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Ergodic Capacity of Doubly Selective Rayleigh Fading MIMO Channels

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Abstract— **The ergodic capacity is investigated for doubly selective (frequency selective and time varying) MIMO Rayleigh fading channels. A closed form formula is derived that quantifies the effect of the ISI fading on the ergodic capacity into an ISI degradation factor. It is discovered that, in general frequency selective MIMO channels, the inter-tap correlations of the ISI fading will reduce the ergodic capacity comparing to the frequency flat fading channel. Only in the special case when the ISI fading does not have inter-tap correlations will the ergodic capacity be the same as that of the frequency flat channel. This new formula is mathematically proved and experimentally verified via Monte-Carlo simulations.**

I. Introduction

Multiple-Input-Multiple-Output (MIMO) wireless communication has recently received significant attention due to its enormous channel capacity potential in rich scattering environment [1]-[3]. The ergodic capacity results have been well established for MIMO Rayleigh fading channels which are spatially correlated (including spatially uncorrelated), time quasi-static, and frequency nonselective [4]-[22], and the references therein. These capacity results are based on the assumption that the MIMO channels have neither Doppler spread nor delay spread, which is not the case in many moderate and high mobility and high date rate mobile communication applications.

The capacity studies for MIMO frequency selective Rayleigh fading channels has also received some attention [8], [23]-[28]. Specifically, in [24], it was reported that OFDM-based MIMO frequency selective (delay spread) channels will in general provide advantages over frequency flat fading channels not only in terms of outage capacity but also in terms of ergodic capacity. However, in [28], it was reported that frequency selectivity does not affect the ergodic capacity of wideband MIMO channels, which is in good agreement with the SISO ergodic capacity results in [29]. Both [28] and [29] are based on the assumption that the discrete-time sampled channel impulse response has no inter-tap correlation. Recently, it was reported by Xiao *et al* [32] and Paulraj *et al* [18] that the sampled fading channel taps are in general inter-tap correlated due to the transmit pulse-shaping and receive matched filters.

In this paper, we consider the ergodic capacity of a MIMO system which undergoes inter-tap correlated (including inter-tap uncorrelated as a special case) frequency selective and time-varying fading. Due to the time variation, we assume that the channel state information is unknown to the transmitter but perfectly known to the receiver. Therefore, the equal power allocation scheme is used at the transmitter. A mathematical formula for the ergodic capacity is derived for Rayleigh fading MIMO channel where the fadings are spatially uncorrelated, frequency selective, and time varying. We find that the inter-tap correlations of frequency selective fading channels can have significant impact on the ergodic capacity. This impact is quantified into an ISI degradation factor in the closed form formula. In a general frequency selective fading channel, the ergodic capacity is reduced by the ISI degradation factor. In the special case when the ISI has no inter-tap correlations, the ISI degradation factor is one, and the ergodic capacity is the same as that of the frequency flat channel. The mathematical proof of our new formula is included, along with verification via extensive simulations using improved Jakes' Rayleigh fading simulator [32], [33].

II. Channel Models and Preliminaries

Consider a wideband MIMO wireless channel shown in Fig. 1. Assume that the transmit pulse shaping filter $p_{T}(t)$ and the receive matched filter $p_{R}(t)$ are normalized with unit energy. Assume also that each physical fading subchannel $g_{m,n}(t,\tau)$ is wide-sense stationary uncorrelated scattering (WSSUS) [30] Rayleigh fading with normalized unit energy. The continuous-time MIMO

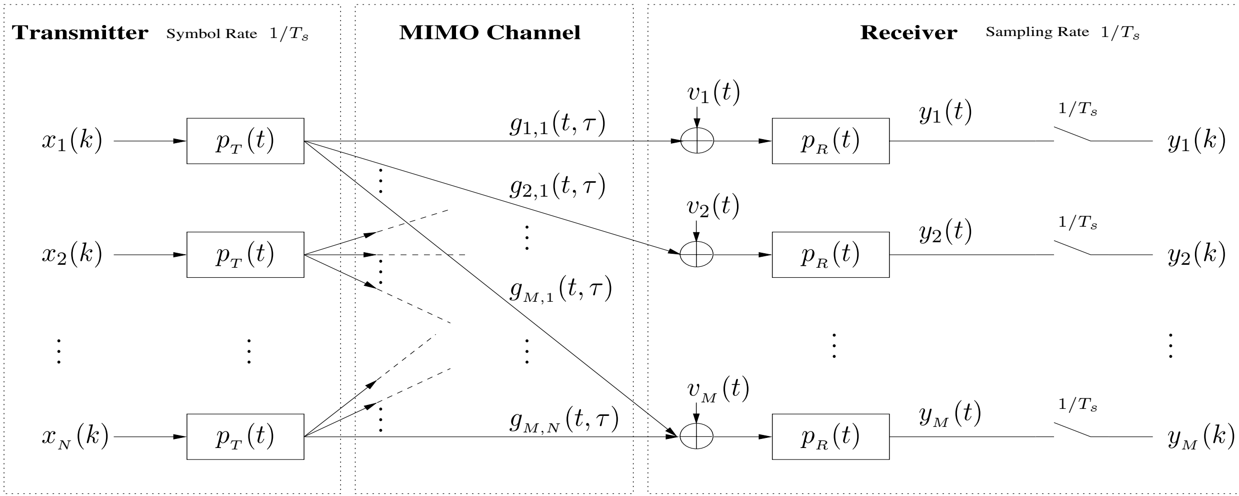

Fig. 1. The baseband block diagram of a MIMO wireless channel, which consists of the transmit filter $p_T(t)$, the physical fading impulse response $g_{m,n}(t,\tau)$, and the receive filter $p_R(t)$ for the (m,n)th-subchannel. The combined MIMO channel impulse response is $h_{m,n}(t,\tau) = p_R(\tau) \otimes g_{m,n}(t,\tau) \otimes p_T(\tau)$. It can be accurately converted to a discrete-time channel model represented by an L-tap FIR $h_{m,n}(l,k), l = 0, 1, \cdots, L-1$.

channel depicted in Fig. 1 can be accurately converted to the following noncausal discrete-time MIMO fading channel model with proper delay [32]

$$\mathbf{y}(k) = \sum_{l=0}^{L} \mathbf{H}(l,k) \cdot \mathbf{x}(k-l) + \mathbf{v}(k), \tag{1}$$

where the input $\mathbf{x}(k) = [x_1(k), x_2(k), \cdots, x_N(k)]^t$, the noise $\mathbf{v}(k) = [v_1(k), v_2(k), \cdots, v_M(k)]^t$, and the output $\mathbf{y}(k) = [y_1(k),\ y_2(k), \cdots, y_M(k)]^t$, with the superscript $(\cdot)^t$ being the transpose; L is the channel length which is depending on the transmit filter, delay spread power profiles and receive filter; and the matrix $\mathbf{H}(l,k)$ is the lT_s delayed channel matrix at time instant k, whose elements are the discrete-time fading coefficients $h_{m,n}(l,k)$.

The correlation function between the channel coefficients is given by [32]

$$E\left[h_{m,n}(l_1,k_1) \cdot h^*_{p,q}(l_2,k_2)\right] = \mathbf{\Psi}_{RX}(m,p) \cdot \mathbf{\Psi}_{TX}(n,q) \cdot \mathbf{\Psi}_{ISI}(l_1,l_2) \cdot \mathbf{\Psi}_{DPR}(k_1,k_2), \tag{2}$$

where the superscript $*$ denotes the conjugate, $E[\cdot]$ denotes the expectation. The matrices $\mathbf{\Psi}_{RX}$, $\mathbf{\Psi}_{TX}$, $\mathbf{\Psi}_{ISI}$ and $\mathbf{\Psi}_{DPR}$ are the receive correlation coefficient matrix, the transmit correlation coefficient matrix, the intersymbol interference (ISI) inter-tap correlation coefficient matrix, and the temporal correlation coefficient matrix, respectively. The first three matrices satisfy $tr\left(\mathbf{\Psi}_{RX}\right) = M$, $tr\left(\mathbf{\Psi}_{TX}\right) = N$ and $tr\left(\mathbf{\Psi}_{ISI}\right) = 1$ [32].

We give two remarks on $\mathbf{\Psi}_{ISI}$ and $\mathbf{\Psi}_{DPR}$. The coefficient $\mathbf{\Psi}_{ISI}(l_1,l_2)$ is related to the channel fading power delay profile, the transmit filter, and the receive filter. Its calculation is given by (17) of [32]. Even if the physical channel $g_{m,n}(t,\tau)$ is WSSUS channel which means no inter-path correlation, the discrete-time sampled channel $h_{m,n}(l,k)$ will generally have inter-tap correlations [32], [18]. Our second remark goes to $\mathbf{\Psi}_{DPR}$. Different fading model will have different $\mathbf{\Psi}_{DPR}$. For the commonly used Clarke's 2-D isotropic scattering model-based Rayleigh fading, $\mathbf{\Psi}_{DPR}(k_1,k_2) = J_0(2\pi F_d(k_1-k_2)T_s)$, with $J_0(\cdot)$ being the zero-order Bessel function of the first kind, F_d the maximum Doppler frequency, and T_s the symbol period.

This discrete-time MIMO channel model (2) is a generalized model describing trebly selective MIMO channels. It contains many existing channel models as special cases. For example, 1), if $L = 1$ and $F_d = 0$, then the channel model becomes the spatially correlated, time quasi-static, and frequency flat model [5]. 2) If $L = 1$, $F_d = 0$, $\mathbf{\Psi}_{TX} = \mathbf{I}_N$, and $\mathbf{\Psi}_{RX} = \mathbf{I}_M$, then the model becomes the spatially uncorrelated, time quasi-static, and frequency flat model [1]. 3) If $M = 1$ and $N = 1$, then our model becomes the doubly selective fading model for SISO systems [31]. 4) If $L = 1$ and $\mathbf{\Psi}_{DPR}$ is an identity matrix, then this model becomes a symbol-wise temporally independent fading model.

When the channel has intersymbol interference (frequency selective), the channel capacity has to be analyzed based on a block of K output symbols $\{\mathbf{y}(k+1), \mathbf{y}(k+2), \cdots, \mathbf{y}(k+K)\}$ at the receiver. The MIMO channel with ISI is then represented by

$$\mathbf{Y}_{K} = \mathcal{H}\mathbf{X}_{K+L-1} + \mathbf{V}_{K}, \quad (3)$$

where $\mathbf{X}_{K+L-1}$, $\mathbf{Y}_{K}$, and $\mathbf{V}_{K}$ are the input (with padded zeros to clear out the ISI memory), output, and additive Gaussian noise vectors, respectively. The K-symbol block channel impulse response matrix $\mathcal{H}$ is given by

$$\mathcal{H}=\begin{bmatrix} \mathbf{H}(L-1,k+1) & \cdots & \mathbf{H}(0,k+1) & 0 & 0 \\ 0 & \ddots & \cdots & \ddots & 0 \\ 0 & 0 & \mathbf{H}(L-1,k+K) & \cdots & \mathbf{H}(0,k+K) \end{bmatrix}. \quad (4)$$

When the channel matrix $\mathcal{H}$ is perfectly known to the receiver and unknown to the transmitter, the equal power allocation scheme is employed. Then the instantaneous mutual information (per input symbol) is defined as

$$\mathcal{I}_{K}(k)=\frac{1}{K+L-1}\cdot\left[\log_2\det\left(\mathbf{I}_{KM}+\frac{\gamma}{N}\mathcal{H}\mathcal{H}^{\dagger}\right)\right], \text{ b/s/Hz} \quad (5)$$

where γ is the SNR, and the superscript $(\cdot)^{\dagger}$ denotes the conjugate transpose. For a large $K \gg L$, the factor $1/(K+L-1)$ in (5) can be approximated by $1/K$. The ergodic capacity $\mathcal{C}^{av}_{MIMO}$ is given by

$$\mathcal{C}^{av}_{MIMO}=E_{\mathcal{H}}\left\{\lim_{K\to\infty}\frac{1}{K}\cdot\left[\log_2\det\left(\mathbf{I}_{KM}+\frac{\gamma}{N}\mathcal{H}\mathcal{H}^{\dagger}\right)\right]\right\}. \quad (6)$$

III. New Results for Ergodic Capacity

This section presents an explicit formula for the ergodic (average) capacity of MIMO channels under doubly selective Rayleigh fading senario. Specialized formulas are given for SISO, SIMO and MISO Rayleigh fading channels. An good approximation for the MIMO channel ergodic capacity is also given when the numbers of transmit and receive antennas are large.

Theorem 1: For a time-varying and frequency-selective MIMO Rayleigh fading channel, if the channel is spatially uncorrelated, meaning that $\mathbf{\Psi}_{TX} = \mathbf{I}_{N}$ and $\mathbf{\Psi}_{RX} = \mathbf{I}_{M}$, then the ergodic capacity is given by

$$\mathcal{C}^{av}_{MIMO} = \int_0^{\infty}\log_2\left(1+\frac{\gamma}{N}\cdot\gamma_{ISI}\cdot\lambda\right)\cdot \sum_{i=0}^{m-1}\frac{i!}{(i+n-m)!}\left[L_i^{n-m}(\lambda)\right]^2\lambda^{n-m}e^{-\lambda}d\lambda \quad (7)$$

where $m = \min\{M,N\}$, $n = \max\{M,N\}$, $L_i^j(\cdot)$ is the associated Laguerre polynomial [1] of order i, and γ_{ISI} is the ISI degradation factor due to the channel ISI inter-tap correlations. It is determined by

$$\gamma_{ISI} = (2^{C_\gamma}-1)/\gamma \quad (8)$$

with

$$C_\gamma = \frac{1}{2\pi}\int_0^{2\pi}\log_2\left[1+\gamma\cdot f(\omega)\right]d\omega,$$

$$f(\omega)=1+2\sum_{i=1}^{L-1}a_i\cos(i\omega), \quad a_i=\sum_{l=0}^{L-1-i}\mathbf{\Psi}_{ISI}(l,l+i). \quad (9)$$

Proof: See the Appendix.

Remark 1: The significance of Theorem 1 is that the effect of the frequency selectivity (or the ISI) on the ergodic capacity is quantified into the simple ISI degradation factor γ_{ISI}. This is also the difference between our formula and the one given in [1] for frequency flat MIMO channels. It will be shown in Fig. 2, that the inter-tap correlation of the ISI channel plays an important role on the ergodic capacity. If the ISI taps have no inter-tap correlation, the ISI degradation factor $\gamma_{ISI} = 1$, meaning that there is no ISI degradation. Thus the ergodic capacity of the frequency-selective channel is the same as that of the frequency flat channel (with no ISI). In this case, our result is in good agreement with the conclusions of [29] and [28] respectively for SISO and MIMO time-quasi-static and frequency-selective Rayleigh fading channels.

Remark 2: It is known that for an inter-tap uncorrelated frequency-selective fading channel, one can always decompose the fading channel into a set of sufficiently narrow orthogonal subbands, each having the same ergodic capacity as the overall channel. Thus this type of frequency-selective fading channel has the same ergodic capacity as frequency flat fading channel [29], [28]. However, as can be seen from the proof of Theorem 1 shown in the appendix, for the general case where the fading channel has inter-tap correlations, the frequency-selective fading channel can not be decomposed into a set of orthogonal subbands. Thus one can not extend the results of the inter-tap uncorrelated special case to the general frequency selective fading channel case. It is noted that in the general case where the ISI taps have inter-tap correlations, $\gamma_{ISI} < 1$, therefore the ergodic capacity is reduced compared to frequency flat fading channels.

Remark 3: The formulation of γ_{ISI} in (7) simplifies the computation from a double-integral to a single-integral for inter-tap correlated frequency selective fading channels.

The ergodic capacity formula given by (7) can be significantly simplified for the cases when m is small, such as that in SISO, SIMO and MISO systems. It can also be approximated with high accuracy when m is large. Details are presented in the following corollaries.

Corollary 1: For a SISO time-varying and frequency-selective Rayleigh fading channel, the ergodic capacity is given by

$$\mathcal{C}^{av}_{SISO} = \int_0^{\infty}\log_2\left(1+\gamma\cdot\gamma_{ISI}\cdot\lambda\right)e^{-\lambda}d\lambda. \quad (10)$$

Corollary 2: For time-varying and frequency-selective MISO and SIMO Rayleigh fading channels, if $\mathbf{\Psi}_{TX}$ and $\mathbf{\Psi}_{RX}$ are identity matrices, then the ergodic capacities are given by

$$c^{av}_{MISO} = \frac{1}{(N-1)!}\int_0^\infty \log_2\left(1+\frac{\gamma}{N}\cdot\gamma_{ISI}\cdot\lambda\right)\lambda^{N-1}e^{-\lambda}d\lambda, \quad (11)$$

$$c^{av}_{SIMO} = \frac{1}{(M-1)!}\int_0^\infty \log_2\left(1+\gamma\cdot\gamma_{ISI}\cdot\lambda\right)\lambda^{M-1}e^{-\lambda}d\lambda. \quad (12)$$

Corollary 3: When the numbers of transmit and receive antennas N and M are large, the ergodic capacity of the MIMO time-varying and frequency-selective Rayleigh fading channel is approximately

$$c^{av}_{MIMO} \approx c^{\text{apprx}}_{MIMO} = \frac{N}{2\pi}\int_a^b \log_2\left(1+\gamma\cdot\gamma_{ISI}\cdot\lambda\right)\frac{\sqrt{(\lambda-a)(b-\lambda)}}{\lambda}d\lambda, \quad (13)$$

where $a = \left(\sqrt{M/N}-1\right)^2$ and $b = \left(\sqrt{M/N}+1\right)^2$.

Remark 4: Eqn. (13) implies that if the number of antennas increases with a fixed ratio $\frac{N}{M}$, then the ergodic capacity increases linearly with N (or M).

IV. Simulation Results

To verify the theoretical ergodic capacity results presented in Section III, we have conducted extensive simulations which employs the discrete-time time-varying frequency-selective Rayleigh fading MIMO channel model described in Section II with different channel conditions such as Doppler spread F_d, channel length L, block length K, and antenna numbers M and N. To keep the paper within the length limit, we only present some of the simulation results.

Figure 2 depicts the ergodic capacity for the SISO, 2×2, and 4×4 systems under three fading channels. It is shown that every simulated curve is in excellent agreement with the corresponding theoretical curve. Comparing the ergodic capacities of the three different MIMO systems, we can see that the ergodic capacity increases as the number of antennas. The inter-tap uncorrelated frequency-selective channel has the same ergodic capacity as that of the frequency flat fading channel. However, when the frequency-selective Rayleigh fading channel has inter-tap correlations, its ergodic capacity is smaller than that of the channel with no inter-tap correlations.

Figure 3 shows the time variation (or Doppler spread) effect on the ergodic capacity with the example of the SISO system. The AWGN channel capacity is included for comparison. All curves were obtained by simulations, which are also in excellent agreement with the theoretically calculated results omitted from the figure. It is clearly shown that the time variations have no effect on the ergodic capacities. The figure also indicates that the ergodic capacities of the fading channels are smaller than that of the AWGN channel.

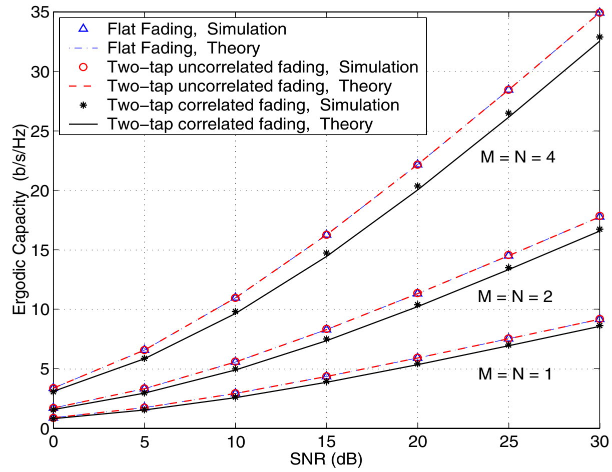

Fig. 2. Ergodic Capacity vs SNR for the SISO, 2×2 and 4×4 systems under: (1) frequency flat fading channel, (2) frequency-selective two-tap uncorrelated fading channel with $\mathbf{\Psi}_{ISI} = [0.5 \;\; 0; \;\; 0 \;\; 0.5]$, (3) frequency-selective two-tap correlated fading channel with $\mathbf{\Psi}_{ISI} = [0.5 \;\; 0.475; \;\; 0.475 \;\; 0.5]$. All fading channels are Rayleigh distributed with $2KF_dT_s = 1$.

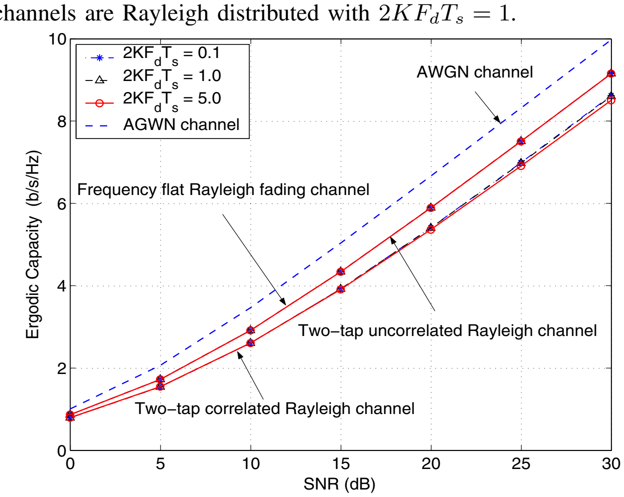

Fig. 3. Time variation (or Doppler spread) effect on ergodic capacity of SISO channels. Three maximum Dopplers are selected such that $2KF_dT_s = 0.1$, $2KF_dT_s = 1.0$ and $2KF_dT_s = 5.0$. The $\mathbf{\Psi}_{ISI}$'s of the fading channels are the same as those in Fig. 2. It is clear that the time variations do not have impact on the ergodic capacity.

V. Conclusion

The ergodic capacity is investigated for doubly selective (frequency selective and time varying) MIMO Rayleigh fading channels. A closed form formula has been derived that quantifies the effect of the ISI fading on the ergodic capacity into an ISI degradation factor γ_{ISI}. In the special case when the ISI fading does not have inter-tap correlations, $\gamma_{ISI} = 1$, and the ergodic capacity is the same as that of the frequency flat channel. In the more general cases of frequency selective MIMO channels, $\gamma_{ISI} < 1$, and the inter-tap correlations of the ISI fading will reduce the ergodic capacity. This new formula has been mathematically proved and experimentally verified via Monte-Carlo simulations.

APPENDIX: PROOF OF THEOREM 1

For stationary and ergodic fading channels, it is known by definition that the time variation does not affect the ergodic capacity. Thus we prove the theorem by considering the slowy time-varying scenario. The SISO case is proved first for better clarity. The SISO channel matrix $\mathcal{H}$ becomes a Toeplitz matrix as follows.

$$\mathcal{H} = \begin{bmatrix} h_{L-1} & h_{L-2} & \cdots & h_0 & 0 & 0 & 0 \\ 0 & h_{L-1} & \cdots & \cdots & h_0 & 0 & 0 \\ 0 & 0 & \ddots & \cdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & h_{L-1} & \cdots & \cdots & h_0 \end{bmatrix}, \quad (14)$$

and the ergodic capacity is given by [34]

$$\begin{aligned} \mathcal{C}^{av}_{SISO} &= E\left[\lim_{K\to\infty} \frac{1}{K} \log_2 \det\left(\mathbf{I}_K + \gamma \cdot \mathcal{H}\mathcal{H}^\dagger\right)\right] \\ &= E\left\{\frac{1}{2\pi}\int_0^{2\pi} \log_2\left[1 + \gamma \cdot |h(\omega)|^2\right] d\omega\right\}, \end{aligned} \quad (15)$$

where

$$h(\omega) = \sum_{l=0}^{L-1} h_l \cdot \exp\left(\sqrt{-1} \cdot l \cdot \omega\right). \quad (16)$$

For Rayleigh fading channels, the channel coefficients h_l are zero mean circularly symmetric complex Gaussian random variables for all l. For a given value ω, the function $h(\omega)$ is also a zero mean circularly symmetric complex Gaussian random variable. Thus $|h(\omega)|^2$ is exponentially distributed at the given value ω. Moreover, the variance of $h(\omega)$ is given by

$$\begin{aligned} f(\omega) &= E\left[h(\omega) \cdot h^*(\omega)\right] \\ &= \sum_{l_1=0}^{L-1}\sum_{l_2=0}^{L-1} E\left[h_{l_1} \cdot h^*_{l_2}\right] \cdot \exp\left[\sqrt{-1} \cdot (l_1 - l_2) \cdot \omega\right] \\ &= \sum_{l_1=0}^{L-1}\sum_{l_2=0}^{L-1} \mathbf{\Psi}_{ISI}(l_1, l_2) \cdot \exp\left[\sqrt{-1} \cdot (l_1 - l_2) \cdot \omega\right] \\ &= 1 + 2\sum_{k=0}^{L-1} a_k \cos(k \cdot \omega), \end{aligned} \quad (17)$$

where $a_k = \sum_{l=0}^{L-1-k} \mathbf{\Psi}_{ISI}(l, l+k)$. Therefore, the PDF of $|h(\omega)|^2$ is equivalent to the PDF of $(\lambda \cdot f(\omega))$ with λ being a unit variance exponentially distributed random variable, and the ergodic capacity is given by

$$\begin{aligned} \mathcal{C}^{av}_{SISO} &= \int_0^\infty \left\{\frac{1}{2\pi}\int_0^{2\pi} \log_2\left[1 + \gamma \cdot f(\omega) \cdot \lambda\right] d\omega\right\} \cdot e^{-\lambda} \cdot d\lambda \\ &= \int_0^\infty \log_2\left[1 + \gamma \cdot f(\xi) \cdot \lambda\right] \cdot e^{-\lambda} \cdot d\lambda, \end{aligned} \quad (18)$$

where the second equality is based on the mean-value theorem of integrals [35], and $\xi \in (0, 2\pi)$.

To determine $f(\xi)$, we fix λ at its mean value which is 1, then calculate the following integral

$$\mathcal{C}_\gamma = \frac{1}{2\pi}\int_0^{2\pi} \log_2\left[1 + \gamma \cdot f(\omega)\right] d\omega. \quad (19)$$

Based on $\mathcal{C}_\gamma = \log_2\left[1 + \gamma \cdot f(\xi)\right]$, we can find $f(\xi)$ and define it as the ISI degradation factor

$$f(\xi) = \frac{2^{\mathcal{C}_\gamma} - 1}{\gamma} = \gamma_{ISI}. \quad (20)$$

This completes the proof for SISO channels with slowly time-varying and frequency-selective Rayleigh fading.

We are now in the position to prove the MIMO case. Again consider the slow time variation case. The channel matrix becomes a block Toeplitz matrix of the form

$$\mathcal{H} = \begin{bmatrix} \mathbf{H}(L-1,k) & \cdots & \mathbf{H}(0,k) & 0 & 0 \\ 0 & \ddots & \cdots & \ddots & 0 \\ 0 & 0 & \mathbf{H}(L-1,k) & \cdots & \mathbf{H}(0,k) \end{bmatrix}. \quad (21)$$

Based on the property of the block Toeplitz matrix, we can reduce the large dimension $(KM \times KM)$ random matrix to a much smaller dimension $(M \times M)$ one as

$$\begin{aligned} \mathcal{C}^{av}_{MIMO} &= E_{\mathcal{H}}\left\{\lim_{K\to\infty} \frac{1}{K} \cdot \left[\log_2 \det\left(\mathbf{I}_{KM} + \frac{\gamma}{N}\mathcal{H}\mathcal{H}^\dagger\right)\right]\right\} \\ &= E\left\{\frac{1}{2\pi}\int_0^{2\pi} \log_2 \det\left[\mathbf{I}_M + \gamma \cdot H(\omega) \cdot H^\dagger(\omega)\right] d\omega\right\}, \end{aligned} \quad (22)$$

where

$$H(\omega) = \sum_{l=0}^{L-1} \mathbf{H}(l, k) \cdot \exp\left(\sqrt{-1} \cdot l \cdot \omega\right). \quad (23)$$

Let $H_{ij}(\omega)$ be the (i,j)th element of $H(\omega)$. For MIMO Rayleigh fading channels, $H_{ij}(\omega)$ are zero mean circularly symmetric complex Gaussian random variables for all i and j. Utilizing the de-coupling property (2) and following the procedures of the SISO case, we can derive the variance of $H_{ij}(\omega)$ being $f(\omega)$ as given by (17). When $\mathbf{\Psi}_{TX} = \mathbf{I}_N$ and $\mathbf{\Psi}_{RX} = \mathbf{I}_M$, the distribution of $H(\omega) \cdot H^\dagger(\omega)$ is the same as the distribution of $f(\omega) \cdot \mathbf{H}_W \cdot \mathbf{H}^\dagger_W$, where $\mathbf{H}_W$ is an $M \times N$ matrix with all elements being i.i.d. zero mean circularly symmetric complex Gaussian random variables. Since $\mathbf{H}_W \cdot \mathbf{H}^\dagger_W$ is a

Wishart matrix, we can use Theorem 2 of [1] and obtain

$$\begin{aligned}
\mathcal{C}^{av}_{MIMO} &= E\left\{\frac{1}{2\pi}\int_0^{2\pi}\log_2\det\left[\mathbf{I}_M+\frac{\gamma}{N}\cdot f(\omega)\cdot\mathbf{H}_W\cdot\mathbf{H}_W^{\dagger}\right]d\omega\right\} \\
&= \int_0^{\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}\log_2\left[1+\frac{\gamma}{N}\cdot f(\omega)\cdot\lambda\right]d\omega\right\}\cdot \\
&\quad \sum_{i=0}^{m-1}\frac{i!}{(i+n-m)!}\left[L_i^{n-m}(\lambda)\right]^2\lambda^{n-m}e^{-\lambda}d\lambda \\
&= \int_0^{\infty}\log_2\left(1+\frac{\gamma}{N}\cdot\gamma_{ISI}\cdot\lambda\right)\cdot \\
&\quad \sum_{i=0}^{m-1}\frac{i!}{(i+n-m)!}\left[L_i^{n-m}(\lambda)\right]^2\lambda^{n-m}e^{-\lambda}d\lambda \qquad (24)
\end{aligned}$$

where $m=\min\{M,N\}$, $n=\max\{M,N\}$, L_i^j is the associated Laguerre polynomial [1] of order i , and γ_{ISI} is the SNR degradation factor determined by (8) and (9). This completes the proof.

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