Multiple-Input Multiple-Output Rayleigh Flat Fading Outage Capacity using Channel Estimation

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***Abstract*— Upper and lower bounds on the outage capacity for a multiple-input multiple-output (MIMO) Rayleigh flat fading wireless baseband communication environment are derived for the case when the channel is completely unknown at the transmitter and the receiver has access to channel state information (CSI) via pilot estimation. The bounds show that the channel estimation error significantly impacts the outage capacity. A change in the outage probability is also shown to have a larger impact on the outage capacity when the channel estimation error is small. To get an idea of how these bounds are affected by a temporally correlated channel, they are calculated when pilot symbol assisted modulation (PSAM) is employed. When the doppler frequency of the channel is increased, the outage capacity deteriorates quicker.**

***Index Terms*— MIMO systems, Channel Capacity, PSAM.**

I. Introduction

An expression for the ergodic capacity for a multiple-input multiple-output (MIMO) system was derived in [1]. The capacity was derived by extending the channel coding theorem for a single-input single-output (SISO) wireless channel [2]. This theorem assumes one has the luxury of being able to send a code with a length which grows exponentially with respect to the uncertainty of the channel. This scenario is not practical for wireless systems since the channel is capable of rapid fluctuation due to scatterers and reflectors. Also, more bits per symbol necessitate a longer symbol period which increases the decoding delay at the receiver. These issues with the ergodic capacity motivated the development of a different way to measure the performance.

A well known alternative to the ergodic capacity is the outage capacity. This was first introduced in [3], where time division multiple access (TDMA) and convolution codes were used while transmitting in a mobile environment. They ran into a problem with the ergodic capacity when the decoding time was constrained. The outage capacity is defined to be the maximum rate that can be achieved given an outage probability. By outage probability, we mean the probability that a given rate is greater than or equal to the instantaneous mutual information.

The channel coding theorem also assumes that the receiver has perfect knowledge of channel state information (CSI). In real time mobile fading environments, this assumption is not valid, and one should include the resulting decrease in capacity. This was considered for a SISO system in [4], where upper and lower bounds for the instantaneous mutual information were obtained with imperfect CSI. These bounds were extended to MIMO systems in [5].

In [6] motivation is provided for using outage capacity as a performance measure for wireless communication. The bounds on the instantaneous mutual information from [4] were used to arrive at upper and lower bounds on the outage capacity of a SISO Rayleigh flat fading channel. The effects of temporal correlation were investigated by deriving an expression for the channel estimation error when using pilot symbol assisted modulation (PSAM).

In this work the results of the SISO case in [6] are extended to arrive at new closed form bounds for the outage capacity of a Rayleigh flat fading MIMO system. As an application we consider how the bounds are affected by a time-varying channel using pilot symbol assisted modulation (PSAM).

The next section mathematically describes the received symbol vectors as well as the MIMO channel. We end the section with previously attained upper and lower bounds for the instantaneous mutual information. In Section III we apply known results on upper and lower bounds for the outage capacity of a SISO system to derive new upper and lower bounds for the outage capacity of a MIMO Rayleigh flat fading system. This is followed by a discussion in Section IV to provide insights to the new bounds. To validate our results, numerical examples are given. The bounds obtained in the previous section are then applied to a MIMO time correlated channel based on Jakes model using PSAM for channel estimation in Section V. Once again numerical examples are given to illustrate our findings. This is followed by our concluding remarks in Section VI.

II. System Model

A MIMO baseband system with N_r receive antennas and N_t transmit antennas is modelled by

$$\boldsymbol{y} = \sqrt{\frac{\rho}{N_t}}\boldsymbol{H}\boldsymbol{x} + \boldsymbol{n}$$

where ρ is the received SNR, $\boldsymbol{y}$ is the $N_r \times 1$ received symbol vector, $\boldsymbol{x}$ is the $N_t \times 1$ transmitted vector, and $\boldsymbol{n}$ is the noise vector of size $N_r \times 1$ with $n_i \overset{iid}{\sim} \mathcal{CN}(0,1)$. The $N_r \times N_t$ channel matrix $\boldsymbol{H} = \{h_{mn}\}$ describes the channel gain between the m^{th} receiver antenna and the n^{th} transmit antenna. For a Rayleigh flat fading environment $h_{mn} \sim \mathcal{CN}(0,1)$.

Suppose the channel can be written in terms of its estimate $\widehat{\boldsymbol{H}}$ as

$$\boldsymbol{H} = \widehat{\boldsymbol{H}} + \boldsymbol{E}$$

which satisfies the following properties

$$\begin{aligned}
\boldsymbol{R}_{\boldsymbol{H}_n\boldsymbol{H}_n} &= N_r \boldsymbol{I}_{N_t} \\
\mathbb{E}\{\boldsymbol{H}|\widehat{\boldsymbol{H}}\} &= \widehat{\boldsymbol{H}} \\
\mathbb{E}\{\boldsymbol{E}\} &= \boldsymbol{0} \\
\boldsymbol{R}_{\boldsymbol{E}\boldsymbol{E}} &= N_r \sigma_e^2 \boldsymbol{I}_{N_t}.
\end{aligned}$$

It is routine to show that if $\widehat{h}_{mn} \sim \mathcal{CN}(0,\sigma_{\widehat{h}}^2)$ then $e_{mn} \sim \mathcal{CN}(0,\sigma_e^2)$. The channel estimation error between the m^{th} receiver and n^{th} transmitter is

$$\sigma_e^2 = 1 - \sigma_{\widehat{h}}^2, \quad \sigma_e^2 < 1.$$

Using a similar approach as in [5], the upper and lower bounds for the instantaneous mutual information $\mathcal{I}(\boldsymbol{x};\boldsymbol{y})$ were extended to multiple antennas at the transmitter and receiver to yield the bounds

$$\begin{aligned}
\mathcal{I}_{lb} &= \log_2 \det\left(\boldsymbol{I}_{N_t} + \frac{\rho\sigma_{\widehat{h}}^2}{N_t(1+\rho\sigma_e^2)} \underline{\widehat{\boldsymbol{H}}}\,\underline{\widehat{\boldsymbol{H}}}^H\right) && (1) \\
\mathcal{I}_{ub} &= \mathcal{I}_{lb} + N_r \mathbb{E}_{\boldsymbol{x}}\left\{\log_2 \frac{N_t\sigma_e^2 + 1}{\|\boldsymbol{x}\|_2^2\sigma_e^2 + 1}\right\} && (2)
\end{aligned}$$

where $\underline{\widehat{\boldsymbol{H}}} \triangleq \widehat{\boldsymbol{H}}/\sigma_{\widehat{h}}$, $\boldsymbol{x}$ is a N_r x 1 vector with $x_n \overset{iid}{\sim} \mathcal{CN}(0,1)$, and $\underline{\widehat{\boldsymbol{H}}}\,\underline{\widehat{\boldsymbol{H}}}^H$ has a Wishart distribution.

III. Outage Capacity for a MIMO Rayleigh Flat Fading Channel

Given a probability P, a lower bound on the outage capacity $\mathcal{C}_{lb}(P)$ is defined to be the lowest value a given rate r can take satisfying $\mathcal{P}(r \geq \mathcal{I}_{lb}(\boldsymbol{x};\boldsymbol{y})) \leq P$. Similarly, an upper bound on the outage capacity $\mathcal{C}_{ub}(P)$ is defined to be the highest value a given rate r can take with $\mathcal{P}(r \geq \mathcal{I}_{ub}(\boldsymbol{x};\boldsymbol{x})) \leq P$. Namely [6],

$$\begin{aligned}
\mathcal{C}_{lb}(P) &\triangleq \arg_r \{P(\mathcal{I}_{lb} \leq r) = P\} && (3) \\
\mathcal{C}_{ub}(P) &\triangleq \arg_r \{P(\mathcal{I}_{ub} \leq r) = P\}. && (4)
\end{aligned}$$

In order to compute (3) and (4), we now attempt to find the probability density function (pdf) for (1) and (2). Letting $k \triangleq \min(N_r, N_t)$ and $d \triangleq \max(N_r N_t) - k$, the pdf of an unordered eigenvalue taken from a Wishart matrix is [1]

$$f_\lambda(\lambda) = \frac{e^{-\lambda}}{k} \sum_{i=0}^{k-1} \frac{i!}{(i+d)!} L_i^{2d}(\lambda)\lambda^d \qquad (5)$$

where $L_i^d(x)$ is the associated Laguerre polynomial defined by

$$L_i^d(x) \triangleq \sum_{p=0}^{i} (-1)^p \frac{(i+d)!}{(i-p)!(d+p)!} x^p. \qquad (6)$$

Observing (5) and (6), it is clear that finding the pdf of (1) and (2) is rather involved compared to the SISO case in [6] where simple pdf transformations were performed. Our derivation for new bounds on the outage capacity will start with [7], where it is shown that $\mathcal{I}(\boldsymbol{x};\boldsymbol{y})$ can be well approximated by a Gaussian random variable. Letting $\rho_{eff} \triangleq \frac{\rho\sigma_{\widehat{h}}^2}{N_t(1+\rho\sigma_e^2)}$ we have $\mathcal{I}_{lb} \sim \mathcal{N}(\mu_{\mathcal{I}_{lb}}, \sigma_{\mathcal{I}}^2)$ and $\mathcal{I}_{ub} \sim \mathcal{N}(\mu_{\mathcal{I}_{ub}}, \sigma_{\mathcal{I}}^2)$ where

$$\begin{aligned}
\mu_{\mathcal{I}_{lb}} &= \int_0^\infty \log_2(\rho_{eff}\lambda + 1)K(\lambda,\lambda)d\lambda \\
\mu_{\mathcal{I}_{ub}} &= \int_0^\infty \log_2(\rho_{eff}\lambda + 1)K(\lambda,\lambda)d\lambda \\
&+ N_r\mathbb{E}_{\boldsymbol{x}}\left\{\log_2 \frac{N_t\sigma_e^2 + 1}{\|\boldsymbol{x}\|_2^2\sigma_e^2 + 1}\right\} \\
\sigma_{\mathcal{I}}^2 &= \int_0^\infty \log_2^2(\rho_{eff}\lambda + 1)K(\lambda,\lambda)d\lambda \\
&- \int_0^\infty \int_0^\infty \log_2(\rho_{eff}\lambda_1 + 1) \\
&\times \log_2(\rho_{eff}\lambda_2 + 1) \\
&\times K^2(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2
\end{aligned}$$

and $K(\lambda,\lambda)$ is a function described by

$$K(x,y) \triangleq \sum_{i=0}^{k-1} \frac{i!}{(i+d)!} L_i^d(x)L_i^d(y)x^{d/2}y^{d/2}e^{(-(x/2+y/2))}.$$

This allows us the write the desired probabilities in (3) and (4) as

$$\begin{aligned}
\mathrm{P}(\mathcal{I}_{lb} \leq r) &= \int_0^r \frac{1}{\sqrt{2\pi}\sigma_{\mathcal{I}_{lb}}} e^{\frac{(x-\mu_{\mathcal{I}_{lb}})^2}{2\sigma_{\mathcal{I}_{lb}}^2}} dx \\
\mathrm{P}(\mathcal{I}_{ub} \leq r) &= \int_0^r \frac{1}{\sqrt{2\pi}\sigma_{\mathcal{I}_{ub}}} e^{\frac{(x-\mu_{\mathcal{I}_{ub}})^2}{2\sigma_{\mathcal{I}_{ub}}^2}} dx.
\end{aligned}$$

Using $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}}\exp(-x^2/2)dx$, its approximation $Q(x) \approx \frac{1}{\sqrt{2\pi x}}\exp(-x^2/2)$ and performing several manipulations, we arrive at the new upper and lower bounds of the outage capacity for a Rayleigh flat fading MIMO channel

$$\begin{aligned}
\mathcal{C}_{lb}(P) &= \mu_{\mathcal{I}_{lb}} + \sigma_{\mathcal{I}} W\left(\frac{1}{2\pi\beta_{lb}^2}\right)^{1/2} && (7) \\
\mathcal{C}_{ub}(P) &= \mu_{\mathcal{I}_{ub}} + \sigma_{\mathcal{I}} W\left(\frac{1}{2\pi\beta_{ub}^2}\right)^{1/2} && (8)
\end{aligned}$$

where $\beta_{lb} \triangleq Q\left(\frac{\mu_{\mathcal{I}_{lb}}}{\sigma_{\mathcal{I}}}\right) + (1-P)$, $\beta_{ub} \triangleq Q\left(\frac{\mu_{\mathcal{I}_{ub}}}{\sigma_{\mathcal{I}}}\right) + (1-P)$ and $W(\cdot)$ is the Lambert W-function.

IV. DISCUSSION

Some insights on the relationships in (7) and (8) are now obtained. For simplicity let $N_r = N_t$. By the monotonicity of the logarithm, it is clear that for any $a < b$

$$\int_0^\infty \log(a\lambda+1)K(\lambda,\lambda)d\lambda < \int_0^\infty \log(b\lambda+1)K(\lambda,\lambda)d\lambda$$

which demonstrates that maximizing ρ_{eff} maximizes $\mu_{\mathcal{I}_{lb}}$. This is also true for $\mu_{\mathcal{I}_{ub}}$ since the distribution for $\boldsymbol{x}$ has been specified. The variance term can be approximated by [8]

$$\sigma_{\mathcal{I}}^2 \approx \mathcal{I}'(\mu_\lambda)^2\sigma_\lambda^2$$

which yields

$$\sigma_{\mathcal{I}}^2 \approx \frac{1}{(\ln 2)^2}\left(\frac{1}{\mu_\lambda + \frac{1}{\rho_{eff}}}\right)^2\sigma_\lambda^2$$

from which it is obvious that increasing ρ_{eff} increases $\sigma_{\mathcal{I}}^2$. Since $W(x)$ is strictly increasing, we see that maximizing $\frac{\mu_{\mathcal{I}_{ub}}}{\sigma_{\mathcal{I}}}$ and P maximizes $W(\frac{1}{2\pi\beta_{ub}^2})^{1/2}$. The same argument follows for $W(\frac{1}{2\pi\beta_{lb}^2})^{1/2}$.

Since decreasing σ_e^2 increases $\mu_{\mathcal{I}_{lb}}$, $\mu_{\mathcal{I}_{ub}}$, and $\sigma_{\mathcal{I}}^2$ it is clear that the outage capacity increases when σ_e^2 decreases. Also, we expect that a change in P will have more impact on the MIMO outage capacity when σ_e^2 is small. This will give a larger ρ_{eff}, assuming all other parameters are fixed, corresponding to a larger $\sigma_{\mathcal{I}}^2$. This will allow $W(\frac{1}{2\pi\beta_{ub}^2})^{1/2}$, which previously mentioned is affected by P, to have a larger impact on the outage capacity.

To validate these observations the MIMO outage capacity bounds were tabulated when varying both P and σ_e^2 for $N_t = N_r = 4$. Looking at Figure 1, it is clear that decreasing σ_e^2 increases the outage capacity as expected. Also, an increase in P has a more substantial impact for smaller σ_e^2, which was discussed.

V. TIME VARYING CHANNEL / PSAM

Up to this point we have assumed that the channel estimation error does not change with time. To understand how the outage capacity is affected in a time varying channel, we will assume $h_{mn}(t)$ is a complex-gaussian random process based on Jakes' model. If there is no spatial correlation between antennas, the autocorrelation function for $h_{mn}(t)$ is [9]

$$R_{h_{mn}h_{mn}}(\tau) = J_o(2\pi f_d T_s \tau)$$

where f_d is the doppler frequency, T_s is the symbol period, and $J_o(\cdot)$ is the zeroth order bessel function of the first kind.

Suppose that during a data burst each transmit antenna sends a block of T symbols, N_t of which are known pilots.

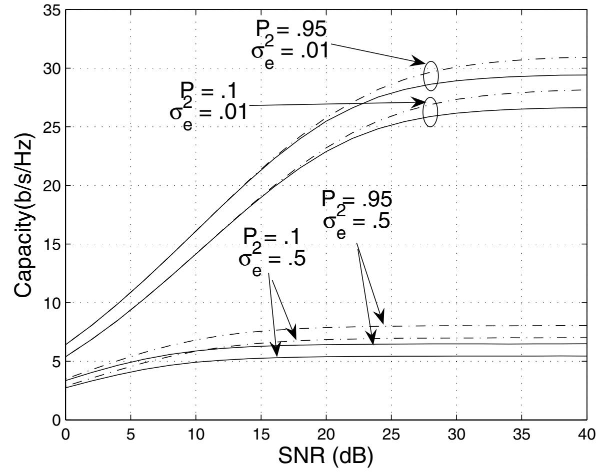

Fig. 1. Upper and lower bounds of the outage capacity for various channel estimation errors and outage probabilities

Let $D \triangleq T - N_t$ denote the number of symbols allocated for data transmission. Assume that $\boldsymbol{H}$ changes noticeably enough to warrant training, yet remains fairly constant during the training interval. Using the same approach as in [5] the optimal weights for a M^{th} order Wiener filter using PSAM are

$$\boldsymbol{w}_o(k) = \boldsymbol{K}^{-1}\boldsymbol{\rho}(k)\,, \quad k = 1, \ldots, D$$

where

$$\begin{aligned} k_{mn} &= \frac{\rho_\tau}{N_t} J_o(2\pi f_d T_s|m-n|T) + \frac{\delta[m-n]}{|x_n(j)|^2} \\ \boldsymbol{\rho}(k) &= \begin{bmatrix} \frac{\rho_\tau}{N_t} J_o(2\pi f_d T_s|\lfloor -(M-1/2)\rfloor T - k|) \\ \vdots \\ \frac{\rho_\tau}{N_t} J_o(2\pi f_d T_s|\lfloor M/2\rfloor T - k|) \end{bmatrix}. \end{aligned}$$

The MSE is found to be

$$\sigma_e^2(k) = 1 - \boldsymbol{\rho}(k)^H\boldsymbol{K}^{-1}\boldsymbol{\rho}(k).$$

To show how time variation in the channel affects the outage capacity, consider the following examples. Let $P = .1$, $f_dT_s = .1$, $N_r = N_t = 4$, $T = 10$, and $M = 10$. Then $D = T - N_t = 6$ data symbols are sent. Figure 2 shows that the outage capacity deviates substantially from the case of perfect CSI when the elapsed time after training is equal to one normalized frequency. At elapsed time $6f_dT_s$, the outage capacity is fairly constant for all SNR, indicating a high estimation error. There is not much point to using PSAM for this condition.

Next let $f_dT_s = .04$ and let the remaining parameters remain the same. Looking at Figure 3 we see that the outage capacity stays very close to the case of perfect CSI up until 30 dB throughout the data stage. Comparing this to Figure 2, we see that when the channel is fast fading there is no benefit to increasing the SNR or spend time

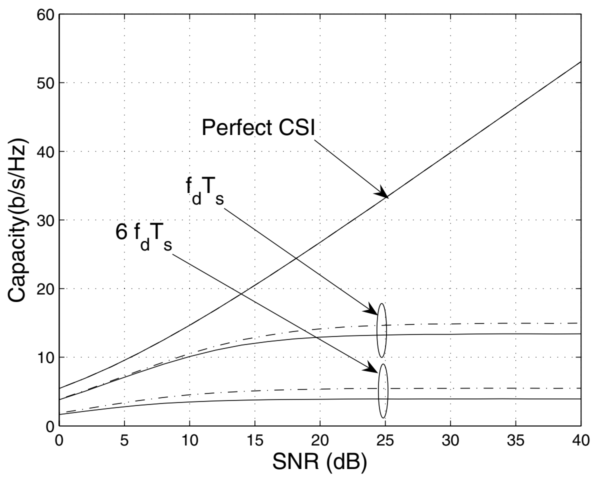

Fig. 2. Upper and lower bounds for the outage capacity using PSAM when $f_d T_s = .1$.

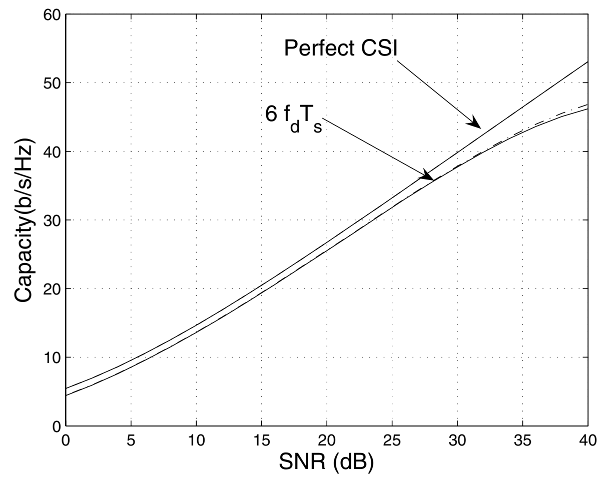

Fig. 3. Upper and lower bounds for the outage capacity using PSAM when $f_d T_s = .04$.

training, unlike the slow fading case. These two examples demonstrate how the MIMO outage capacity is sensitive to time variation in the channel.

VI. Conclusion

Upper and lower bounds for the outage capacity of a MIMO Rayleigh flat fading wireless channel were derived. We showed that an increase in channel estimation error negatively impacts the outage capacity. A change in the outage probability was found to vary the outage capacity more when the channel estimation error was small. This is due to the fact that an decrease in the channel estimation error decreased the variance of the upper and lower bounds on the instantaneous mutual information.

We next considered the effect on the outage capacity for a temporally fading channel using PSAM. The outage capacity was shown to decrease dramatically during data transmission in a fast fading channel. For this case it was determined that neither increasing the training time or SNR had a positive impact on the outage capacity. For a slow fading channel, it was shown that the outage capacity remained close to the perfect CSI case for a wide range of SNR during data transmission.

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