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Gravitational Larmor formula in higher dimensions

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The Larmor formula for scalar and gravitational radiation from a pointlike particle is derived in any even higher-dimensional flat spacetime. General expressions for the field in the wave zone and the energy flux are obtained in closed form. The explicit results in four and six dimensions are used to illustrate the effect of extra dimensions on linear and uniform circular motion. Prospects for detection of bulk gravitational radiation are briefly discussed.

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I. INTRODUCTION

The interest in higher-dimensional scenarios has increased ever since the first attempts by Kaluza and Klein to unify gravity and electromagnetism [1]. Recent proposals include large extra-dimensional models [2] and braneworlds [3,4], which have been advocated as a possible solution to the hierarchy problem of gauge couplings. Moreover, higher-dimensional theories generally possess more degrees of freedom, thus providing a richer arena to describe physical phenomena.

Higher-dimensional models of gravity generally exhibit two potentially testable characteristics: (i) Newton's force at short distances no longer scales as r^{-2} [2–5] and (ii) generation and emission of gravitational radiation differ from the four-dimensional analogues, leading to observable effects [6–8]. Searches for deviations from Newton's inverse-squared law are currently in progress [9]. Gravitational events in particle colliders and cosmic ray extensive airshowers [10,11] may also provide indirect evidence of large extra dimensions. The physics of generation and emission of gravitational waves in higher-dimensional scenarios has hardly been explored at all.

The aim of this paper is to derive an exact formula for the radiation field of a charge moving in an even higher-dimensional spacetime. Here, charge will stand either for scalar or gravitational charge. There are several motivations for this study. The most popular models of extra dimensions allow for gravitational and scalar degrees of freedom in the bulk (e.g. the radion [12]). Observable effects of bulk gravitational and scalar radiation on the visible brane could provide a valuable signature of the existence of extra dimensions. Moreover, most radiation phenomena can be analyzed in flat geometries by means of scalar fields, provided that careful cutoffs are imposed. (See for instance Mironov and Morozov [8].) A full tensorial analysis of Einstein equations in higher dimensions shows indeed that the gravitational degrees of freedom are either equivalent to a massless scalar equation in flat spacetime with an appropriate source term [6,13], or to a massless scalar field plus a massive field with source term including brane contributions [5,14]. Therefore, scalar fields can be used as a simple model mimicking the more complex tensor field.

This paper is organized as follows. In Sec. II the field in the radiation zone and the Larmor formula are derived. Section III deals with some special cases (linear and uniform circular motion). Conclusions are presented in Sec. IV. Unless explicitly stated, throughout the paper the speed of light is set equal to unity.

II. COMPUTATION OF THE LARMOR FORMULA

A massless scalar field in D dimensions ($x = 0 \ldots D-1$) with source $S(x)$ satisfies the wave equation

$$\Box \varphi_D = S(x). \tag{1}$$

The source is a minimally-coupled pointlike particle with nonzero mass

$$S(x) = \frac{\alpha_D}{\sqrt{g(x)}} \int d\tau \delta^{(D)}[x^\mu - x_p^\mu(\tau)], \tag{2}$$

where the particle worldline is defined by $x^\mu - x_p^\mu(\tau) = 0$, τ is the proper time along a geodesic, $g(x)$ is the determinant of the metric, and α_D is the coupling constant. The (retarded) solution of Eq. (1) is

$$\varphi_D(x) = \int d^D x' \mathcal{G}_D(x, x') S(x'), \tag{3}$$

where $\mathcal{G}_D(x, x')$ is the retarded D-dimensional Green function of the wave operator, $\Box \mathcal{G}_D(x, x') = \delta^{(D)}(x - x')$. According to the discussion in the introduction, the metric $g_{\mu\nu}$ is replaced with the Minkowski metric. The Green functions in $D = 2$ and $D = 3$ are

$$\mathcal{G}_2(z) = \tfrac{1}{2}\theta(z), \tag{4}$$

and

$$\mathcal{G}_3(z) = \frac{1}{2\pi} \frac{\theta(z)}{\sqrt{(x^0 - x_p^0)^2 - R^2}}, \tag{5}$$

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respectively. Here $z = x^0 - x_p^0(t') - |\mathbf{x} - \mathbf{x}_p(t')|$, $R = |\mathbf{x} - \mathbf{x}_p(t')|$, and $\theta(z)$ is the Heaviside step function. The Green function in $D \geq 4$ dimensions can be obtained from the Green function in $D - 2$ dimensions through the recursive relation [15,16]

$$\mathcal{G}_D(z) = \frac{1}{2\pi R}\frac{d}{dz}\mathcal{G}_{D-2}(z), \qquad D \geq 4. \tag{6}$$

Equations (4)–(6) show that the Green functions for even-dimensional spacetimes ($D > 2$) have support on the past light cone. The Green function for $D = 2$ and for odd-dimensional spacetimes also have support inside the past light cone, because of their dependence on $\theta(z)$. Therefore, Huygens' principle is not satisfied in these spacetimes [16]. (This is also the reason for the appearance of wakes behind a boat sailing on the two-dimensional surface of a lake.) Since the appearance of wake phenomena in odd dimensions makes the problem very complex to handle [6], the analysis below will be limited to even-dimensional spacetimes. The leading term of the Green function in the far zone is

$$\mathcal{G}_{2k}(z) = \left(-\frac{1}{2\pi R}\frac{\partial}{\partial R}\right)^{k-1}\mathcal{G}_2(z), \tag{7}$$

where $k = D/2$. From Eqs. (4) and (7), it follows that the dominant term is of order R^{-k+1}. The Green function in the far zone is

$$\mathcal{G}_{2k}(z) = \frac{\delta^{(k-1)}(z)}{2(2\pi R)^{k-1}} + \mathcal{O}(R^{-k}). \tag{8}$$

The field in the far zone is found by substituting Eqs. (2) and (8) in Eq. (3):

$$\varphi_{2k}(x) = \alpha_{2k}\int_{-\infty}^{+\infty} d\tau\left[\frac{\delta^{(k-2)}(z)}{2(2\pi R)^{k-1}} + \mathcal{O}(R^{-k})\right], \tag{9}$$

where z and R are defined as below Eq. (5) with $t' \rightarrow \tau$. Integrating by parts, Eq. (9) reads

$$\varphi_{2k}(x) = \frac{\alpha_{2k}}{2(2\pi R)^{k-1}}\left(\frac{1}{B}\frac{d}{d\tau}\right)^{k-2}\frac{1}{B} + \mathcal{O}(R^{-k}), \tag{10}$$

where

$$B \overset{\text{def}}{=} -\frac{dz}{d\tau} = \gamma(1 - \mathbf{n}\cdot\mathbf{v}). \tag{11}$$

The above result can be rewritten in the useful form

$$\varphi_{2k}(x) = \frac{1}{2\pi R B}\frac{d}{d\tau}(\varphi_{2k-2}) + \mathcal{O}(R^{-k}). \tag{12}$$

The field in the wave zone can be computed recursively with Eq. (12) in any even $D \geq 4$ dimension. The Larmor formula can be easily derived from the stress energy-momentum tensor of the field. In the asymptotic region, the energy flux per unit time (Poynting vector) is

$$\mathbf{T}_{2k} = -\dot{\varphi}_{2k}\nabla\varphi_{2k}. \tag{13}$$

Substitution of Eq. (10) in Eq. (13) yields

$$\mathbf{T}_{2k} = \frac{\alpha_{2k}^2}{4(2\pi R)^{2k-2}}\left[\left(\frac{1}{B}\frac{d}{d\tau}\right)^{k-1}\frac{1}{B}\right]^2\mathbf{n} + \mathcal{O}(R^{-2k+1}), \tag{14}$$

where $\mathbf{n}$ is the unit vector in the direction of $\mathbf{x} - \mathbf{x}_p$. The power emitted per unit of solid angle in the direction $\mathbf{n}$ is

$$\frac{dP_{2k}}{d\Omega_{2k-2}} = \frac{\alpha_{2k}^2}{4(2\pi)^{2k-2}}\frac{B}{\gamma}\left[\left(\frac{1}{B}\frac{d}{d\tau}\right)^{k-1}\frac{1}{B}\right]^2. \tag{15}$$

Equation (15) is an exact expression.

III. LINEAR AND CIRCULAR MOTION

It is instructive to consider some special cases of Eq. (15). The power loss in four ($k = 2$) and six ($k = 3$) dimensions are

$$\frac{dP_4}{d\Omega_2} = \frac{\alpha_4^2}{16\pi^2}\frac{[\mathbf{a}\cdot(\gamma^2(1 - \mathbf{v}\cdot\mathbf{n})\mathbf{v} - \mathbf{n})]^2}{\gamma^2(1 - \mathbf{v}\cdot\mathbf{n})^5}, \tag{16}$$

$$\frac{dP_6}{d\Omega_4} = \frac{\alpha_6^2}{64\pi^4}\frac{[CF - 3E^2/F]^2}{\gamma^8(1 - \mathbf{v}\cdot\mathbf{n})^7}, \tag{17}$$

respectively. In the above equations, $\mathbf{a}$ and $\mathbf{v}$ are the acceleration and velocity of the particle and

$$\begin{aligned} C &= \gamma^4(\mathbf{a}\cdot\mathbf{v})^{\cdot} - \frac{\mathbf{n}\cdot(\gamma^2\mathbf{a})^{\cdot} + \gamma^4(\mathbf{n}\cdot\mathbf{a})(\mathbf{a}\cdot\mathbf{v})}{1 - \mathbf{n}\cdot\mathbf{v}}, \\ E &= \gamma^2[\gamma^2\mathbf{a}\cdot\mathbf{v}(1 - \mathbf{n}\cdot\mathbf{v}) - \mathbf{n}\cdot\mathbf{a}], \\ F &= \gamma(1 - \mathbf{n}\cdot\mathbf{v}), \end{aligned} \tag{18}$$

where dot denotes differentiation w.r.t. x_0. As is expected from simple relativistic considerations, it is straightforward to check that there is no radiation for linear uniform motion. This remains true if the bulk has finite volume and its spatial boundary is flat, such as the simple ADD scenario with a smooth brane. However, if the latter is inhomogeneous, radiation is generated [7]. This phenomenon is analogous to electromagnetic diffraction of an electric charge moving near a metal grating (Smith-Purcell effect). It can be understood by replacing the brane with a set of oscillating image charges. The image configuration is time dependent because of the brane inhomogeneities and diffraction radiation is generated by the reflection of the boosted static field on the nearby wall.

The total power emitted from a particle in planar uniform circular motion with radius R_0 and angular frequency ω is

$$P_{\text{circ},4} = \frac{\alpha_4^2}{12\pi}\gamma^4\omega^4R_0^2, \tag{19}$$

$$P_{\mathrm{circ},6} = \frac{\alpha_6^2}{120\pi^2}\gamma^8\omega^6 R_0^2(1 + 4\omega^2 R_0^2). \tag{20}$$

Equations (19) and (20) describe scalar synchrotron power loss in four and six dimensions, respectively. Assuming that the field coupling constant does not vary too much with D, the synchrotron loss in six dimensions is larger than in four dimensions for angular frequencies greater than $\omega \sim L^{-1}$, where L is the fundamental length scale. Thus a particle radiates more in higher dimensions. Equations (19) and (20) can be used, at least in principle, for indirect detection of extra dimensions by measuring the increase in the power loss of a particle on the brane as function of the Lorentz factor γ. A power loss scaling as $\gamma^8\omega^6$ would signal the presence of two additional dimensions.

The gravitational analogue of Eqs. (19) and (20) can be estimated by setting $\alpha_{2k} = \sqrt{G_D}m\gamma^2$ [13], where m is the mass of the particle and G_D is the Newton constant in D dimensions. It should be stressed that this procedure is strictly valid only for nonrelativistic geodesic motion, where it agrees with previous results [6]. For ultrarelativistic motion, effects due to the structure of the source and to gravitational stresses should also be included. (See, for instance, Ref. [17].) However, as shown by Price and Sandberg [18], these effects can be safely neglected for a point particle in ultrarelativistic circular orbit. With this word of caution, let us discuss a scenario with $D - 4$ large extra dimensions of size L [2]. In this model, G_D is related to the four-dimensional Newton constant G_4 by $G_D \sim L^{D-4}G_4$. Restoring the speed of light c, the expressions for the gravitational power loss in four and six dimensions are

$$P_{\mathrm{circ},4} \sim \frac{G_4}{12\pi c^3}m^2\gamma^8\omega^2(\omega R_0)^2, \tag{21}$$

$$P_{\mathrm{circ},6} \sim \frac{G_4}{120\pi^2 c^5}m^2\gamma^{12}\omega^2(L\omega)^2(\omega R_0)^2(1 + 4\omega^2 R_0^2/c^2). \tag{22}$$

The energy loss for physical systems can be roughly estimated by comparing the above results to the ordinary four-dimensional synchrotron radiation, P_{sync}, which is emitted by a particle with electric charge e [19]

$$\frac{P_{\mathrm{circ},4}}{P_{\mathrm{sync}}} \sim \frac{G_4 m^2}{e^2}\gamma^4 \sim 10^{-36}\gamma^4\left(\frac{m}{\mathrm{GeV}}\right)^2, \tag{23}$$

$$\begin{aligned}\frac{P_{\mathrm{circ},6}}{P_{\mathrm{sync}}} &\sim \frac{G_4 m^2}{e^2 c^2}\gamma^8(L\omega)^2 \\ &\sim 10^{-59}\gamma^8\left(\frac{m}{\mathrm{GeV}}\right)^2\left(\frac{L}{\mathrm{mm}}\right)^2\left(\frac{\omega}{\mathrm{Hz}}\right)^2.\end{aligned} \tag{24}$$

For a proton, $m \sim 1$ GeV, thus the gravitational emission becomes comparable to the synchrotron emission for $\gamma \sim 10^9$ (four dimensions) and $\gamma \sim 10^8(\omega/\mathrm{Hz})^{-2}$ (six dimensions). Thus the gravitational emission is negligible in any current or near-future Earth-based experiment. For instance, the Large Hadron Collider at CERN will collide protons with $\gamma \sim 10^4$ at a frequency of $\omega \sim 10^4$ Hz [20], which yields $P_{\mathrm{circ},4} \sim 10^{-20}P_{\mathrm{sync}}$ and $P_{\mathrm{circ},6} \sim 10^{-20}(L/\mathrm{mm})^2 P_{\mathrm{sync}}$. However, gravitational emission could become relevant in astrophysical processes. Magnetic fields larger than 10^{12} G are thought to occur in neutron stars, active galactic nuclei and other sources [21]. This implies very high frequencies and large γ factors. Therefore, indirect detection of extra dimensions by gravitational synchrotron radiation could be possible.

IV. CONCLUSIONS

We derived a simple expression for scalar and gravitational radiation by point particles in a generic (even) number of spacetime dimensions. The power loss in scalar and gravitational waves becomes significant for high frequencies and large Lorentz factors. The enhancement of radiation in extra-dimensional scenarios may lead to detectable astrophysical effects, such as higher radiation damping around neutron stars and active galactic nuclei. These results are limited to flat spacetimes. An interesting refinement of the above calculations would be to consider gravitational backreaction and stresses, which could lead to a modification of the gravitational power loss for ultrarelativistic geodesic motion [17,18]. It would also be interesting to consider gravitational radiation in warped scenarios, including possible Kaluza-Klein mode excitation.

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[1] English translations of the original articles as well as a detailed exposition of these theories can be found in T. Appelquist, A. Chodos, and P. G. O. Freund, *Modern Kaluza-Klein Theories* (Addison-Wesley, Menlo Park, 1987); See also P. Halpern, *The Great Beyond: Higher Dimensions, Parallel Universes and the Extraordinary Search for a Theory of Everything* (John Wiley and Sons, New Jersey, 2004).

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