is not analytically related to the EM field in hyperbolic space by the simple transformation $t \to i\tau$, but in general we have

$$E^2_{\rm hyp} = \varepsilon E^2_{\rm Eucl},\; H^2_{\rm hyp} = -\varepsilon H^2_{\rm Eucl} \;. \tag{2}$$

Since we wish to deal with real analytical continuations of (real) hyperbolic fields in Euclidean space, we allow for a different sign in front of the F^2 term in the action, according to the configuration of the EM field. We will choose $\varepsilon = -1$ for a purely magnetic configuration and $\varepsilon = 1$ for a purely electric one.[1]

Now let us consider a four-dimensional Riemannian manifold described by a line element of the form

$$ds^2 = A^2(r)dt^2 + B^2(r)d\chi^2 + r^2 d\Omega_2^2 \;, \tag{3}$$

where χ is the coordinate of the one-sphere, $0 \leq \chi < 2\pi$, and $d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$ represents the line element of the two-sphere S^2. Choosing for the EM field the magnetic monopole configuration on S^2 (and thus $\varepsilon = -1$),

$$F = Q_m \sin\theta\, d\theta \wedge d\varphi \;, \tag{4}$$

[1] Also duality invariance arguments support this prescription (see [1] for details). These arguments are similar to those used in Ref. [10] for the case of the axion. The key point is that $F \to {}^*F$ and the continuation to Euclidean space do not commute.

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the solution of the field equations derived from (1) is

$$ds^2 = \left(1 - \frac{Q^2}{r^2}\right)^{-1} dr^2 + Q^2\left(1 + \frac{Q}{r}\right)^{k-1}\left(1 - \frac{Q^2}{r^2}\right) d\chi^2 + r^2 d\Omega_2^2 \ , \quad (5)$$

$$e^{2(\phi-\phi_0)} = \left(1 + \frac{Q}{r}\right)^{(k-1)/2} \ , \quad (6)$$

where the magnetic charge Q_m has been redefined through

$$Q_m = \tfrac{1}{2}\sqrt{1-k}Q \ . \quad (7)$$

The crucial point for the identification of (4)–(6) with a vacuum decay process is the analytical continuation of the line element to hyperbolic space. Therefore, let us discuss the geometric and topological properties of the Euclidean manifold described by (5). Since the latter has by definition signature $(+,+,+,+)$, r can take values only in the range $[Q,\infty[$. For $r \to \infty$ the space is asymptotically flat with topology $R^3 \times S^1$. For $r = Q$ the metric tensor is singular. However, in $r = Q$ the manifold is smooth, as can be shown by putting $r = \sqrt{Q^2+\tau^2}(\tau \in]-\infty,\infty[)$ and defining χ as a periodic variable with period $2\pi \times 2^{1-k}$ [1]. This conclusion seems to indicate that the coordinate system (r,χ,θ,φ) does not cover the whole manifold. In order to obtain the maximal extension of the Euclidean metric (5), we have to perform an appropriate coordinate transformation:

$$r = \frac{(x^2+t^2)+Q^2}{2\sqrt{x^2+t^2}}, \quad \tan\theta = \frac{x}{t} \ . \quad (8)$$

The inverse of (8) is

$$x = f(r)\sin\theta, \quad t = f(r)\cos\theta \ , \quad (9)$$

where

$$f(r) = \sqrt{x^2+t^2} = Q\exp[\operatorname{arccosh}(r/Q)] \ . \quad (10)$$

The coordinate transformation (9) is never singular. Using (8), the Euclidean solution (4)–(6) becomes

$$ds^2 = \frac{1}{4}\left(1 + \frac{Q^2}{f^2}\right)^2 [dt^2 + dx^2 + x^2 d\varphi^2] + Q^2\left(1 - \frac{2Q^2}{f^2+Q^2}\right)^2\left(1 + \frac{2Qf}{f^2+Q^2}\right)^{k-1} d\chi^2 \ , \quad (11)$$

$$e^{2(\phi-\phi_0)} = \left(1 + \frac{2Qf}{f^2+Q^2}\right)^{(k-1)/2} \ , \quad (12)$$

$$F = \frac{1}{2}\sqrt{1-k}Q\frac{x}{f^3}[x\,dt\wedge d\varphi - t\,dx\wedge d\varphi] \ . \quad (13)$$

Equation (11) represents the maximal extension of (5). As before, when $x,t \to \infty$ the manifold is asymptotically flat with topology $R^3 \times S^1$. The critical surfaces are two: $x^2+t^2 = Q^2$ and $x^2+t^2 = 0$. Using the coordinate transformation, it is easy to verify that the first one corresponds to $r = Q$. The second critical surface corresponds to $r = \infty$. Hence Eq. (11) describes two asymptotically flat regions smoothly joined through the surface $r = Q$. This strange structure is related to the existence of a conformal equivalence between the region inside $x^2+t^2 = Q^2$ and the region outside. In fact, the Euclidean line element (11) is invariant under the transformation

$$y^\mu \to \frac{Q^2}{y^2}O^\mu{}_\nu y^\nu, \quad \mu = 1,2,3 \ , \quad (14)$$

where y^μ are Cartesian coordinates of three-dimensional space (t,x,φ), $y^1 = t$, $y^2 = x\cos\varphi$, $y^3 = x\sin\varphi$, and $O^\mu{}_\nu$ is a 3×3 rotation matrix. Hence solution (11) represents a Hawking-type wormhole [2] with a minimum radius equal to Q connecting two asymptotically flat spaces with topology $R^3 \times S^1$. Note that (14) is an invariance of the entire solution (11)–(13), not only of the metric (11). Indeed, also the expression (12), (13) for the dilaton and EM field do not change under the transformation (14).

How can we recover the vacuum decay interpretation? In order to answer this question, we have to go back to (5) and continue analytically the Euclidean solution to a hyperbolic spacetime. In Ref. [1] the analytical continuation was performed first by defining $\tau = \sqrt{r^2-Q^2}$, thereafter by the complexification of τ, $\tau \to i\tau$. The resulting hyperbolic manifold was interpreted as a baby universe of spatial topology $S^2 \times S^1$ nucleated at $\tau = 0$. However, the latter is not the only analytic continuation we can perform. For instance, we can complexify the θ coordinate of the two-sphere S^2. In this case, since $\theta = 0$ is a coordinate singularity of the metric, it is convenient to choose as a symmetry plane the surface $\theta = \pi/2$ and to put

$$\theta \to \frac{\pi}{2} + i\xi \ . \quad (15)$$

After the replacement (15) we obtain the hyperbolic solution

$$ds^2 = \left(1 - \frac{Q^2}{r^2}\right)^{-1} dr^2 + Q^2\left(1 + \frac{Q}{r}\right)^{k-1}\left(1 - \frac{Q^2}{r^2}\right) d\chi^2 - r^2 d\xi^2 + r^2\cosh^2\xi\, d\varphi \ , \quad (16)$$

$$e^{2(\phi-\phi_0)} = \left(1 + \frac{Q}{r}\right)^{(k-1)/2} \ . \quad (17)$$

The EM two-form is now

$$F = Q_m\cosh\xi\, d\xi \wedge d\varphi \ . \quad (18)$$

The EM field is real, as a result of the choice $\varepsilon = -1$ in the action (1). For $r \geq Q$ this spacetime is nonsingular, the coordinate singularity at $r = Q$ being as harmless as it was for the Euclidean space (5). The solution (16) for $r \geq Q$ represents the spacetime in which the $R^3 \times S^1$ vacuum decays. The topology of the initial $\xi = 0$ surface is $R^2 \times S^1$. Note that the analytic continuation to the hyperbolic space of Ref. [1], even though it was obtained from the Euclidean instanton (5), has instead a spatial topology $S^2 \times S^1$.

The topology of the analytic continuation to hyperbolic space depends thus on the coordinate chosen to complexify. A better understanding of the features of this space can be achieved starting from a hyperbolic line element that covers only the region $r \geq Q$. Using the coordinate transformation

$$x = f(r)\cosh\xi, \quad t = f(r)\sinh\xi \ , \tag{19}$$

where $f(r) = \sqrt{x^2 - t^2}$ is defined as a function of r as in Eq. (10), we obtain

$$ds^2 = \frac{1}{4}\left(1 + \frac{Q^2}{f^2}\right)^2 [-dt^2 + dx^2 + x^2 d\varphi^2] + Q^2\left(1 - \frac{2Q^2}{f^2 + Q^2}\right)^2 \left(1 + \frac{2Qf}{f^2 + Q^2}\right)^{k-1} d\chi^2 \ , \tag{20}$$

$$e^{2(\phi - \phi_0)} = \left(1 + \frac{2Qf}{f^2 + Q^2}\right)^{(k-1)/2} \ , \tag{21}$$

$$F = \frac{1}{2}\sqrt{1-k}\,Q\frac{x}{f^3}[x\, dt \wedge d\varphi - t\, dx \wedge d\varphi] \ . \tag{22}$$

Since $-1 \leq t/x \leq 1$, the new coordinates (x, t) do not cover the whole plane. They cover only the region outside the light cone, $x = \pm t$, corresponding to the physical region. As for the Euclidean case, the critical surfaces are two: $x^2 - t^2 = Q^2$, corresponding to $r = Q$, and $x^2 - t^2 = 0$, representing infinity (see Fig. 1). Of course, the manifold described by (20) is *geodesically complete* and its topology is $R^3 \times S^1$. Regions I and II in Fig. 1

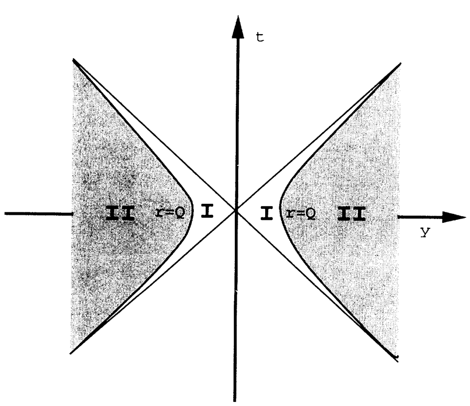

FIG. 1. Two-dimensional section of hyperbolic space described by the metric (20). The physical region corresponds to the shaded part (region II) of the picture enclosed by the hyperbola $y^2 - t^2 = Q^2$.

are analogous to the Euclidean ones, and their conformal equivalence can be proved using a coordinate transformation similar to (14).

Region II is the starting point for the vacuum decay interpretation of the Euclidean instanton. As one can easily verify, the origin of the Euclidean plane (x, t), coinciding with an asymptotically flat infinity, is not the only surface we can use to perform the analytic continuation in hyperbolic space. At $t = 0$ we can join the Euclidean manifold described by (11) with a hyperbolic spacetime, namely, the region $x^2 - t^2 > Q^2$ of the spacetime (20) (region II in Fig. 1). Indeed, at $t = 0$ the metric, dilaton field, and EM field assume a minimal configuration, and so the extrinsic curvature vanishes and the joining is possible. The hyperbolic spacetime in which the vacuum decays is region II in Fig. 1. Let us explore in detail its properties. Because of the maximal analytic extension, the regions on the left and right of the plane (x, t) are identical, and so we will focus our attention on one of them. Choosing for simplicity $\chi =$ const, the line element (20) becomes conformally equivalent to a R^3 flat Minkowskian spacetime. Of course, the manifold is not geodesically complete, since there exist geodesics crossing the boundary $x^2 - t^2 = Q^2$. The meaning of the boundary can be understood following its time evolution. Starting at $t = 0$, as t becomes larger and larger, the coordinate x of the boundary grows according to $x = \sqrt{Q^2 + t^2}$. Since the coordinate x corresponds to a radius in the cylindrical system of coordinates (t, x, φ), the boundary can be interpreted as a hole in space starting with radius Q at $t = 0$ and growing up for $t > 0$. At $t = 0$ the EM field is a purely electric field in the φ direction, $E_\varphi = Q_m/x$; as the time t flows and E_φ changes in intensity, the latter generates a magnetic field in the perpendicular χ direction. Finally, when $x, t \to \infty$, the EM field vanishes, as expected because the spacetime is asymptotically flat. The Euclidean line element (11) represents thus the decay process of the flat spacetime of topology $R^3 \times S^1$ in a spacetime with a growing hole.

In conclusion, the Euclidean instanton we are dealing with represents either a wormhole or a vacuum decay process according to the null-extrinsic curvature surface used for the analytic continuation to hyperbolic spacetime.

The previous results can be straightforwardly extended to the purely electric EM field configuration. Choosing $\varepsilon = 1$ in the action (1) and using an electric field along the χ direction, we obtain a line element which differs from the previous one for the purely magnetic case only through the conformal factor $e^{4\phi_0}(1 - Q/r)^{1-k}$, and so all conclusions remain unchanged.

At this stage we can ask ourselves if the semiclassical vacuum decay process is consistent with energy conservation. Since the $R^3 \times S^1$ vacuum has zero energy, the space (16) in which it decays must also have zero energy. Using the Arnowitt-Reser-Misner (ADM) formula generalized to dilaton-gravity theories, the total energy of (16)–(18) can be calculated as usual by means of a surface integral depending on the asymptotic behavior of the gravitational and dilaton fields. The line element (20) is not static with respect to t, and so the integral must be evaluated at the initial $t = 0$ surface, corresponding in (16) to $\xi = 0$. The result of the integration is zero. Indeed, the terms of the gravitational and dilaton fields which contribute to the total energy of the solution are those of order $1/r$. However, in our case these terms give a null contribution to the energy, owing to the $R^2 \times S^1$ topology of the $\xi = 0$ surface. The space described by (16) has therefore zero energy. This feature makes the $R^3 \times S^1$ vacuum not stable for the theory defined by (1), since there exists a solution with zero energy and the same asymptotic behavior as the $R^3 \times S^1$ vacuum. An important consequence of this result is that the positive energy theorem [11] does not hold for the theory (1) if one considers vacua with topology $R^3 \times S^1$. The positive energy theorem states that every nonflat, asymptotically Minkowskian solution of the Einstein equations has zero energy. However, its validity for spaces with arbitrary topology and for theories such as (1) is difficult to prove. In the case under consideration, the failure of the positive energy theorem seems related to the presence of the EM field: In the $R^3 \times S^1$ vacuum there exists excitations of the EM field for which the positive energy theorem does not hold.

The interpretation of the Euclidean solution (5) as an instability of the vacuum has been established using the analytical continuation (15). Considering a second analytical continuation to a hyperbolic spacetime, we have also seen that the instanton can be interpreted as a Hawking-type wormhole. The latter has an intrinsically three-dimensional nature because its topology is $R^3 \times S^1$ and the radius of S^1 is equal to Q in the two asymptotic regions $f = \infty$, $f = 0$ and shrinks to zero for $r = Q$. Hence the most natural interpretation of this solution can be found in the context of a 3+1 Kaluza-Klein theory.

Starting from the action (1) with $\varepsilon = -1$, setting to zero the components of the EM field along the χ direction, and splitting the four-dimensional line element as

$$ds^{(4)} = ds^{(3)} + Q^2 e^{-2\psi} d\chi^2 \ , \tag{23}$$

after some manipulations we obtain the three-dimensional action

$$S_E = \int_\Omega d^3x \sqrt{|g^{(3)}|}\, e^{-2\sigma} \Bigg[-R^{(3)} + \frac{k-1}{2}[4(\nabla\sigma)^2 - (\nabla\eta)^2] - \frac{3+k}{1-k} F^2 \Bigg] , \tag{24}$$

where $\sigma = \phi + \psi/2$, $\eta = \psi + 2\phi(k+1)/(k-1)$, and we have dropped the boundary terms.

A solution of the ensuing equations of motion is

$$ds^2 = \frac{1}{4}\left(1 + \frac{Q^2}{f^2}\right)^2 [dt^2 + dx^2 + x^2 d\varphi^2] \ ,$$
$$e^{2(\sigma-\sigma_0)} = \frac{f^2+Q^2}{f^2-Q^2}, \quad e^{2(\eta-\eta_0)} = \left(\frac{f+Q}{f-Q}\right)^2 \ , \tag{25}$$

where $f = \sqrt{x^2+t^2}$ and we have chosen the EM tensor F as in (13). The solution of the three-dimensional theory is thus a Hawking-type wormhole connecting two asymptotic regions of topology R^3.

Now let us calculate the decay rate of the vacuum. Evaluating the action (1) on the Euclidean solution (4)–(6), we have

$$S_E = 4\pi^2 e^{-2\phi_0} Q^2 (k+1) \ . \tag{26}$$

This result has been obtained by integrating r and θ in the range $Q \le r < \infty$, $0 \le \theta \le \pi/2$, the appropriate one for the vacuum decay process. The rate of decay of the $R^3 \times S^1$ vacuum is

$$\Gamma_{\rm VD} = \exp\left[-4\pi^2 e^{-2\phi_0} Q^2 (k+1)\right] \ . \tag{27}$$

The vacuum is long lived for values of Q much greater than the Planck length l_P and becomes unstable when Q is of the same order of magnitude of l_P. Finally, it is interesting to compare the vacuum decay rate $\Gamma_{\rm VD}$ with the probability for the nucleation of a baby universe $\Gamma_{\rm BU}$ (see Ref. [1]):

$$\Gamma_{\rm BU} = (\Gamma_{\rm VD})^2 \ . \tag{28}$$

Hence the probability of nucleation of a baby universe is smaller than the probability of the vacuum decay.

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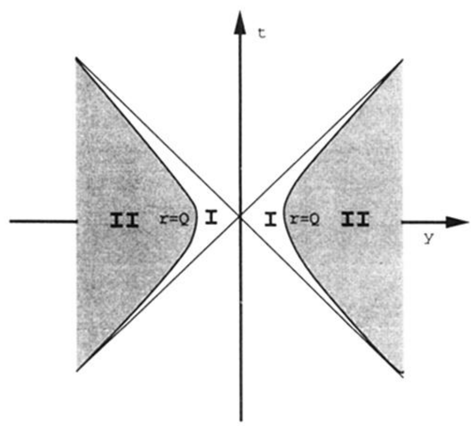

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