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Approximately self-similar critical collapse in 2+1 dimensions

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Critical collapse of a self-gravitating scalar field in a (2+1)-dimensional spacetime with negative cosmological constant seems to be dominated by a continuously self-similar solution of the field equations without cosmological constant. However, previous studies of linear perturbations in this background were inconclusive. We extend the continuously self-similar solutions to solutions of the field equations with negative cosmological constant, and analyze their linear perturbations. The extended solutions are characterized by a continuous parameter. A suitable choice of this parameter seems to improve the agreement with the numerical results. We also study the dynamics of the apparent horizon in the extended background.

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Following the numerical work of Choptuik and Pretorius on the critical collapse of scalar matter field in 2+1 dimensions [1] (see also Ref. [2]), there has been debate about understanding and analytically reproducing their results [3–6]. Garfinkle found a one-parameter (n) family of continuously self-similar (CSS) solutions, and proposed that one of these is the critical solution for scalar field collapse in 2+1 dimensional AdS spacetime [3]. Numerical comparisons suggest that the critical value is $n=4$. Subsequently, Garfinkle and Gundlach [4] performed the linear perturbation analysis in this background and found that the only solution exhibiting a single growing mode is the $n=2$ solution. Because of this discrepancy, they characterized their work as "inconclusive." A weak point of their approach is that it neglects the negative cosmological constant Λ, although the latter is essential for the existence of black hole solutions in three dimensions (the BTZ black hole [7]). This was motivated in Ref. [3] by the following arguments: (i) self-similarity requires $\Lambda=0$; (ii) close to the singularity, the Λ contribution to the full solution is negligible. Although these arguments seem reasonable, we expect the cosmological constant to play a crucial role in black hole formation [1]. Therefore, the inclusion of Λ in the above analysis may solve the above contradiction on the critical value of n.

The Garfinkle family of CSS solutions is

$$ds^2=A(v^q+u^q)^{4c^2}dudv-\frac{(v^{2q}-u^{2q})^2}{4}d\theta^2,$$

$$\phi=-2c\ln(v^q+u^q), \tag{1}$$

where $A=2^{2(1-q)/q}q^2$ and $c^2=1-1/2q$. These solutions satisfy the three-dimensional Einstein equations

$$G_{ab}-\Lambda g_{ab}=\nabla_a\phi\nabla_b\phi-\frac{1}{2}g_{ab}(\nabla\phi)^2 \tag{2}$$

with $\Lambda=0$. The source term in Eq. (2) is the stress-energy tensor of the minimally-coupled massless scalar field ϕ. [Note that the scalar field in Eq. (2) and Ref. [3] differ by a factor $-1/\sqrt{4\pi}$.] The Garfinkle CSS solutions are singular at $u=v=0$. If q is a positive integer n, the initial region $u\geqslant 0$, $v\geqslant 0$ can be extended across the surface $v=0$, which plays the role of an apparent horizon. ($q=n$ will be assumed below.)

We first extend Eq. (1) to solutions of the Einstein equations with $\Lambda<0$ and then consider the perturbation analysis in this background. Since the cosmological constant breaks the self-similarity, the appropriate variables are a scaling variable, for instance u, and a similarity variable, which we choose as $y=(v/u)^n$. The metric coefficients and the scalar field are expanded in terms of the dimensionless combination Λu^{4n}

$$r\equiv\sqrt{-g_{\theta\theta}}=r_0+\Lambda u^{6n}F(y)+\cdots,$$

$$\sigma\equiv\frac{1}{2}\ln(2g_{uv})=\sigma_0+\Lambda u^{4n}G(y)+\cdots,$$

$$\phi\equiv\phi_0+\Lambda u^{4n}H(y)+\cdots, \tag{3}$$

where r_0, σ_0 and ϕ_0 are the background contributions in Eq. (1). At each order in the expansion the functions F, G, and H satisfy a system of second-order coupled ordinary differential equations. We only consider the truncation of the expansion (3) to the first-order. The relevant equations are

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$$-yF''+5F'=\frac{A}{4n^2}y^{(1-n)/n}(1-y)(1+y)^{(5n-2)/n}, \quad (4)$$

$$-y(1-y^2)H''+2(2-y^2)H'-4yH-2c\frac{1-y}{1+y}F' -4c\frac{2+3y}{(1+y)^2}F=0, \quad (5)$$

$$-yF''+\frac{1-n+(3n-1)y}{n(1+y)}F'-\frac{4c^2y}{(1+y)^2}F-2y^2G' +2cy(1-y)H'=0, \quad (6)$$

plus two first-order constraints that reduce the moduli space of the initial conditions. Below we briefly discuss the solutions of Eqs. (4)–(6). More details will be given in Ref. [8]. The solution of Eq. (4) regular at the center $y=1$ $(u=v)$ is

$$F(y)=\int_1^y y^5 f(y)dy+\alpha(1-y^6), \quad (7)$$

where

$$f(y)=-\frac{A}{4n^2}\int_1^y y^{(1-7n)/n}(1-y)(1+y)^{(5n-2)/n}dy. \quad (8)$$

The constant α can be set to zero by the gauge transformation

$$u\to u\left(1-\frac{\Lambda\alpha}{n}u^{4n}\right), \quad v\to v\left(1-\frac{\Lambda\alpha}{n}v^{4n}\right). \quad (9)$$

H can be obtained from Eq. (5). The two independent solutions of the homogeneous equation are

$$H_1=3+2y^2+3y^4,$$

$$H_2=(3+2y^2+3y^4)\ln\left|\frac{1+y}{1-y}\right| -6y(1+y^2). \quad (10)$$

The regular solution of the inhomogeneous equation is

$$H=C_1H_1+C_2H_2, \quad (11)$$

where

$$C_1=\int_1^y XH_2+c\beta, \quad C_2=-\int_1^y XH_1, \quad (12)$$

$$X=\frac{c}{64y^5(1+y)^2}[(1-y^2)F'+2(2+3y)F]. \quad (13)$$

Finally, $G(y)$ is obtained by integration of Eq. (6) with the boundary condition $G(1)=-F'(1)=0$. This condition follows from one of the constraints and implies the absence of conical singularities:

$$g^{\mu\nu}r_{,\mu}r_{,\nu}|_{y=0}=4e^{-2\sigma}r_{,u}r_{,v}|_{y=0}=-1. \quad (14)$$

The functions F, G and H are shown to be analytic in $z\equiv y^{1/n}$.

This first-order extension of the Garfinkle solutions is not uniquely defined, as we have found a one-parameter (β) family of regular, analytic in z, solutions:

$$(F,G,H)=(F,\overline{G},\overline{H})+c\beta(0,G_\beta,H_\beta), \quad (15)$$

where $\overline{G}$ and $\overline{H}$ are the $\beta=0$ solutions, and $G_\beta=-c(1-y)^2(3+2y+3y^2)$, $H_\beta=H_1$. It can be shown that a new integration constant will appear at each successive order in the Λ-expansion. Thus, there is a manifold of exact solutions of Eqs. (4)–(6) asymptotic to the Garfinkle solutions near the singularity $u=0$.

Let us determine the effect of the first-order Λ-corrections on the location of the apparent horizon. From the definition of the apparent horizon $(\nabla r)^2=0$, we obtain, to first-order in Λ,

$$\left(\frac{y}{(1+y)^2}\right)^{2-1/n}(1-\Lambda u^{4n}\psi(y))=0, \quad (16)$$

where

$$\psi(y)\equiv 2G-6F+\frac{(1+y^2)}{y}F'. \quad (17)$$

For $\Lambda=0$, the apparent horizon is the past light cone $y=0$ of the singularity $u=v=0$. For $\Lambda<0$, the behavior of the functions F and G near $y=0$ is

$$F(y)\sim F(0)+\frac{A}{4(6n-1)}y^{1/n},$$

$$G(y)\sim G(0)+\frac{A(8n-1)}{4(5n-1)(6n-1)}y^{1/n}, \quad (18)$$

where $F(0)$ and $G(0)$ are determined numerically. On the apparent horizon, Eqs. (18) imply

$$u^{-4n}=\Lambda\psi(y)\simeq\frac{\Lambda A}{4n(6n-1)}y^{1/n-2}. \quad (19)$$

Therefore, the apparent horizon recedes into the region $z=y^{1/n}<0$ and becomes spacelike. This feature is essentially due to the term F'/y in Eq. (17) and does not depend on β.

The linear perturbation analysis in this background can be performed by expanding r, σ and ϕ as

$$r=r_0+\Lambda u^{6n}F(y)+\epsilon u^{2n-2kn}[f_0(y)+\Lambda u^{4n}f_1(y)],$$

$$\sigma=\sigma_0+\Lambda u^{4n}G(y)+\epsilon u^{-2kn}[g_0(y)+\Lambda u^{4n}g_1(y)],$$

$$\phi=\phi_0+\Lambda u^{4n}H(y)+\epsilon u^{-2kn}[h_0(y)+\Lambda u^{4n}h_1(y)], \tag{20}$$

where ϵ is a small parameter that controls the strength of the perturbation, and we have truncated the expansion to first-order in Λu^{4n}. The growing modes are given by $Re(k)>0$. The critical solutions have a single growing mode [9].

The analysis of the zeroth-order perturbations f_0, g_0 and h_0 was carried out in Ref. [4]. Here, we only recall the main points of this analysis. The regular solution of the differential equation for f_0 ($f_0(1)=0$),

$$-yf_0''+(1-2k)f_0'=0, \tag{21}$$

is

$$f_0=c_1(1-y^{2-2k}). \tag{22}$$

This solution is pure gauge, i.e., it can be generated from the unperturbed solution $r_0(u,v)=(u^{2n}-v^{2n})/2$ by the coordinate transformation

$$u\rightarrow u\left(1+\frac{\epsilon c_1}{n}u^{-2kn}\right),\quad v\rightarrow v\left(1+\frac{\epsilon c_1}{n}v^{-2kn}\right). \tag{23}$$

In the gauge $f_0=0$, the scalar field perturbation $u^{-2kn}h_0(y)$ solves the massless Klein-Gordon equation for the $\Lambda=0$ background spacetime. The solution, in terms of hypergeometric functions, depends on two integration constants. The first one is fixed by the regularity condition, i.e., the absence of logarithmic divergence for $y=1$. The second integration constant is fixed by the condition of smoothness on the null line $y=0$, i.e., the analyticity (in at least one gauge) of h_0 as a function of $y^{1/n}=v/u$. A necessary condition is $2kn=m$, where m is a positive integer. For $m<n$ this condition is also sufficient. For $n<m<2n$ one can find a gauge, i.e., a value of c_1, such that h_0 is analytic. By contrast, for $m=n$ and $m\geqslant 2n$ there is no gauge in which h_0 is analytic. The second-order equation for g_0 shows that g_0 is generically divergent on the null line $y=0$. However, for $1<m<n$ there is a gauge in which g_0 is analytic, and for the value $m=2n-1>n$ ($n>1$) g_0 and h_0 are analytic in the same gauge. Therefore, regularity at the origin and analyticity in $y^{1/n}$ require $k=m/2n$ and either (i) $1<m<n$ or (ii) $m=2n-1$ ($n>1$). It can be easily seen that only the solution with $n=2$ has a single unstable mode, namely $m=3$ ($k=3/4$).

The only debatable question in this analysis is whether the requirement that the non-scalar quantity g_0 is analytic at $y=0$ might not be too strong. In principle, it should be enough to demand that the perturbation of a scalar quantity, such as the Ricci scalar, is analytical at $y=0$. At zeroth-order in Λ, the Ricci scalar is

$$R=R_0(u,y)(1-2\epsilon u^{-2kn}\rho_0(y)), \tag{24}$$

where [8]

$$\rho_0=g_0+\frac{1+y}{4c}[(1-y)h_0'-2kh_0]. \tag{25}$$

In the gauge $c_1=0$, g_0 and h_0' diverge for $y\rightarrow 0$ as $y^{-m/n}$. Therefore, ρ_0 diverges. However, the zeroth-order Ricci scalar R_0 goes to zero as $y^{1-1/n}$ and the perturbation

$$\delta R\propto R_0\rho_0\sim u^{-4n-m}y^{1-(m+1)/n}, \tag{26}$$

remains finite at $y=0$ for $m<n$, including $m=1$. If the mode $m=1$ ($k=1/2n$) were allowed, none of the Garfinkle solutions would be critical: for $n=1$ there would be no growing mode, for $n=2$ there would be two growing modes, $m=1$ and $m=3$, etc. However, as we now show, the extra modes with $m=1$ do not survive the first-order extension in Λ.

The first-order perturbation f_1 solves the inhomogeneous differential equation

$$-yf_1''+(5-2k)f_1'=\frac{A}{2n^2}y^{(1-n)/n}(1+y)^{2(2n-1)/n}\times[f_0+(1-y^2)g_0]. \tag{27}$$

In the gauge $f_0=0$, $g_0\sim y^{-m/n}$ implies

$$f_1\sim y^{(1-m)/n}\quad(m>1)\quad\text{or}\quad\ln y\quad(m=1) \tag{28}$$

for $y\rightarrow 0$. If $m>1$, the divergence of f_1 can be gauged away by the zeroth-order gauge transformation (23). The logarithmic divergence of the first-order contribution to the $m=1$ mode cannot be gauged away; this mode is never analytic at $y=0$. A detailed analysis of the first-order perturbations will be presented elsewhere [8]. Here, let us just note that the analytic and numerical integrations of the first-order perturbations indicate that all the modes found in the analysis of the zeroth-order perturbations satisfy the boundary conditions of regularity and analyticity at first-order for any value of β. This shows that the analysis of Garfinkle and Gundlach [4] is robust, i.e., it survives extension to first-order in Λ.

Now let us discuss the effect of the extension on the behavior of the apparent horizon for the perturbed critical solution ($n=2$, $k=3/4$). The apparent horizon satisfies the equation

$$\frac{y^{3/2}}{(1+y^3)}(1-\Lambda u^8\psi-\epsilon u^{-3}\chi-\epsilon\Lambda u^5\eta)=0, \tag{29}$$

where ψ is defined in Eq. (17) and

$$\chi=2g_0-\frac{1}{2}f_0+\frac{1+y^2}{y}f_0'. \tag{30}$$

(The exact form of η is inessential for the following discussion and will be given in Ref. [8].) For $y\rightarrow 0$, χ is dominated by the last term in Eq. (30):

$$\chi\simeq-\frac{c_1}{2}y^{-3/2}. \tag{31}$$

The zeroth-order approximation of Eq. (29) with $\Lambda=0$ is

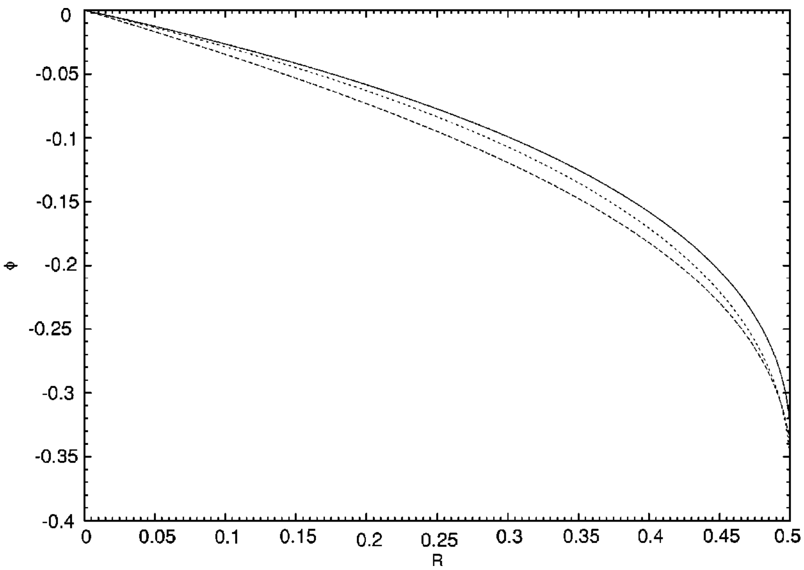

FIG. 1. The scalar field ϕ as a function of R for the $n=2$ CSS solution (solid), the $n=2$ extended CSS solution (dashed), and the $n=4$ CSS solution (dotted). The $n=2$ extended CSS solution is computed for $\beta=1$ and $T_0=1.9$. The scalar field has been rescaled by a factor $-1/\sqrt{4\pi}$ to facilitate comparison with the results of Ref. [3].

$$v^3 \simeq -\frac{\epsilon c_1}{2}. \tag{32}$$

The apparent horizon is null, and exists for both signs of ϵ. (The singularity $u=v=0$ is hidden by the apparent horizon only for $\epsilon c_1<0$.) The situation changes dramatically when we take into account the first-order contributions. Neglecting the term η, we see from Eq. (19) and Eq. (31) that near $y=0$ the shape of the apparent horizon is determined by a balance between the Λ and ϵ contributions. The leading behavior is

$$u \simeq u_0\left(1-\frac{y}{3}\right), \quad u_0 \equiv \left(\frac{22\epsilon c_1}{\Lambda}\right)^{1/11}. \tag{33}$$

The apparent horizon, which exists only for $\epsilon c_1<0$, is spacelike for small positive y and becomes null ($u=u_0$) for $y=0$. The numerical solution of Eq. (29) shows that on the apparent horizon u is everywhere bounded by u_0. This confirms *a posteriori* that the η contribution to Eq. (29) can be neglected for small ϵ. The existence and the shape of the apparent horizon, which hides the singularity $u=v=0$, do not depend on the parameter β.

Finally, we present some evidence that the $O(\Lambda)$ corrections improve the agreement with the numerical simulations of near-critical collapse. Following Ref. [3], we introduce the coordinates (T,R)

$$T=-2n\ln u, \quad R=u^{-2n}r=\frac{1-y^2}{2}+\Lambda e^{-2T}F(y). \tag{34}$$

The expression of the extended Garfinkle scalar field at some fixed T_0 is

$$\phi_n(y,T_0)=-2c\ln\frac{(1+y)}{2}+\Lambda e^{-2T_0}(H(y)-H(1)), \tag{35}$$

where ϕ has been shifted by a constant to make it vanish at $y=1$. In Ref. [3] Garfinkle shows that the nonextended solution with $n=4$ agrees with the numerical critical solution of Ref. [1] at an intermediate time $T_0\sim 9$. For such a large T_0, the extended ϕ_n (35) reduces to that of the CSS solution. However, the calibration of the numerical T_0 involves some ambiguity. In Ref. [1], T is defined by $T=-\ln t_c$, where $t_c=0$ ($T=+\infty$) at the accumulation point (the singularity). Even a tiny error in the determination of this zero from near-critical simulations will translate into a large error on the corresponding value of T_0. Therefore, the latter has to be considered as an unknown parameter. A second unknown parameter in Eq. (35) is β. T_0 and β can be set by comparing $\phi_2(0,T_0)$ for the critical solution $n=2$ with $\phi_4(0,\infty)$. From the numerical solution of Eqs. (4)–(6), we find $\overline{H}(0)\sim 0.016 \ll c\beta H_\beta(0)=3c\beta$, provided that β is not too small. So $\phi_2(0,T_0)$ depends only on the product βe^{-2T_0}. By comparing the latter with the numerical value of $\phi_4(0,\infty)$, we obtain $\beta e^{-2T_0}\sim 0.022$. For the (arbitrary) choice $\beta=1$ this gives $T_0\sim 1.9$. In Fig. 1 we plot in terms of R the CSS solution with $n=2$, the extended solution for $n=2$ and the CSS solution with $n=4$. The $O(\Lambda)$ corrections seem to improve the agreement of the $n=2$ critical solution with the numerical results. This conclusion is strengthened by Fig. 2, where we plot the mass aspect in terms of R

$$M_n(y,T_0) \equiv -\Lambda r^2+4e^{-2\sigma}r_{,u}r_{,v}=-\left[\frac{4y}{(1+y)^2}\right]^{(2n-1)/n} -\Lambda u^{4n}\left[\frac{(1-y^2)^2}{4}-\left(\frac{4y}{(1+y)^2}\right)^{(2n-1)/n}\psi(y)\right], \tag{36}$$

for the same values of β and T_0. However, it is clear that the first-order extended $n=2$ solution agrees with the numerics only over a small range of T_0, as opposed to the $n=4$ CSS solutions, which agree over a large range of intermediate

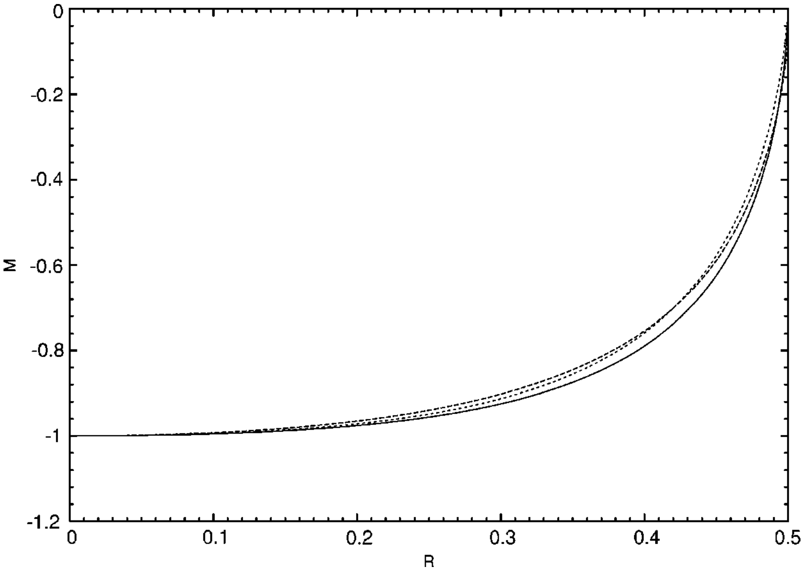

FIG. 2. The mass aspect M as a function of R for the $n=2$ CSS solution (solid), the $n=2$ extended CSS solution (dashed), and the $n=4$ CSS solution (dotted).

times [3]. This suggests that the question of the agreement between the analytical and numerical critical solutions is still an open problem.

To conclude, our analysis shows that in the near-critical regime the shape of the apparent horizon is determined by a balance between the Λ and ϵ contributions. This is evidence that the cosmological constant plays a role in black hole formation. We have also shown that the apparent contradiction between the results of Ref. [1] and Ref. [3] can partly be solved by including $O(\Lambda)$ terms. Another result of our analysis is that, at this order, there seems to be a one-parameter family of critical solutions, rather than a single critical solution. This parameter is not connected with gauge transformations (the gauge parameter is α, which has been set to zero). Rather, as will be discussed in more detail in Ref. [8], the first-order terms linear in β in Eq. (15), which solve the homogeneous equations (4)–(6), can be reinterpreted as zeroth-order $k=-2$ perturbations. It follows that the extended $n=2$ critical solution is unique modulo the addition of a decaying perturbation.

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