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# BRIEF COMMUNICATIONS

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# Some observations of bispectral behavior of large ensembles of exact solutions to the Burgers equation for random initial conditions*

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Accurate bispectra for the Burgers equation have been computed into the high wave-number domain. Although only the imaginary part $ib(k,l)$ contributes to spectral transfer, new insights having implications for three-dimensional turbulence can be gained by incorporating the real part $a(k,l)$. For fixed $l$, for instance, the modulus of the bispectrum of the Burgers equation manifests a power-law subrange corresponding to the inertial subrange of the energy spectrum and an exponential subrange corresponding to the viscous cutoff of the energy spectrum.

The simplest measure of inertial effects on the turbulence energy spectrum is the spectral transfer function $T(k,t)$, and it is not simple. It has nonetheless been studied both for Navier–Stokes turbulence[1–4] and for random solutions to the Burgers equation.[5,6]

What $T(k)\Delta k$ provides is the resultant of the nonlinear interactions of all other wave numbers with those in $k$, $k+\Delta k$ as they contribute to the energy spectrum $E(k)$ between $k,k+\Delta k$. What $T(k)\Delta k$ does not provide is any information on the variety of origins or destinations of wave-number subranges as they make individual contributions.

That information is essentially contained in the bispectrum, $B(k,l) = \langle \hat{u}(k)\hat{u}(l)\hat{u}^*(k+l)\rangle$, which is related to the spectral transfer function by

$$T(k) = i\int_0^\infty (k+l)[B(k,l) - B^*(k,l)]dl \qquad (1)$$

in one dimension. In both one and three dimensions, the spectral version of the von Kármán–Howarth equation,

$$\frac{\partial E(k,t)}{\partial t} = T(k,t) - 2\,\mathrm{Re}^{-1}\,k^2E(k,t), \qquad (2)$$

has the following physical content. In the range $k,k+\Delta k$, the turbulent energy change is $[\partial E(k,t)/\partial t]\Delta k$, and the viscous dissipation rate decreasing the energy there is $2\,\mathrm{Re}^{-1}\,k^2E(k,t)\Delta k$; $T(k,t)\Delta k$ then represents the redistribution of energy over the spectrum by inertial effects.

The details of the definitions and calculations of the terms obviously depend upon the dimension. The experimental determination, in particular, of the bispectral *tensor* in even isotropic turbulence is forbidding.[7–10] It is little easier computationally.[11,12] As the representation of $\langle u(x)u(x+r_1)u(x+r_2)\rangle$ in wave-number space, an extensive measurement/computational program of $B(k,l)$ would have to be undertaken for even this most symmetric of homogeneous turbulent flows. For this reason, considerations in the literature have been limited almost exclusively to bispectra of longitudinal velocity components separated only in longitudinal directions.



The bispectra presented here are also one dimensional, being determined from exact solutions to the Burgers equation for random Thomas initial conditions.[13] An algorithm has been developed for the numerical evaluation of exact solutions, moreover, which is both accurate and efficient.[6,14] Thus our velocities $u(x,t)$ are accurate to nine or ten decimal places, and we routinely compute 50 realizations per ensemble. This enables us to have a degree of confidence in our bispectra that would be difficult for counterparts, either from experimental data or from computations based on direct numerical simulations.

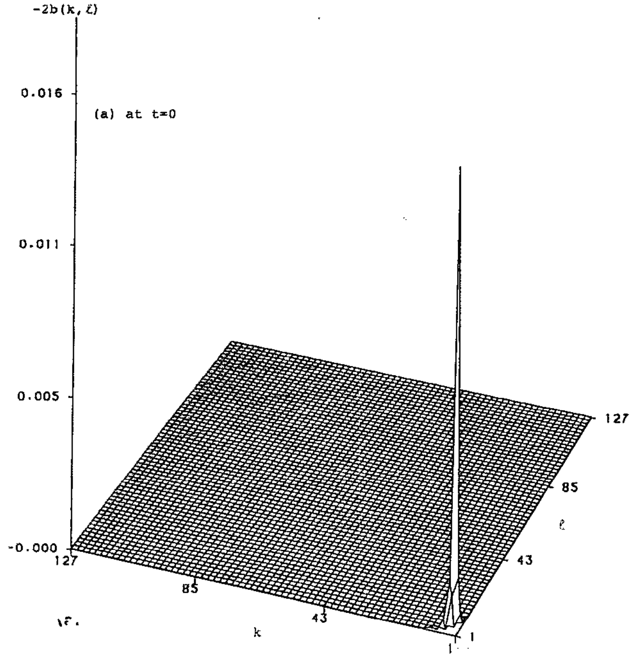
Our computational approach to bispectra takes into account concerns and precautionary measures expressed by earlier workers, in particular the averaging over wave-number squares emphasized by Lii *et al.*[7] The bispectrum $B(k,l)$ has been both ensemble averaged over 50 realizations and locally averaged in the wave-number domain. With $2^{16}$ evaluation points in the spatial domain, and with local averages over $64\times64$ point squares, our data points for $B(k,l)$ are for values of $k$, $l$ satisfying

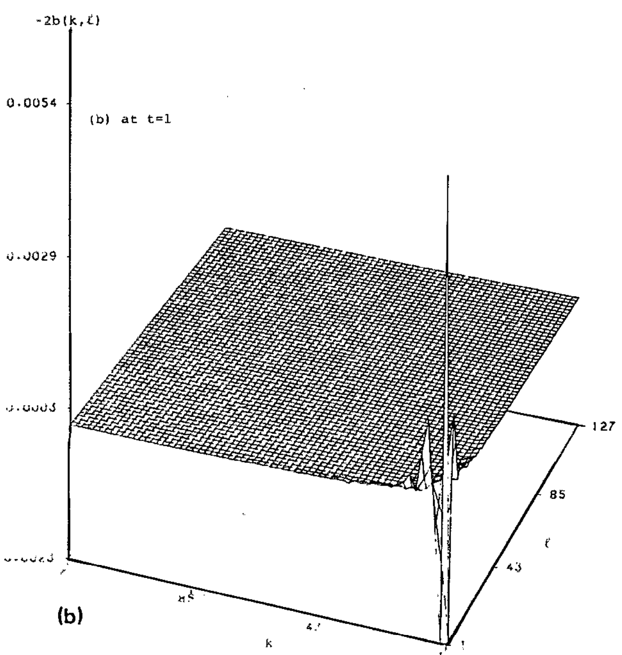
$$1\lesssim k,\ l\lesssim 2^{16}\pi/200\simeq 1029.$$

For reasons that are evident from the plots, our three-dimensional (3-D) perspectives are shown for a reduced domain. All computations have been carried out for a turbulence Reynolds number of $\mathrm{Re}_0(=u_0l_0/\nu)=400$ to provide comparison with energy spectra and spectral transfer for modified Thomas initial conditions.[6]

The bispectrum is complex, say $B(k,l)=a(k,l)+ib(k,l)$, but only $b(k,l)$ makes a contribution to spectral



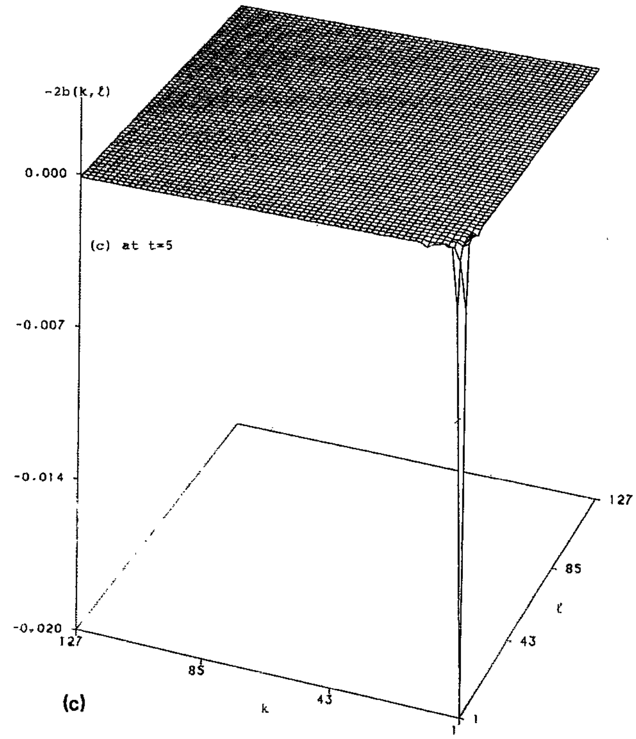




FIG. 1. Three-dimensional perspective plots of $B(k,l)-B^*(k,l) = -2b(k,l)$: (a) at $t=0$, (b) at $t=1$.

transfer [Eq. (1)], viz., $-2b(k,l)$. This contribution is shown in Figs. 1(a) and 1(b) for $t=0,1$.

At $t=0$, there is a pronounced spike, with $-2b>0$, for small values of $k,l$.

At $t=1$, there is a much smaller positive spike for similar values of $k,l$, but now there are negative spikes symmetrically located about $k=l$, just outside that region. Still further out, there are smaller amplitude positive spikes as the local surface resembles a decaying oscillation for fixed $k \gtrsim 1$ as $l$ increases and for fixed $l \gtrsim 1$ as $k$ increases. This behavior contrasts with those for both $t=0$ and $t=5$. The striking difference is peculiar to Thomas initial conditions,[13] for which characteristic length and velocity scales both are identically unity, so that a typical "turnover" time is also of order unity.

At $t=5$, the $(k,l)$ neighborhood of the origin has a dominant *negative* spike, comparable in all other respects to the *positive* spike at $t=0$. The figure is omitted for brevity.

In physical terms, there is an initial increase in the energy spectrum for large spatial scales due to the random crosscut sawtooth initial conditions (see the initial velocity profiles in Shih and Reed[13]). This is essentially an inviscid phenomenon as shocks are formed for all initial segments with negative slopes, all of which represent large scales at $t=0$. This initial energy *increase* at low wave numbers from low-wave-number interactions gives way by $t=5$ to a *decrease* at low wave numbers due to low wave-number interactions. In between, there is a nonlinear "scrambling," with a portion of it representing local transfer $(k \simeq l \simeq 1)$

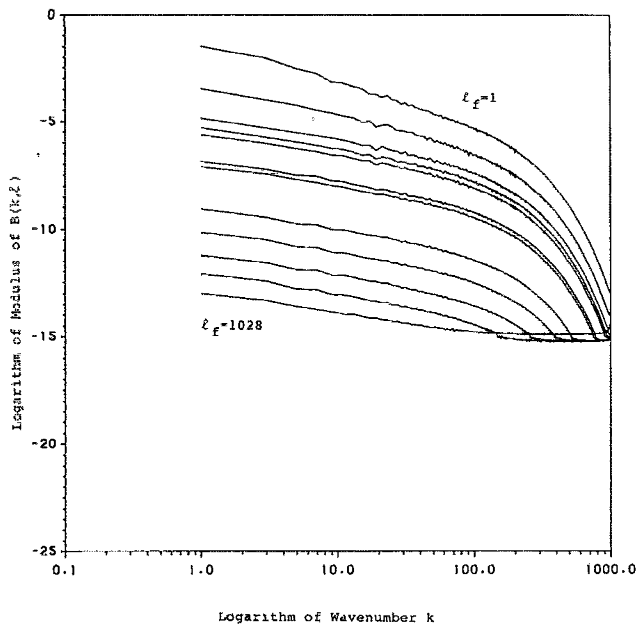



FIG. 2. Log–log plot of the modulus of $B(k,l_f)$ for sequential values of $l_f$ = 1, 16, 64, 100, 128, 256, 384, 512, 640, 768, 876, 1028, at $t = 1$.

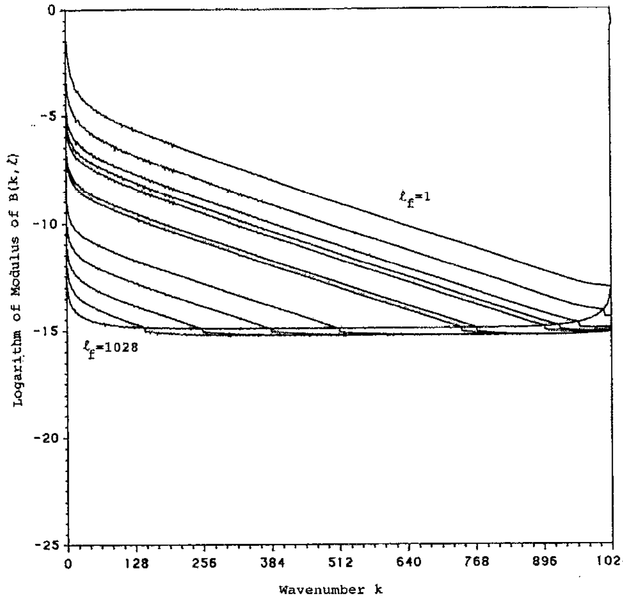



FIG. 3. Log–semi plot of the modulus of $B(k,l_f)$ for sequential values of $l_f$ = 1, 16, 64, 100, 128, 256, 384, 512, 640, 768, 876, 1028, at $t = 1$.

and other parts representing nonlocal transfer [$k \simeq 1, l$ increasing from 1; $l \simeq 1$, $k$ increasing from 1; $B(k,l)$ is symmetric about $k = l$]. The large "eddies" initially become larger, on average, but ultimately their energy starts to decay. That decay is dominated by inviscid mechanisms.[15,16] At a typical "eddy turnover" time, there are varied nonlinear interactions, with some modes interacting to contribute negatively, others to contribute positively, to the overall loss in total turbulent energy.

The interacting triads $k$, $l$, and $k + l$ that dominate in $T(k)$ are actually determined by $(k + l)[-2b(k,l)]$, and not by $-2b$ alone. In the wave-number range shown, there are no essential (i.e., no qualitative) differences in the plots of $(k + l)(-2b)$, which are therefore omitted for brevity.

The real part of $B(k,l)$, namely $a(k,l)$, makes no direct contribution to spectral transfer. It is nevertheless neither identically zero nor negligibly small when compared to $b(k,l)$.

These two observations, as important as they are, overlook a most important role for $a(k,l)$. If we express $B(k,l)$ in terms of its modulus $\sqrt{(a^2 + b^2)}$ and phase $\tan^{-1}(b/a)$, then dramatic new insights come to light. These are especially relevant to three-dimensional turbulence because $a(k,l)$ may be of smaller than as well as the same order of magnitude as $b(k,l)$, suggesting that it may not be a matter of $a(k,l)$ being unresolvable.[3,10]

As but two examples will show, reanalyzing some of the longitudinal bispectra for three-dimensional turbulence could yield overlooked relationships. For instance, if we fix $l$ at the successive values of 1, 16, 64, 100, 128, 256, 384, 512, 640, 768, 876, 1028, then a sequence of straight line sections with slopes $\simeq -5/4$ occurs on log–log plots of modulus of $B(k,l = l_f)$, with $l_f$ the specific fixed value of $l$. This may be seen for $t = 1$ in Fig. 2. This subrange may be associated with the well-known $k^{-2}$ inertial subrange of the energy spectrum.

In the high wave-number range of the energy spectrum, straight lines occur on log–semi plots, indicating that the viscous cutoff is exponential,[6] with $\alpha = \alpha(t)$ in $\exp(-\alpha k)$. In an associated wave-number range, log–semi plots of modulus of $B(k,l_f)$ also exhibit straight lines, indictating that modulus of $B(k,l_f) \sim \exp[-\gamma(t)k]$. This can be seen in Fig. 3. The slope $\gamma$ does not depend upon $l_f$.

We are presently engaged in a much larger study of the evolution of several statistical properties related to the bispectral ones shown or mentioned here. The insights strongly suggest, however, that valuable insights into the behavior of interacting wave-number triads in three-dimensional turbulence could be gained from reanalyzing earlier measurements.



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