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Neuro Emission Controller for Minimizing Cyclic Dispersion in Spark Ignition Engines¹

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Abstract: A novel neural network (NN) controller is developed to control Spark Ignition (SI) engines at extreme lean conditions. The purpose of neurocontroller is to reduce the cyclic dispersion at lean operation even when the engine dynamics are unknown. The stability analysis of the closed-loop control system is given and the boundedness of all the signals is ensured. Results demonstrate that the cyclic dispersion is reduced significantly using the proposed controller. The neuro controller can also be extended to minimize engine emissions with high EGR levels, where similar complex cyclic dynamics are observed. Further, the proposed approach can be applied to control nonlinear systems that have a similar structure as that of the engine dynamics.

I. INTRODUCTION

Today's automobiles utilize sophisticated microprocessor-based engine control system to meet stringent Federal regulations governing fuel economy and the emission of carbon monoxide (CO), oxides of nitrogen (NO_x) and hydrocarbons (HC). The control efforts are tailored to decrease the total amount of emissions and to minimize the fuel consumption. To address these two requirements, lean combustion control technology receives increased preference [7]. The major difficulty of lean engine operation is cyclic dispersion [4] of heat release, which causes significant performance deterioration.

Inoue et al. [7] designed a lean combustion engine control system using a combustion pressure sensor. However, pressure sensors are expensive and not reliable. Davis et al. [2] developed a feedback control approach to reduce the cyclic dispersion at lean conditions. However, only the fuel system is controlled and the air system is not. Consequently, significant cyclic dispersion is still left. He et al. [6] proposed a nonlinear backstepping controller to keep a stable operation of the SI engine at lean conditions by altering the fuel intake (control variable) based on the air intake. All of these methods require the precise mathematical model of the cyclic dispersion and engine dynamics. Moreover, the cyclic dispersion process is sensitive to the variations in the delivery of air and fuel into the cylinder, the fluid dynamics effects during engine intake, residual gas fraction, which are typically uncertain. Differences between model and real engine dynamics could jeopardize the controller performance.

In this paper, a direct adaptive NN controller is proposed for stable operation of the SI engine at extreme lean conditions. A nonlinear system of the form $x_1(k+1) = f_1(x_1(k), x_2(k)) + g_1(x_1(k), x_2(k))x_2(k) + d_1'(k)$, $x_2(k+1) = f_2(x_1(k), x_2(k)) + g_2(x_1(k), x_2(k))u(k) + d_2'(k)$ (see Section 3.1) can be used to describe the engine dynamics at lean operation, where $f_1(x_1(k), x_2(k))$, $g_1(x_1(k), x_2(k))$, $f_2(x_1(k), x_2(k))$ and $g_2(x_1(k), x_2(k))$ are unknown nonlinear functions. The control objective is to reduce the cyclic dispersion in heat release by minimizing variations in equivalence ratio ($\phi(k) = \frac{1}{R} \frac{x_2(k)}{x_1(k)}$, where R is a constant).

Controlling such a class of non strict feedback nonlinear systems is extremely difficult because the control input cannot directly influence both the states $x_1(k)$ and $x_2(k)$. Moreover, unlike standard backstepping control scheme [5], the objective here is to show the boundedness of both the states close to their respective targets so that the actual equivalence ratio is close to its target and bounded tightly, then the cyclic dispersion can be reduced significantly.

Two NNs are employed to learn the unknown nonlinear dynamics since the residual gas and combustion efficiency are unknown. Backstepping approach is utilized to design the control input (injected fuel) to the total fuel system. The total fuel is then treated as the virtual control signal to the air system so that both the states are bounded tightly to their respective targets. Consequently, the equivalence ratio error can be bounded tightly and its variations are minimized. As a result, the cyclic dispersion is reduced and the engine is stable. The stability analysis of the closed-loop control system is given and the boundedness of the closed-loop signals is shown.

The proposed NN controller design is even applicable to a class of nonlinear systems that have a similar structure as that of the engine dynamics. Compared with standard backstepping schemes [5], our approach is not limited to the control of strict feedback nonlinear systems since the engine dynamics during lean operation are only expressed as non-strict feedback nonlinear system.

The paper is organized as follows. Section 2 discusses background on neural network, and engine dynamics at lean conditions. The proposed NN controller is presented in Section 3. Section 4 describes the simulations and Section 5 carries the conclusions.

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II. BACKGROUND

2.1 Properties of the NN Systems

A general function $f(x) \in C^s$ can be approximated using the two-layer neural network as

$$f(x(k)) = W^T \phi_2 \left(V^T \phi_1(x(k)) \right) + \varepsilon(k) \quad (1)$$

where W and V are constant weights and $\phi_2(V^T \phi_1(x(k)))$, $\phi_1(x(k))$ denote the vectors of activation functions at the instant k , with $\varepsilon(k)$ an NN functional reconstruction error vector. The net output is defined as

$$\hat{f}(x(k)) = \hat{W}^T \phi_2 \left(V^T \phi_1(x(k)) \right) + \varepsilon(k) \quad (2)$$

From now on $\phi_1(x(k))$ is denoted as $\phi_1(k)$ and $\phi_2(V^T \phi_1(x(k)))$ is denoted as $\phi_2(k)$. It is well known [1] that if the inputs to the hidden layer weights are selected at random and held constant, $\phi(k)$ forms a basis, then any smooth nonlinear function can be approximated to a sufficient degree.

2.2 Single Cylinder Combustion Model

Daw et al. [4] developed a mathematical representation, which has a striking resemblance to experimental data, to investigate nonlinear cycle dynamics in SI engines under lean conditions. The engine model is expressed as

$$a(k+1) = F(k) \times \left[a(k) - \frac{1}{R} \times CE(k) \times m(k) \right] + (1 - F(k)) \times (AF) + d'_1(k), \quad (3)$$

$$m(k+1) = m(k) \times (1 - CE(k)) \times F(k) + (1 - F(k)) \times (MF + \delta MF(k)) + d'_2(k), \quad (4)$$

$$CE(k) = \frac{CE_{max}}{1 + 100 \left(\frac{1}{R} \times \frac{m(k)}{a(k)} - \varphi_m \right)^{1/(\varphi_u - \varphi_l)}} \varphi_m = \frac{\varphi_u + \varphi_l}{2} \quad (5)$$

where $m(k)$ is the mass of fuel before k th burn, $a(k)$ is the mass of air before k th burn, $\delta MF(k)$ is the small changes in mass of fresh fuel per cycle, MF is the mass of fresh fuel per cycle, AF is the mass of fresh air fed per cycle, F is the fraction of cylinder gas remaining, R is the stoichiometric air-fuel ratio, ~ 14.6 , $CE(k)$ is the combustion efficiency, $\varphi_m, \varphi_l, \varphi_u$ are system parameters, and $d'_1(k)$ and $d'_2(k)$ are unknown but bounded disturbances. From the above equations, it is clear that the dynamics are highly nonlinear with combustion efficiency and residual gas fraction being unknown.

III. NN CONTROLLER DESIGN

The overall goal is to minimize the cyclic dispersion, i.e., to minimize the heat release variations at lean conditions, without needing to know its dynamics. The heat release, $Q(k) = m(k) \times CE(k)$, is proportional to the mass of the fuel burnt and the combustion efficiency. From (5), the $CE(k)$ is a function of equivalence ratio

$\left(\varphi(k) = \frac{1}{R} \frac{m(k)}{a(k)} \right)$ alone. So, to reduce the heat release

variations and to keep a constant mass of the fuel injected $m(k)$, the equivalence ratio variations must be decreased. In other words, the objective is to attain, a constant $m(k)$, and to reduce the variations in $\varphi(k)$. This objective is accomplished by driving the mass of total fuel and air approach close to their respective targets simultaneously.

Here a NN backstepping type approach is employed to design the controller. A virtual control signal is designed to control the air intake system and the actual control input is selected to control the fuel intake system. When both the total fuel and air are tightly controlled, the equivalence ratio and the heat release variations are reduced, the engine dynamics is kept stable and the engine can operate smoothly at lean conditions.

3.1 Controller Design

Since it is difficult to force both the states to approach their respective targets in the presence of disturbances and uncertainties, the control objective is accomplished when the two states $x_1(k)$ and $x_2(k)$ are bounded close (tightly) to their respective targets X_{1d} and X_{2d} , where $x_1(k)$ and $x_2(k)$ represent respectively, before k th burn, as

$$x_1(k) = a(k), \quad (6)$$

$$x_2(k) = m(k), \quad (7)$$

Step 1: State space model of the system

Let us denote

$$\delta MF(k) = u(k). \quad (8)$$

Substituting (6), (7) and (8) in (3), (4) and (5) results in

$$x_1(k+1) = F(k) \times x_1(k) + (1 - F(k)) \times AF - (F(k)/R) \times CE(k) \times x_2(k) + d'_1(k) \quad (9)$$

$$x_2(k+1) = F(k) \times (1 - CE(k)) \times x_2(k) + (1 - F(k)) \times MF + (1 - F(k)) \times u(k) + d'_2(k) \quad (10)$$

and

$$CE(k) = \frac{CE_{max}}{1 + 100 \left(\frac{1}{R} \times \frac{x_2(k)}{x_1(k)} - \varphi_m \right)^{1/(\varphi_u - \varphi_l)}} \varphi_m = \frac{\varphi_u + \varphi_l}{2} \quad (11)$$

By assigning

$$f_1(x_1(k), x_2(k)) = F(k) \times x_1(k) + (1 - F(k)) \times AF, \quad (12)$$

$$g_1(x_1(k), x_2(k)) = -(F(k)/R) \times CE(k), \quad (13)$$

$$f_2(x_1(k), x_2(k)) = F(k) \times (1 - CE(k)) \times x_2(k) + (1 - F(k)) \times MF, \quad (14)$$

$$g_2(x_1(k), x_2(k)) = 1 - F(k). \quad (15)$$

Then the nonlinear dynamics are expressed in the non strict feedback form

$$x_1(k+1) = f_1(x_1(k), x_2(k)) + g_1(x_1(k), x_2(k))x_2(k) + d'_1(k), \quad (16)$$

$$x_2(k+1) = f_2(x_1(k), x_2(k)) + g_2(x_1(k), x_2(k))u(k) + d'_2(k). \quad (17)$$

or in compact form

$$x_1(k+1) = f_1(k) + g_1(k)x_2(k) + d'_1(k), \quad (18)$$

$$x_2(k+1) = f_2(k) + g_2(k)u(k) + d'_2(k). \quad (19)$$

3.2 Nonlinear Stability Analysis

It is required to show the errors $e_1(k)$ and $e_2(k)$ and the NN weights $\hat{w}_1(k)$ and $\hat{w}_2(k)$ are bounded in order to ensure that the cyclic dispersion is reduced. First, we present bounds on the ideal weights and activation functions. Second, a discrete-time weight tuning algorithm is given so that closed-loop stability is inferred. **Assumption 2:** Both the ideal weights and the activation functions for all NNs are bounded by known positive values so that

$$\|w_1\| \leq w_{1\max}, \|w_2\| \leq w_{2\max}, \|\phi(\cdot)\| \leq \phi_{\max}$$

$$\text{and } \|\sigma(\cdot)\| \leq \sigma_{\max}. \quad (36)$$

Theorem 3.1: Consider the system given by equations (18) and (19). Assume that the Assumption 1 and 2 hold. Let the disturbance and NN approximation errors be bounded. Take the first NN weight tuning be

$$\hat{w}_1(k+1) = \hat{w}_1(k) - \alpha_1 \phi(k) (\hat{w}_1^T(k) \phi(k) + k_1 e_1(k)). \quad (37)$$

with the second NN weight tuning be provided by

$$\hat{w}_2(k+1) = \hat{w}_2(k) - \frac{\alpha_2}{k_2} \sigma(k) (\hat{w}_2^T(k) \sigma(k) + k_2 e_2(k)). \quad (38)$$

where $\alpha_1, \alpha_2, k_1,$ and k_2 are design parameters. The error $e_1(k)$ and $e_2(k)$, the NN weights estimates, $\hat{w}_1(k)$ and $\hat{w}_2(k)$ are bounded, with the bounds specifically given by (A.8) through (A.11) provide the design parameters are selected as:

$$(1) 0 < \alpha_1 \|\phi(\cdot)\|^2 < 1, \quad (39)$$

$$(2) 0 < \alpha_2 \|\sigma(\cdot)\|^2 < k_2 \quad (40)$$

$$(3) 0 < |k_1| < 1/\sqrt{12g_{1M}}, \quad (41)$$

$$(4) 0 < k_2 < (-1 + \sqrt{4 + g_{2M}^2})/(6g_{2M}) \quad (42)$$

Moreover, the equivalence ratio error is bounded and the actual equivalence ratio is bounded.

Proof: See Appendix.

Remark 1: Controller singularity problem ($\hat{g}(\cdot) \rightarrow 0$) that is commonly noticed in other works is avoided.

Remark 2: It is important to note that in this theorem there is no certainty equivalence (CE) assumption or the need for persistency of excitation condition.

Remark 3: Our control scheme requires the need for the measurement of the total mass of the fuel, $x_2(k)$ and the air, $x_1(k)$. There are several sensors (oxygen sensor and Universal Exhaust Gas Oxygen (UEGO) sensor) available to measure the equivalence ratio of the exhaust. For details see [3]. The information available from an air intake sensor is the mass of the new air. The mass of the new fuel injected is also available. Using this information and by performing certain experiments on an engine, one can infer the values of $x_2(k)$ and $x_1(k)$.

IV. SIMULATION

The purpose of simulation is to verify that the cyclic dispersion of a SI engine using the proposed closed-loop controller is indeed small at very lean operation conditions. The simulation parameters are selected as the following: 1000 cycles are considered at equivalence ratio of 0.71 with $R=14.6, F=0.14,$ mass of new air = 1.0, the standard deviation of mass of new fuel is 0.007, $\phi_u = 0.685, \phi_l = 0.665,$ the desired mass of air is taken as $X_{1d} = 0.9058$ and the desired mass of fuel is calculated as $X_{2d} = R \times 0.71 \times X_{1d} = 9.3895$. A 5% unknown noise is added to the residual gas fraction as a way to emulate stochastic perturbation. The gains of controllers are selected as $k_1 = k_2 = 0.1,$ respectively. Both NNs, NN1 $\hat{w}_1^T \phi(k)$ and NN2 $\hat{w}_2^T \sigma(k)$ are configured to possess 15 nodes in the hidden layer. For weight updating, the learning rate is selected as $\alpha_1 = 0.01$ and $\alpha_2 = 0.001$. The inputs to the NNs, NN1 $\hat{w}_1^T \phi(k)$ and NN2 $\hat{w}_2^T \sigma(k)$ are taken as $[x_1(k), x_2(k), X_{1d}]^T$ and $[x_1(k), x_2(k), \hat{w}_1(k), e_1(k)]$ respectively. The initial weights are selected uniformly within an interval of $[0, 1]$ and all the activation functions are selected as hyperbolic tangent sigmoid functions. The NN weights are initialized at random.

The cyclic dispersion observed at a lean equivalence ratio of 0.71 is presented in Figure 2 when no control scheme is employed. It is clear that without any control, the engine is unstable due to unacceptable amount of dispersion. Figure 3 illustrates that the performance of the NN controller where the heat release appears to exhibit some dispersion. The dispersion is small and bounded and may be tolerable. Figure 4 shows that the error between actual and desired equivalence ratio is bounded with the NN controller. For comparison, PD controller alone is used with the same simulation parameters and from Figure 5, it is clear that the unacceptable amount of dispersion still exists.

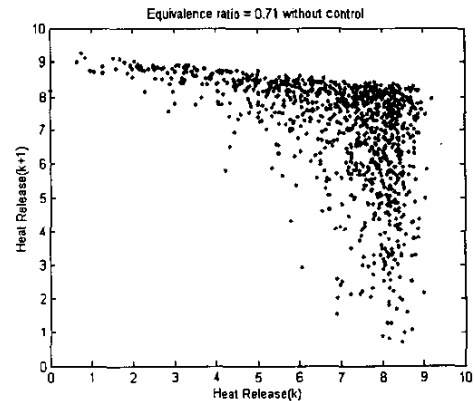


Fig. 2: Cyclic dispersion without control.

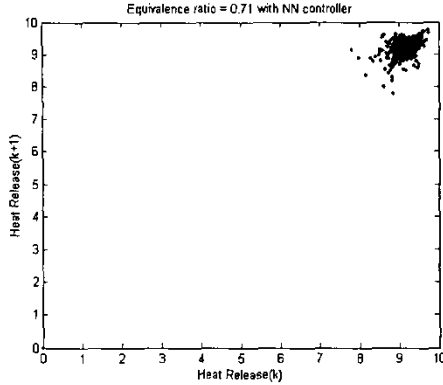


Fig. 3: Heat release with NN controller.

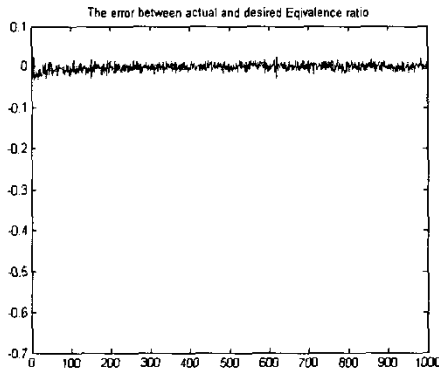


Fig. 4: Error in equivalence ratio.

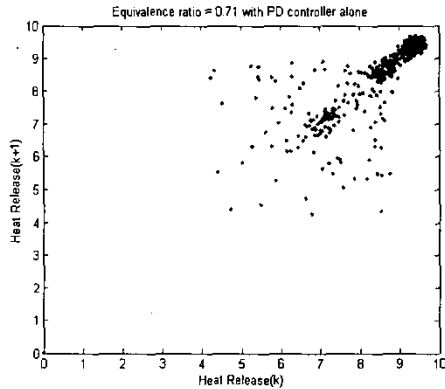


Fig. 5: Heat release with PD controller alone.

V. CONCLUSION

A novel NN controller scheme is presented to reduce the cyclic dispersion in engine combustion at lean conditions. The proposed control scheme utilizes both NN approximation property and the backstepping approach for maintaining certain air to fuel ratio by altering the fuel

injected into the cylinder as the control input. The stability analysis of the closed-loop control system was proven and the boundedness of the closed-loop signals was shown. Results show that the performance of the proposed controller is highly satisfactory while meeting the closed loop stability eventhough the dynamics are not known beforehand. Since the dynamics of an engine with high EGR levels exhibits similar behavior as that of a lean operation, the proposed controller can be easily extended to EGR operation to achieve further reduction in emissions.

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APPENDIX

Proof of Theorem 3.1: Define the Lyapunov function candidate

$$J(k) = e_1^2(k)/4g_{1M}^2 + e_2^2(k)/3k_2g_{2M}^2 + (1/\alpha_1)\tilde{w}_1^T(k)\tilde{w}_1(k) + (1/\alpha_2)\tilde{w}_2^T(k)\tilde{w}_2(k) \quad (A.1)$$

whose first difference is

$$\Delta J(k) = \Delta J_1(k) + \Delta J_2(k) + \Delta J_3(k) + \Delta J_4(k) \quad (A.2)$$

The first difference $\Delta J_1(k)$ is obtained using (27)

$$\Delta J_1(k) = (1/4g_{1M}^2) \left((g_1(k)(k, e_1(k) + e_2(k) + \zeta_1(k) + d_1(k)))^2 - e_1^2(k) \right)$$

$$\leq (k_1^2 - 1/4g_{1M}^2)e_1^2(k) + e_2^2(k) + \zeta_1^2(k) + d_1^2(k) \quad (\text{A.3})$$

Now taking the second term in the first difference (A.1) and substituting (33) into it

$$\begin{aligned} \Delta J_2(k) &= (1/3k_2g_{2M}^2)(g_2(k)(k_2e_2(k) + \zeta_2(k) + d_2(k)))^2 - e_2^2(k) \\ &\leq (1/k_2)(k_2^2 - 1/3g_{2M}^2)e_2^2(k) + \zeta_2^2(k)/k_2 + d_2^2(k)/k_2 \quad (\text{A.4}) \end{aligned}$$

Taking the third term in (A.1) and substituting the weights updates from (37) and simplifying to get

$$\begin{aligned} \Delta J_3(k) &= -(2 - \alpha_1\phi^T(k)\phi(k))\zeta_1^2(k) \\ &\quad - 2(1 - \alpha_1\phi^T(k)\phi(k))(w_1^T\phi(k) + k_1e_1(k))\zeta_1(k) \\ &\quad + \alpha_1\phi^T(k)\phi(k)(w_1^T\phi(k) + k_1e_1(k))^2 \quad (\text{A.5}) \end{aligned}$$

Taking the fourth term in (A.1) and substituting the weights updates from (38) and simplifying to get

$$\begin{aligned} \Delta J_4(k) &= -(1/k_2)\left(2 - \frac{\alpha_2}{k_2}\sigma^T(k)\sigma(k)\right)\zeta_2^2(k) \\ &\quad - 2(1/k_2)\left(1 - \frac{\alpha_2}{k_2}\sigma^T(k)\sigma(k)\right)(w_2^T\sigma(k) + k_2e_2(k))\zeta_2(k) \\ &\quad + \frac{\alpha_2}{k_2^2}\sigma^T(k)\sigma(k)(w_2^T\sigma(k) + k_2e_2(k))^2 \quad (\text{A.6}) \end{aligned}$$

Combining (A.3), (A.4), (A.5) and (A.6) to get the first difference and simplifying to get

$$\begin{aligned} \Delta J &\leq (3k_1^2 - 1/4g_{1M}^2)e_1^2(k) + \\ &\quad + (1/k_2)(3k_2^2 + k_2 - 1/3g_{2M}^2)e_2^2(k) \\ &\quad - (1 - \alpha_1\phi^T(k)\phi(k))(w_1^T\phi(k) + k_1e_1(k))^2 \\ &\quad - (1/k_2)\left(1 - \frac{\alpha_2}{k_2}\sigma^T(k)\sigma(k)\right)(w_2^T\sigma(k) + k_2e_2(k))^2 \\ &\quad + d_1^2(k) + d_2^2(k)/k_2 + 2w_{1\max}^2\phi_{\max}^2 + w_{2\max}^2\sigma_{\max}^2/k_2 \quad (\text{A.7}) \end{aligned}$$

This implies that $\Delta J \leq 0$ as long as (39) through (42) hold and

$$|e_1(k)| \geq \frac{2D_M}{\sqrt{1 - 12k_1^2g_{1M}^2}} \quad (\text{A.8})$$

or

$$|e_2(k)| \geq \frac{\sqrt{3k_2g_{2M}D_M}}{\sqrt{(1 - 3k_2g_{2M}^2 - 9k_2^2g_{2M}^2)}} \quad (\text{A.9})$$

or

$$\|\zeta_1(k)\| \geq \frac{D_M}{\sqrt{(1 - \alpha_1\|\phi(k)\|^2)}} + \frac{2k_1D_M}{\sqrt{1 - 12k_1^2g_{1M}^2}} + w_{1\max}\phi_{\max} \quad (\text{A.10})$$

or

$$\begin{aligned} \|\zeta_2(k)\| &\geq \frac{k_2D_M}{(k_2 - \alpha_2\|\sigma(k)\|^2)} + w_{2\max}\sigma_{\max} \\ &\quad + \frac{\sqrt{3k_2k_2g_{2M}D_M}}{\sqrt{(1 - 3k_2g_{2M}^2 - 9k_2^2g_{2M}^2)}} \quad (\text{A.11}) \end{aligned}$$

where

$$D_M^2 = d_1^2(k) + d_2^2(k)/k_2 + 2w_{1\max}^2\phi_{\max}^2 + w_{2\max}^2\sigma_{\max}^2/k_2 \quad (\text{A.12})$$

According to a standard Lyapunov extension theorem, this demonstrates that the system errors and the errors in

weight estimates are bounded. The boundedness of $\|\zeta_1(k)\|$ and $\|\zeta_2(k)\|$ implies that $\|\tilde{w}_1(k)\|$ and $\|\tilde{w}_2(k)\|$ are bounded, or equivalently the weight estimates $\hat{w}_1(k)$ and $\hat{w}_2(k)$ are bounded.

From (A.8) and (A.9), $e_1(k)$ and $e_2(k)$ are bounded. Using (20) and (30), $x_1(k)$ and $x_2(k)$ are bounded to X_{1d} and $\hat{x}_{2d}(k)$, respectively. However, to minimize equivalence ratio variations, $x_1(k)$ and $x_2(k)$ have to be bounded tightly to their respective targets X_{1d} and X_{2d} . Then the equivalent ratio will be close to its target ($\phi_d = \frac{1}{R} \frac{X_{1d}}{X_{2d}}$), and the combustion efficiency $CE(k)$ is

held constant. Consequently, heat release $Q(k) = x_2(k) \times CE(k)$, will be bounded and its variations are reduced provided the error bounds are small and tight. Since the error bounds are a function of the design parameters k_1, k_2, α_1 and α_2 , by suitably selecting these, the equivalence variations can be reduced.

So far, it has been shown that $x_1(k)$ is bounded close to X_{1d} and $x_2(k)$ is bounded close to $\hat{x}_{2d}(k)$. In order to prove $x_2(k)$ is bounded close to X_{2d} , the difference between $\hat{x}_{2d}(k)$ and $x_{2d}(k)$ has to be considered as

$$\hat{x}_{2d}(k) - x_{2d}(k) = (\hat{w}_1(k) - w_1(k))^T \phi(k) - \varepsilon_1(k) = \zeta_1(k) - \varepsilon_1(k). \quad (\text{A.14})$$

Since $\zeta_1(k)$ and $\varepsilon_1(k)$ are bounded, $\hat{x}_{2d}(k)$ is bounded close to $x_{2d}(k)$. Since $e_2(k)$ is bounded, and $x_2(k)$ is bounded close to $\hat{x}_{2d}(k)$, and $\hat{x}_{2d}(k)$ is bounded close to $x_{2d}(k)$, it can be concluded that $x_2(k)$ is bounded close to $x_{2d}(k)$. By suitably selecting X_{1d} according the following equation

$$\begin{aligned} x_{2d}(k) &= \frac{1}{g_1(k)}(-f_1(k) + X_{1d}) + k_1e_1(k) \\ &= X_{2d} + \delta(k) = R \times \phi_d \times X_{1d} + \delta(k) \quad (\text{A.15}) \end{aligned}$$

where $\delta(k)$ is a small and known bounded value and $x_{2d}(k)$ is forced to be close to X_{2d} at steady state. Then, it follows that $x_2(k)$ is bounded close to X_{2d} .

Since both $x_1(k)$ and $x_2(k)$ are bounded tightly to their targets X_{1d} and X_{2d} respectively, the equivalence ratio is bounded close to its desired value. Consequently, the combustion efficiency, $CE(k)$, is held constant, the heat release is close to its target and the heat release dispersion is reduced provided the design parameters are selected appropriately to result in tight error bounds.